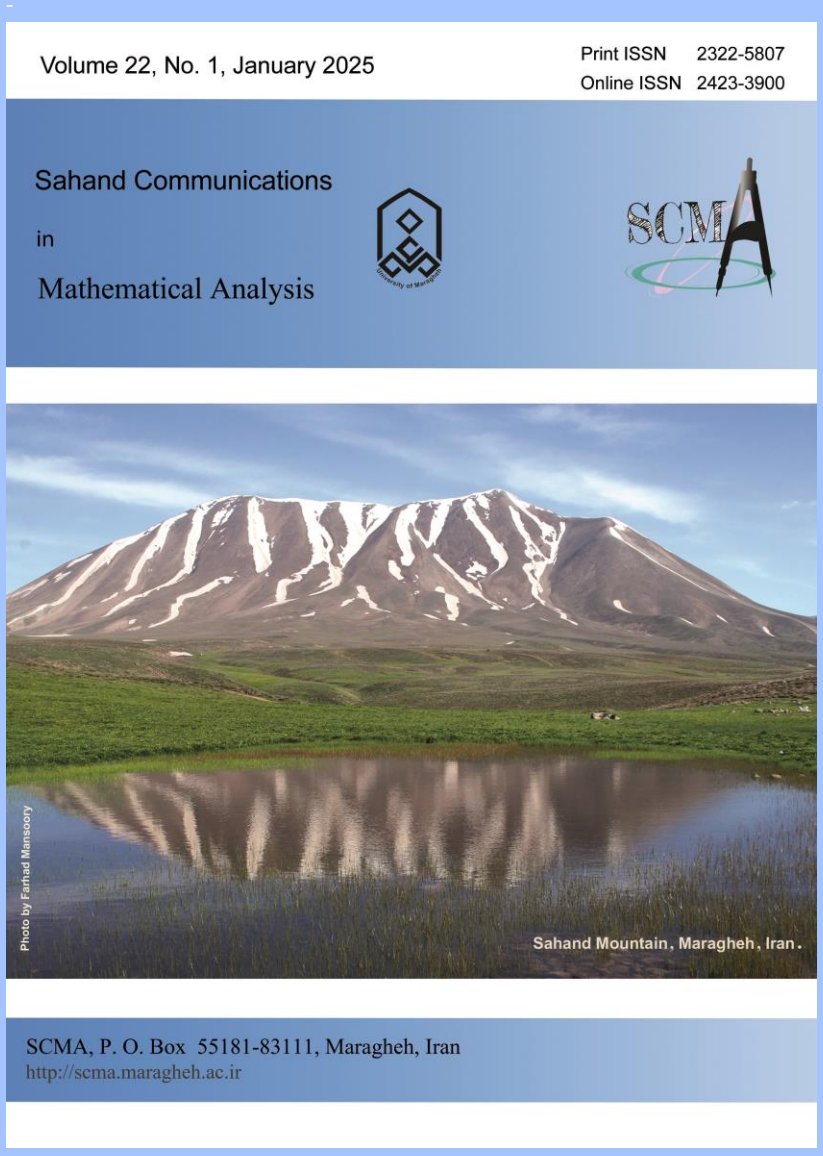
Essential Norm of the Weighted Composition Operators Between Growth Space

Ebrahim Abbasi and Mostafa Hassanlou

Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 22
Number: 1
Pages: 137-147

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2024.2026204.1694



Sahand Communications in Mathematical Analysis (SCMA) Vol. Vol. 22 No. 1 (2025), 137-147
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2024.2026204.1694

Essential Norm of the Weighted Composition Operators Between Growth Space

Ebrahim Abbasi[1,*] and Mostafa Hassanlou[2]

Abstract. For $\alpha > 0$, the growth space $\mathcal{A}^{-\alpha}$ is the space of all function $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{A}^{-\alpha}} = \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{\alpha} |f(z)| < \infty.$$

In this work, we obtain exact formula for the norm of weighted composition operators from $\mathcal{A}^{-\alpha}$ into $\mathcal{A}^{-\beta}$. Especially, we show that

$$\|uC_\varphi\|_{\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}} = \sup_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)^{\beta}|u(z)|}{\left(1-|\varphi(z)|^2\right)^{\alpha}}.$$

As a corollary, we show that $C_\varphi : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\alpha}$ is isometry if and only if φ is rotation. Then the exact formula for the essential norm $uC_\varphi : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\beta}$ is given as follow

$$\|uC_\varphi\|_{e,\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}} = \left(\frac{e}{2\alpha}\right)^{\alpha} \limsup n^{\alpha} \left\|u\varphi^{n-1}\right\|_{\mathcal{A}^{-\beta}}.$$

Also, some equivalence conditions for compactness of such operators operator between difference growth spaces are given.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane, $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$ and $S(\mathbb{D})$ the set of all analytic self-maps of $\mathbb{D}$. For $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$, the weighted composition operator induced by u and φ, uC_φ is the operator acting on functions f with domain $\mathbb{D}$ defined as

$$uC_\varphi f = u\,(f \circ \varphi)\,.$$

2020 *Mathematics Subject Classification.* 47B38,47B33, 30H99 .
Key words and phrases. Essential norm, Growth space, Isometry, Norm.
Received: 2024-04-08, Accepted: 2024-07-14.
* Corresponding author.

When $u \equiv 1$, we get the composition operator C_φ, and for $\varphi(z) = z$, we obtain the multiplication operator M_u. The main problem concerning weighted composition operators is to study the properties of the operators acting on some classes of analytic function spaces in terms of properties of the functions u and φ. For some research on this area, see, for example, [1–3, 6–8, 14]. Another area in this subject is to study the topological structure of the space of these operators and as a starting point, the difference or linear combination of elements of the space is considered. This line of research was first started in the setting of Hardy spaces by Berkson and Shapiro and Sundberg (see [10]). After that, such related problems were studied in several analytic function spaces, see [4, 11, 12] and references therein.

Let $\alpha > 0$. The growth space $\mathcal{A}^{-\alpha}$ consists of all function $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{A}^{-\alpha}} = \sup_{z\in\mathbb{D}} \left(1 - |z|^2\right)^\alpha |f(z)| < \infty.$$

This space is Banach with norm $\|\cdot\|_{\mathcal{A}^{-\alpha}}$. The little growth space $\mathcal{A}_0^{-\alpha}$, a closed subspace of $\mathcal{A}^{-\alpha}$, contains all $f \in \mathcal{A}^{-\alpha}$ such that

$$\lim_{|z|\to 1} \left(1 - |z|^2\right)^\alpha |f(z)| = 0.$$

Let $\alpha > 0$. The Bloch type space $\mathcal{B}^\alpha$ consists of all function $f \in H(\mathbb{D})$ such that

$$\|f\|_{s,\mathcal{B}^\alpha} = \sup_{z\in\mathbb{D}} \left(1 - |z|^2\right)^\alpha \left|f'(z)\right| < \infty.$$

This space Banach with the norm $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \|f\|_{s,\mathcal{B}^\alpha}$. More information about growth spaces or Bloch-type spaces can be found in [15].

Let X and Y be normed linear spaces. The linear operator $T : X \to Y$ is called an isometry if for every $x \in X$,

$$\|Tx\|_Y = \|x\|_X .$$

For continuous linear operator T, from Banach spaces X into Banach spaces Y, the essential norm is the distance of T from the space of all compact operators, that is

$$\|T\|_{e,X\to Y} = \inf \{\|T - K\| : K : X \to Y \text{ is compact}\} .$$

So, $\|T\|_{e,X\to Y} = 0$ if and only if T is compact. For more information about essential norm see[1, 5, 9, 13]

In Section 2, we investigate the boundedness of weighted composition operators between difference growth spaces and we obtain the exact

formula for the norm of these operators as follow

$$\|uC_\varphi\|_{\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}} = \sup_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)^\beta |u(z)|}{\left(1-|\varphi(z)|^2\right)^\alpha}.$$

Then, we consider composition isometry from $\mathcal{A}^{-\alpha}$ into $\mathcal{A}^{-\beta}$.

In [13], Zhao proved that the essential norm of composition operators between Bloch-type spaces can be expressed as

$$\|C_\varphi\|_e = \left(\frac{e}{2\alpha}\right)^\alpha \limsup_{n\to\infty} n^{\alpha-1} \|\varphi^n\|_{\mathcal{B}^\beta}.$$

In Section 3, we find the exact formula for the essential norm $uC_\varphi : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\beta}$ as follows

$$\|uC_\varphi\|_{e,\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}} = \left(\frac{e}{2\alpha}\right)^\alpha \limsup n^\alpha \left\|u\varphi^{n-1}\right\|_{\mathcal{A}^{-\beta}}.$$

Using the above formula, we present some equivalence conditions for the compactness of such operators.

2. Boundedness

In this section, we obtain exact formula for the norm of weighted composition operators from $\mathcal{A}^{-\alpha}$ into $\mathcal{A}^{-\beta}$. Then we consider composition isometry on growth spaces.

Theorem 2.1. *Let $\alpha, \beta > 0$, $\varphi \in S(\mathbb{D})$ and $u \in H(\mathbb{D})$, then*

$$\|uC_\varphi\|_{\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}} = \sup_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)^\beta |u(z)|}{\left(1-|\varphi(z)|^2\right)^\alpha}. \tag{2.1}$$

Proof. For any $f \in \mathcal{A}^{-\alpha}$, we have

$$\begin{aligned}
\|uC_\varphi(f)\|_{\mathcal{A}^{-\beta}} &= \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\beta |u(z)|\,|f(\varphi(z))| \\
&= \sup_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)^\beta |u(z)|}{\left(1-|\varphi(z)|^2\right)^\alpha} \left(1-|\varphi(z)|^2\right)^\alpha |f(\varphi(z))| \\
&\le \sup_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)^\beta |u(z)|}{\left(1-|\varphi(z)|^2\right)^\alpha} \|f\|_{\mathcal{A}^{-\alpha}}.
\end{aligned}$$

So,

$$\|uC_\varphi\|_{\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}} \leq \sup_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)^\beta |u(z)|}{\left(1-|\varphi(z)|^2\right)^\alpha}.$$

For any $w \in \mathbb{D}$, we set $g_w(z) = (\psi_w')^\alpha(z)$, where $\psi_w(z) = \frac{w-z}{1-\overline{w}z}$. Hence,

$$\begin{aligned}\|g_w\|_{\mathcal{A}^{-\alpha}} &= \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left|\left(\psi_w'\right)(z)\right|^\alpha \\ &= \sup_{z\in\mathbb{D}} \left(1-|\psi_w(z)|^2\right)^\alpha \\ &= 1.\end{aligned}$$

For each $w, \eta \in \mathbb{D}$, we get

$$\begin{aligned}\|uC_\varphi\|_{\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}} &\geq \sup_{w\in\mathbb{D}} \|uC_\varphi(g_w)\|_{\mathcal{A}^{-\beta}} \\ &= \sup_{w\in\mathbb{D}}\sup_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)^\beta |u(z)|}{\left(1-|\varphi(z)|^2\right)^\alpha} \left(1-|\psi_w(\varphi(z))|^2\right)^\alpha \\ &\geq \sup_{w\in\mathbb{D}} \frac{\left(1-|\eta|^2\right)^\beta |u(\eta)|}{\left(1-|\varphi(\eta)|^2\right)^\alpha} \left(1-|\psi_w(\varphi(\eta))|^2\right)^\alpha \\ &\geq \frac{\left(1-|\eta|^2\right)^\beta |u(\eta)|}{\left(1-|\varphi(\eta)|^2\right)^\alpha}.\end{aligned}$$

Taking the supremum over all $\eta \in \mathbb{D}$, we obtain

$$\sup_{\eta\in\mathbb{D}} \frac{\left(1-|\eta|^2\right)^\beta |u(\eta)|}{\left(1-|\varphi(\eta)|^2\right)^\alpha} \leq \|uC_\varphi\|_{H^\infty_\alpha\to\mathcal{A}^{-\beta}}.$$

The proof is completed. □

Putting $u \equiv 1$ in the previous theorem, we obtain the following corollary.

Corollary 2.2. *Let $\alpha, \beta > 0$ and $\varphi \in S(\mathbb{D})$. Then*

$$\|C_\varphi\|_{\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}} = \sup_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)^\beta}{\left(1-|\varphi(z)|^2\right)^\alpha}. \tag{2.2}$$

Corollary 2.3. *Let $\alpha, \beta > 0$ and $\varphi \in S(\mathbb{D})$.*

(i) If $\varphi(0) \neq 0$, *then* $\|C_\varphi\|_{\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}} > 1$.
(ii) If $\|C_\varphi\|_{\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\alpha}} \leq 1$, *then* $|\varphi(z)| \leq |z|$.

Proof. (i) Applying (2.2), we get

$$\|C_\varphi\|_{\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}} \geq \frac{1}{\left(1-|\varphi(0)|^2\right)^\alpha} > 1.$$

(ii) Applying (2.2) for $\alpha = \beta$, we have $\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^\alpha \leq 1$, therefore

$$1-|z|^2 \leq 1-|\varphi(z)|^2,$$

so $|\varphi(z)| \leq |z|$. □

Corollary 2.4. *Let* $\alpha > 0$. *Then the operator* $C_\varphi : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\alpha}$ *is isometry if and only if* φ *is rotation.*

Proof. When φ is rotation, it is clear, C_φ is isometry on $\mathcal{A}^{-\alpha}$. Conversely, we assume $C_\varphi : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\alpha}$ be an isometry. Hence,

$$\begin{aligned}\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha|\varphi(z)| &= \|C_\varphi(z)\|_{\mathcal{A}^{-\alpha}}\\ &= \|z\|_{\mathcal{A}^{-\alpha}}\\ &= \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha|z|.\end{aligned}$$

Since by previous corollary, $|\varphi(z)| \leq |z|$, we obtain

$$\lim_{|z|\to 1}\left(1-|z|^2\right)^\alpha|\varphi(z)| \leq \lim_{|z|\to 1}\left(1-|z|^2\right)^\alpha|z| = 0.$$

Therefore, there exists $0 \neq z_0 \in \mathbb{D}$ such that

$$\begin{aligned}\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha|\varphi(z)| &= \left(1-|z_0|^2\right)^\alpha|\varphi(z_0)|\\ &\leq \left(1-|z_0|^2\right)^\alpha|z_0|\\ &\leq \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha|z|.\end{aligned}$$

So, $|\varphi(z_0)| = |z_0|$, therefore φ must be rotation. □

In the following example, we show that there is no composition isometry from $\mathcal{A}^{-\alpha}$ into $\mathcal{A}^{-\beta}$, when $\varphi(z)$ is rotation and $\alpha \neq \beta$.

Example 2.5. Let $\alpha, \beta > 0$, $\alpha \neq \beta$ and $\varphi(z)$ is rotation. Then there is no composition isometry from $\mathcal{A}^{-\alpha}$ into $\mathcal{A}^{-\beta}$.

Proof. When $\alpha > \beta$ by using (2.2),

$$\begin{aligned}\|C_\varphi\|_{\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}} &= \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{\beta-\alpha}\\ &= \infty.\end{aligned}$$

For case $\beta > \alpha$, $\|C_\varphi\|_{\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}} = 1$, but we show $C_\varphi : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\beta}$ is not isometry. Let $C_\varphi : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\beta}$ be isometry, so for any $n \in \mathbb{N}$, we have

$$\begin{aligned}\|z^n\|_{\mathcal{B}^\beta} &= n\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{\beta}|z|^{n-1}\\ &= n\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{\beta}|\varphi(z)|^{n-1}\\ &= \left\|C_\varphi\left(nz^{n-1}\right)\right\|_{\mathcal{A}^{-\beta}}\\ &= \left\|nz^{n-1}\right\|_{\mathcal{A}^{-\alpha}}\\ &= n\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{\alpha}|z|^{n-1}\\ &= \|z^n\|_{\mathcal{B}^\alpha}.\end{aligned}$$

From equation (2.3) of [13], we have $\lim_{n\to\infty} n^{\alpha-1}\|z^n\|_{\mathcal{B}^\alpha} = \left(\frac{2\alpha}{e}\right)^\alpha$. Therefore,

$$\begin{aligned}\left(\frac{2\beta}{e}\right)^\beta &= \lim_{n\to\infty} n^{\beta-1}\|z^n\|_{\mathcal{B}^\beta}\\ &= \lim_{n\to\infty} n^{\beta-\alpha}n^{\alpha-1}\|z^n\|_{\mathcal{B}^\alpha}\\ &= \infty\times\left(\frac{2\alpha}{e}\right)^\alpha\\ &= \infty.\end{aligned}$$

We have reached a contradiction. □

3. Essential norm

In this section, we find the exact formula for the norm of weighted composition operators from $\mathcal{A}^{-\alpha}$ into $\mathcal{A}^{-\beta}$. Applying the formula obtained, we give some equivalence conditions for the compactness of operator uC_φ between different growth spaces. Firstly, we begin with the following theorem.

We get the following theorem by putting $\nu(z) = \left(1-|z|^2\right)^\alpha$ in Proposition 2.1 of [9].

Theorem 3.1. *There exists a sequence L_n of compact operators on $\mathcal{A}_0^\alpha$ such that*

(i) *For any* $0 \leq t < 1$, *we have* $\lim_{n\to\infty} \sup_{\|f\|_{\mathcal{A}^{-\alpha}}\leq 1} \sup_{|z|\leq t} |(I - L_n) f(z)| = 0$

(ii) $\limsup_{n\to\infty} \|I - L_n\| \leq 1$.

Furthermore, the same is true for the sequence of biadjoints $L_n^{**} : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\alpha}$.

Lemma 3.2 ([13]). *Let* $\alpha > 0$ *and* $\varphi \in S(\mathbb{D})$.

(i) *The sequence* $\{r_n\} \subset (0,1)$, *defined as* $r_1 = 0$ *and, for* $n \geq 2$, $r_n = \left(\frac{n-1}{n-1+2\alpha}\right)^{1/2}$, *is increasing and* $\lim_{n\to\infty} r_n = 1$.

(ii) *For any* $n \in \mathbb{N}$, *let* $D_n = \{z \in \mathbb{D} : \quad r_n \leq |\varphi(z)| \leq r_{n+1}\}$. *If* $\|\varphi\|_\infty = 1$ *then there exists smallest positive integer* m *such that* $D_m \neq \emptyset$ *and for any* $n \geq m$, $D_n \neq \emptyset$.

(iii) *For each* $z \in D_k$,

$$k^\alpha |\varphi(z)|^{k-1} \left(1 - |\varphi(z)|^2\right)^\alpha \geq \left(\frac{2k\alpha}{k+2\alpha}\right)^\alpha \left(\frac{k}{k+2\alpha}\right)^{\frac{k-1}{2}}.$$

(iv) *For each* $\varepsilon > 0$ *there exists* $N > m+1$ *(with* m *obtained in part* (b)*) such that for any integer* $k \geq N$,

$$\left(\left(\frac{2k\alpha}{k+2\alpha}\right)^\alpha \left(\frac{k}{k+2\alpha}\right)^{\frac{k-1}{2}}\right)^{-1} < \left(\frac{e}{2\alpha}\right)^\alpha + \varepsilon.$$

Theorem 3.3. *Let* $\alpha, \beta > 0$, $\varphi \in S(\mathbb{D})$, $u \in H(\mathbb{D})$ *and* $uC_\varphi : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\beta}$ *be bounded. Then*

$$\|uC_\varphi\|_{e,\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}} = \left(\frac{e}{2\alpha}\right)^\alpha \limsup n^\alpha \left\|u\varphi^{n-1}\right\|_{\mathcal{A}^{-\beta}}.$$

Proof. Let $f_n(z) = \frac{z^{n-1}}{\|z^{n-1}\|_{\mathcal{A}^{-\alpha}}}$, hence $\|f_n\|_{\mathcal{A}^{-\alpha}} = 1$ and converge to zero uniformly on compact subsets of $\mathbb{D}$. So for any compact operator $K : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\beta}$, we have

$$\lim_{n\to\infty} \|K(f_n)\|_{\mathcal{A}^{-\beta}} = 0.$$

Therefore,

$$\begin{aligned}
\|uC_\varphi\|_{e,\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}} &\geq \limsup_{n\to\infty} \|(uC_\varphi - K) f_n\|_{\mathcal{A}^{-\beta}} \\
&= \limsup_{n\to\infty} \|uC_\varphi(f_n)\|_{\mathcal{A}^{-\beta}} - \limsup_{n\to\infty} \|K(f_n)\|_{\mathcal{A}^{-\beta}} \\
&= \limsup_{n\to\infty} \frac{\left\|u\varphi^{n-1}\right\|_{\mathcal{A}^{-\beta}}}{\|z^{n-1}\|_{\mathcal{A}^{-\alpha}}} \\
&= \limsup_{n\to\infty} \frac{n^\alpha \left\|u\varphi^{n-1}\right\|_{\mathcal{A}^{-\beta}}}{n^{\alpha-1} \|z^{n-1}\|_{\mathcal{B}^\alpha}}
\end{aligned} \tag{3.1}$$

$$= \frac{\limsup_{n\to\infty} n^{\alpha} \left\|u\varphi^{n-1}\right\|_{\mathcal{A}^{-\beta}}}{\limsup_{n\to\infty} n^{\alpha-1} \left\|z^{n-1}\right\|_{\mathcal{B}^{\alpha}}}$$
$$= \left(\frac{e}{2\alpha}\right)^{\alpha} \limsup n^{\alpha} \left\|u\varphi^{n-1}\right\|_{\mathcal{A}^{-\beta}}.$$

We next prove the reverse inequality. If $\|\varphi\|_{\infty} < 1$ then

$$\|uC_{\varphi}\|_{e,\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}} = 0$$

and we are done, so assume $\|\varphi\|_{\infty} = 1$. Let L_n be the sequence of compact operators in Theorem 3.1. Then

$$\|C_{\varphi}\|_{e,\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}} \le \limsup_{n\to\infty} \|uC_{\varphi} - uC_{\varphi}L_n\|_{\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}}$$
$$= \limsup_{n\to\infty} \sup_{\|f\|_{\mathcal{A}^{-\alpha}}\le 1} \|uC_{\varphi}(I-L_n)f\|_{\mathcal{A}^{-\beta}} \tag{3.2}$$
$$= \limsup_{n\to\infty} \sup_{\|f\|_{\mathcal{A}^{-\alpha}}\le 1} \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{\beta} |u(z)|\,|(I-L_n)\,f(\varphi(z))|$$
$$=: J$$

Fix $\varepsilon > 0$. Then there exist D_k, m and N satisfying the conclusions of Lemma 3.2 such that

$$J \le \limsup_{n\to\infty} j_1 + \limsup_{n\to\infty} j_2,$$

where the quantities j_1 and j_2 dependent on n are defined as

$$j_1 = \sup_{\|f\|_{\mathcal{A}^{-\alpha}}\le 1} \sup_{m\le k\le N-1} \sup_{z\in D_k} \left(1-|z|^2\right)^{\beta} |u(z)|\,|(I-L_n)\,f(\varphi(z))|,$$

$$j_2 = \sup_{\|f\|_{\mathcal{A}^{-\alpha}}\le 1} \sup_{k\ge N} \sup_{z\in D_k} \left(1-|z|^2\right)^{\beta} |u(z)|\,|(I-L_n)\,f(\varphi(z))|.$$

Again applying part (i) of Theorem 3.1, we obtain

$$\limsup_{n\to\infty} j_1 \le \sup_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)^{\beta}|u(z)|}{\left(1-|\varphi(z)|^2\right)^{\alpha}} \limsup_{n\to\infty} \sup_{\|f\|_{\mathcal{A}^{-\alpha}}\le 1} \tag{3.3}$$
$$\times \sup_{r_k\le|\varphi(z)|\le r_{N-1}} \left(1-|\varphi(z)|^2\right)^{\alpha} |(I-L_n)\,f(\varphi(z))|$$
$$\le \sup_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)^{\beta}|u(z)|}{\left(1-|\varphi(z)|^2\right)^{\alpha}} \limsup_{n\to\infty} \sup_{\|f\|_{\mathcal{A}^{-\alpha}}\le 1}$$
$$\times \sup_{|z|\le r_{N-1}} \left(1-|\varphi(z)|^2\right)^{\alpha} |(I-L_n)\,f(\varphi(z))|$$
$$= 0.$$

Using parts (iii) and (iv) of Lemma 3.2, for $k \geq N$ and $z \in D_k$, we have

$$\begin{aligned}
&\left(1-|z|^2\right)^{\beta} |u(z)|\, |(I-L_n)\, f(\varphi(z))| \\
&= \frac{\left(1-|z|^2\right)^{\beta} |u(z)|\, k^{\alpha}\, |\varphi(z)|^{k-1}}{k^{\alpha}\, |\varphi(z)|^{k-1} \left(1-|\varphi(z)|^2\right)^{\alpha}} \left(1-|\varphi(z)|^2\right)^{\alpha} |(I-L_n)\, f(\varphi(z))| \\
&\leq \left(\left(\frac{e}{2\alpha}\right)^{\alpha} + \varepsilon\right) k^{\alpha} \left\|u\varphi^{k-1}\right\|_{\mathcal{A}^{-\beta}} \|(I-L_n)\, f\|_{\mathcal{A}^{-\beta}}\,.
\end{aligned}$$

From part (ii) of Theorem 3.1, it follows that

$$\limsup_{n\to\infty} j_2 \leq \left(\left(\frac{e}{2\alpha}\right)^{\alpha} + \varepsilon\right) \sup_{k\geq N} k^{\alpha} \left\|u\varphi^{k-1}\right\|_{\mathcal{A}^{-\beta}}\,. \tag{3.4}$$

Hence, by applying (3.2), (3.3) and (3.4), we obtain

$$\|C_{\varphi}\|_{e,\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}} \leq \left(\frac{e}{2\alpha}\right)^{\alpha} \limsup_{k\to\infty} k^{\alpha} \left\|u\varphi^{k-1}\right\|_{\mathcal{A}^{-\beta}}\,.$$

The proof is completed. □

By using (3.1), we get the following corollary.

Corollary 3.4. *Let $\alpha, \beta > 0$ and $\varphi \in S(\mathbb{D})$. Then*

$$\|uC_{\varphi}\|_{e,\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}} = \limsup_{n\to\infty} \frac{\|uC_{\varphi}(z^{n-1})\|_{\mathcal{A}^{-\beta}}}{\|z^{n-1}\|_{\mathcal{A}^{-\alpha}}} = \limsup_{n\to\infty} \frac{\|u\varphi^{n-1}\|_{\mathcal{A}^{-\beta}}}{\|z^{n-1}\|_{\mathcal{A}^{-\alpha}}}.$$

Applying Theorem 3.3 and Theorem 2.1 of [9], we obtain the following corollary.

Corollary 3.5. *Let $\alpha, \beta > 0$ and $\varphi \in S(\mathbb{D})$. Then the following conditions are equivalence.*

(a) *The operator $uC_{\varphi} : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\beta}$ is compact.*

(b) $\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^{\beta}|u(z)|}{(1-|\varphi(z)|^2)^{\alpha}} = 0.$

(c) $\lim_{n\to\infty} n^{\alpha} \|u\varphi^{n-1}\|_{H^{\infty}_{\beta}} = 0.$

(d) $\limsup_{n\to\infty} \frac{\|u\varphi^{n-1}\|_{\mathcal{A}^{-\beta}}}{\|z^{n-1}\|_{\mathcal{A}^{-\alpha}}} = 0.$

Remark 3.6. Setting $u(z) \equiv 1$ in Corollary 3.5 and 3.7, we get similar results for $C_{\varphi} : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\beta}$.

The following corollary was proved by R. Zhao (see Theorem 1.1 [13])

Corollary 3.7. *Let $\alpha, \beta > 0$ and $\varphi \in S(\mathbb{D})$. Then*

$$\|C_{\varphi}\|_{e,\mathcal{B}^{\alpha}\to\mathcal{B}^{\beta}} = \left(\frac{e}{2\alpha}\right)^{\alpha} \limsup n^{\alpha-1} \|\varphi^n\|_{s,\mathcal{B}^{\beta}}.$$

Proof. Since $\|C_\varphi\|_{e,\mathcal{B}^\alpha\to\mathcal{B}^\beta} = \|\varphi' C_\varphi\|_{e,\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}}$([9]), by using Theorem 3.3, we have

$$\begin{aligned}
\|C_\varphi\|_{e,\mathcal{B}^\alpha\to\mathcal{B}^\beta} &= \|\varphi' C_\varphi\|_{e,\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}} \\
&= \left(\frac{e}{2\alpha}\right)^\alpha \limsup n^\alpha \|\varphi'\varphi^{n-1}\|_{\mathcal{A}^{-\beta}} \\
&= \left(\frac{e}{2\alpha}\right)^\alpha \limsup n^{\alpha-1} \left(n \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\beta \left|\varphi'(z)\right| \left|\varphi^{n-1}(z)\right| \right) \\
&= \left(\frac{e}{2\alpha}\right)^\alpha \limsup n^{\alpha-1} \|\varphi^n\|_{s,\mathcal{B}^\beta}.
\end{aligned}$$

The proof is completed. □

References

1. E. Abbasi and M. Hassanlou, *Generalized Stević-Sharma type operators on spaces of fractional Cauchy transforms*, Mediterr. J. Math., 21(1) (2024), pp. 40.
2. J.S. Choa and H.O. Kim, *Composition operators from the space of Cauchy transforms into its Hardy-type subspaces*, Rocky Mountain J. Math., 31 (2001), pp. 95–113.
3. J.A. Cima, A. Matheson and W.T. Ross, *The Backward Shift on the Space of Cauchy Transforms*, Proc. Amer. Math. Soc., 132 (3) (2004), pp. 745–754.
4. X. Guo and M. Wang, *Linear combination of composition operators on Cauchy transform type spaces*, Sci. China Math., 50 (2020), pp. 1733–1744.
5. M. Hassanlou, E. Abbasi, M.K. Arpatapeh and S. Nasresfahani, *Product-type operators between minimal Möbius invariant spaces and Zygmund type spaces*, Sahand Commun. Math. Anal., 20(3) (2023), pp. 69–50.
6. R.A. Hibschweiler and T.H. MacGregor, *Fractional Cauchy Transforms*, Chapman and Hall. Boca Raton, FL: CRC, 2006.
7. R.A. Hibschweiler, *Composition operators on spaces of fractional Cauchy transforms*, Complex Anal. Oper. Theory, 6 (2012), pp. 897–911.
8. T.H. MacGregor, *Fractional Cauchy transforms*, J. Comput. Appl. Math., 105 (1-2) (1999), pp. 93–108.
9. A.M. Rodríguez, *Weighted composition operators on weighted Banach spaces of analytic functions*, J. London Math. Soc., 61(3) (2000), pp. 872–884.
10. J. Shapiro and C. Sundberg, *Isolation amongst the composition operators*, Pac. J. Math., 145 (1990), pp. 117-152.

11. A. Sharma, R. Krishan and E. Subhadarsini, *Difference of composition operators from the space of Cauchy integral transforms to Bloch-type spaces*, Integral Transform Spec. Funct., (28) (2017), pp. 145–155.
12. M. Wang and X. Guo, *Difference of differentiation composition operators on the fractional Cauchy transforms spaces*, Num. Func. Anal. Optim., 39 (2018), pp. 1291–1315.
13. R. Zhao, *Essential norms of composition operators between Bloch type spaces*, Proc. Amer. Math. Soc., 138(7) (2010), pp. 2537–2546.
14. X. Zhu, E. Abbasi and D. Molaei, *Weighted composition operators from the Besov space into nth weighted type spaces*, Math. Slovaca, 72(4) (2022), pp. 977–992.
15. K. Zhu, *Bloch Type Spaces of Analytic Functions*, Rocky Mount. J. Math., 23 (1993), pp. 1143–1177.

---

[1]Department of Mathematics, Mahabad Branch, Islamic Azad University, Mahabad, Iran.
Email address: ebrahimabbasi81@gmail.com

[2]Engineering Faculty of Khoy, Urmia University of Technology, Urmia, Iran.
Email address: m.hassanlou@urmia.ac.ir