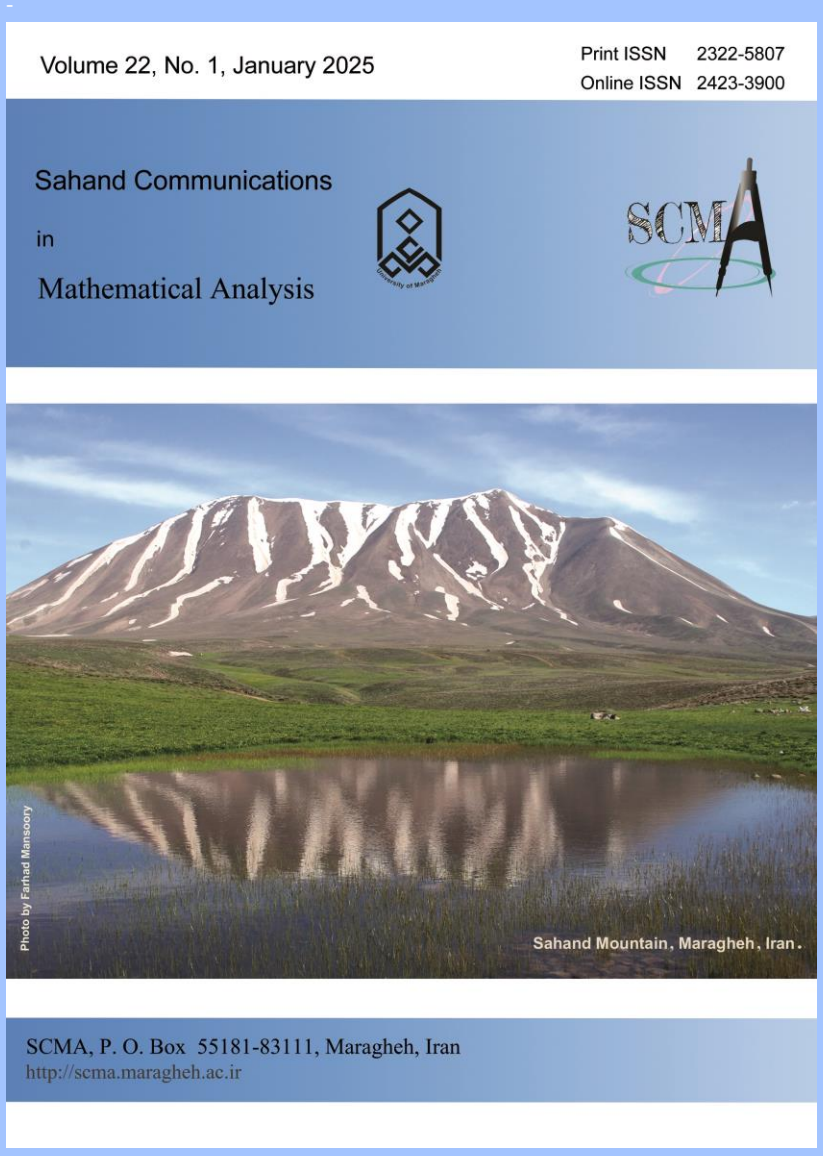
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Recognition of Finite Bitopological Spaces with Unique Isolated Point

Devi Selvam[1*] and Sivaramakrishnan Monikandan[2]

Abstract. A mapping f from a bitopological space (X, τ_1, τ_2) into a bitopological space $(Y, \tau_1^{'}, \tau_2^{'})$ is said to be a pair-homeomorphism if and only if the induced functions $f_1 : (X, \tau_1) \to (Y, \tau_1^{'})$ and $f_2 : (X, \tau_2) \to (Y, \tau_2^{'})$ are homeomorphisms. The deck of a bitopological space (X, τ_1, τ_2) is the set $\mathscr{D}(X) = \{[X_x] : x \in X\}$, where $[Z]$ denotes the pair-homeomorphism class of Z. A bitopological space X is reconstructible if whenever $\mathscr{D}(X) = \mathscr{D}(Y)$ then (X, τ_1, τ_2) is pair-homeomorphic to $(Y, \tau_1^{'}, \tau_2^{'})$. A property $\mathscr{P}$ of a bitopological space X is recognizable if $\mathscr{D}(X) = \mathscr{D}(Y)$ implies "X has $\mathscr{P}$ if and only if Y has $\mathscr{P}$". It is shown that every finite bitopological space, with a unique isolated point and at most two non pair-homeomorphic cards, are recognizable.

1. Introduction

A card $G - v$ of a graph G is obtained by deleting the vertex v and all edges incident with v. The collection of all cards of G is called the deck of G and denoted by $\mathscr{D}(G)$. The graph reconstruction conjecture, posed by Kelly and Ulam [11] in 1941, asserts that every graph G on n (≥ 3) vertices is reconstructible. More precisely, if G and H are finite graphs with at least three vertices such that $\mathscr{D}(H) = \mathscr{D}(G)$, then G and H are isomorphic.

In 2016, Pitz and Suabedissen [9] introduced the concept of reconstruction in topological spaces as follows. For a topological space X, the subspace X_x is called a card of X. The set $\mathscr{D}(X) = \{[X_x] \ : \ x \in X\}$ of

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subspaces of X is called the deck of X, where $[X_x]$ denotes the homeomorphism class of the card X_x. Given topological spaces X and Z, we say that Z is a reconstruction of X if their decks agree. A topological space X is said to be reconstructible if the only reconstructions of it are the spaces homeomorphic to X. Formally, a space X is reconstructible if $\mathscr{D}(X) = \mathscr{D}(Z)$ implies $X \cong Z$. A parameter (or a property)$\mathscr{P}$ of a topological space X is reconstructible (or recognizable respectively) if $\mathscr{D}(X) = \mathscr{D}(Z)$ implies "X has $\mathscr{P}$ if and only if Z has $\mathscr{P}$.

Gartside et al. [5, 6, 9] have proved that the space of real numbers, the space of rational numbers, the space of irrational numbers, every compact Hausdorff space that has a card with a maximal finite compactification, and every Hausdorff continuum X with weight $\omega(X) < |X|$ are reconstructible. In their paper, they also proved certain properties of a space, namely all hereditary separation axioms and all cardinal invariants are reconstructible.

Kelly [7] studied a set with two topologies, called a bitopological space, which is a natural topological structure associated with a quasi-pseudometric on the set. Anat Jaslin Jini [1] has studied reconstruction of finite topological spaces. Anat Jaslin Jini and Monikandan [2] have proved certain bitopological spaces and few of their properties are reconstructible. Devi and Monikandan [3] have proved that finite bitopological spaces with no isolated point are recognizable. By the order of a set, we mean the number of elements in the set. Terms not defined here are taken as in [4].

We recall a few definitions in bitopological spaces, which are based on [8, 10]. Let (X, τ_1, τ_2) and $(Y, \tau_1^{'}, \tau_2^{'})$ be any two bitopological spaces. Any map $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^{'}, \tau_2^{'})$ is said to be a pair-homeomorphism (respectively, pair one-one, pair-onto, pair-continuous function) if and only if the induced functions $f : (X, \tau_1) \to (Y, \tau_1^{'})$ and $f : (X, \tau_2) \to (Y, \tau_2^{'})$ are homeomorphisms (respectively, one-to-one, onto, continuous functions). A subset U of X is said to be pairwise-open if U is both τ_1-open and τ_2-open set of X. If the pairwise-open set is of order k, then it is called pairwise k-open set. A point x of a bitopological space (X, τ_1, τ_2) is said to be an isolated point of X if $\{x\}$ is pairwise-open.

In this paper, we show that every finite bitopological space of order at least four with unique isolated point and at most two non pair-homeomorphic cards is recognizable. The condition $n \geq 4$ is needed because there are non reconstructible bitopological spaces of order 2 and 3. For illustration, consider the space $(\{a, b\}, \tau_1, \tau_2)$ where $\tau_1 = \{\emptyset, \{a\}, X\}$, $\tau_2 = \{\emptyset, X\}$ and the space $(\{a, b\}, \tau_1^{'}, \tau_2^{'})$ where $\tau_1^{'} = \{\emptyset, \{a\}, \{b\}, X\}$, $\tau_2^{'} = \{\emptyset, \{a\}, X\}$. Clearly these two spaces have same deck, but not pair-homeomorphic. For $n = 3$, consider the space $(\{a, b, c\}, \tau_1, \tau_2)$ where

$\tau_1 = \{\emptyset, \{a, b\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, X\}$ and the space $(\{a, b, c\}, \tau_1^{'}, \tau_2^{'})$ where $\tau_1^{'} = \{\emptyset, \{c\}, X\}$, $\tau_2^{'} = \{\emptyset, \{b, c\}, X\}$. Clearly these two spaces have same deck, but not pair-homeomorphic.

2. Spaces Having Unique Isolated Point

In this section we aim to present our main result. By m, we refer the number of cards, which are not mutually pair-homeomorphic, in the deck of the bitopological space X.

The bitopological space (X, τ_1, τ_2) is simply denoted by X, if no confusion arises. The card, obtained by deleting isolated point of X, is called i-card. If (X, τ_1, τ_2) has a τ_2-open set $\{x_j\}$ such that x_j belongs to some τ_1-open set of order two, then the set $\{x_j\}$ is said to be 1_2-open set. The point x_j is called 1_2-point. The τ_1-open set of order two containing 1_2-point is called 2_1-open set and it is denoted by $W_j = \{x_j, x_j^{'}\}$. The point $x_j^{'}$ is called 2_1-point.

We define pairwise 2-open set not containing any isolated point as p_2-set and pairwise 2-open set containing only one isolated point as ip_2-set. Note that p_2-sets are always disjoint. A pairwise 3-open set, not containing any isolated point, is said to be p_3-set if it is disjoint from every pairwise 2-open set and every 2_1-open set. A pairwise 3-open set is said to be ip_3-set if it contains only one isolated point.

The card, obtained by deleting a point in a p_2-set but not in a 2_1-open set, is called α-card. The card, obtained by deleting a non-isolated point in an ip_2-set but not in a 2_1-open set, is called α_i-card. The card, obtained by deleting a point of a p_3-set, is called β-card. The card, obtained by deleting the 1_2-point, is called 1_2-card. The card, obtained by deleting a point in $W_j - \{x_j\}$ but not in a pairwise 2-open set, is called 2_1-card. Note that a pairwise 2-open set can intersect a 2_1-open set only at the point $x_j^{'}$ (otherwise, either 2_1-open set is pairwise 2-open or 1_2-point is isolated point, giving a contradiction). The card $X_{x_j^{'}}$, obtained by deleting the point $x_j^{'}$ which is present in both 2_1-open set and pairwise 2-open set, is called γ-card.

Note that the isolated point does not belong to any 2_1-open set and 1_2-open set (otherwise, it would contradict the definition of 2_1-open set and 1_2-open set).

We illustrate these definitions with an example. Consider the bitopological space (X, τ_1, τ_2), where

$$X = \{a, b, c, d, e, f, g, h\},$$

$$\tau_1 = \Big\{\emptyset, \{a\}, \{b, c\}, \{d, e\}, \{a, b, c\}, \{a, d, e\}, \{f, g, h\}, \{b, c, d, e\},$$
$$\{a, f, g, h\}, \{a, b, c, d, e\}, \{b, c, f, g, h\}, \{d, e, f, g, h\}, \{a, b, c, f, g, h\},$$

$$\{a,d,e,f,g,h\},\{b,c,d,e,f,g,h\},X\Big\},$$

$$\tau_2=\Big\{\emptyset,\{a\},\{e\},\{a,e\},\{b,c\},\{a,b,c\},\{b,c,e\},\{f,g,h\},\{a,b,c,e\},$$
$$\{a,f,g,h\},\{e,f,g,h\},\{a,e,f,g,h\},\{b,c,f,g,h\},\{a,b,c,f,g,h\},$$
$$\{b,c,e,f,g,h\},\{a,b,c,e,f,g,h\},X\Big\}.$$

Here the point a is the isolated point. The pairwise 2-open set $\{b,c\}$ is the p_2-set and the pairwise 3-open set $\{f,g,h\}$ is the p_3-set and $\{a,b,c\}$ is the ip_3-set. The τ_1-open set $\{d,e\}$ is 2_1-open set and the τ_2-open set $\{e\}$ is 1_2-open set.

A topological space (X,τ) is said to have an ascending chain if all the open sets of X together form an ascending chain. For example, let $X=\{a,b,c,d,e\}$ and put $\tau_1=\{\emptyset,\{a\},\{a,b,c\},\{a,b,c,d\},X\}$.

Lemma 2.1. *Let (X,τ_1,τ_2) be a bitopological space with $m=1$. Then (X,τ_1) is discrete and (X,τ_2) has an ascending chain consisting of an isolated point and only one proper non empty τ_2-open set of order r, for every $r=2,3,\ldots,n-1$, if and only if the deck consists of only one card $(X_x,\tau_{1x},\tau_{2x})$, where (X_x,τ_{1x}) is discrete and (X_x,τ_{2x}) has an ascending chain consisting of an isolated point and only one proper non empty τ_{2x}-open set of order r, for every $r=2,3,\ldots,n-2$.*

Proof. Necessity: Since every subspace of the discrete topological space is discrete, it is clear that (X_x,τ_{1x}) is discrete. Now assume that (X,τ_2) has an ascending chain consisting of an isolated point and only one proper non empty τ_2-open set of order r, for every $r=2,3,\ldots,n-1$. Let V_1, V_2,...,V_{n-1} be the proper τ_2-open sets in the chain where $V_i=\{x_1,x_2,\ldots,x_i\}$ and $V_1\subseteq V_2\subseteq\cdots V_{n-1}$. It is obvious that the subspace (X_x,τ_{2x}) has an ascending chain consisting of an isolated point and only one proper non empty τ_{2x}-open set of order r, for every $r=2,3,\ldots,n-2$.

Now we have to prove that $m=1$. Consider the cards $(X_{x_y},\tau_{1x_y},\tau_{2x_y})$ and $(X_{x_z},\tau_{1x_z},\tau_{2x_z})$, where (X_y,τ_{1y}) and (X_{x_z},τ_{1x_z}) are discrete;

$$\tau_{2x_y}=\{\emptyset,V_1,V_2,\ldots,V_{y-1}=V_y-\{x_y\},\ldots,V_n-\{x_y\}\}$$
$$\tau_{2x_z}=\{\emptyset,V_1,V_2,\ldots,V_{z-1}=V_z-\{x_z\},\ldots,V_n-\{x_z\}\},\quad 0<y<z\le n.$$

The mapping $f:X_{x_y}\to X_{x_z}$ defined by

$$f(x_k)=\begin{cases}x_k & \text{if } k<y \text{ and } k>z\\ x_{k-1} & \text{if } y<k\le z\end{cases}$$

is the required pair-homeomorphism from X_{x_y} to X_{x_z}, completing the necessity part.

Sufficiency: Assume that the deck consists of only one card $(X_x,\tau_{1x},\tau_{2x})$

where (X_x, τ_{1x}) is discrete and (X_x, τ_{2x}) has an ascending chain consisting of an isolated point and only one proper non empty τ_{2x}-open set of order r, for every $r = 2, 3, \ldots, n-2$. Suppose, to the contrary that, (X, τ_1) is not discrete. Then there exists a point $y \in X$ such that $\{y\} \notin \tau_1$. If y is contained in τ_1-open set of order at least three, then (X_z, τ_{1z}), $z \neq y$, is not discrete, giving a contradiction. If y is contained in some τ_1-open set, say U of order two, then (X_p, τ_{1p}), $p \notin U$, is not discrete, again a contradiction. Hence (X, τ_1) is discrete.

Now suppose, to the contrary that, (X, τ_2) has no ascending chain. That is, τ_2-open sets do not form an ascending chain. Then there exist two proper non empty τ_2-open sets, say P and Q such that $P \nsubseteq Q$ and $Q \nsubseteq P$. Then there exist two elements, say p and q such that $p \in P - Q$ and $q \in Q - P$. If $P \cup Q \neq X$, then there exists an element, say r, where $r \notin P \cup Q$. Since τ_{2r}-open sets do not form an ascending chain, the card (X_r, τ_{2r}) has no ascending chain, giving a contradiction. Hence, assume that $P \cup Q = X$. Either P or Q has at least two elements. Without loss of generality, let us take $| P | \geq 2$. Let $p'(\neq p)$ be an element of P. Then, in the card $(X_{p'}, \tau_{2p'})$, the $\tau_{2p'}$-open sets $P - \{p'\}$ and $Q - \{p'\}$ are not contained in each other and so the card does not form an ascending chain, again giving a contradiction. Hence (X, τ_2) has an ascending chain.

Now we have to prove that the ascending chain of (X, τ_2) consists of an isolated point and only one proper non empty τ_2-open set of order r, for every $r = 2, 3, \ldots, n-1$. Suppose, to the contrary that, X has no isolated point. Since (X, τ_2) has an ascending chain, there exists at least two proper non empty τ_2-open sets of order at least two, say P and Q such that either $P \subseteq Q$ or $Q \subseteq P$. Without loss of generality, assume that $P \subseteq Q$. Let q be a point of X such that $q \in Q - P$. We can observe that the card X_q is not homeomorphic to X_p, $p \in P$. This leads to the existence of two non pair-homeomorphic cards, giving a contradiction. Hence the ascending chain consists of only one proper non empty τ_2-open sets of order r, where $r = 1, 2, \ldots, n-1$. Since (X, τ_1) is discrete, $\{x\} \in \tau_1$ for every $x \in X$. This implies the τ_2-open set of order one becomes pairwise-open and so X has an isolated point. □

Lemma 2.2. *Let X be a finite bitopological space with $m = 1$. Then X has a unique isolated point if and only if (X, τ_1) is discrete and (X, τ_2) has an ascending chain consisting of one isolated point and only one proper non empty τ_2-open set of order r for every $r = 2, 3, \ldots, n-1$.*

Proof. Necessity: Assume that the space X with $m = 1$ has a unique isolated point, say x. Since $m = 1$, all cards $(X_z, \tau_{1z}, \tau_{2z})$, for all $z \in X$, are mutually pair-homeomorphic. This implies that (X_z, τ_{1z}), for all

$z \in X$, are mutually homeomorphic whereas (X_z, τ_{2z}), for all $z \in X$, are mutually homeomorphic.

Now we have to prove that the following two conditions hold:

(i) (X, τ_1) is discrete.
(ii) (X, τ_2) has an ascending chain of proper non empty τ_2-open sets of order r, $r = 1, 2, \ldots, n-1$.

If the isolated point x does not belong to any proper non empty τ_1(or τ_2)-open set of order at least two, then we have at least two cards X_x and X_y, $y \neq x \in X$ since the card X_x has fewer number of τ_1(or τ_2)-open sets of order one than the card X_y. This gives a contradiction to $m = 1$. If the isolated point x belongs to some, but not all, proper non empty τ_1(or τ_2)-open sets, say U_i, of order at least two, then we have at least two cards X_x and X_y, $y \neq x \notin U_i$ since the card X_x has fewer number of τ_1(or τ_2)-open sets U_i than the card X_y. This again gives a contradiction to $m = 1$. Hence the isolated point x belongs to every τ_1(or τ_2)-open set.

Let U and V be any two non empty proper τ_1(or τ_2)-open sets such that $x \in U \cap V$. If $U \cap V = \{x\}$, then we get two non pair-homeomorphic cards X_x and X_u, $u \in U - \{x\}$ since the card X_x contains no set V whereas the card X_u contains the set V, again giving a contradiction to $m = 1$. Hence either $U \subseteq V$ or $V \subseteq U$.

Since the isolated point x belongs to every τ_1(or τ_2)-open set, we get an ascending chain of τ_1(or τ_2)-open sets. If the ascending chain does not contain τ_1(or τ_2)-open sets of some order, then we would get $m \geq 2$, since X_x is not pair-homeomorphic to X_y, $y \neq x \in X$. This is again a contradiction to $m = 1$. Hence either τ_1 or τ_2 or both of them contain an ascending chain of open sets having the isolated point of order r, $r = 1, 2, \ldots, n$. Since $\tau_1 \neq \tau_2$, if both τ_1 and τ_2 contains an ascending chain having the isolated point, the chains should be different. That is, some τ_1-open set, say P of order r is different from some τ_2-open set, say Q of same order. In that case, we obviously get at least three non pair-homeomorphic cards since $X_x \ncong X_p \ncong X_q$ where $p \neq q \neq x$, $p \in P$ and $q \in Q$. Hence both τ_1 and τ_2 cannot contain the ascending chain of open sets. Without loss of generality, we have (X, τ_2) has an ascending chain of proper non empty τ_2-open sets of order r, $r = 1, 2, \ldots, n-1$. Thus (ii) holds.

Now considering the (X, τ_1), since X has the isolated point x, $\{x\} \in \tau_1$. As previously discussed, (X, τ_1) should have proper non empty τ_1-open sets, but it cannot have yet another ascending chain containing the isolated point. Since $m = 1$, we have $(X_x, \tau_{1x}) \cong (X_y, \tau_{1y})$ for all non isolated points $y \in X$. The only possibility for (X, τ_1) is to be discrete and therefore (i) holds (otherwise, we will have the following: There

exists a point $y \neq x \in X$ such that $\{y\} \notin \tau_1$. Since the card (X_x, τ_{1x}) has fewer number of one-point τ_1-open sets than the card (X_y, τ_{1y}), the card (X_x, τ_{1x}) is not homeomorphic to (X_y, τ_{1y}). This implies that the card X_x is not pair-homeomorphic to the card X_y, giving a contradiction to $m = 1$).
Sufficiency: Let (X, τ_1, τ_2) be a bitopological space such that (X, τ_1) is discrete and (X, τ_2) has an ascending chain of proper non empty τ_2-open sets of order r, $r = 1, 2, \ldots, n-1$. Since τ_1 is discrete, every τ_2-open set becomes pairwise-open and the only τ_2-open set of order one is none other than the isolated point of the bitopological space X. Also by Lemma 2.1, all cards of X are pair-homeomorphic. □

Theorem 2.3. *Let X be a finite bitopological space with isolated points and $m = 1$. Then X has a unique isolated point if and only if the deck consists of a card $(X_x, \tau_{1x}, \tau_{2x})$ has a unique isolated point where (X_x, τ_{1x}) is discrete and (X_x, τ_{2x}) has an ascending chain of proper non empty τ_{2x}-open sets of order $r-1$, $r = 2, 3, \ldots, n$.*

Proof. Follows by Lemmas 2.1 and 2.2. □

Theorem 2.4. *Let X be a finite bitopological space with isolated points and $m = 2$. Then X has a unique isolated point if and only if exactly one of the following holds.*

(i) *One card has no isolated point with more than one p_2-set whereas the other card has one isolated point with no p_2-set and one ip_2-set.*
(ii) *One card has no isolated point with no p_2-set and no 1_2-open set whereas the other card has one isolated point with at most one p_2-set and no ip_2-set and no 1_2-open set.*
(iii) *One card has no isolated point whereas the other card has two isolated points.*
(iv) *Both the cards have an isolated point such that one card contains no p_2-set and no ip_2-set whereas the other card contains at most one p_2-set and one ip_2-set.*
(v) *Both the cards have an isolated point such that one card contains no p_2-set and more than one ip_2-set whereas the other card contains no p_2-set and one ip_2-set.*
(vi) *One card has k isolated points and the other card has an isolated point and at least $k-1$ ip_2-sets, $k \geq 3$.*
(vii) *One card has two isolated points with one ip_2-set, no p_2-set and no proper ip_3-set and the other card has an isolated point with one p_2-set and no ip_2-set.*

(viii) *One card has an isolated point with at least two p_2-sets and no ip_2-set and the other card has two isolated points with one ip_2-set and at least one p_2-set.*

Proof. Necessity: Assume that X, with $m = 2$, has only one isolated point x_i. If there is no pairwise 2-open set and no 1_2-open set, then the i-card has no isolated point and the other card has isolated point x_i. If there exist at least two (otherwise, $m \geq 3$) pairwise 3-open sets containing the isolated point, then i-card has more than one p_2-set and the card, obtained by deleting the non isolated point of pairwise 3-open set, has no p_2-set and one ip_2-set. Hence (i) holds. If there exist at least one pairwise 3-open set not containing the isolated point, then i-card has no p_2-set and the card, obtained by deleting the point of pairwise 3-open set, has one p_2-set and no ip_2-set. Also both the cards cannot have 1_2-open set. Hence (ii) holds. If there is no pairwise 3-open set, then both cards have no p_2-set and no 1_2-open set, (ii) holds. If there exist at least one 1_2-open set, then we get $m \geq 3$, a contradiction. Hence the space X has only pairwise 2-open set, not any 1_2-open set.

Suppose there exist k pairwise 2-open sets, say U_r, $(r = 1, 2, \ldots, k)$.
Case 1. $x_i \notin U_r$ for any $r = 1, 2, \ldots, k$.

Since $x_i \notin U_r$ for any $r = 1, 2, \ldots, k$, every pairwise 2-open set must be disjoint (otherwise, the space X would have more than one isolated point, a contradiction). Now, the i-card has no isolated point. Also α-cards are pair-homeomorphic (otherwise, $m > 2$ which is not the case under consideration). This card has two isolated points, (iii) holds.
Case 2. $x_i \in U$, where U is the only pairwise 2-open set of X.

Now, α_i-card and i-card are pair-homeomorphic (otherwise, $m \geq 3$, a contradiction). The i-card has an isolated point and the card other than i-card has only one isolated point x_i. If there is no pairwise 3-open set, then both cards have no p_2-set such that the i-card has no ip_2-set and the other card has one ip_2-set, (iv) holds. If there exist at least one pairwise 3-open set not containing the set U_1, then i-card has no p_2-set and no ip_2-set and the card, obtained by deleting the point of pairwise 3-open set, has one p_2-set and one ip_2-set. Hence (iv) holds. If there exist at least two (since $|X| \geq 4$ and $m = 2$) pairwise 3-open sets containing the set U_1, then i-card has more than one ip_2-set and the card, obtained by deleting the point of pairwise 3-open set, which is not in U_1, has one ip_2-set. Also both the cards have no p_2-set. Hence (v) holds.
Case 3. X has at least two pairwise 2-open sets and $x_i \in U_r$ for all $r = 1, 2, \ldots, k$.

Since $m = 2$ and $|X| \geq 4$, the space X has at least three ip_2-sets. Now $U_i \cap U_j = \{x_i\}$, $i \neq j$. Then i-card has at least three isolated

points. Since $m = 2$, the cards other than i-card are pair-homeomorphic and so α_i-cards are pair-homeomorphic. This card has only one isolated point x_i and at least two ip_2-sets, (vi) holds.
Case 4. $x_i \in U_r$ for some $r = 1, 2, \ldots, k$.

If x_i belongs to more than one pairwise 2-open set, then we get $m > 2$. So, assume that x_i belongs to exactly one pairwise 2-open set. Every other pairwise 2-open set not containing x_i (that is, p_2-set) must be disjoint (otherwise, the space X would have two isolated points, a contradiction). Here the α_i-card and i-card must be pair-homeomorphic (otherwise, $m \geq 3$ due to the presence of α-card, a contradiction) and this card has only one isolated point. Since $m = 2$, the α-cards are pair-homeomorphic and this card has two isolated points; If X has exactly one p_2-set, then we have the i-card with one p_2-set and no ip_2-set and we have the α-card with one ip_2-set, no p_2-set and no proper ip_3-set, (vii) holds; If X has more than one p_2-set, then we have the i-card with at least two p_2-sets and no ip_2-set and we have the α-card with one ip_2-set and at least one p_2-set, (viii) holds.

Sufficiency: Suppose that X has more than one isolated point. Then every card would contain at least one isolated point, which contradicts (i), (ii) and (iii). If X had more than one isolated point, then at least one card would contain more than one isolated point, a contradiction to (iv) and (v). Suppose that (vi) holds and that X has more than one isolated point.

If X had more than two isolated points, then each card would contain at least two isolated points, a contradiction. So, assume that X has exactly two isolated points, say x_i, $x_i^{'}$. Since $m = 2$, exactly one card of X has only one isolated point and hence X_{x_i} and $X_{x_i'}$ are pair-homeomorphic. Since there is only one card with $k(> 1)$ isolated points, all cards other than X_{x_i} are pair-homeomorphic. By (vi), the card X_{x_i} with an isolated point has at least $k - 1(k > 3)$ ip_2-sets, say U_r, $r \geq 1$. If one of U_r is pairwise 2-open in X, then X_{x_i} has two isolated points, a contradiction. If none of the U_r is pairwise 2-open in X, then $U_r \cup \{x_i\}$ $(r \geq 1)$ is pairwise 3-open in X. That is, $U_r \cap \{x_i, x_i^{'}\} = \{x_i, x_i^{'}\}$ $(r \geq 1)$. In this case, since $m = 2$, the only possibility for X to have one topology as discrete topology. Since X has two isolated points and has no p_2-sets (otherwise, $m > 2$ due to the presence of pairwise 3-open set), the only possibility of cards to have either one or two isolated points, but not more than two, which leads to a contradiction to (vi).

Suppose that (vii) or (viii) holds and that X has more than one isolated point. In view of the above discussion, we can assume that X has exactly two isolated points, say x_i, $x_i^{'}$. Since $m = 2$, the cards, having one isolated point, X_{x_i} and $X_{x_i'}$ are pair-homeomorphic.

By (vii), consider the card X_{x_i} containing one isolated point, one p_2-set and no ip_2-set. If this p_2-set is pairwise 2-open in X, then the card obtained by deleting some point of that set has three isolated points, a contradiction; If it is not pairwise 2-open in X, then the p_2-set along with deleted point x_i is pairwise 3-open in X. So X has an ip_3-set containing x_i. If there exists an ip_3-set containing $x_i^{'}$, then the card, obtained by deleting the non isolated point of one of the ip_3-sets, has two isolated points and one proper ip_3-set, contradicting (vii). Hence X has only one ip_3-set containing x_i and so the deck consists of at least three cards X_{x_i}, $X_{x_i^{'}}$ and X_{x_t}, where $x_t \neq x_i, x_i^{'}$ and x_t belongs to pairwise 3-open set, which gives a contradiction to $m = 2$.

By (viii), consider the card X_{x_i} containing one isolated point and at least two p_2-sets. Now the space X cannot have p_2-sets since otherwise α-card contains more than two isolated points. Hence the card can have one isolated point and at least two p_2-sets only by deleting the isolated point from two ip_3-sets. So, the card, obtained by deleting the non isolated point of one of the ip_3-sets, has two isolated points x_i and $x_i^{'}$ and one ip_2-set and no p_2-set, giving a contradiction to (viii). □

Theorem 2.5. *The property that a finite bitopological space X, with at most two non pair-homeomorphic cards to have exactly one isolated point, is recognizable.*

Proof. Follows by Theorems 2.3 and 2.4. □

3. Spaces with Bi-Ascending Chain

A bitopological space (X, τ_1, τ_2) is said to have a bi-ascending chain if all the τ_1-open sets of X together form an ascending chain and all the τ_2-open sets of X together form an ascending chain. This section deals with some results on spaces with bi-ascending chain. The following lemma implies that a finite topological space X has an ascending chain if and only if every card of X has an ascending chain.

Lemma 3.1. *Let (X, τ_1, τ_2) be finite bitopological space. Then X has a bi-ascending chain if and only if every card has a bi-ascending chain.*

Proof. Necessity is obvious. Sufficiency: Assume that every card in the deck has a bi-ascending chain and so, in each card, both τ_1-open sets and τ_2-open sets form their ascending chains respectively. Suppose, to the contrary, X does not have a bi-ascending chain and so either τ_1-open sets or τ_2-open sets, say τ_1-open sets, do not form an ascending chain. Then there exist two τ_1-open sets P and Q in X such that $P \nsubseteq Q$ and $Q \nsubseteq P$. Hence none of P and Q is equal to $\emptyset$ or X. Also there exist two elements p and q such that $p \in P - Q$ and $q \in Q - P$. If $P \cup Q \neq X$, then

P and Q are τ_1-open sets in the card X_t, where $t \in X - (P \cup Q)$ and hence all the τ_1-open sets in the card X_t do not form ascending chain. This leads to a contradiction. So assume that $P \cup Q = X$. Now, one of P and Q, say P has at least two elements. Let $p^{'}$ be an element in P such that $p^{'} \neq p$. In the card, $X_{p'}$, the sets $P - \{p^{'}\}$ and $Q - \{p^{'}\}$ are τ_1-open and none of them are contained in the other. Hence τ_1-open sets in the card $X_{p'}$ would not form an ascending chain. It leads to a contradiction. □

Lemma 3.2. *A finite topological space X of order n has an ascending chain which consists of only one open set of order k, for every $k = 1, 2, \ldots, n-1$ if and only if the deck of X contains only one card having an ascending chain which consists of only one open set of order k, for every $k = 1, 2, \ldots, n-2$.*

Proof. Necessity: Assume that X has an ascending chain which consists of only one open set of order k, for every $k = 1, 2 \ldots, n-1$. Let $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_{n-1} \subseteq U_n = X$ be the ascending chain of open sets where $U_k = \{x_1, x_2, \ldots, x_k\}$. Now each card of X has only one open set of order k, for every $k = 1, 2, \ldots, n-2$. Consider the cards X_{x_a} and X_{x_b} of X obtained by deleting points x_a and x_b $(0 < a < b \leq n)$ of X where

$$\tau_{X_{x_a}} = \Big\{\emptyset, U_1, U_2, \ldots, U_{a-1} = U_a - \{x_a\}, \ldots, U_{b-1} = U_b - \{x_a\}, \ldots, U_{n-1} = U_n - \{x_a\} = X_{x_a}\Big\},$$

$$\tau_{X_{x_b}} = \Big\{\emptyset, U_1, U_2, \ldots, U_a, \ldots, U_{b-1} = U_b - \{x_b\}, \ldots, U_{n-1} = U_n - \{x_b\} = X_{x_b}\Big\}.$$

The mapping $h : (X_{x_a}, \tau_{X_{x_a}}) \to (X_{x_b}, \tau_{X_{x_b}})$ by

$$h(x_i) = \begin{cases} x_i & \text{if } i < a \text{ and } i > b \\ x_{i-1} & \text{if } a < i \leq b \end{cases}$$

is a homeomorphism between X_{x_a} and X_{x_b}. Since there is homeomorphism between any two cards, the deck contains exactly one card and it is also known that every card of X has an ascending chain, necessity follows.

Sufficiency: Assume that the deck of X contains only one card having an ascending chain which consists of only one open set of order k, for every $k = 1, 2, \ldots, n-2$. Then it is clear that X has an ascending chain. Suppose, to the contrary that, X has no open set of some order r for $1 \leq r \leq n-1$. If $r = 1$, then the card X_x, obtained by deleting the point, which is not in the open set of order two, has no open set of order one,

giving a contradiction. If $r = 2$, then the card X_x, obtained by deleting the point of open set of order $r-1$, has no open set of order one, giving a contradiction. If $r = n-1$, then the card X_x, obtained by deleting the point of open set of order at most $r-1$, has no open set of order $n-2$, giving a contradiction. If $2 < r < n-1$, then the card X_x, obtained by deleting the point of open set of order at most $r-1$, has no open set of order $r-1$, giving a contradiction. □

Theorem 3.3. *The space X has an isolated point and only one τ_1-open set of order r for every $r = 2, 3, .., n-1$ and no τ_2-open set of order s for any $s = 2, 3, \ldots, n-1$ and X has a bi-ascending chain if and only if $m = 2$, among the two cards having bi-ascending chain, one card X_x has an isolated point and only one τ_{1x}-open set of order r for every $r = 2, 3, \ldots, n-2$ and no τ_{2x}-open set of order s for any $s = 2, 3, \ldots, n-2$ and another card X_y has only one τ_{1y}-open set of order r for every $r = 1, 2, 3, \ldots, n-2$ and τ_{2y} is indiscrete.*

Proof. Necessity: Assume that X has an isolated point, say x_1, and only one τ_1-open set U_r of order r for every $r = 2, 3, .., n-1$ and no τ_2-open set of order s for any $s = 2, 3, \ldots, n-1$ and X has a bi-ascending chain. Let $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_{n-1} \subseteq U_n$ be the ascending chain of τ_1-open sets and $V_1 \subseteq V_n$ be the ascending chain of τ_2-open sets where $U_i = \{x_1, x_2, \ldots, x_i\}$ and $U_1 = V_1 = \{x_1\}$ and $U_n = V_n = X$. By Lemma 3.1, every card of X has a bi-ascending chain. Now, in the card $(X_{x_1}, \tau_{1x_1}, \tau_{2x_1})$, obtained by deleting the isolated point of X, $\tau_{1x_1} = \{\emptyset, U_2 - \{x_1\}, \ldots, U_n - \{x_1\}\}$ and τ_{2x_1} is indiscrete. This implies that the card $(X_{x_1}, \tau_{1x_1}, \tau_{2x_1})$ has only one τ_{1x_1}-open set of order r for every $r = 1, 2, 3, \ldots, n-2$ and τ_{2x_1} is indiscrete. Let X_{x_a} and X_{x_b} be two cards of X obtained by deleting the non isolated points x_a and x_b $(1 < a < b \leq n)$ of X where

$$\tau_{1x_a} = \Big\{\emptyset, U_1, U_2, \ldots, U_{a-1} = U_a - \{x_a\}, \ldots, U_{b-1} = U_b - \{x_a\}, \ldots, U_{n-1} = U_n - \{x_a\}\Big\}$$
$$\tau_{2x_a} = \{\emptyset, V_1, V_{n-1} = V_n - \{x_a\}\}$$
$$\tau_{1x_b} = \{\emptyset, U_1, U_2, \ldots, U_a, \ldots, U_{b-1} = U_b - \{x_b\}, \ldots, U_{n-1} = U_n - \{x_b\}\}$$
$$\tau_{2x_b} = \{\emptyset, V_1, V_{n-1} = V_n - \{x_b\}\}.$$

This implies that the cards X_{x_a} and X_{x_b} have an isolated point and only one τ_{1x}-open set of order r for every $r = 2, 3, \ldots, n-2$ and no τ_{2x}-open set of order s for any $s = 2, 3, \ldots, n-2$. Now it is enough to prove that the cards X_{x_a} and X_{x_b} are pair-homeomorphic. The mapping $h : (X_{x_a}, \tau_{1x_a}, \tau_{2x_b}) \to (X_{x_b}, \tau_{1x_b}, \tau_{2x_b})$ by

$$h(x_k) = \begin{cases} x_k & \text{if } k < a \text{ and } k > b \\ x_{k-1} & \text{if } a < k \leq b \end{cases}$$

is a pair-homeomorphism between X_{x_a} and X_{x_b}. Since there is pair-homeomorphism between any two cards other than X_{x_1}, the deck contains exactly two cards, necessity follows.
Sufficiency: Assume that the deck contains exactly two cards having bi-ascending chain and among those two cards, One card X_x has an isolated point and only one τ_{1x}-open set of order r for every $r = 2, 3, \ldots, n-2$ and no τ_{2x}-open set of order s for any $s = 2, 3, \ldots, n-2$ and another card X_y has only one τ_{1y}-open set of order r for every $r = 1, 2, 3, \ldots, n-2$ and τ_{2y} is indiscrete. By Lemma 3.1, the space X has a bi-ascending chain. By theorem 2.5, the space X has exactly one isolated point, say x_1. By Lemma 3.2, the space (X, τ_1) has an isolated point and only one τ_1-open set of order r for every $r = 2, 3, \ldots, n-1$. Suppose, to the contrary that, X has some proper τ_2-open set of order s, $2 \leq s \leq n-1$. Since X has an isolated point and has a bi-ascending chain, the card (X_x, τ_{2x}) cannot be indiscrete for any $x \in X$, giving a contradiction. □

Theorem 3.4. *Let (X, τ_1, τ_2) be a finite bitopological space with $m \geq 2$. The space X has a bi-ascending chain with only one τ_1-open set of order r, for every $r = 1, 2, \ldots, n-1$, only one τ_2-open set of order one and only one pairwise k-open set for $2 \leq k \leq n-2$ if and only if every card X_x, for $x \in X$, has a bi-ascending chain with only one τ_{1x}-open set of order r, for every $r = 1, 2, \ldots, n-2$ such that only one card, say X_y has exactly one τ_{2y}-open set of order one and only one pairwise k-open set whereas the remaining $m-1$ cards, say X_z, have at most one τ_{2z}-open set of order one and only one pairwise $k-1$-open set for $2 \leq k \leq n-2$.*

Proof. Necessity: Assume that the space X has a bi-ascending chain with only one τ_1-open set of order r, for every $r = 1, 2, \ldots, n-1$, only one τ_2-open set of order one and only one pairwise k-open set for $2 \leq k \leq n-2$. Let $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_{n-1} \subseteq U_n$ be the ascending chain of τ_1-open sets where $U_i = \{x_1, x_2, \ldots, x_i\}$. Let $U = \{x_1\}$ be the τ_2-open set of X and let V be the pairwise k-open set containing the point x_1. Since $2 \leq k \leq n-2$, there exists at least two points which are not in V. By Lemma 3.1, every card of X has a bi-ascending chain. By Lemma 3.2, every card of X has only one τ_1-open set of order r, for every $r = 1, 2, \ldots, n-2$. Any card X_{x_r} obtained by deleting the point which is not in V, has only one τ_{2x}-open set U and only one pairwise k-open set V. We need to prove such cards are pair-homeomorphic. Let X_{x_a} and X_{x_b} be the two cards, obtained by deleting the points which are not in V where $1 < a < b \leq n$. The mapping $h : (X_{x_a}, \tau_{1x_a}, \tau_{2x_a}) \to (X_{x_b}, \tau_{1x_b}, \tau_{2x_b})$ by

$$h(x_k) = \begin{cases} x_k & \text{if } k < a \text{ and } k > b \\ x_{k-1} & \text{if } a < k \leq b \end{cases}$$

is a pair-homeomorphism between X_{x_a} and X_{x_b}. Hence there is only one card, say X_{x_r} has exactly one τ_{2x_r}-open set of order one and only one pairwise k-open set. Remaining $m-1$ cards, say X_{x_p} obtained by deleting the points of V, have at most one τ_{2x_p}-open set of order one and only one pairwise $k-1$-open set.
Sufficiency: Assume that every card X_x, for $x \in X$, has a bi-ascending chain with only one τ_{1x}-open set of order r, for every $r = 1, 2, \ldots, n-2$ such that only one card, say X_y has exactly one τ_{2y}-open set of order one and only one pairwise k-open set whereas the remaining $m-1$ cards, say X_z, have at most one τ_{2z}-open set of order one and only one pairwise $k-1$-open set. By Lemma 3.1, the space X has a bi-ascending chain. By Lemma 3.2, the space X has only one τ_1-open set of order r, for every $r = 1, 2, \ldots, n-1$. Since (X, τ_2) has an ascending chain, τ_2 is not indiscrete. If there exist only pairwise-open sets, not any proper τ_2-open sets, then every card follows the same. This leads to a contradiction. So there exist a proper τ_2-open set, say P, of some order $r \geq 1$. Suppose that r lies between 3 and $n-1$. Then there exists a card, say X_x, having a proper τ_{2x}-open set of order $r-1$ where $2 \leq r-1 \leq n-2$, giving a contradiction; If $r = 2$, then the card, obtained by deleting the point y which is not in P (since $\mid X \mid \geq 4$,) has a proper τ_{2y}-open set of order two, giving a contradiction. Hence the proper τ_2-open set is of order 1. If there is more than one proper pairwise-open set of order at least two, we get contradiction to that each card having only one pairwise-open set. Since X has a bi-ascending chain, only one proper pairwise-open set, say V, exists (otherwise, we get contradiction to the card having pairwise k-open set and the cards having pairwise $k-1$-open set where $2 \leq k \leq n-2$) is of order at least two. If the order of V is $n-1$, then only one card has no proper pairwise k-open set, giving a contradiction. Hence the order of V is at most $n-2$. This completes the proof. □

In the space (X, τ_1, τ_2), let A be a proper τ_1-open set of order p and let B be a proper τ_2-open set of order q where $1 \leq p \leq n-1$ and $q = p-1$. The set A is said to be p_q-open set if B is contained in A. Let $A - B = \{x\}$. The point x is called p_q-point.

Theorem 3.5. *The space X has a bi-ascending chain with only one τ_1-open set of order r and only one τ_2-open set of order r, for every $r = 1, 2, \ldots, n-1$ such that X has no proper pairwise k-open set and no p_q-open set, $1 \leq k \leq n-1$ if and only if the cards X_x, for any $x \in X$ have a bi-ascending chain with only one τ_{1x}-open set of order s, only*

one τ_{2x}-open set of order s, for every $s = 1, 2, \ldots, n-2$ such that there is no proper pairwise k-open set, $1 \leq k \leq n-2$.

Proof. Necessity: Assume that X has a bi-ascending chain with only one τ_1-open set of order r and only one τ_2-open set of order r, for every $r = 1, 2, \ldots, n-1$ such that X has no pairwise k-open set and no p_q-open set. By Lemma 3.1, every card of X has a bi-ascending chain. By Lemma 3.2, every card of X has only one τ_{1x}-open set of order s, for every $s = 1, 2, \ldots, n-2$ and every card of X has only one τ_{2x}-open set of order s, for every $s = 1, 2, \ldots, n-2$. A pairwise-open set in the card can be obtained by deleting the points of either pairwise-open set of X or p_q-point of a p_q-open set. Since X has no proper pairwise k-open set and no p_q-open set, the cards have no proper pairwise k-open set, necessity follows.
Sufficiency: Assume that every card X_x, for any $x \in X$ has a bi-ascending chain with only one τ_{1x}-open set of order s, only one τ_{2x}-open set of order s, for every $s = 1, 2, \ldots, n-2$ such that there is no proper pairwise k-open set, $1 \leq k \leq n-2$. By Lemma 3.1, the space X has a bi-ascending chain. By Lemma 3.2, X has only one τ_1-open set of order r, for every $r = 1, 2, \ldots, n-1$ and X has only one τ_2-open set of order r, for every $r = 1, 2, \ldots, n-1$. Suppose, to the contrary that, X has a proper pairwise k-open set of order at most $n-1$. Then at least one card of X has a proper pairwise $k-1$-open set, giving a contradiction. Suppose, to the contrary that, X has a p_q-open set. Then the card, obtained by deleting the p_q-point, has a proper pairwise-open set, giving a contradiction. □

Concluding Remarks

In this paper, it was shown that all finite bitopological spaces with unique isolated point and at most two non pair-homeomorphic cards are recognizable. Regarding the further direction of research in this paper, the reconstruction of the number of isolated points for spaces with at least one isolated point is still an open question. If this is solved, then it may quite easy to reconstruct all bitopological spaces.

The multideck of a bitopological space X is the collection of all cards of X. A space X that can be obtained uniquely (upto pair-homeomorphism) by the multideck of X is said to be weakly reconstructible. The deck of X is known from its multideck, which gives more information than the deck. Thus, Reconstructing the number of isolated points of a space X or the space X from its multideck may lead us to reconstruct them from its deck. The above remarks are given for the purpose of future scope of works on this paper.

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