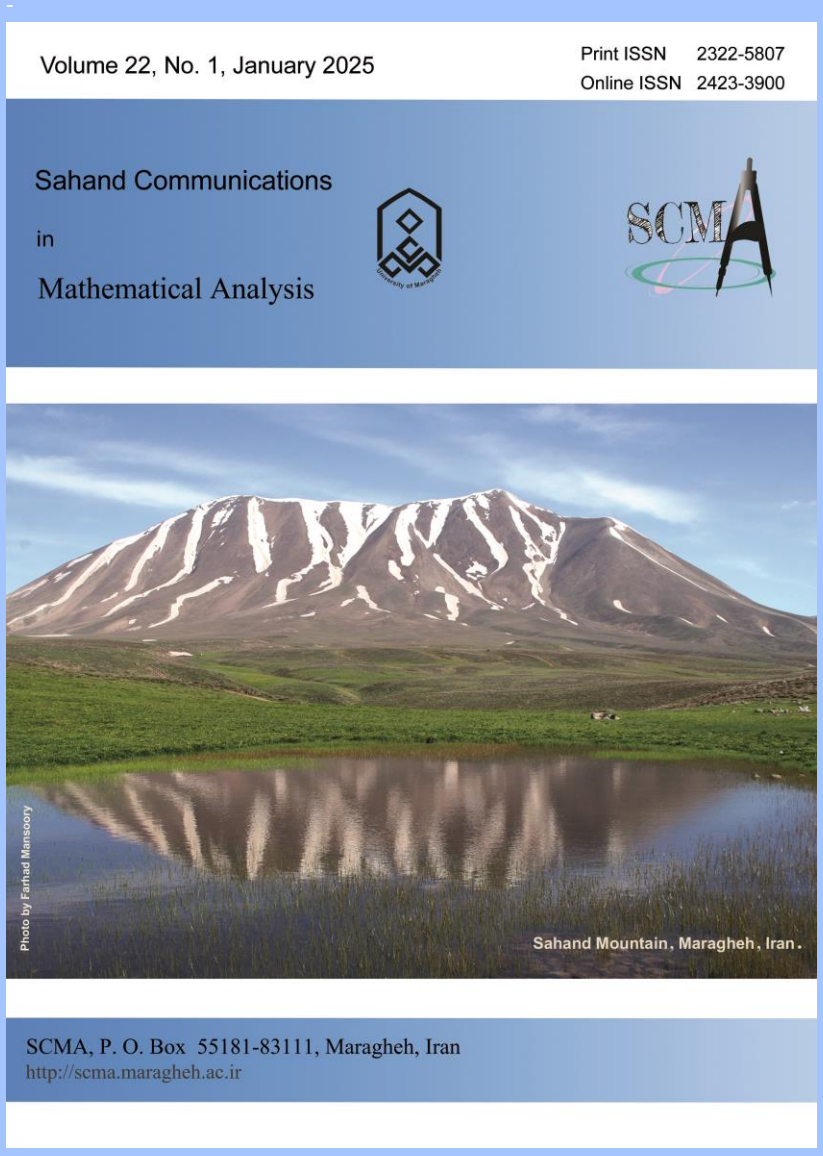
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A Note on Frames in Quaternionic Hilbert Spaces

Sumit Kumar Sharma[1] and Raksha Sharma[2*]

Abstract. In this paper, we define a modified version of frame in quaternionic Hilbert spaces (QHS) and give a necessary condition for a block Bessel sequence in terms of a bounded right linear operator. Also, a necessary and sufficient condition for a sequence to be a block Bessel sequence is given. Further, a method to construct a desired block frame using a given block frame is discussed. Finally, a characterization of block frame in terms of a right linear operator is given.

1. Introduction

Frames were officially defined by Duffin and Schaeffer [11] in 1952 within the framework of non-harmonic Fourier series. After the significant article by Daubechies et al. [10] published in 1986, the potential and utility of frames were understood by the researchers who then developed the theory in various branches of mathematics and other allied topics. Frames offer a stable representation (not necessarily unique) of each vector within the space. Today frames have applications in almost all branches of applied mathematics. The reader may consult [2–4, 8, 13, 14, 16] for further information related to frames and allied concepts.

Throught this paper, we denote $\mathbb{N}$ as the set of natural numbers, $\{M_j\}_{j\in\mathbb{N}}$ as finite subsets of $\mathbb{N}$ with $M_i \cap M_j = \phi$, $i \neq j$ such that $\mathbb{N} = \bigcup_{j=1}^{\infty} M_j$, $\mathfrak{Q}$ as the non-commutative field of quaternions and $\mathbb{H}^R(\mathfrak{Q})$ denote a right quaternionic Hilbert space (QHS). The term right linear

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operator refers to a right $\mathfrak{Q}$-linear operator, and the set of all bounded (right $\mathfrak{Q}$-linear) operators of $\mathbb{H}^R(\mathfrak{Q})$ is denoted by $\mathfrak{B}\left(\mathbb{H}^R(\mathfrak{Q})\right)$.

Quaternions forms a four-dimensional non-commutative real algebra that are generated by the numbers $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are referred to as imaginary units. Since quaternions are an extension of complex number systems, operations on the non-commutative field of quaternionics $\mathfrak{Q}$ are restricted to the complex number system $\mathbb{C}$. Refer [1, 15] for information on various quaternionic operations and properties. For additional readings on various other related ideas in quaternionic Hilbert spaces (QHS) refer [5–7, 9, 15].

Definition 1.1 ([15])**.** A right quaternionic pre-Hilbert space or right quaternionic inner product space $\mathbb{V}_R(\mathfrak{Q})$ is a right quaternionic vector space together with the binary mapping $\langle .|. \rangle : \mathbb{V}_R(\mathfrak{Q}) \times \mathbb{V}_R(\mathfrak{Q}) \to \mathfrak{Q}$ (called the Hermitian quaternionic inner product) which satisfies the following properties:

(a) $\overline{\langle x_1|x_2\rangle} = \langle x_2|x_1\rangle$, for all $x_1, x_2 \in \mathbb{V}_R(\mathfrak{Q})$.
(b) $\langle x|x\rangle > 0$, if $x \neq 0$.
(c) $\langle x|x_1 + x_2\rangle = \langle x|x_1\rangle + \langle x|x_2\rangle$, for all $x, x_1, x_2 \in \mathbb{V}_R(\mathfrak{Q})$.
(d) $\langle x|yq\rangle = \langle x|y\rangle q$, for all $x, y \in \mathbb{V}_R(\mathfrak{Q})$ and $q \in \mathfrak{Q}$.

Define a function $\|.\|$ as $\|u\| = \sqrt{\langle u|u\rangle}$, for all $u \in \mathbb{V}_R(\mathfrak{Q})$. Then $\|.\|$ forms a norm on $\mathbb{V}_R(\mathfrak{Q})$. Also, if $\mathbb{V}_R(\mathfrak{Q})$ is complete under this norm, then it is called a right quaternionic Hilbert space and is denoted by $\mathbb{H}^R(\mathfrak{Q})$.

Frames for separable right quaternionic Hilbert spaces were introduced and studied in [19].

Definition 1.2 ([19])**.** Let $\mathbb{H}^R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in\mathcal{I}}$ be a sequence in $\mathbb{H}^R(\mathfrak{Q})$. Then $\{u_i\}_{i\in\mathcal{I}}$ is said to be a frame for $\mathbb{H}^R(\mathfrak{Q})$, if there exist two finite real constants r_1 and r_2 with $0 < r_1 \leq r_2$ such that

$$r_1\|u\|^2 \leq \sum_{i\in\mathcal{I}} |\langle u_i|u\rangle|^2 \leq r_2\|u\|^2, \quad \text{for all } u \in \mathbb{H}^R(\mathfrak{Q}). \tag{1.1}$$

Frame inequality is the term used to describe the inequality (1.1). Also, the constants r_1 and r_2 are termed as lower and upper frame bounds of the frame $\{u_i\}_{i\in\mathcal{I}}$. In case the sequence $\{u_i\}_{i\in\mathcal{I}}$ satisfies only the upper inequality in (1.1), then it is termed as a Bessel sequence.

Discrete frames for finite dimensional left quaternionic Hilbert spaces were first studied by Khokulan, Thirulogasanthar and Srisatkunarajah [17]. After that lots of work on the theory of discrete frames for quaternionic Hilbert spaces (QHS) have been done by many researchers. For instance, one may refer [19, 20] for dual frames and other related concepts in QHS. Various properties of K- frames in QHS has been studied

in [12] and woven frames were studied in [18]. For aditional information on frames and Riesz bases in QHS one may refer [19–21]. In this note, we define a modified version of frame termed as block frame in a quaternionic Hilbert space and give examples to support its existence. Also, we prove a result related to its existence and give a condition for a sequence to be a block Bessel sequence. Further, construction of a block frame from a given block frame is discussed. Finally, we discuss Parseval block frames in a right quaternionic Hilbert space and give a characterization of block frame using a right linear operator.

2. Main Results

In this section, we shall study a modified version of frames called block frames in quaternionic Hilbert spaces. We begin with the following definition:

Definition 2.1. Let $\{M_j\}_{j\in\mathbb{N}}$ be finite subsets of $\mathbb{N}$ with $M_i \cap M_j = \phi$, $i \neq j$ such that $\mathbb{N} = \bigcup_{j=1}^{\infty} M_j$, $\{x_{ij}\}_{i\in M_j, j\in\mathbb{N}}$ be a block sequence of vectors in $\mathbb{H}^R(\mathfrak{Q})$ and $\{\mathfrak{q}_{ij}\}_{i\in M_j, j\in\mathbb{N}}$ be a block sequence of quaternions. Then, $\{x_{ij}\}_{i\in M_j, j\in\mathbb{N}}$ is said to be a block frame with respect to $\big(\{M_j\}_{j\in\mathbb{N}}; \{\mathfrak{q}_{ij}\}_{i\in M_j, j\in\mathbb{N}}\big)$ if there exist real constants r_1, r_2 $(0 < r_1 \leq r_2 < \infty)$ such that

$$r_1\|x\|^2 \leq \sum_{j\in\mathbb{N}} \sum_{i\in M_j} |\mathfrak{q}_{ij}|^2 |\langle x_{ij}|x\rangle|^2 \leq r_2\|x\|^2, \quad \text{for all } \ x \in \mathbb{H}^R(\mathfrak{Q}). \tag{2.1}$$

If $r_1 = r_2$, then $\{x_{ij}\}_{i\in M_j, j\in\mathbb{N}}$ is called a tight block frame w.r.t. $\big(\{M_j\}_{j\in\mathbb{N}}; \{\mathfrak{q}_{ij}\}_{i\in M_j, j\in\mathbb{N}}\big)$ and if $r_1 = r_2 = 1$, then $\{x_{ij}\}_{i\in M_j, j\in\mathbb{N}}$ is called a Parseval block frame w.r.t. $\big(\{M_j\}_{j\in\mathbb{N}}; \{\mathfrak{q}_{ij}\}_{i\in M_j, j\in\mathbb{N}}\big)$. If only the right side inequality in (2.1) holds, then we say that $\{x_{ij}\}_{i\in M_j, j\in\mathbb{N}}$ form a block Bessel sequence for $\mathbb{H}^R(\mathfrak{Q})$ w.r.t. $\big(\{M_j\}_{j\in\mathbb{N}}; \{\mathfrak{q}_{ij}\}_{i\in M_j, j\in\mathbb{N}}\big)$.

In the following example, we observe that each frame in a quaternionic Hilbert space is also a block frame.

Example 2.2. Let $\mathbb{H}^R(\mathfrak{Q})$ be a quaternionic Hilbert space and $\{x_j\}_{j\in\mathbb{N}}$ be a frame for $\mathbb{H}^R(\mathfrak{Q})$. For each $j \in \mathbb{N}$, let $M_j = \{j\}$. Then $\{x_{ij}\}_{i\in M_j, j\in\mathbb{N}} = \{x_{jj}\}_{j\in\mathbb{N}} = \{x_j\}_{j\in\mathbb{N}}$. Choose $\{\mathfrak{q}_{ij}\}_{i\in M_j, j\in\mathbb{N}} = \{\mathfrak{q}_{jj}\}_{j\in\mathbb{N}} = \{\mathfrak{q}_j = \mathfrak{q}\}_{j\in\mathbb{N}}$, where $\mathfrak{q}$ is a non-zero quaternionic. Then $\{x_{ij}\}_{i\in M_j, j\in\mathbb{N}}$ is a block frame w.r.t. $\big(\{M_j\}_{j\in\mathbb{N}}; \{\mathfrak{q}_{ij}\}_{i\in M_j, j\in\mathbb{N}}\big)$. □

However, it is important to note that a block frame does not necessarily constitute a frame, as demonstrated in the following example:

Example 2.3. Let $\mathbb{H}^R(\mathfrak{Q})$ be a quaternionic Hilbert space and $\{e_j\}_{j=1}^{\infty}$ be the standard orthonormal basis for $\mathbb{H}^R(\mathfrak{Q})$. For each $j \in \mathbb{N}$, let $M_j = \{j\}$. Then $\{x_{ij}\}_{i\in M_j, j\in\mathbb{N}} = \{x_{jj}\}_{j\in\mathbb{N}} = \left\{\frac{e_j}{j}\right\}_{j\in\mathbb{N}}$. Choose $\{\mathfrak{q}_{ij}\}_{i\in M_j, j\in\mathbb{N}} = \{\mathfrak{q}_{jj}\}_{j\in\mathbb{N}} = \left\{\frac{1}{\sqrt{j}}\right\}_{j\in\mathbb{N}}$. Then $\{x_{ij}\}_{i\in M_j, j\in\mathbb{N}}$ is not a block frame w.r.t. $\left(\{M_j\}_{j\in\mathbb{N}}; \{\mathfrak{q}_{ij}\}_{i\in M_j, j\in\mathbb{N}}\right)$ as it does not satisfy the lower frame inequality. □

Let $\{M_j\}_{j\in\mathbb{N}}$ be a sequence of finite subsets of $\mathbb{N}$ with $M_i \cap M_j = \phi$, $i \neq j$ such that $\mathbb{N} = \bigcup_{j=1}^{\infty} M_j$ and let $\{e_j\}_{j=1}^{\infty}$ be the standard orthonormal basis of $\ell^2(\mathfrak{Q})$. For each $j \in \mathbb{N}$, define

$$\begin{aligned}\ell^2_{M_j} &= \left\{\{\mathfrak{q}_i\}_{i\in M_j} : \mathfrak{q}_i \in \mathfrak{Q}, j \in \mathbb{N}\right\} \\ &= \left\{\sum_{i\in M_j} e_i\mathfrak{q}_i : \mathfrak{q}_i \in \mathfrak{Q}, j \in \mathbb{N}\right\}\end{aligned}$$

and

$$\left(\sum_{j\in\mathbb{N}} \oplus \ell^2_{M_j}\right)_{\ell^2(\mathfrak{Q})} = \left\{\{\mathfrak{q}_{ij}\}_{i\in M_j, j\in\mathbb{N}} : \sum_{j\in\mathbb{N}}\sum_{i\in M_j} |\mathfrak{q}_{ij}|^2 < \infty\right\}.$$

For each $j \in \mathbb{N}$, $\ell^2_{M_j}$ is a subset of $\ell^2(\mathfrak{Q})$. Define an inner product on $\left(\sum_{j\in\mathbb{N}} \oplus \ell^2_{M_j}\right)_{\ell^2(\mathfrak{Q})}$ as $\left\langle \{\mathfrak{q}_{ij}\}_{i\in M_j, j\in\mathbb{N}} \middle| \{\mathfrak{r}_{ij}\}_{i\in M_j, j\in\mathbb{N}} \right\rangle = \sum_{j\in\mathbb{N}}\sum_{i\in M_j} \langle \mathfrak{q}_{ij} | \mathfrak{r}_{ij} \rangle$, where $\{\mathfrak{q}_{ij}\}_{i\in M_j, j\in\mathbb{N}}$, $\{\mathfrak{r}_{ij}\}_{i\in M_j, j\in\mathbb{N}} \in \left(\sum_{j\in\mathbb{N}} \oplus \ell^2_{M_j}\right)_{\ell^2(\mathfrak{Q})}$. Clearly $\left(\sum_{j\in\mathbb{N}} \oplus \ell^2_{M_j}\right)_{\ell^2(\mathfrak{Q})}$ is a right quaternionic Hilbert space.

In the following lemma, we give a necessary condition for a block Bessel sequence in terms of the existence of a bounded right linear operator.

Lemma 2.4. *Let $\{x_{ij}\}_{i\in M_j, j\in\mathbb{N}}$ be a block Bessel sequence for $\mathbb{H}^R(\mathfrak{Q})$ with respect to $\left(\{M_j\}_{j\in\mathbb{N}}; \{\mathfrak{q}_{ij}\}_{i\in M_j, j\in\mathbb{N}}\right)$. Then, there exists a bounded*

right linear operator $T_{\mathfrak{Q}}$ *from* $\left(\sum_{j\in\mathbb{N}}\oplus\ell^2_{M_j}\right)_{\ell^2(\mathfrak{Q})}$ *into* $\mathbb{H}^R(\mathfrak{Q})$ *for which* $T_{\mathfrak{Q}}\left(\{\mathfrak{r}_{ij}\}_{i\in M_j,j\in\mathbb{N}}\right)=\sum_{j=1}^{\infty}\sum_{i\in M_j}x_{ij}\mathfrak{q}_{ij}\mathfrak{r}_{ij}$.

Proof. Let r_2 be a bound of the block Bessel sequence $\{x_{ij}\}_{i\in M_j,j\in\mathbb{N}}$ and let $k_1,k_2\in\mathbb{N}$ with $k_1>k_2$. Then

$$\begin{aligned}
&\left\|\sum_{j=1}^{k_1}\sum_{i\in M_j}x_{ij}\mathfrak{q}_{ij}\mathfrak{r}_{ij}-\sum_{j=1}^{k_2}\sum_{i\in M_j}x_{ij}\mathfrak{q}_{ij}\mathfrak{r}_{ij}\right\|^2\\
&\quad=\left\|\sum_{j=k_2}^{k_1}\sum_{i\in M_j}x_{ij}\mathfrak{q}_{ij}\mathfrak{r}_{ij}\right\|^2\\
&\quad=\sup_{\|x\|=1}\left|\left\langle\sum_{j=k_2}^{k_1}\sum_{i\in M_j}x_{ij}\mathfrak{q}_{ij}\mathfrak{r}_{ij}|x\right\rangle\right|^2\\
&\quad=\sup_{\|x\|=1}\left|\sum_{j=k_2}^{k_1}\sum_{i\in M_j}\overline{\mathfrak{q}_{ij}\mathfrak{r}_{ij}}\left\langle x_{ij}|x\right\rangle\right|^2\\
&\quad\leq\sum_{j=k_2}^{k_1}\sum_{i\in M_j}|\mathfrak{r}_{ij}|^2\sup_{\|x\|=1}\left(\sum_{j=k_2}^{k_1}\sum_{i\in M_j}|\mathfrak{q}_{ij}|^2|\left\langle x_{ij}|x\right\rangle|^2\right)\\
&\quad\leq r_2\sum_{j=1}^{\infty}|\mathfrak{r}_{ij}|^2.
\end{aligned}$$

Since $\{\mathfrak{r}_{ij}\}_{i\in M_j,j\in\mathbb{N}}\in\left(\sum_{j\in\mathbb{N}}\oplus\ell^2_{M_j}\right)_{\ell^2(\mathfrak{Q})}$, $\left\{\sum_{j=1}^{k}\sum_{i\in M_j}x_{ij}\mathfrak{q}_{ij}\mathfrak{r}_{ij}\right\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{H}^R(\mathfrak{Q})$. Hence, $T_{\mathfrak{Q}}$ is a well-defined operator satisfying $\|T_{\mathfrak{Q}}\|\leq\sqrt{r_2}$. □

If $\{x_{ij}\}_{i\in M_J,j\in\mathbb{N}}$ is a block Bessel sequence of $\mathbb{H}^R(\mathfrak{Q})$ with respect to $\left(\{M_j\}_{j\in\mathbb{N}};\{\mathfrak{q}_{ij}\}_{i\in M_j,j\in\mathbb{N}}\right)$, then the synthesis operator for $\{x_{ij}\}_{i\in M_j,j\in\mathbb{N}}$ is a right linear operator $T_{\mathfrak{Q}}:\left(\sum_{j\in\mathbb{N}}\oplus\ell^2_{M_j}\right)_{\ell^2(\mathfrak{Q})}\to\mathbb{H}^R(\mathfrak{Q})$ defined by

$$T_{\mathfrak{Q}}\left(\{\mathfrak{r}_{ij}\}_{i\in M_j,j\in\mathbb{N}}\right)=\sum_{j=1}^{\infty}\sum_{i\in M_j}x_{ij}\mathfrak{q}_{ij}\mathfrak{r}_{ij},$$

where $\{\mathfrak{r}_{ij}\}_{i\in M_j, j\in\mathbb{N}} \in \left(\sum_{j\in\mathbb{N}} \oplus l^2_{M_j}\right)_{l^2(\mathfrak{Q})}$. Also, the analysis operator $T^*_{\mathfrak{Q}} : \mathbb{H}^R(\mathfrak{Q}) \to \left(\sum_{j\in\mathbb{N}} \oplus \ell^2_{M_j}\right)_{\ell^2(\mathfrak{Q})}$ is given by

$$T^*_{\mathfrak{Q}}(x) = \{\langle x_{ij}\mathfrak{q}_{ij}|x\rangle\}_{i\in M_j, j\in\mathbb{N}}, \quad x \in \mathbb{H}^R(\mathfrak{Q}).$$

For any $x \in \mathbb{H}^R(\mathfrak{Q})$ and $\{\mathfrak{r}_{ij}\}_{i\in M_j, j\in\mathbb{N}} \in \left(\sum_{j\in\mathbb{N}} \oplus \ell^2_{M_j}\right)_{\ell^2(\mathfrak{Q})}$, we compute

$$\begin{aligned}
\left\langle T^*_{\mathfrak{Q}}(x)|\{\mathfrak{r}_{ij}\}_{i\in M_j, j\in\mathbb{N}}\right\rangle &= \sum_{j=1}^{\infty}\sum_{i\in M_j} \langle x|x_{ij}\rangle\mathfrak{q}_{ij}\mathfrak{r}_{ij} \\
&= \sum_{j=1}^{\infty}\sum_{i\in M_j} \overline{\langle x_{ij}\mathfrak{q}_{ij}|x\rangle}\mathfrak{r}_{ij} \\
&= \left\langle \left\{\langle x_{ij}\mathfrak{q}_{ij}|x\rangle\right\}_{i\in M_j, j\in\mathbb{N}} | \left\{\mathfrak{r}_{ij}\right\}_{i\in M_j, j\in\mathbb{N}}\right\rangle.
\end{aligned}$$

This yields $T^*_{\mathfrak{Q}}(x) = \{\langle x_{ij}\mathfrak{q}_{ij}|x\rangle\}_{i\in M_j, j\in\mathbb{N}}$, $x \in \mathbb{H}^R(\mathfrak{Q})$. Further, the frame operator is a right linear operator $S_{\mathfrak{Q}} : \mathbb{H}^R(\mathfrak{Q}) \to \mathbb{H}^R(\mathfrak{Q})$ given by

$$\begin{aligned}
S_{\mathfrak{Q}}(x) &= T_{\mathfrak{Q}}T^*_{\mathfrak{Q}}(x) \\
&= \sum_{j\in\mathbb{N}}\sum_{i\in M_j} x_{ij}\mathfrak{q}_{ij}\langle x_{ij}\mathfrak{q}_{ij}|x\rangle, \quad x \in \mathbb{H}^R(\mathfrak{Q}).
\end{aligned}$$

In the following result, we give a necessary and sufficient condition for a block sequence in $\mathbb{H}^R(\mathfrak{Q})$ to be a block Bessel sequence.

Theorem 2.5. *A block sequence $\{x_{ij}\}_{i\in M_j, j\in\mathbb{N}}$ in $\mathbb{H}^R(\mathfrak{Q})$ is a block Bessel sequence w.r.t $\left(\{M_j\}_{j\in\mathbb{N}}; \{\mathfrak{q}_{ij}\}_{i\in M_j, j\in\mathbb{N}}\right)$ if and only if $T_{\mathfrak{Q}}$ is a bounded right linear operator from $\left(\sum_{j\in\mathbb{N}} \oplus \ell^2_{M_j}\right)_{\ell^2(\mathfrak{Q})}$ into $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. Necessary part follow from Lemma 2.4. For the sufficient part, suppose there exist a real number r_2 that satisfy $\|T_{\mathfrak{Q}}\| \leq \sqrt{r_2}$. Then

$$\begin{aligned}
\sum_{j\in\mathbb{N}}\sum_{i\in M_j} |\mathfrak{q}_{ij}|^2|\langle x_{ij}|x\rangle|^2 &= \|T^*_{\mathfrak{Q}}(x)\|^2 \\
&\leq r_2\|x\|^2, \quad x \in \mathbb{H}^R(\mathfrak{Q}).
\end{aligned}$$

□

Next, we discuss a property of the frame operator corresponding to a block frame in $\mathbb{H}^R(\mathfrak{Q})$.

Theorem 2.6. *Let $\{x_{ij}\}_{i\in M_j, j\in\mathbb{N}}$ be a block frame in $\mathbb{H}^R(\mathfrak{Q})$ with respect to $\big(\{M_j\}_{j\in\mathbb{N}}; \{\mathfrak{q}_{ij}\}_{i\in M_j, j\in\mathbb{N}}\big)$ with lower and upper bounds r_1 and r_2 respectively. Then the frame operator $S_{\mathfrak{Q}}$ is positive, bounded, invertible and self adjoint right linear operator on $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. For any $x \in \mathbb{H}^R(\mathfrak{Q})$, we compute

$$\begin{aligned}\langle S_{\mathfrak{Q}}(x)|x\rangle &= \left\langle \sum_{j\in\mathbb{N}}\sum_{i\in M_j} x_{ij}\mathfrak{q}_{ij}\langle x_{ij}\mathfrak{q}_{ij}|x\rangle \Big| x \right\rangle \\ &= \sum_{j\in\mathbb{N}}\sum_{i\in M_j} |\mathfrak{q}_{ij}|^2 |\langle x_{ij}|x\rangle|^2.\end{aligned}$$

The fact that $\{x_{ij}\}_{i\in M_j, j\in\mathbb{N}}$ is a block frame with respect to $\big(\{M_j\}_{j\in\mathbb{N}}; \{\mathfrak{q}_{ij}\}_{i\in M_j, j\in\mathbb{N}}\big)$ yields that $S_{\mathfrak{Q}}$ is a positive, bounded and invertible right linear operator. Also, for any $x, x^{'} \in \mathbb{H}^R(\mathfrak{Q})$, we have

$$\begin{aligned}\left\langle S_{\mathfrak{Q}}x|x^{'}\right\rangle &= \left\langle \sum_{j\in\mathbb{N}}\sum_{i\in M_j} x_{ij}\mathfrak{q}_{ij}\langle x_{ij}\mathfrak{q}_{ij}|x\rangle \Big| x^{'} \right\rangle \\ &= \left\langle x \Big| \sum_{j\in\mathbb{N}}\sum_{i\in M_j} x_{ij}\mathfrak{q}_{ij}\langle x_{ij}\mathfrak{q}_{ij}|x^{'}\rangle \right\rangle \\ &= \langle x|S_{\mathfrak{Q}}x^{'}\rangle.\end{aligned}$$

Thus, $S_{\mathfrak{Q}}$ is a self adjoint right linear operator. □

The reconstruction formula for the block frame is given in the following corollary.

Corollary 2.7. *Let $\{x_{ij}\}_{i\in M_j, j\in\mathbb{N}}$ be a block frame for $\mathbb{H}^R(\mathfrak{Q})$ with respect to $\big(\{M_j\}_{j\in\mathbb{N}}; \{\mathfrak{q}_{ij}\}_{i\in M_j, j\in\mathbb{N}}\big)$. Then, every $x \in \mathbb{H}^R(\mathfrak{Q})$ can be written as*

$$x = \sum_{j\in\mathbb{N}}\sum_{i\in M_j} S_{\mathfrak{Q}}^{-1}(x_{ij})\mathfrak{q}_{ij}\langle x_{ij}\mathfrak{q}_{ij}|x\rangle.$$

Proof. Follows in view of Theorem 2.6. □

Next, we give a method for constructing a block frame using a given block frame in a right quaternionic Hilbert space. More precisely, we prove the following result:

Theorem 2.8. *Let $\{x_{ij}\}_{i\in M_j, j\in\mathbb{N}}$ be a block frame for $\mathbb{H}^R(\mathfrak{Q})$ with respect to $\big(\{M_j\}_{j\in\mathbb{N}}; \{\mathfrak{q}_{ij}\}_{i\in M_j, j\in\mathbb{N}}\big)$, with lower and upper bounds r_1 and r_2, respectively and with frame operator $S_{\mathfrak{Q}}$. Then $\big\{S_{\mathfrak{Q}}^{-1}(x_{ij})\big\}_{i\in M_j, j\in\mathbb{N}}$ is a block frame for $\mathbb{H}^R(\mathfrak{Q})$ with respect to $\big(\{M_j\}_{j\in\mathbb{N}}; \{\mathfrak{q}_{ij}\}_{i\in M_j, j\in\mathbb{N}}\big)$ with*

lower and upper bounds r_2^{-1} and r_1^{-1}, respectively with frame operator $S_{\mathfrak{Q}}^{-1}$.

Proof. By hypothesis, we have

$$\sum_{j=1}^{\infty}\sum_{i\in M_j}|\mathfrak{q}_{ij}|^2\left|\langle S_{\mathfrak{Q}}^{-1}x_{ij}|x\rangle\right|^2\leq r_2\left\|S_{\mathfrak{Q}}^{-1}\right\|^2\|x\|^2,\quad \text{for all}\ \ x\in\mathbb{H}^R(\mathfrak{Q}).$$

This implies that $\left\{S_{\mathfrak{Q}}^{-1}x_{ij}\right\}_{i\in M_j,j\in\mathbb{N}}$ is a block Bessel sequence for $\mathbb{H}^R(\mathfrak{Q})$ with respect to $\left(\{M_j\}_{j\in\mathbb{N}};\{\mathfrak{q}_{ij}\}_{i\in M_j,j\in\mathbb{N}}\right)$. So the frame operator corresponding to block Bessel sequence $\{S_{\mathfrak{Q}}^{-1}x_{ij}\}_{i\in M_j,j\in\mathbb{N}}$ is well defined. Therefore, we compute

$$\begin{aligned}\sum_{j\in\mathbb{N}}\sum_{i\in M_j}S_{\mathfrak{Q}}^{-1}(x_{ij})\mathfrak{q}_{ij}\left\langle S_{\mathfrak{Q}}^{-1}x_{ij}\mathfrak{q}_{ij}|x\right\rangle &= S_{\mathfrak{Q}}^{-1}S_{\mathfrak{Q}}(S_{\mathfrak{Q}}^{-1}x)\\ &= S_{\mathfrak{Q}}^{-1}x,\quad \text{for all } x\in\mathbb{H}^R(\mathfrak{Q}).\end{aligned}$$

This yields that the right frame operator for $\{S_{\mathfrak{Q}}^{-1}x_{ij}\}_{i\in M_j,j\in\mathbb{N}}$ is $S_{\mathfrak{Q}}^{-1}$. Rest is Straightforward. □

In view of Theorem 2.8, we have the following definition

Definition 2.9. Let $\{x_{ij}\}_{i\in M_j,j\in\mathbb{N}}$ be a block frame for $\mathbb{H}^R(\mathfrak{Q})$ with respect to $\left(\{M_j\}_{j\in\mathbb{N}};\{\mathfrak{q}_{ij}\}_{i\in M_j,j\in\mathbb{N}}\right)$ with frame operator $S_{\mathfrak{Q}}$. Then the block frame $\{S_{\mathfrak{Q}}^{-1}x_{ij}\}_{i\in M_j,j\in\mathbb{N}}$ with respect to $\left(\{M_j\}_{j\in\mathbb{N}};\{\mathfrak{q}_{ij}\}_{i\in M_j,j\in\mathbb{N}}\right)$ is called the canonical dual block frame of the block frame $\{x_{ij}\}_{i\in M_j,j\in\mathbb{N}}$.

Next, we construct a Parseval block frame using a given block frame in a right quaternionic Hilbert space.

Theorem 2.10. *Let $\{x_{ij}\}_{i\in M_j,j\in\mathbb{N}}$ be a block frame for $\mathbb{H}^R(\mathfrak{Q})$ with respect to $\left(\{M_j\}_{j\in\mathbb{N}};\{\mathfrak{q}_{ij}\}_{i\in M_j,j\in\mathbb{N}}\right)$ and frame operator $S_{\mathfrak{Q}}$. Then $\left\{S_{\mathfrak{Q}}^{-1/2}x_{ij}\right\}_{i\in M_j,j\in\mathbb{N}}$ is a Parseval block frame for $\mathbb{H}^R(\mathfrak{Q})$ with respect to $\left(\{M_j\}_{j\in\mathbb{N}};\{\mathfrak{q}_{ij}\}_{i\in M_j,j\in\mathbb{N}}\right)$.*

Proof. It can be worked out using [15, Theorem 2.18] and [19, Theorem 3.9]. □

In the following result, we characterize Parseval block frames for a right quaternionic Hilbert space.

Theorem 2.11. *Let $\{x_{ij}\}_{i\in M_j,j\in\mathbb{N}}$ be a block frame for $\mathbb{H}^R(\mathfrak{Q})$ w.r.t. $\left(\{M_j\}_{j\in\mathbb{N}};\{\mathfrak{q}_{ij}\}_{i\in M_j,j\in\mathbb{N}}\right)$ with frame operator $S_{\mathfrak{Q}}$. Then $\{x_{ij}\}_{i\in M_j,j\in\mathbb{N}}$ is a Parseval block frame with respect to $\left(\{M_j\}_{j\in\mathbb{N}};\{\mathfrak{q}_{ij}\}_{i\in M_j,j\in\mathbb{N}}\right)$ for $\mathbb{H}^R(\mathfrak{Q})$ if and only if $S_{\mathfrak{Q}}$ is the identity operator on $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. Follows in view of the fact that for any $x \in \mathbb{H}^R(\mathfrak{Q})$

$$\langle S_{\mathfrak{Q}}(x)|x\rangle = \sum_{j\in\mathbb{N}} \sum_{i\in M_j} |\mathfrak{q}_{ij}|^2 |\langle x_{ij}|x\rangle|^2. \qquad \square$$

Finally, we give a characterization of a block frame in terms of an operator for the right quaternionic Hilbert space.

Theorem 2.12. *A block sequence $\{x_{ij}\}_{i\in M_j, j\in\mathbb{N}}$ in $\mathbb{H}^R(\mathfrak{Q})$ is a block frame with respect to $\left(\{M_j\}_{j\in\mathbb{N}}; \{\mathfrak{q}_{ij}\}_{i\in M_j, j\in\mathbb{N}}\right)$ if and only if the operator $T_{\mathfrak{Q}} : \left(\sum\limits_{j\in\mathbb{N}} \oplus \ell^2_{M_j}\right)_{\ell^2(\mathfrak{Q})} \to \mathbb{H}^R(\mathfrak{Q})$ given by*

$$T_{\mathfrak{Q}}\left(\{q_{ij}\}_{j\in\mathbb{N}, i\in M_j}\right) = \sum_{j\in\mathbb{N}} \sum_{i\in M_j} x_{ij}\mathfrak{q}_{ij}q_{ij},$$

where $\{q_{ij}\}_{i\in M_j, j\in\mathbb{N}} \in \left(\sum\limits_{j\in\mathbb{N}} \oplus \ell^2_{M_j}\right)_{\ell^2(\mathfrak{Q})}$, is a well defined right linear operator from $\left(\sum\limits_{j\in\mathbb{N}} \oplus \ell^2_{M_j}\right)_{\ell^2(\mathfrak{Q})}$ onto $\mathbb{H}^R(\mathfrak{Q})$.

Proof. Let $\{x_{ij}\}_{i\in M_j, j\in\mathbb{N}}$ be a block frame with frame operator $S_{\mathfrak{Q}}$. Then $\{x_{ij}\}_{i\in M_j, j\in\mathbb{N}}$ is a block Bessel sequence and so $T_{\mathfrak{Q}}$ is a well defined bounded right linear operator. Also, $S_{\mathfrak{Q}}$ being an invertible right linear operator implies that $T_{\mathfrak{Q}}$ is an onto right linear operator. Conversely, let $T_{\mathfrak{Q}}$ be a well defined, bounded, onto right linear operator. Then, there exits a right linear operator $T^{\dagger}_{\mathfrak{Q}} : \mathbb{H}^R(\mathfrak{Q}) \to \left(\sum\limits_{j\in\mathbb{N}} \oplus \ell^2_{M_j}\right)_{\ell^2(\mathfrak{Q})}$ such that for each $x \in \mathbb{H}^R(\mathfrak{Q})$

$$\begin{aligned} x &= T_{\mathfrak{Q}} T^{\dagger}_{\mathfrak{Q}} x \\ &= \sum_{j\in\mathbb{N}} \sum_{i\in M_j} x_{ij}\mathfrak{q}_{ij} \left(T^{\dagger}_{\mathfrak{Q}} x\right)_{ij}, \end{aligned}$$

where

$$\left\{\left(T^{\dagger}_{\mathfrak{Q}} x\right)_{ij}\right\}_{i\in M_j, j\in\mathbb{N}} \in \left(\sum_{j\in\mathbb{N}} \oplus \ell^2_{M_j}\right)_{\ell^2(\mathfrak{Q})}.$$

Also, for $x \in \mathbb{H}^R(\mathfrak{Q})$, we compute

$$\|x\|^4 = |\langle x|x\rangle|^2$$

$$
\begin{aligned}
&= \left| \left\langle \sum_{j\in\mathbb{N}} \sum_{i\in M_j} x_{ij}\mathfrak{q}_{ij} \left(T_{\mathfrak{Q}}^{\dagger}x\right)_{ij} |x \right\rangle \right|^2 \\
&\leq \sum_{j\in\mathbb{N}} \sum_{i\in M_j} \left| \left(T_{\mathfrak{Q}}^{\dagger}x\right)_{ij} \right|^2 \sum_{j\in\mathbb{N}} \sum_{i\in M_j} |\mathfrak{q}_{ij}|^2 |\langle x_{ij}|x\rangle|^2 \\
&\leq \left\| T_{\mathfrak{Q}}^{\dagger} \right\|^2 \|x\|^2 \sum_{j\in\mathbb{N}} \sum_{i\in M_j} |\mathfrak{q}_{ij}|^2 |\langle x_{ij}|x\rangle|^2.
\end{aligned}
$$

Hence $\{x_{ij}\}_{i\in M_j, j\in\mathbb{N}}$ is a block frame of $\mathbb{H}^R(\mathfrak{Q})$ with respect to $\left(\{M_j\}_{j\in\mathbb{N}}; \{\mathfrak{q}_{ij}\}_{i\in M_j, j\in\mathbb{N}}\right)$ with bounds $\left\| T_{\mathfrak{Q}}^{\dagger} \right\|^{-2}$ and $\|T_{\mathfrak{Q}}\|^2$. □

Conclusion. Ellouz [12] studied some properties of K-frames in quaternionic Hilbert spaces (QHS). Various other properties of frames were studied by many other researchers [17–21] in QHS setting. In the present article, keeping more applications of frames in QHS in mind, we have defined a modified version of frame in QHS called block frame and give examples to support its existence. Also, a necessary and sufficient condition for a sequence to be a block Bessel sequence is given. A method for the construction of a desired block frame from a given block frame is discussed. Finally, we give a characterization of block frames in terms of an operator in QHS.

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