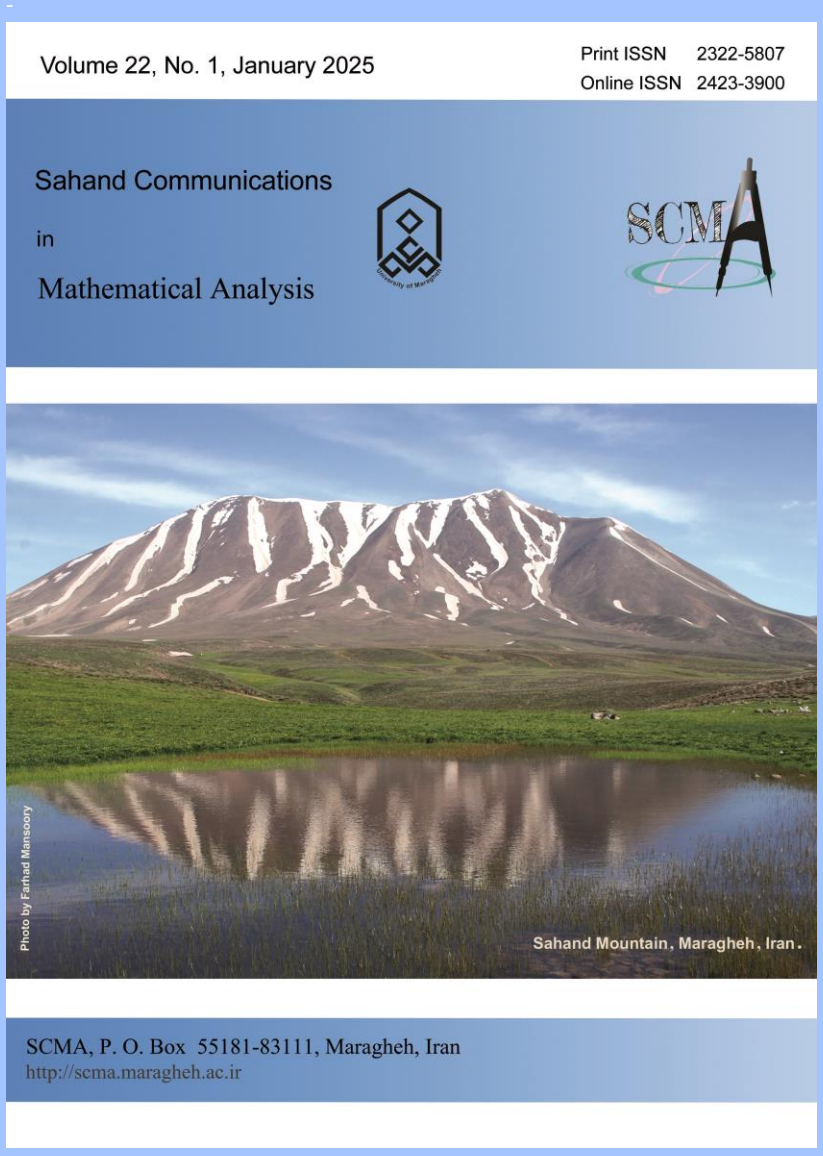
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Alienation of the Set-Valued Drygas-Additive and Quadratic-Additive Functional Equations

Elham Mohammadi[1], Abbas Najati[2*] and Iz-iddine EL-Fassi[3]

Abstract. Consider Y as a real Hausdorff topological vector space and $(G,+)$ as a Abelian group uniquely divisible by 2. In this paper, the solutions and stability of the Pexiderized set-valued functional equations

$$f(x+y)+f(x-y)+g(x+y)=2f(x)+f(y)+f(-y)+g(x)+g(y),$$
$$f(x+y)+f(x-y)+g(x+y)=2f(x)+2f(y)+g(x)+g(y),$$

are investigated, where f and g are unknown functions from G to $cc(Y)$.

1. Introduction and Preliminaries

The concept of alienation in functional equations was first proposed by J. Dhombres [4]. In brief, if we have two functional equations $E_1(f)=0$ and $E_2(g)=0$ involving two functions, which may be the same, f and g, we combine them to yield $E_1(f)+E_2(g)=0$. We investigate whether equation $E_1(f)+E_2(g)=0$ can be decomposed into the system of two functional equations:

$$\begin{cases} E_1(f)=0, \\ E_2(g)=0. \end{cases}$$

If the answer is "yes," the two functional equations $E_1(f)=0$ and $E_2(g)=0$ are termed as strongly alien. Numerous authors have investigated the concept of alienation (see for example [1, 5, 6, 9, 10, 14]). For further insights into the concept of alienation in the theory of functional

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equations, we recommend consulting the survey article [7] written by Ger and Sablik.

Consider $(G,+)$ as a uniquely 2-divisible Abelian group and Y as a real Hausdorff topological vector space. Both the zero element of G and the origin of Y are denoted by 0. We denote the set of all nonempty subsets of Y by $P_0(Y)$ and the set of nonempty compact and convex subsets of Y by $cc(Y)$. Furthermore, the collection of nonempty subsets of Y that are bounded, convex, and closed is denoted by $bccl(Y)$. For given $A, B \in P_0(Y)$ and $r \in \mathbb{R}$, we define the operations as follows:

$$rA := \{ra : \ a \in A\}, \qquad A + B := \{a + b : \ a \in A, \ b \in B\}.$$

Clearly, for a convex set A and non-negative numbers r and s, it follows that

$$rA + sA = (r + s)A.$$

In this paper, we examine the solutions and the Hyers-Ulam stability of the Pexiderized set-valued functional equations

(1.1)
$$f(x+y) + f(x-y) + g(x+y) = 2f(x) + f(y) + f(-y) + g(x) + g(y),$$

(1.2)
$$f(x+y) + f(x-y) + g(x+y) = 2f(x) + 2f(y) + g(x) + g(y),$$

for unknown functions $f, g : G \to cc(Y)$. Certainly, the functional equation (1.1) arises from combining additive and the Drygas functional equations, whereas equation (1.2) is obtained by adding the additive and quadratic functional equations.

Below, we introduce certain symbols and introduce results that will be referenced later on. Let's consider Rådström's cancellation law in the context of a topological vector space Y. Rådström originally formulated this principle in the setting of normed spaces [12]. However, the proof provided applies to topological vector spaces as well.

Lemma 1.1 (Rådström's cancellation law)**.** *Suppose A, B and C are subsets of a topological vector space Y such that B is both convex and closed. If $C \neq \emptyset$ is bounded and $A + C \subseteq B + C$, then $A \subseteq B$.*

Corollary 1.2. *Suppose A, B and C are subsets of a topological vector space Y, where A and B are both convex and closed sets and $C \neq \emptyset$ is a bounded set. If $A + C = B + C$, then $A = B$.*

Consider A, B as elements of $cc(Y)$. If there exists a set $C \in cc(Y)$ such that $A = B + C$, this set C is termed the Hukuhara difference of A and B, denoted by $C = A \ominus B$ (see [8]). The uniqueness of such a difference is assured by Corollary 1.2.

2. Solutions of (1.1) and (1.2)

In this section, we address the issue of the alienation problem concerning functional equations (1.1) and (1.2). The following straightforward example illustrates that, in general, Drygas (quadratic) and additive functional equations do not exhibit strong alienation from each other.

Example 2.1. Let $a \in Y$ be a fixed non-zero element. Define $f, g : G \to cc(Y)$ by $f(x) = \{a\}$ and $g(x) = \{-2a\}$ for all $x \in G$. Then it is evident that these functions satisfy equations (1.1) and (1.2). However, f does not satisfy the Drygas (quadratic) functional equation and g does not meet the criteria of the additive functional equation.

In the following theorems, we demonstrate a significant form of alienation between under consideration functional equations. Specifically, we establish that Drygas (quadratic) and additive functional equations exhibit strong alienation from each other, modulo a constant.

Theorem 2.2. *Set-valued functions $f, g : G \to cc(Y)$ satisfy (1.1) if and only if there exists a set-valued Drygas function $D : G \to cc(Y)$ and an additive function $A : G \to Y$ such that*

$$f(x) = D(x) + f(0), \qquad g(x) = A(x) + g(0), \quad x \in G, \tag{2.1}$$

where $2f(0) + g(0) = \{0\}$.

Proof. Let f, g satisfy (1.1). Substituting $x = y = 0$ in (1.1), we obtain

$$2f(0) + g(0) = \{0\}. \tag{2.2}$$

Consequently, it follows that both $f(0)$ and $g(0)$ are singleton sets. Let $f(0) = \{a\}$, $g(0) = \{b\}$, $F(x) = f(x) - a$, and $H(x) = g(x) - b$ for $x \in G$. Hence, we can express (1.1) in the following manner:

$$F(x+y)+F(x-y)+H(x+y) = 2F(x)+F(y)+F(-y)+H(x)+H(y), \tag{2.3}$$

for $x, y \in G$.

Obviously, $F(0) = \{0\}$ and $H(0) = \{0\}$. By replacing y with $-y$ in (2.3) and then adding $H(x + y)$ to both sides of the resulting equation, we obtain

$$\begin{aligned} &F(x + y) + F(x - y) + H(x + y) + H(x - y) \\ &\quad = 2F(x) + F(y) + F(-y) + H(x) + H(-y) + H(x + y), \quad x, y \in G. \end{aligned}$$

Using (2.3) and the cancellation law, we derive

$$H(y) + H(x - y) = H(-y) + H(x + y), \quad x, y \in G. \tag{2.4}$$

Setting $y = x$ in (2.4), we obtain

$$H(x) = H(2x) + H(-x), \quad x \in G.$$

As a result $H(2x) + H(-2x) = \{0\}$ for all $x \in G$. Consequently, the function H is both single-valued and odd. We are now able to present equation (2.4) in the following form:

$$H(y+x) + H(y-x) = 2H(y), \quad x, y \in G.$$

Setting $x = y$ in the recent equation, we get $H(2y) = 2H(y)$. This implies

$$H(y+x) + H(y-x) = H(2y), \quad x, y \in G.$$

Therefore H is additive and (2.3) yields

$$F(x+y) + F(x-y) = 2F(x) + F(y) + F(-y), \quad x, y \in G. \tag{2.5}$$

This means F is a set-valued Drygas function. Hence (2.1) can be derived by introducing $D = F$ and $A = H$.

The converse is evident. □

The following result is deduced directly from [2, Theorem 5], [11] and [13, Theorem 3].

Corollary 2.3. *If $f, g : \mathbb{R} \to cc(Y)$ are continuous set-valued functions satisfying (1.1), then $f(x) = x^2M + cx + f(0)$; and $g(x) = dx + g(0)$ for some M in $cc(Y)$ and $c, d \in Y$, where $2f(0) + g(0) = \{0\}$.*

Theorem 2.4. *Set-valued functions $f, g : G \to cc(Y)$ satisfy (1.2) if and only if there exists a quadratic set-valued function $Q : G \to cc(Y)$ and an additive function $A : G \to Y$ such that*

$$f(x) = Q(x) + f(0), \qquad g(x) = A(x) + g(0), \quad x \in G, \tag{2.6}$$

where $2f(0) + g(0) = \{0\}$.

Proof. Suppose that (f, g) satisfies (1.2). Put $x = y = 0$ in (1.2) to infer $2f(0) + g(0) = \{0\}$. Then $f(0)$ and $g(0)$ are singleton. Set

$$f(0) = \{a\}, \quad g(0) = \{b\}, \quad F(x) = f(x) - a, \quad H(x) = g(x) - b, \ x \in G.$$

Hence, we can express (1.2) in the following manner:

$$F(x+y)+F(x-y)+H(x+y) = 2F(x)+2F(y)+H(x)+H(y), \quad x, y \in G. \tag{2.7}$$

It is evident that $F(0) = \{0\}$ and $H(0) = \{0\}$. Substituting $x = 0$ in (2.7), we deduce that $F(y) = F(-y)$ for each $y \in G$. If we substitute $-y$ for y in (2.7) and subsequently add $H(x+y)$ to both sides of the resulting equation, we are able to derive the following functional equation

$$\begin{aligned} &F(x+y) + F(x-y) + H(x+y) + H(x-y) \\ &\quad = 2F(x) + 2F(y) + H(x) + H(-y) + H(x+y), \quad x, y \in G. \end{aligned}$$

Using (2.7) and the cancellation law in the equation above, one obtains

$$H(y) + H(x-y) = H(-y) + H(x+y), \quad x, y \in G.$$

Using the same argument as in the proof of Theorem 2.2, we can infer that H is a single valued additive function. Therefore (2.7) yields

$$F(x+y)+F(x-y)=2F(x)+2F(y), \quad x,y\in G.$$

This implies that F is a quadratic set-valued function. Hence, we obtain the desired result (2.6) by letting $Q=F$ and $A=H$.

Verifying the converse is straightforward. □

The subsequent outcome is directly inferred from [2, Theorem 5] and [11].

Corollary 2.5. *If $f,g:\mathbb{R}\to cc(Y)$ are continuous set-valued functions satisfying* (1.2)*, then $f(x)=x^2M+f(0)$; and $g(x)=dx+g(0)$ for some $M\in cc(Y)$ and $d\in Y$, where $2f(0)+g(0)=\{0\}$.*

One can apply the proof technique of Theorem 2.4 and, with a different and, using a simpler approach compared to the proof in [1, Theorem 2.1], derive the following result.

Corollary 2.6. *Suppose $(K,+)$ is an Abelian group. Functions $f,g: G\to K$ satisfy* (1.2) *if and only if there exists a quadratic function $q:G\to K$ and an additive function $a:G\to K$ such that*

$$f(x)=q(x)+f(0), \qquad g(x)=a(x)+g(0), \quad x\in G,$$

where $2f(0)+g(0)=0$.

3. Stability of Set-Valued Functional Equations (1.1) and (1.2).

In this section, we investigate the Hyers-Ulam stability of set-valued functional equations (1.1) and (1.2). Let Y denote a normed linear space. Within $bcl(Y)$, the collection of all nonempty bounded and closed subsets of Y, we introduce a distance measure known as the Hausdorff distance. This distance function, denoted by d_H, is defined as follows: for any pair of subsets A and B belonging to $bcl(Y)$,

$$d_H(A,B):=\max\left\{\sup_{x\in A}d(x,B),\sup_{y\in B}d(A,y)\right\},$$

where $d(x,B):=\inf\{\|x-y\|:\ y\in B\}$ and $d(A,y):=\inf\{\|x-y\|:\ x\in A\}$. Alternatively, we can express $d_H(A,B)$ as:

$$d_H(A,B):=\inf\left\{r>0:\ A\subseteq B+rU \text{ and } B\subseteq A+rU\right\},$$

where U denotes the closed unit ball in Y. It is noteworthy that $bcl(Y)$, equipped with the Hausdorff distance, constitutes a metric space. Clearly, that if $A=\{a\}$ and $B=\{b\}$, then $d_H(A,B)=\|a-b\|$.

The following lemma summarizes the key properties of the Hausdorff distance.

Lemma 3.1 ([3, 12])**.** *For any $A, B, C \in cc(Y)$ and any real number λ, the following hold:*

(*i*) $d_H(A + C, B + C) = d_H(A, B)$,
(*ii*) $d_H(\lambda A, \lambda B) = |\lambda| d_H(A, B)$,
(*iii*) *if Y is a Banach space, then $(cc(Y), d_H)$ is a complete metric space.*

Proposition 3.2. *Let $A, B, C, D \in cc(Y)$. Then*

(*i*) $d_H(A + B, C + D) \leq d_H(A, C) + d_H(B, D)$,
(*ii*) *if the Hukuhara differences $A \ominus B$ and $C \ominus D$ exist, then*

$$d_H(A \ominus B, C \ominus D) \leq d_H(A, C) + d_H(B, D).$$

Proof. (*i*) Since d_H is an invariant metric on $cc(Y)$ as stated in part (*i*) of Lemma 3.1, we have

$$\begin{aligned} d_H(A + B, C + D) &\leq d_H(A + B, B + C) + d_H(B + C, C + D) \\ &= d_H(A, C) + d_H(B, D). \end{aligned}$$

(*ii*) Let $E = A \ominus B$ and $F = C \ominus D$. Then $A = B + E$ and $C = D + F$. Consequently, we obtain

$$\begin{aligned} d_H(A \ominus B, C \ominus D) &= d_H(E, F) \\ &= d_H(B + E, B + F) \\ &\leq d_H(B + E, D + F) + d_H(D + F, B + F) \\ &= d_H(A, C) + d_H(B, D). \end{aligned}$$

□

Now, the main findings regarding the Hyers-Ulam stability of set-valued functional equations (1.1) and (1.2) are ready to be presented.

Theorem 3.3. *Let Y be a Banach space. Take $\varepsilon \geq 0$ and suppose $f, g : G \to cc(Y)$ are set-valued functions satisfying*

$$\begin{aligned} d_H\big[&f(x + y) + f(x - y) + g(x + y), \\ &2f(x) + f(y) + f(-y) + g(x) + g(y)\big] \leq \varepsilon \end{aligned} \tag{3.1}$$

for each $x, y \in G$. Then there exist a unique additive function $A : G \to Y$ such that

$$d_H\big[A(x) + g(0), g(x)\big] \leq 6\varepsilon, \quad x \in G. \tag{3.2}$$

Moreover if $f(x) \ominus f(-x)$ exists for each $x \in G$, then there is a unique set-valued Drygas function $D : G \to cc(Y)$ such that

$$d_H\big[f(x), D(x) + f(0)\big] \leq 8\epsilon, \quad x \in G. \tag{3.3}$$

Proof. Substitute $x = y = 0$ in (3.1) to get

$$d_H\big[2f(0) + g(0), \{0\}\big] \leq \varepsilon. \tag{3.4}$$

By substituting $-y$ for y in (3.1), we obtain

$$d_H\big[f(x+y) + f(x-y) + g(x-y), 2f(x) + f(y) + f(-y) + g(x) + g(-y)\big] \leq \varepsilon$$

for all $x, y \in G$. Since d_H is invariant under translation, we get

$$\begin{aligned} d_H\big[&f(x+y) + f(x-y) + g(x+y) + g(x-y), \\ &2f(x) + f(y) + f(-y) + g(x) + g(-y) + g(x+y)\big] \leq \varepsilon \end{aligned} \tag{3.5}$$

for all $x, y \in G$. By applying the triangle inequality, we can conclude from (3.1) and (3.5) that

$$d_H\big[g(x-y) + g(y), g(x+y) + g(-y)\big] \leq 2\varepsilon, \quad x, y \in G. \tag{3.6}$$

If we substitute $y = x$ in (3.6), we acquire

$$d_H\big[g(2x) + g(-x), g(0) + g(x)\big] \leq 2\varepsilon, \quad x \in G. \tag{3.7}$$

Then

$$d_H\big[g(-2x) + g(x), g(0) + g(-x)\big] \leq 2\varepsilon, \quad x \in G.$$

From the last inequalities and Proposition 3.2 (i), we obtain

$$d_H\big[g(2x) + g(-2x), 2g(0)\big] \leq 4\varepsilon, \quad x \in G.$$

Thus

$$d_H\big[g(x) + g(-x), 2g(0)\big] \leq 4\varepsilon, \quad x \in G. \tag{3.8}$$

By (3.7), (3.8) and Proposition 3.2 (i), it can be deduced that

$$d_H\big[g(2x) + g(0), 2g(x)\big] \leq 6\varepsilon, \quad x \in G.$$

Hence

$$d_H\left[\frac{1}{2^{n+1}}g(2^{n+1}x) + \sum_{k=m}^{n}\frac{1}{2^{k+1}}g(0), \frac{1}{2^m}g(2^m x)\right] \leq \sum_{k=m}^{n}\frac{3\varepsilon}{2^k}, \quad x \in G \tag{3.9}$$

for all non-negative integers m, n with $m \leq n$. Consequently, the sequence $\{\frac{1}{2^n}g(2^n x)\}_n$ is a Cauchy sequence in $cc(Y)$. Since $(cc(Y), d_H)$ is complete, we can define

$$T : X \to cc(Y), \quad T(x) = \lim_n \frac{1}{2^n}g(2^n x).$$

By (3.6), we acquire

$$\begin{aligned} &d_H\big[T(x-y) + T(y), T(x+y) + T(-y)\big] \\ &= \lim_n d_H\left[\frac{1}{2^n}g(2^n x - 2^n y) + \frac{1}{2^n}g(2^n y), \frac{1}{2^n}g(2^n x + 2^n y) + \frac{1}{2^n}g(-2^n y)\right] \end{aligned}$$

$$\leq \lim_n \frac{2\varepsilon}{2^n} = 0, \quad x, y \in G.$$

Therefore $T(x-y)+T(y) = T(x+y)+T(-y)$, $x, y \in G$.

Applying the same argument as presented in the proof of Theorem 2.2, it can be inferred that T is a single-valued additive function. Let $T(x) = \{a_x\}$ and define $A : X \to Y$ by $A(x) = a_x$ for all $x \in G$. So, A is additive. By substituting $m = 0$ and letting n tend to infinity in (3.9), we derive (3.2).

We now suppose that $F_e(x) = \frac{1}{2}[f(x)+f(-x)]$ for each $x \in G$. Letting $y = -x$ in (3.1) and using (3.4), (3.8) and Proposition 3.2 (i), we obtain

$$d_H\big[f(2x)+3f(0), 3f(x)+f(-x)\big] \leq 6\varepsilon, \quad x \in G. \tag{3.10}$$

So $d_H\big[F_e(2x)+3F_e(0), 4F_e(x)\big] \leq 6\varepsilon$, $x \in G$.

Consequently, we derive

(3.11)

$$d_H\left[\frac{1}{4^{n+1}}F_e(2^{n+1}x)+\sum_{k=m}^{n}\frac{1}{4^{k+1}}3F_e(0), \frac{1}{4^m}F_e(2^m x)\right] \leq \frac{3}{2}\sum_{k=m}^{n}\frac{\varepsilon}{4^k}, \quad x \in G$$

for all non-negative integers m, n with $m \leq n$. Hence, the sequence $\{\frac{1}{4^n}F_e(2^n x)\}_n$ forms a Cauchy sequence within the space $cc(Y)$. As $(cc(Y), d_H)$ is a complete metric space, we are able to define

$$Q : X \to cc(Y), \quad Q(x) = \lim_n \frac{1}{4^n}F_e(2^n x).$$

Because $\lim_n \frac{1}{4^n}g(2^n x) = 0$, (3.1) implies that Q is a quadratic function. By substituting $m = 0$ and letting n tend towards infinity in (3.11), we obtain

$$d_H\left[Q(x)+F_e(0), F_e(x)\right] \leq 2\varepsilon, \quad x \in G. \tag{3.12}$$

Additionally, let us put $F_o(x) = \frac{1}{2}[f(x) \ominus f(-x)]$ for each $x \in G$. By (3.10) and Proposition 3.2 (ii), we deduce

$$d_H\big[F_o(2x), 2F_o(x)\big] \leq 6\varepsilon, \quad x \in G.$$

Therefore

$$d_H\left[\frac{1}{2^{n+1}}F_o(2^{n+1}x), \frac{1}{2^m}F_o(2^m x)\right] \leq 3\sum_{k=m}^{n}\frac{\varepsilon}{2^k}, \quad x \in G \tag{3.13}$$

for all non-negative integers m, n with $m \leq n$. Consequently, the sequence $\{\frac{1}{2^n}F_o(2^n x)\}_n$ will be a Cauchy sequence and we can define the function

$$S : X \to cc(Y), \qquad S(x) = \lim_n \frac{1}{2^n}F_o(2^n x),$$

which satisfies

$$d_H\big[F_o(x), S(x)\big] \leq 6\varepsilon, \quad x \in G. \tag{3.14}$$

By (3.1), (3.2), (3.4) and using the additivity of A, we obtain

$$d_H\big[f(x+y)+f(x-y)+2f(0), 2f(x)+f(y)+f(-y)\big] \leq 20\varepsilon, \quad x, y \in G.$$

So,

$$d_H\big[F_o(x+y)+F_o(x-y), 2F_o(x)\big] \leq 20\varepsilon, \quad x, y \in G.$$

This implies that $S(x+y)+S(x-y) = 2S(x)$ for each $x, y \in G$. Since $S(0) = \{0\}$, we conclude that S is a single valued additive function.

By defining $D := S+Q$, (3.3) can be deduced from (3.12) and (3.14). The uniqueness of A and D follows from (3.2) and (3.3) (see [13, Theorem 3]). □

Theorem 3.4. *Consider Y as a Banach space. Assume $\varepsilon \geq 0$ and let $f, g : G \to cc(Y)$ be set-valued functions satisfying*

$$d_H\big[f(x+y)+f(x-y)+g(x+y), 2f(x)+2f(y)+g(x)+g(y)\big] \leq \varepsilon, \tag{3.15}$$

for each $x, y \in G$. Then there exists a unique additive function $A : G \to Y$ and a unique quadratic set-valued function $Q : G \to cc(Y)$ satisfying

$$d_H\big[A(x)+g(0), g(x)\big] \leq 12\varepsilon, \quad x \in G,$$
$$d_H\big[Q(x)+f(0), f(x)\big] \leq 4\varepsilon, \quad x \in G.$$

Proof. Substitute $x = y = 0$ into (3.15) to derive (3.4). By replacing y with $-y$ in (3.15), we acquire

$$d_H\big[f(x+y)+f(x-y)+g(x-y), 2f(x)+2f(-y)+g(x)+g(-y)\big] \leq \varepsilon,$$

for all $x, y \in G$. As d_H remains unchanged under translation, we derive

$$\begin{aligned} d_H\big[&f(x+y)+f(x-y)+g(x+y)+g(x-y),\\ &2f(x)+2f(-y)+g(x)+g(-y)+g(x+y)\big] \leq \varepsilon, \end{aligned} \tag{3.16}$$

for all $x, y \in G$. Applying the triangle inequality, it can be deduced from Proposition 3.2, (3.15) and (3.16) that

$$d_H\big[g(x-y)+g(y)+2f(y), g(x+y)+g(-y)+2f(-y)\big] \leq 2\varepsilon, \ x, y \in G. \tag{3.17}$$

If we substitute $x = 0$ in (3.17), we acquire

$$d_H\big[2f(y), 2f(-y))\big] \leq 2\varepsilon, \quad y \in G. \tag{3.18}$$

From Proposition 3.2, (3.17) and (3.18), it can be concluded that

$$d_H\big[g(x-y)+g(y), g(x+y)+g(-y)\big] \leq 4\varepsilon, \quad x, y \in G.$$

By a similar argument as presented in the proof of Theorem 3.3, one can infer

$$d_H\big[g(x)+g(-x), 2g(0)\big] \leq 8\varepsilon, \quad x \in G,$$
$$d_H\big[g(2x)+g(0), 2g(x)\big] \leq 12\varepsilon, \quad x \in G. \tag{3.19}$$

Therefore

$$d_H\left[\frac{1}{2^{n+1}}g(2^{n+1}x)+\sum_{k=m}^{n}\frac{1}{2^{k+1}}g(0),\frac{1}{2^m}g(2^m x)\right]\leq\sum_{k=m}^{n}\frac{6\varepsilon}{2^k},\quad x\in G$$

for all non-negative integers m, n with $m \leq n$. Using the same reasoning as in the proof of Theorem 3.3, one can establish the existence of a unique additive function $A : X \to Y$ satisfying

$$d_H\big[A(x)+g(0),g(x)\big]\leq 12\varepsilon,\quad x\in G.$$

Put $y = -x$ in (3.15) to obtain

(3.20)

$$d_H\big[f(2x)+f(0)+g(0),2f(x)+2f(-x)+g(x)+g(-x)\big]\leq\varepsilon,\quad x\in G.$$

Adding (3.4), (3.18), (3.19) and (3.20), we get

$$d_H\big[f(2x)+3f(0),4f(x)\big]\leq 12\varepsilon,\quad x\in G.$$

The remainder of the proof proceeds through analogous steps to those in the proof of Theorem 3.3 and we omit the specifics. □

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Statements and Declarations

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