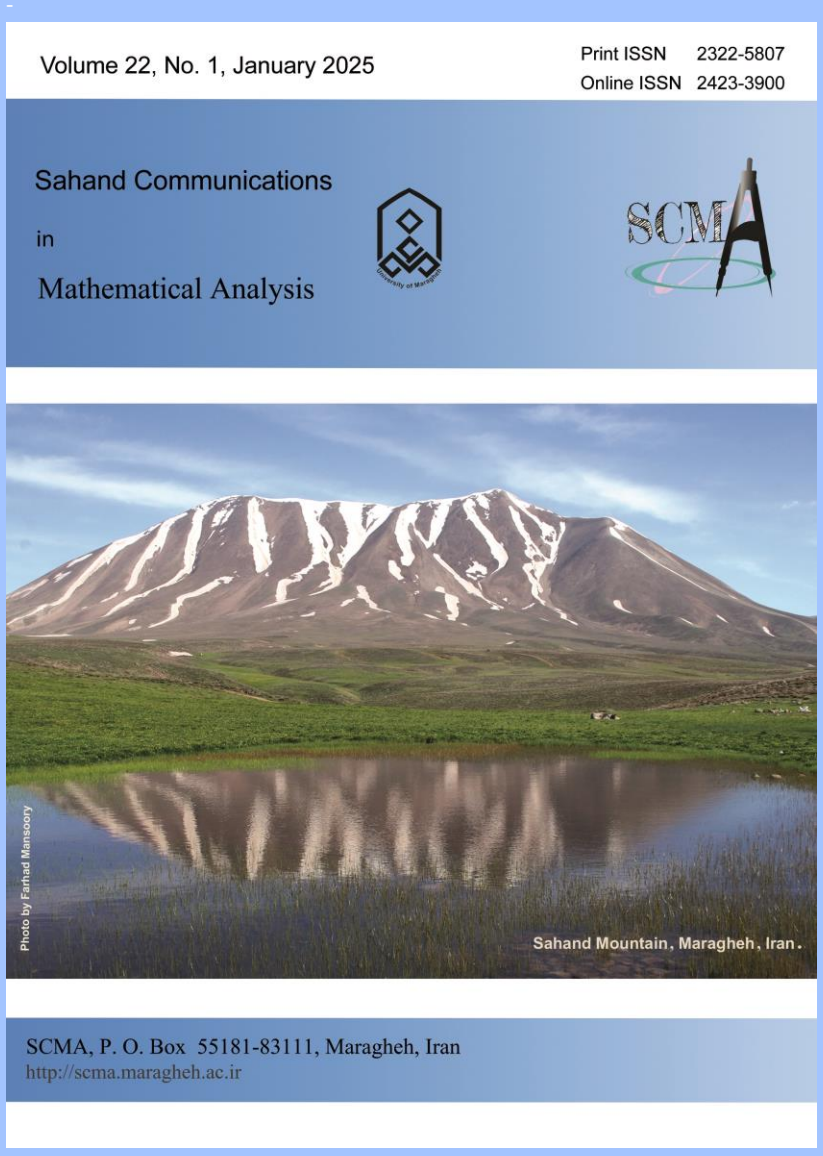
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Lipschitz Analysis of g-Phase Retrievable Frames

Mohammad Ali Hasankhani Fard

Abstract. A g-phase retrievable frame is a λ-phase retrievable frame in finite dimensional Hilbert space $\mathcal{H}_n$, where λ is an special function, which is called phase coefficient function. In this paper we study the Lipschitz analysis of the nonlinear map $\alpha_{\lambda,\mathcal{F}} : \widehat{\mathcal{H}_n} \longrightarrow \mathbb{F}^m, \quad \alpha_{\lambda,\mathcal{F}}(\hat{x}) := \left[\lambda\left(\langle x, f_k\rangle\right)\right]_{1\leq k\leq m}$, where $\widehat{\mathcal{H}_n}$ is the quotient space corresponding to a special equivalence relation on $\mathcal{H}_n$ with respect to phase coefficient function λ, $\mathcal{F} = \{f_k\}_{k=1}^m$ is a λ-phase retrievable frame for $\mathcal{H}_n$, $\mathbb{F} = \mathbb{R}$ for real Hilbert space $\mathcal{H}_n$ and $\mathbb{F} = \mathbb{C}$ for complex Hilbert space $\mathcal{H}_n$.

1. Introduction

The concept of frames in a Hilbert space $\mathcal{H}_n$ space was originally introduced by Duffin and Schaeffer in the context of the non-harmonic Fourier series [11]. Over the past decade, various generalizations of the frames have been proposed such as pseudo-frames, oblique frames, continuous frames, fusion frames, g-frames, and so on. The concept of equal norm Parseval frames in finite-dimensional Hilbert spaces was first introduced by Casazza and Leonhard in [8, 9] and it has developed rapidly over the last ten years, especially in the context of wavelets and Gabor systems.

Given an n-dimensional Hilbert space $\mathcal{H}_n$ on $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$ with inner product $\langle .,. \rangle$, a sequence $\mathcal{F} = \{f_k\}_{k=1}^m$ is called a frame for $\mathcal{H}_n$ if there exist constants $A > 0$ and $B < \infty$ such that for all $f \in \mathcal{H}_n$,

$$A \|f\|^2 \leq \sum_{k=1}^{m} |\langle f, f_k\rangle|^2 \leq B \|f\|^2, \tag{1.1}$$

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where A, B are respectively the lower and upper frame bounds. For more information concerning frames refer to[5, 6, 10, 13, 16, 17].

Phase retrievable frames are introuduced in [2]. Also see [1, 3, 4, 7, 12, 15] for their applications.

Generalized phase retrievable frames or simply g-phase retrievable frames in real or complex n-dimensional Hilbert space $\mathcal{H}_n$, which include ordinary phase retrievable frames in $\mathcal{H}_n$ are λ-phase retrievable frames, for some phase coefficient function λ, [14]. G-phase retrievable frames are defined by following special functions:

Definition 1.1 ([14])**.** A phase coefficient function on $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$ is a function $\lambda : \mathbb{F} \to \mathbb{F}$ such that the set

$$\mathcal{N}(\lambda) := \left\{c \in \mathbb{F} \setminus \{0\} : \lambda(ct) = \lambda\left(c^{-1}t\right) = \lambda(t), \forall t \in \mathbb{F}\right\}$$

contains at least a $c \neq 1$. A phase coefficient function λ is called admissible phase coefficient function if $\lambda(t) = 0 \Leftrightarrow t = 0$.

See [14, Example 2.4.] for some examples of phase coefficient functions. Associated with the phase coefficient function λ, the set $\mathcal{N}(\lambda)$ is multiplicative subgroup of $\mathbb{F}$ [14, Proposition 2.3.].

Let the phase coefficient function λ be given. We define the equivalence relation $\sim$ associated to λ on $\mathcal{H}_n$ by

$$x \sim y \quad \Leftrightarrow \quad y = cx, \text{ for some } c \in \mathcal{N}(\lambda).$$

For any $x \in \mathcal{H}_n$ the equivalence class of x is denoted by $\hat{x}$. We have $\hat{x} = \{cx : c \in \mathcal{N}(\lambda)\}$. Let $\widehat{\mathcal{H}_n} := \frac{\mathcal{H}_n}{\sim} = \{\hat{x} : x \in \mathcal{H}_n\}$.

Definition 1.2 ([14])**.** Let $\lambda : \mathbb{F} \longrightarrow \mathbb{F}$ be a phase coefficient function.

(a) A frame $\mathcal{F} = \{f_k\}_{k=1}^m$ in $\mathcal{H}_n$ is called λ-phase retrievable frame if for all $x, y \in \mathcal{H}_n$ satisfying $\lambda(\langle x, f_k\rangle) = \lambda(\langle y, f_k\rangle)$ for all $k = 1, 2, \ldots, m$, then $y = cx$ for some $c \in \mathcal{N}(\lambda)$.

(b) A frame $\mathcal{F}$ in $\mathcal{H}_n$ is called a generalized phase retrievable frame or simply a g-phase retrievable frame if there exists a phase coefficient function λ such that $\mathcal{F}$ is λ-phase retrievable frame in $\mathcal{H}_n$.

Let $\mathcal{F} = \{f_k\}_{k=1}^m$ be a frame for $\mathcal{H}_n$ and let $\lambda : \mathbb{F} \longrightarrow \mathbb{F}$ be a phase coefficient function. The map $\alpha_{\lambda,\mathcal{F}}$ defined by

$$\alpha_{\lambda,\mathcal{F}} : \widehat{\mathcal{H}_n} \longrightarrow \mathbb{F}^m, \qquad \alpha_{\lambda,\mathcal{F}}(\hat{x}) := \left[\lambda(\langle x, f_k\rangle)\right]_{1 \leq k \leq m}$$

is well-defined.

Obviously, every ordinary phase retrievable frame is λ_1-phase retrievable frame, where λ_1 is the phase coefficient function defined by $\lambda_1(t) := |t|, \forall t \in \mathbb{F}$. Hence, every ordinary phase retrievable frame is

a g-phase retrievable frame. However, there exists a g-phase retrievable frames for $\mathcal{H}_n$, which is not an ordinary phase retrievable frame for $\mathcal{H}_n$[14, Example 2.6.].

A continuous map $\phi : (X, d_X) \to (Y, d_Y)$ is called Lipschitz continuous if

$$Lip(\phi) := \sup_{x,x' \in X} \frac{d_Y(\phi(x), \phi(x'))}{d_X(x, x')} < \infty.$$

Moreover, the map ϕ is called bi-Lipschitz with lower Lipschitz constant C and upper Lipschitz constant D if for every $x, x' \in X$,

$$Cd_X(x, x') \leq d_Y(\phi(x), \phi(x')) \leq Dd_X(x, x').$$

In this paper two metrics on the quotient space corresponding to a special equivalence relation on $\mathcal{H}_n$ with respect to phase coefficient function λ are introduced and the lower and upper Lipschitz constants of the the nonlinear map $\alpha_{\lambda,\mathcal{F}} : \widehat{\mathcal{H}_n} \longrightarrow \mathbb{F}^m, \quad \alpha_{\lambda,\mathcal{F}}(\hat{x}) := \left[\lambda\left(\langle x, f_k\rangle\right)\right]_{1\leq k\leq m}$ associated to the λ-phase retrievable frame $\mathcal{F} = \{f_k\}_{k=1}^m$ in $\mathcal{H}_n$, with respect to these metrics are investigated.

2. Lipschitz Analysis

In this section, the Lipschitz analysis of the nonlinear map $\alpha_{\lambda,\mathcal{F}}$ is investigated.

To study the Lipschitz properties of nonlinear map $\alpha_{\lambda,\mathcal{F}}$ associated to the λ-phase retrievable frame $\mathcal{F} = \{f_k\}_{k=1}^m$ for $\mathcal{H}_n$, we present two metrics on $\widehat{\mathcal{H}_n}$. Let $\mathbb{S} := \{x \in \mathcal{H}_n;\ \|x\| = 1\}$.

Proposition 2.1. *Suppose that $\lambda : \mathbb{F} \longrightarrow \mathbb{F}$ is a continuous phase coefficient function such that $\mathcal{N}(\lambda)$ is compact subset of $\mathbb{F}$.*

(a) *The map*

$$d_0(\hat{x}, \hat{y}) := \begin{cases} \sup_{c,d\in\mathcal{N}(\lambda)} \|cx - dy\|, & \text{if } \hat{x} \neq \hat{y}, \\ 0, & \text{if } \hat{x} = \hat{y}, \end{cases}$$

defines a metric on $\widehat{\mathcal{H}_n}$.

(b) *If λ is admissible and*

$$|\lambda(s+t)| \leq |\lambda(s)| + |\lambda(t)|, \quad \text{for all } s, t \in \mathbb{F}, \tag{2.1}$$

then the map

$$d(\hat{x}, \hat{y}) := \inf_{c\in\mathcal{N}(\lambda)} \sup_{z\in\mathbb{S}} |\lambda(\langle y - cx, z\rangle)|$$

defines a metric on $\widehat{\mathcal{H}_n}$.

Proof. It is easy to show that d_0 is a metric on $\widehat{\mathcal{H}_n}$.

Let $\hat{x}, \hat{y}, \hat{z} \in \widehat{\mathcal{H}_n}$. It is obvious that $d(\hat{x}, \hat{y}) \geq 0$. If $\hat{x} = \hat{y}$, then $x \sim y$ and hence $y = c_0 x$, for some $c_0 \in \mathcal{N}(\lambda)$. Using admissibility of λ

$$\begin{aligned} d(\hat{x}, \hat{y}) &= \inf_{c \in \mathcal{N}(\lambda)} \sup_{z \in \mathbb{S}} |\lambda(\langle y - cx, z\rangle)| \\ &\leq \sup_{z \in \mathbb{S}} |\lambda(\langle y - c_0 x, z\rangle)| \\ &= 0. \end{aligned}$$

Also, if $d(\hat{x}, \hat{y}) = 0$, then there exists a $c_0 \in \mathcal{N}(\lambda)$ such that $\sup_{z \in \mathbb{S}} |\lambda(\langle y - c_0 x, z\rangle)| = 0$. Using admissibility of λ we have $\langle y - c_0 x, z\rangle = 0$, for all $z \in \mathbb{S}$ and hence $y = c_0 x$. Thus $\hat{x} = \hat{y}$.

There exists a $c_0 \in \mathcal{N}(\lambda)$ such that

$$\begin{aligned} d(\hat{x}, \hat{y}) + \epsilon &\geq \sup_{z \in \mathbb{S}} |\lambda(\langle y - c_0 x, z\rangle)| \\ &= \sup_{-z \in \mathbb{S}} |\lambda(\langle y - c_0 x, z\rangle)| \\ &= \sup_{z \in \mathbb{S}} |\lambda(\langle y - c_0 x, -z\rangle)| \\ &= \sup_{z \in \mathbb{S}} |\lambda(\langle c_0 x - y, z\rangle)| \\ &= \sup_{z \in \mathbb{S}} \left|\lambda\left(c_0 \left\langle x - c_0^{-1} y, z\right\rangle\right)\right| \\ &= \sup_{z \in \mathbb{S}} \left|\lambda\left(\left\langle x - c_0^{-1} y, z\right\rangle\right)\right| \\ &\geq \inf_{c \in \mathcal{N}(\lambda)} \sup_{z \in \mathbb{S}} |\lambda(\langle x - cy, z\rangle)| \\ &= d(\hat{y}, \hat{x}). \end{aligned}$$

Letting $\epsilon \to 0$, we have $d(\hat{x}, \hat{y}) \geq d(\hat{y}, \hat{x})$. Similarly, we have $d(\hat{y}, \hat{x}) \geq d(\hat{x}, \hat{y})$ and hence $d(\hat{x}, \hat{y}) = d(\hat{y}, \hat{x})$.

There exist a $c_1, c_2 \in \mathcal{N}(\lambda)$ such that

$$d(\hat{x}, \hat{y}) + \frac{\epsilon}{2} \geq \sup_{u \in \mathbb{S}} |\lambda(\langle y - c_1 x, u\rangle)|, \qquad d(\hat{y}, \hat{z}) + \frac{\epsilon}{2} \geq \sup_{u \in \mathbb{S}} |\lambda(\langle z - c_2 y, u\rangle)|.$$

Using inequality (2.1), for any $u \in \mathbb{S}$ we have

$$\begin{aligned} |\lambda(\langle z - c_1 c_2 x, u\rangle)| &= |\lambda(\langle z - c_2 y + c_2 y - c_1 c_2 x, u\rangle)| \\ &= |\lambda(\langle z - c_2 y, u\rangle + c_2 \langle y - c_1 x, u\rangle)| \\ &\leq |\lambda(\langle z - c_2 y, u\rangle)| + |\lambda(c_2 \langle y - c_1 x, u\rangle)| \\ &= |\lambda(\langle z - c_2 y, u\rangle)| + |\lambda(\langle y - c_1 x, u\rangle)| \\ &\leq d(\hat{x}, \hat{y}) + d(\hat{y}, \hat{z}) + \epsilon. \end{aligned}$$

Now we have $\sup_{u\in\mathbb{S}} |\lambda(\langle z - c_1c_2x, u\rangle)| \leq d(\hat{x},\hat{y}) + d(\hat{y},\hat{z}) + \epsilon$ and hence

$$\begin{aligned} d(\hat{x},\hat{z}) &= \inf_{c\in\mathcal{N}(\lambda)} \sup_{u\in\mathbb{S}} |\lambda(\langle z - cx, u\rangle)| \\ &\leq \sup_{u\in\mathbb{S}} |\lambda(\langle z - c_1c_2x, u\rangle)| \\ &\leq d(\hat{x},\hat{y}) + d(\hat{y},\hat{z}) + \epsilon. \end{aligned}$$

Using $\epsilon \to 0$, we obtain $d(\hat{x},\hat{z}) \leq d(\hat{x},\hat{y}) + d(\hat{y},\hat{z})$. Thus d is a metric on $\widehat{\mathcal{H}_n}$. □

For Lipschitz continuous phase coefficient function λ such that $\mathcal{N}(\lambda)$ is compact subset of $\mathbb{F}$, an upper Lipschitz constant of $\alpha_{\lambda,\mathcal{F}}$ with respect to metric d_0 on $\widehat{\mathcal{H}_n}$ is presented in the next theorem.

Theorem 2.2. *Suppose that $\lambda : \mathbb{F} \longrightarrow \mathbb{F}$ is a Lipschitz continuous phase coefficient function such that $\mathcal{N}(\lambda)$ is compact subset of $\mathbb{F}$ and suppose $\mathcal{F} = \{f_k\}_{k=1}^m$ is a λ-phase retrievable frame for $\mathcal{H}_n$. Then*

$$\|\alpha_{\lambda,\mathcal{F}}(\hat{x}) - \alpha_{\lambda,\mathcal{F}}(\hat{y})\| \leq D\sqrt{B}d_0(\hat{x},\hat{y}), \quad \forall \hat{x},\hat{y} \in \hat{\mathcal{H}},$$

where B is the upper frame bound of $\mathcal{F}$ and D, the upper Lipschitz constant of λ.

Proof. Let $\hat{x},\hat{y} \in \hat{\mathcal{H}}$. If $\hat{x} \neq \hat{y}$, then

$$\begin{aligned} \|\alpha_{\lambda,\mathcal{F}}(\hat{x}) - \alpha_{\lambda,\mathcal{F}}(\hat{y})\|^2 &= \sum_{k=1}^m |\lambda(\langle x, f_k\rangle) - \lambda(\langle y, f_k\rangle)|^2 \\ &\leq D^2 \sum_{k=1}^m |\langle x, f_k\rangle - \langle y, f_k\rangle|^2 \\ &= D^2 \sum_{k=1}^m |\langle x - y, f_k\rangle|^2 \\ &\leq D^2 B \|x - y\|^2 \\ &\leq D^2 B d_0^2(\hat{x},\hat{y}). \end{aligned}$$

Thus

$$\|\alpha_{\lambda,\mathcal{F}}(\hat{x}) - \alpha_{\lambda,\mathcal{F}}(\hat{y})\| \leq D\sqrt{B}d_0(\hat{x},\hat{y}).$$

It is obvious that the above inequality holds in the case where $\hat{x} = \hat{y}$. □

The following example shows that it may be the map $\alpha_{\lambda,\mathcal{F}}$ has no lower Lipschitz constant, even $\lambda : \mathbb{F} \longrightarrow \mathbb{F}$ is a Lipschitz continuous phase coefficient function such that $\mathcal{N}(\lambda)$ is compact subset of $\mathbb{F}$ and $\mathcal{F} = \{f_k\}_{k=1}^m$ is a λ-phase retrievable frame for $\mathcal{H}_n$.

Example 2.3. Consider the function $\lambda : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $\lambda(t) = \sqrt{1+t^2}$ and the frame $\mathcal{F} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^2$. It is easy to show that λ is Lipschitz continuous phase coefficient function with upper Lipschitz constant 1, $\mathcal{N}(\lambda) = \{-1, 1\}$ and $\mathcal{F}$ is a λ-phase retrievable frame for $\mathbb{R}^2$.

Let $C > 0$ be given. There exists $n \in \mathbb{N}$ such that $n > \frac{1}{C}$. Thus for vectors vector $x := \begin{bmatrix} \frac{1}{n} \\ 0 \end{bmatrix}, y := \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \mathbb{R}^2$ we have

$$\begin{aligned} \|\alpha_{\lambda,\mathcal{F}}(\hat{x}) - \alpha_{\lambda,\mathcal{F}}(\hat{y})\|^2 &= \sum_{k=1}^{3} |\lambda(\langle x, f_k\rangle) - \lambda(\langle y, f_k\rangle)|^2 \\ &= 2\left|\lambda\left(\frac{1}{n}\right) - \lambda(0)\right|^2 \\ &= \frac{2}{n^4\left(\sqrt{1+\frac{1}{n^2}}+1\right)^2} \\ &< \frac{1}{n^4} \\ &= \frac{1}{n^2} d_0^2(\hat{x}, \hat{y}) \\ &< C^2 d_0^2(\hat{x}, \hat{y}) \end{aligned}$$

and hence $\|\alpha_{\lambda,\mathcal{F}}(\hat{x}) - \alpha_{\lambda,\mathcal{F}}(\hat{y})\| < C d_0(\hat{x}, \hat{y})$.

If $\lambda : \mathbb{F} \longrightarrow \mathbb{F}$ is a phase coefficient function, then it has no lower Lipschitz constant. Indeed, if $\lambda : \mathbb{F} \longrightarrow \mathbb{F}$ is a function with lower Lipschitz constant C and there is a $c \in \mathbb{F}$ such that $\lambda(ct) = \lambda(t)$ for all $t \in \mathbb{F}$, then

$$C|ct - t| \leq |\lambda(ct) - \lambda(t)| = 0, \quad \forall\, t \in \mathbb{F}$$

and hence $c = 1$. Thus λ is not phase coefficient function, which is a contradiction.

Definition 2.4. A phase coefficient function $\lambda : \mathbb{F} \longrightarrow \mathbb{F}$ is weakly bi-lipschitz if there are consant $0 < C \leq D < \infty$ such for any $s, t \in \mathbb{F}$, there exists $c = c_{s,t} \in \mathcal{N}(\lambda)$ such that

$$C|s - ct| \leq |\lambda(s) - \lambda(t)| \leq D|s - t|.$$

The following theorem presents a class of admissible weakly bi-lipschitz phase coefficient function.

Theorem 2.5. *Suppose that $\phi : [0, \infty) \longrightarrow [0, \infty)$ is a continuous function, for which there are $0 < C \leq D < \infty$ such that $C \leq \phi(r) \leq D$, for*

all $r \in [0,\infty)$. Then the function λ_ϕ defined by

$$\lambda_\phi : \mathbb{F} \longrightarrow \mathbb{F}$$
$$\lambda_\phi(t) := \int_0^{|t|} \phi(r)\, dr$$

is an admissible weakly bi-lipschitz phase coefficient function with weakly bi-lipschitz constants C, D.
Moreover, if ϕ is also decreasing function, then λ_ϕ is also satisfied in inequality (2.1).

Proof. For any $c \in \mathbb{F}$ such that $|c| = 1$ we have $\lambda_\phi(ct) = \int_0^{|ct|} \phi(r)\, dr = \int_0^{|t|} \phi(r)\, dr = \lambda_\phi(t) = \lambda_\phi\left(c^{-1}t\right)$, for all $t \in \mathbb{F}$. On the other hand, suppose $c \in \mathbb{F}$ and $|c| \neq 1$. With less of the generality we can assume that $|c| < 1$. Then for $t \neq 0$

$$\begin{aligned}
\lambda_\phi(t) &= \int_0^{|t|} \phi(r)\, dr \\
&= \int_0^{|ct|} \phi(r)\, dr + \int_{|ct|}^{|t|} \phi(r)\, dr \\
&\geq \lambda_\phi(ct) + C(|t| - |ct|) \\
&> \lambda_\phi(ct).
\end{aligned}$$

Thus λ_ϕ is a phase coefficient function with $\mathcal{N}(\lambda) = \{c \in \mathbb{F};\ |c| = 1\}$ which is compact subset of $\mathbb{F}$. It is straightforward to show that λ_ϕ is also admissible.

Also for any $s, t \in \mathbb{F}$ we have

$$\begin{aligned}
|\lambda_\phi(s) - \lambda_\phi(t)| &= \left| \int_0^{|s|} \phi(r)\, dr - \int_0^{|t|} \phi(r)\, dr \right| \\
&\leq D\,||s| - |t|| \\
&\leq D\,|s - t|.
\end{aligned}$$

Suppose $s, t \in \mathbb{F}$ and $s \neq t$. With less of the generality we can assume that $|s| < |t|$. Thus

$$\begin{aligned}
|\lambda_\phi(s) - \lambda_\phi(t)| &= \left| \int_0^{|s|} \phi(r)\, dr - \int_0^{|t|} \phi(r)\, dr \right| \\
&= \left| \int_{|s|}^{|t|} \phi(r)\, dr \right| \\
&= \int_{|s|}^{|t|} \phi(r)\, dr
\end{aligned}$$

$$\begin{aligned} &\geq C\left||s|-|t|\right| \\ &= C\left|s-ct\right|, \end{aligned}$$

where $c = c_{s,t} := \frac{|s|\bar{t}}{|t|\bar{s}} \in \mathcal{N}(\lambda)$ for $s,t \neq 0$ and $c = c_{s,t} := 1 \in \mathcal{N}(\lambda)$ for $s = 0$ or $t = 0$.

Now suppose that ϕ is also decreasing function and $s,t \in \mathbb{F}$.

If $|s| \leq |t|$, then using mean value theorem for integrals, there are numbers $0 \leq d_1 \leq |s|$ and $|t| \leq d_2 \leq |s|+|t|$ such that

$$\int_0^{|s|} \phi(r)\,dr = |s|\,\phi(d_1), \qquad \int_{|t|}^{|s|+|t|} \phi(r)\,dr = |s|\,\phi(d_2).$$

Thus

$$\begin{aligned} \int_0^{|s|+|t|} \phi(r)\,dr &= \int_0^{|t|} \phi(r)\,dr + \int_{|t|}^{|s|+|t|} \phi(r)\,dr \\ &= \int_0^{|t|} \phi(r)\,dr + |s|\,\phi(d_2) \\ &\leq \int_0^{|t|} \phi(r)\,dr + |s|\,\phi(d_1) \\ &= \int_0^{|t|} \phi(r)\,dr + \int_0^{|s|} \phi(r)\,dr. \end{aligned}$$

If $|t| \leq |s|$, then using mean value theorem for integrals, there are numbers $0 \leq d_1 \leq |t|$ and $|s| \leq d_2 \leq |s|+|t|$ such that

$$\int_0^{|t|} \phi(r)\,dr = |t|\,\phi(d_1), \qquad \int_{|s|}^{|s|+|t|} \phi(r)\,dr = |t|\,\phi(d_2).$$

Thus

$$\begin{aligned} \int_0^{|s|+|t|} \phi(r)\,dr &= \int_0^{|s|} \phi(r)\,dr + \int_{|s|}^{|s|+|t|} \phi(r)\,dr \\ &= \int_0^{|s|} \phi(r)\,dr + |t|\,\phi(d_2) \\ &\leq \int_0^{|s|} \phi(r)\,dr + |t|\,\phi(d_1) \\ &= \int_0^{|s|} \phi(r)\,dr + \int_0^{|t|} \phi(r)\,dr. \end{aligned}$$

Thus

$$\int_0^{|s|+|t|} \phi(r)\,dr \leq \int_0^{|s|} \phi(r)\,dr + \int_0^{|t|} \phi(r)\,dr, \quad \forall s,t \in \mathbb{F}. \tag{2.2}$$

Using $|s+t| \leq |s| + |t|$ and inequality (2.2), we have

$$\begin{aligned} |\lambda_\phi (s+t)| &= \int_0^{|s+t|} \phi (r)\, dr \\ &\leq \int_0^{|s|+|t|} \phi (r)\, dr \\ &\leq \int_0^{|s|} \phi (r)\, dr + \int_0^{|t|} \phi (r)\, dr \\ &= |\lambda_\phi (s)| + |\lambda_\phi (t)| . \end{aligned}$$ □

Example 2.6. The function ϕ_1 defined by $\phi_1 (r) = 2 + \sin r,\ \forall r \in [0, \infty)$, satisfied in Theorem 2.5 with $C = 1$ and $D = 3$ and hence the associated function λ_{ϕ_1} is an admissible weakly bi-lipschitz phase coefficient function with weakly bi-lipschitz constants $C = 1, D = 3$.

Also the function ϕ_2 defined by $\phi_2 (r) := \frac{r+2}{r+1},\ \forall r \in [0, \infty)$ is a continuous decreasing function such that $1 \leq \phi_2 (r) \leq 2,\ \forall r \in [0, \infty)$ and hence the associated function λ_{ϕ_2} is an admissible weakly bi-lipschitz phase coefficient function with weakly bi-lipschitz constants $C = 1, D = 2$, which satisfied in inequality (2.1).

If λ is an admissible weakly bi-lipschitz phase coefficient function with weakly bi-lipschitz constants C, D, then for any $s, t \in \mathbb{F}$ we have

$$\begin{aligned} |\lambda (s+t)| &= |\lambda (s+t) - \lambda (0)| \\ &\leq D\, |s+t| \\ &\leq D\, |s| + D\, |t| \\ &= D\, |s - c_{s,0} 0| + D\, |t - c_{t,0} 0| \\ &\leq \frac{D}{C} |\lambda (s) - \lambda (0)| + \frac{D}{C} |\lambda (t) - \lambda (0)| \\ &= \frac{D}{C} (|\lambda (s)| + |\lambda (t)|) \end{aligned}$$

and hence $|\lambda (s+t)| \leq |\lambda (s)| + |\lambda (t)|$, for the case $C = D$.

For admissible weakly bi-lipschitz phase coefficient function λ, satisfied in (2.1) such that $\mathcal{N} (\lambda)$ is compact subset of $\mathbb{F}$, lower and upper Lipschitz constants of $\alpha_{\lambda,\mathcal{F}}$ with respect to metric d on $\widehat{\mathcal{H}_n}$ is presented in the next theorem.

Theorem 2.7. *Suppose $\lambda : \mathbb{F} \longrightarrow \mathbb{F}$ is an admissible weakly bi-lipschitz phase coefficient function, which is satisfied in (2.1) and $\mathcal{N} (\lambda)$ is compact subset of $\mathbb{F}$. Suppose $\mathcal{F} = \{f_k\}_{k=1}^m$ is a λ-phase retrievable frame for $\mathcal{H}_n$. Then*

$$\frac{C}{D} \sqrt{A} d(\hat{x}, \hat{y}) \leq \|\alpha_{\lambda,\mathcal{F}} (\hat{x}) - \alpha_{\lambda,\mathcal{F}} (\hat{y})\| \leq \frac{D}{C} \sqrt{B} d(\hat{x}, \hat{y}), \quad \forall \hat{x}, \hat{y} \in \hat{\mathcal{H}},$$

where A, B are the lower and upper frame bound of $\mathcal{F}$, respectively and C, D are the lower and upper weakly bi-lipschitz constants of λ, respectively.

Proof. Using admissibility of λ, for any $s \in \mathbb{F}$ we have

$$\begin{aligned} C|s| &= C|s - c_{s,o}0| \leq |\lambda(s) - \lambda(0)| \\ &= |\lambda(s)| \\ &= |\lambda(s) - \lambda(0)| \\ &\leq D|s - 0| \\ &= D|s| \end{aligned}$$

and hence

$$\begin{aligned} \|\alpha_{\lambda,\mathcal{F}}(\hat{x}) - \alpha_{\lambda,\mathcal{F}}(\hat{y})\|^2 &= \sum_{k=1}^{m} |\lambda(\langle x, f_k\rangle) - \lambda(\langle y, f_k\rangle)|^2 \\ &= \sum_{k=1}^{m} |\lambda(\langle x, f_k\rangle) - \lambda(\langle cy, f_k\rangle)|^2 \\ &\leq D^2 \sum_{k=1}^{m} |\langle x, f_k\rangle - \langle cy, f_k\rangle|^2 \\ &= D^2 \sum_{k=1}^{m} |\langle x - cy, f_k\rangle|^2 \\ &\leq D^2 B \|x - cy\|^2 \\ &= D^2 B \sup_{z\in\mathbb{S}} |\langle x - cy, z\rangle|^2 \\ &\leq \frac{D^2}{C^2} B \sup_{z\in\mathbb{S}} |\lambda(\langle x - cy, z\rangle)|^2, \end{aligned}$$

for any $\hat{x}, \hat{y} \in \hat{\mathcal{H}}$ and $c \in \mathcal{N}(\lambda)$. Thus

$$\begin{aligned} \|\alpha_{\lambda,\mathcal{F}}(\hat{x}) - \alpha_{\lambda,\mathcal{F}}(\hat{y})\| &\leq \frac{D}{C}\sqrt{B} \inf_{c\in\mathcal{N}(\lambda)} \sup_{z\in\mathbb{S}} |\lambda(\langle x - cy, z\rangle)| \\ &= \frac{D}{C}\sqrt{B} d(\hat{x}, \hat{y}). \end{aligned}$$

On the other hand, we have $|\lambda(\langle v, z\rangle)| \leq D|\langle v, z\rangle| \leq D\|v\|$, for any $v \in \mathcal{H}$ and $z \in \mathbb{S}$. Thus for any $\hat{x}, \hat{y} \in \hat{\mathcal{H}}$, there exists $c_k = c_{x,y,k} \in \mathcal{N}(\lambda)$ such that for all $z \in \mathbb{S}$ we have

$$\|\alpha_{\lambda,\mathcal{F}}(\hat{x}) - \alpha_{\lambda,\mathcal{F}}(\hat{y})\|^2 = \sum_{k=1}^{m} |\lambda(\langle x, f_k\rangle) - \lambda(\langle y, f_k\rangle)|^2$$

$$
\begin{aligned}
&\geq C^2 \sum_{k=1}^{m} |\langle x, f_k\rangle - c_k \langle y, f_k\rangle|^2 \\
&\geq C^2 \sum_{k=1}^{m} |\langle x, f_k\rangle - c_0 \langle y, f_k\rangle|^2 \\
&= C^2 \sum_{k=1}^{m} |\langle x - c_0 y, f_k\rangle|^2 \\
&\geq C^2 A \|x - c_0 y\|^2 \\
&\geq \frac{C^2}{D^2} A |\lambda(\langle x - c_0 y, z\rangle)|^2,
\end{aligned}
$$

where $c_0 = c_0(x, y) := \inf_{c \in \mathcal{N}(\lambda)} \Lambda_{x,y}(c)$ and $\Lambda_{x,y}$ is the continuous function defined on $\mathbb{F}$ by $\Lambda_{x,y}(c) := \sum_{k=1}^{m} |\langle x, f_k\rangle - c \langle y, f_k\rangle|^2$, for all $c \in \mathbb{F}$. Thus

$$
\begin{aligned}
\|\alpha_{\lambda,\mathcal{F}}(\hat{x}) - \alpha_{\lambda,\mathcal{F}}(\hat{y})\| &\geq \frac{C}{D}\sqrt{A} \sup_{z \in \mathbb{S}} |\lambda(\langle x - c_0 y, z\rangle)| \\
&\geq \frac{C}{D}\sqrt{A} \inf_{c \in \mathcal{N}(\lambda)} \sup_{z \in \mathbb{S}} |\lambda(\langle x - c y, z\rangle)| \\
&= \frac{C}{D}\sqrt{A} d(\hat{x}, \hat{y}). \qquad \square
\end{aligned}
$$

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