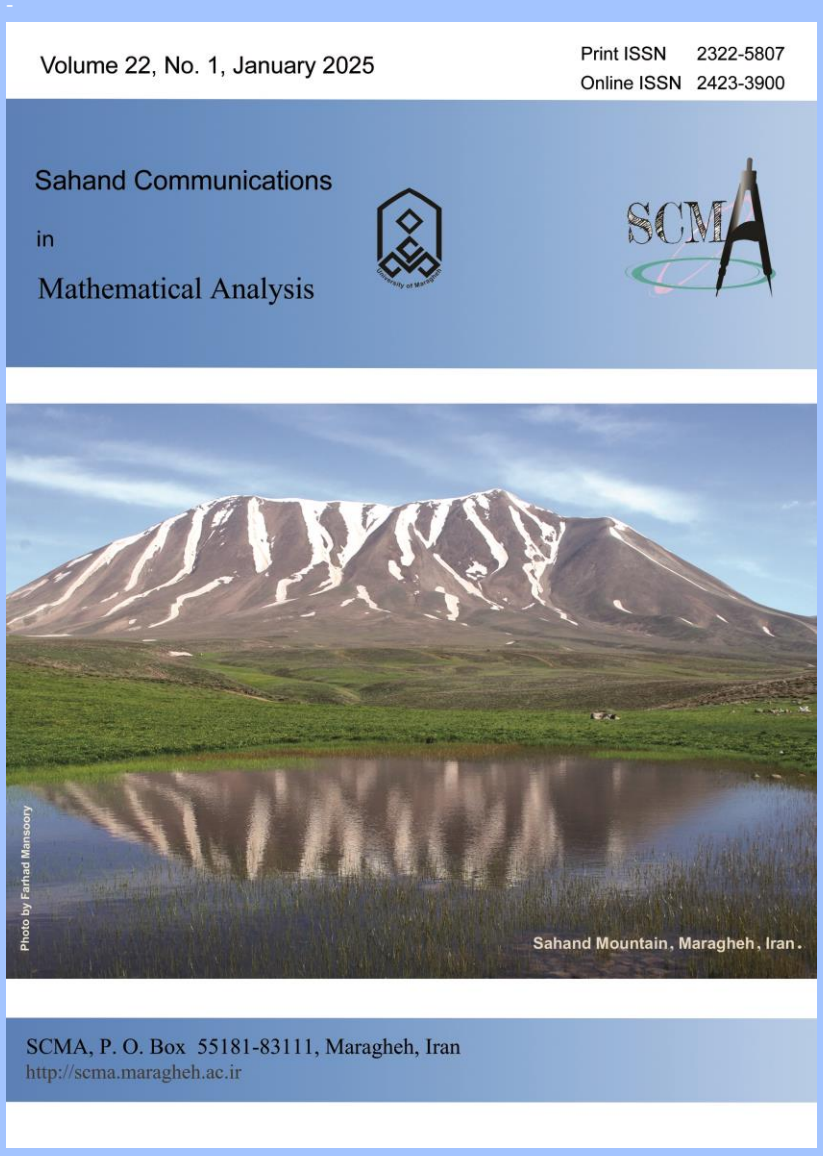
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Existence of Mild Solutions for Fuzzy Fractional Evolution Equations via Krasnosel'skii Fixed Point Theorem

Ismail Airou[1,*], Ali El Mfadel[2] and M'hamed Elomari[3]

Abstract. The primary purpose of this paper is to investigate the existence of two distinct types of fuzzy mild solutions for fuzzy fractional evolution equations under Caputo's $gH-$differentiability. The proofs are based on fuzzy strongly continuous semigroups, a new Krasnoselskii fixed point theorem appropriate for fuzzy metric spaces, and some elementary fuzzy fractional calculus tools.

1. Introduction

Fuzzy fractional evolution equations are a natural approach to modeling dynamic systems in the presence of unknown factors. In recent years, there has been considerable work written by many different authors in the theory of fuzzy fractional evolution equations (see [1–3, 5, 20]). The original method employs the Hukuhara derivative (H-derivative) of a fuzzy valued function. However, certain fuzzy differential equations in this context show an infinite diameter of the equation's solution as time passes, which distinguishes them from crisp cases. To overcome the previous problem to a certain extent, Bede et al. [5, 6] introduced a novel concept of fuzzy derivatives called generalized Hukuhara differentiability (gH differentiability) for fuzzy valued functions. Researchers including Mosleh and Otadi [19], Alikhani and Bahrami [3], and Khastan et al [16] have researched and used this notion to fuzzy evolution equations.

Recently, there has been significant attention directed towards fuzzy fractional evolution equations, garnering interest in both mathematical studies and various practical applications. These applications include

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a wide range of topics, including electromagnetic wave theory, biology, thermodynamics, mechanics, geology, astrophysics, economics, chemistry, wave propagation, and signal processing.

Evolution equations are understood as the differential description of a system's temporal development. The derivation of explicit solutions for an evolution equation in a Banach space is based principally on local compactness and on internal estimations in normal linear spaces, employing methods involving semigroups as outlined in references [12, 14]. Unfortunately, these features are not applicable within the realm of fuzzy number spaces, where the absence of separability and linearity poses significant challenges for mathematicians attempting to achieve outcomes analogous to those in standard situations[17]. The development of evolution equations within fuzzy metric spaces remains in the beginning stages.

In [13], Gal and Gal completed the first formal investigation of semigroups of operators on fuzzy valued function spaces. Son [21] used Schauder's fixed point theorem and Caputo's gH-differentiability to show that fuzzy fractional evolution equations have two types of mild solutions.

Inspired by the previous works, we investigate, in the current paper, the existence of two types of fuzzy mild solutions in the space of triangular fuzzy numbers $\mathcal{T}$ for the following fuzzy fractional evolution equation

$$\begin{aligned} &{}^{C}_{gH}\mathcal{D}^{p}x(t) = Ax(t) + f\left(t, x(t)\right), \quad t \in [0, a], \\ &x(0) = x_0 \end{aligned} \tag{1.1}$$

where $x : [0, a] \to \mathcal{T}$ takes values in $\mathcal{T}$, a is a finite constant, $A : D(A) \subset \mathcal{T} \to \mathcal{T}$ is a closed linear operator that generating a semigroup $\{T(t)\}_{t\geq 0}$ that is strongly continuous on $\mathcal{T}$, ${}^{C}_{gH}\mathcal{D}^{p}$ is the fuzzy Caputo fractional gH-derivative of order $p \in (0, 1)$, and $f : [0, a] \times \mathcal{T} \to \mathcal{T}$ is a nonlinear function satisfying some assumptions.

The structure of this paper is as follows. Section 2 gives a general overview of fuzzy sets, fuzzy number operations, and some specific definitions of fuzzy fractional integral and Caputo fractional gH-derivative, which are used throughout the rest of the paper. Before some conclusions are given in Section 4, the existence of two types of fuzzy mild solutions to fuzzy fractional evolution equations is studied in Section 3.

2. Preliminaries

This section contains the notations, definitions, and introductory information used throughout the study.

The class of all fuzzy subsets $v : \mathbb{R} \to [0,1]$ that satisfy the following conditions can be represented by the notation E^1 :

(i) v is normal, that is, $\exists y_0 \in \mathbb{R}$ such that $v(y_0) = 1$;
(ii) v is fuzzy convex, that is, $v(ty_1 + (1-t)y_2) \geq \min\{v(y_1), v(y_2)\}$, for $t \in [0,1]$ and $y_1, y_2 \in \mathbb{R}$;
(iii) v is upper semi-continuous on $\mathbb{R}$, that is, $\forall \varepsilon > 0,\ \exists \delta > 0$ such that $v(y) - v(y_0) < \varepsilon$, whenever $|y - y_0| < \delta$;
(iv) $\overline{\{y \in \mathbb{R}; v(y) > 0\}}$ is compact;

E^1 is also called the fuzzy numbers space.

The set $[v]^\alpha = \{y \in \mathbb{R} \mid v(y) \geq \alpha\}$ for $\alpha \in [0,1]$ is called α-cuts of $v \in E^1$.

The mapping $v : \mathbb{R} \to [0,1]$ assumes a special form, as the α-cuts of v transform into intervals characterized by a parametric structure $v = (v^-, v^+)$ such that $[v(y)]^\alpha = [v(y)^-_\alpha, v(y)^+_\alpha]$ for $\alpha \in [0,1]$ and $y \in \mathbb{R}$. The zero fuzzy numbers $\widehat{0}$ is defined by $[\widehat{0}]^\alpha = [0,0]$ for all $\alpha \in [0,1]$. The following are the fuzzy numbers addition and scalar multiplication operations on E^1:

$$[v \oplus u]^\alpha = [v]^\alpha + [u]^\alpha, \qquad [\beta \odot v]^\alpha = \beta[v]^\alpha, \quad \forall \alpha \in [0,1]$$

$\beta \in \mathbb{R}$, $[v]^\alpha + [u]^\alpha$ represents the usual addition of two real intervals, and $\beta[v]^\alpha$ represents the normal product of a scalar β and an interval of $\mathbb{R}$.

Definition 2.1 ([6])**.** Let $v, u \in E^1$. The fuzzy number w is defined as the generalized Hukuhara differential of v and u (gH-difference for brief), where

$$u \ominus_{gH} v = w \iff \begin{cases} (i)\ u = v \oplus w \\ (ii)\ v = u \oplus (-1) \odot w. \end{cases}$$

Denote $D_\infty : E^1 \times E^1 \to [0, +\infty[$,

$$D_\infty(v,u) = \sup_{\alpha \in [0,1]} \left\{ D_H\left([v]^\alpha, [u]^\alpha\right) \right\},$$

where D_H represents the Hausdorff metric in the set of all nonempty, convex, and compact subsets of $\mathbb{R}$. For fuzzy numbers v and u, $D_\infty(v,u)$ is known as the sup-metric. It is well known that $\left(E^1, D_\infty\right)$ is a complete metric space.

Generally, with $v \in E^1$, it doesn't exist $u \in E^1$ such that $v \oplus u = \widehat{0}$. Then E^1 is not a vector space. As a result, $\left(E^1, \| \, . \, \|\right)$ is not a Banach space, where $\| \, v \, \| = D_\infty(v, \widehat{0}), v \in E^1$.

$\mathcal{T}$ refers to all triangular fuzzy numbers in E^1. The complete metric space $\left(E^1, D_\infty\right)$ contains the subspace $(\mathcal{T}, D_\infty)$. Bede [7] demonstrated that given $v, u \in \mathcal{T}$, the difference $v \ominus_{gH} u$ always exists in $\mathcal{T}$. Let $a > 0$, we define $C_a^{\mathcal{T}} := C([0,a], \mathcal{T})$ as the complete metric space of continuous

mapping from the compact interval $[0,a]$ into $\mathcal{T}$ with the sup-metric $D_a(x,y) = \sup_{0\leq t\leq a} D_\infty\left(x(t),y(t)\right)$.

The next definition presents the gH-differentiability of fuzzy-valued functions. Gal [6].

Definition 2.2 ([6])**.** Let $f \in C_a^{\mathcal{T}}$, $a_0 \in (0,a)$ and r a real number such that $a_0 + r \in [0,a]$. If there is $\mathcal{D}_{gH} f\left(a_0\right) \in \mathcal{T}$ where

$$\mathcal{D}_{gH} f\left(a_0\right) = \lim_{r\to 0} \frac{1}{r} \odot \left[f\left(a_0+r\right) \ominus_{gH} f\left(a_0\right)\right],$$

consequently, f is gH-differentiable at a_0.

If $\mathcal{D}_{gH} f(t) \in \mathcal{T}$ exists for every $t \in [0,a]$, then f is gH-differentiable on $[0,a]$. The function $\mathcal{D}_{gH} f : [0,a] \to \mathcal{T}$ is called the gH-derivative of f on $[0,a]$.

Definition 2.3 ([3, 6])**.** Consider $f : [0,a] \to \mathcal{T}$ with $a_0 \in (0,a)$.

We say that f is [(i)-gH]-differentiable at a_0 if there is $\mathcal{D}_{gH} f\left(a_0\right) \in \mathcal{T}$ such that

(i) For any $r > 0$ that is sufficiently small, there exist $f\left(a_0+r\right) \ominus_H f\left(a_0\right)$, $f\left(a_0\right) \ominus_H f\left(a_0-r\right)$, and the limits as follows :

$$\mathcal{D}_{gH} f\left(a_0\right) = \lim_{r\to 0} \frac{f\left(a_0+r\right) \ominus_H f\left(a_0\right)}{r} = \lim_{r\to 0} \frac{f\left(a_0\right) \ominus_H f\left(a_0-r\right)}{r}.$$

We say that f is [(ii)-gH]-differentiable at a_0 if there exists $\mathcal{D}_{gH} f\left(a_0\right) \in \mathcal{T}$ such that

(ii) For any $r > 0$ that is sufficiently small, there exist $f\left(a_0\right) \ominus_H f\left(a_0+r\right)$, $f\left(a_0-r\right) \ominus_H f\left(a_0\right)$, and the limits as follows :

$$\begin{aligned}\mathcal{D}_{gH} f\left(a_0\right) &= \lim_{r\to 0} \frac{f\left(a_0\right) \ominus_H f\left(a_0+r\right)}{-r} \\ &= \lim_{r\to 0} \frac{f\left(a_0-r\right) \ominus_H f\left(a_0\right)}{-r}.\end{aligned}$$

(r and $-r$ mean respectively $\frac{1}{r}\odot$ and $-\frac{1}{r}\odot$).

Definition 2.4 ([8])**.** Consider $f : [0,a] \to \mathcal{T}$ as a gH-differentiable function on $[0,a]$, with the parametric form $f(t,\alpha) = [f^-(t,\alpha), f^+(t,\alpha)]$, where $t \in [0,a]$ and $\alpha \in [0,1]$.

(i) If $\left[\mathcal{D}_{gH} f(a_0)\right]_\alpha = \left[\left(f^-\right)'\left(a_0,\alpha\right), \left(f^+\right)'\left(t_0,\alpha\right)\right]$, then f is $[(i)-gH]$-differentiable at $a_0 \in [0,a]$.

(ii) If $\left[\mathcal{D}_{gH} f(a_0)\right]_\alpha = \left[\left(f^+\right)'\left(a_0,\alpha\right), \left(f^-\right)'\left(a_0,\alpha\right)\right]$, then f is $[(ii)-gH]$-differentiable at $a_0 \in [0,a]$.

We denote the $[(i)-gH]$-derivative and the $[(ii)-gH]$-derivative for the fuzzy-valued function f at $a_0 \in [0,a]$ as $\mathcal{D}^{i}_{gH} f(a_0)$ and $\mathcal{D}^{ii}_{gH} f(a_0)$, respectively.

Definition 2.5 ([17])**.** Assume that $g : [0,a] \to \mathcal{T}$ is a fuzzy-valued function with the parametric form $g(t) = (g^-(t), g^+(t))$, where $t \in [0,a]$, such that g^-, g^+ are measurable and Lebesgue integrable on $[0,a]$. Then, in parametric form, we define $\int_0^a g(t)dt$ as follows:

$$\int_0^a g(t)dt = \left(\int_0^a g^-(t)dt, \int_0^a g^+(t)dt \right).$$

Thus, it follows that

$$\left[\int_0^a g(t)dt \right]^\alpha = \left[\int_0^a g_\alpha^-(t)dt, \int_0^a g_\alpha^+(t)dt \right], \quad \text{for all } \alpha \in [0,1]..$$

Then we establish that on $[0,a]$, g is Lebesgue integrable.

$L([0,a],\mathcal{T})$ represents the space of all fuzzy-valued functions that take values on $\mathcal{T}$ and are Lebesgue integrable on $[0,a]$.

For a fuzzy-valued function, the Riemann-Liouville fuzzy integral and the Caputo gH-derivative are defined as follows :

Definition 2.6 ([4, 18])**.** Let $x \in L([0,a],\mathcal{T})$ and $[x(t)]^\alpha = [x_\alpha^-(t), x_\alpha^+(t)]$, for all $t \in [0,a]$, $\alpha \in [0,1]$. The fractional Riemann-Liouville integral of order $p \in (0,1]$ for a fuzzy-valued function x, with a lower limit at zero, can be written as follows:

$$\overset{RL}{\mathcal{F}}\mathcal{I}^p_{0^+} x(t) = \frac{1}{\Gamma(p)} \odot \int_0^t (t-m)^{p-1} \odot x(m)dm$$

Considering α-cuts:

$$\left[\overset{RL}{\mathcal{F}}\mathcal{I}^p_{0^+} x(t)\right]^\alpha = \left[{}^{RL}\mathcal{I}^p_{0^+} x^-_\alpha(t), {}^{RL}\mathcal{I}^p_{0^+} x^+_\alpha(t)\right], \quad \alpha \in [0,1], \quad t \in [0,a],$$

where the fractional Riemann-Liouville integral operator of order $p \in (0,1]$ for a real-valued function is denoted by ${}^{RL}\mathcal{I}^p_{0^+}$.

Definition 2.7 ([4, 18])**.** A function $u \in C_a^{\mathcal{T}}$ has a Caputo gH-derivative of order $p \in [0,1)$ that is defined by

$$\overset{C}{}_{gH}\mathcal{D}^p u(t) = \overset{RL}{\mathcal{F}}\mathcal{I}^{1-p}_{0^+} \left(\mathcal{D}_{gH} u(t)\right), \quad t \in [0,a]$$

as long as the expression on the right-hand side is defined.

In concrete terms, if

$$\overset{C}{}_{gH}\mathcal{D}^p_i u(t) = \overset{RL}{\mathcal{F}}\mathcal{I}^{1-p}_{0^+} \left(\mathcal{D}^i_{gH} u(t)\right),$$

then u is Caputo $[(i)-gH]$-differentiable, and if

$$\overset{C}{}_{gH}\mathcal{D}^p_{ii} u(t) = \overset{RL}{\mathcal{F}}\mathcal{I}^{1-p}_{0^+} \left(\mathcal{D}^{ii}_{gH} u(t)\right),$$

then u is Caputo $[(ii)-gH]$-differentiable.

Let $L(\mathcal{T})$ be the set of all bounded linear operators $\rho : \mathcal{T} \to \mathcal{T}$, such that

$$\|\rho\|_{L(\mathcal{T})} := \inf\left\{C \mid D_\infty\left(\rho a, \widehat{0}\right) \leq C\ D_\infty\left(a, \widehat{0}\right), \forall a \in \mathcal{T}\right\}$$

is the norm of the operator ρ.

It's easy to verify that

$$D_\infty\left(\rho a, \widehat{0}\right) \leq\ \|\rho\|_{L(\mathcal{T})}\ D_\infty\left(a, \widehat{0}\right),$$

and

$$D_\infty\left(\rho a, \rho b\right) \leq\ \|\rho\|_{L(\mathcal{T})}\ D_\infty\left(a, b\right), \quad \text{for all } a, b \in \mathcal{T}.$$

Definition 2.8 ([13])**.** A family of bounded linear operators $\{T(t)\}_{t\geq 0}$ is considered a semigroup on $\mathcal{T}$ if it meets the following conditions

(i) $T(0) = id_{\mathcal{T}}$,
(ii) $T(s+h) = T(s)T(h)$ for all $s, h \geq 0$.

Furthermore, if $\lim\limits_{t\to 0^+} T(t)x = x$ holds true for all $x \in \mathcal{T}$, then we characterize $\{T(t)\}_{t\geq 0}$ as a strongly continuous semigroup on $\mathcal{T}$.

Definition 2.9 ([21])**.** Suppose $\{T(t)\}_{t\geq 0} \subset L(\mathcal{T})$ is a strongly continuous semigroup on $\mathcal{T}$. This implies that the map $T(\cdot)x : [0, +\infty) \to \mathcal{T}$ is continuous for any $x \in \mathcal{T}$.

Lemma 2.10 ([21])**.** *Let $\{T(t)\}_{t\geq 0} \subset L(\mathcal{T})$ be a strongly continuous semigroup on $\mathcal{T}$. Then,*

$$C := \sup_{t\in[0,a]} \|T(t)\|_{L(\mathcal{T})} \geq 1.$$

In the rest of this paper, we consider $\{T(t)\}_{t\geq 0}$ a strongly continuous semigroup of uniformly bounded linear operators on $\mathcal{T}$, such that for every $t \in [0, +\infty)$, $\|T(t)\|_{L(\mathcal{T})} \leq C$.

Consider the density probability functions defined on $(0, \infty)$ for $p \in (0, 1)$ and $n \in \mathbb{N}$ as follows:

$$\psi_p(\varphi) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\,\varphi^{-pn-1}\frac{\Gamma(np+1)}{n!}\sin(n\pi p),$$

$$\phi_p(\varphi) = \frac{1}{p}\varphi^{-1-\frac{1}{p}}\psi_p\left(\varphi^{\frac{-1}{p}}\right), \qquad \varphi \in (0, \infty).$$

For $p \in (0, 1)$, $\nu \in \mathcal{T}$, we define the following two integral operators:

$$U_p(t)\nu = \int_0^\infty \phi_p(\varphi) \odot T\left(t^p\varphi\right)\nu d\varphi, \tag{2.1}$$

and

$$V_p(t)\nu = p \odot \int_0^\infty \varphi\phi_p(\varphi) \odot T\left(t^p\varphi\right)\nu d\varphi. \tag{2.2}$$

Lemma 2.11 ([21]). *$U_p(t)$ and $V_p(t)$ are bounded linear operators on $\mathcal{T}$ that are continuous on $[0,\infty)$ for any fixed $t \geq 0$.*

Definition 2.12 ([13]). Let $\{T(t)\}_{t\geq 0} \subset L(\mathcal{T})$ a strongly continuous semigroup on $\mathcal{T}$. The infinitesimal generator $A : D(A) \subset \mathcal{T} \to \mathcal{T}$ is defined as follows.

$$Ax = \lim_{t\to 0^+} \frac{1}{t} \odot \left(T(t)x \ominus_{gH} x\right),$$

$$D(A) = \left\{ x \in \mathcal{T} : \lim_{t\to 0^+} \frac{1}{t} \odot \left(T(t)x \ominus_{gH} x\right) \text{ exists in } \mathcal{T} \right\}.$$

Next, we present a similar result of Krasnosel'skii's fixed point theorem in the fuzzy domain.

Theorem 2.13 ([15]). *Let F be a non-empty, convex, and closed subset of $C\left([0,a], E^1\right)$. Assume $P_1, P_2 : F \longrightarrow C\left([0,a], E^1\right)$ are two mappings, such that:*

(i) *$P_1x + P_2y \in F$ for any $x, y \in F$;*
(ii) *P_2 is a contraction, i.e.,*

$$\exists k \in]0;1[, \quad D_a\left(P_2x, P_2y\right) \leq k\ D_a(x,y) \quad \forall x, y \in F.$$

(iii) *P_1 is continuous and compact;*

Then, there is a fixed point for $P_1 + P_2$ in F, which means there exists $y \in F$ such that $P_1y + P_2y = y$.

3. Main Results

This section will show the existence of solutions to the fuzzy fractional evolution equation (1.1).

Lemma 3.1. *The fractional fuzzy evolution equation* (1.1) *corresponds to one of these fuzzy integral equations:*
If x is Caputo $[(i)-gH]$-differentiable,

$$x(t) = U_p(t)x_0 + \int_0^t (t-h)^{p-1} \odot V_p(t-h) f\left(h, x(h)\right) dh. \tag{3.1}$$

If x is Caputo $[(ii)-gH]$-differentiable,

$$x(t) = U_p(t)x_0 \ominus_H (-1) \odot \int_0^t (t-h)^{p-1} \odot V_p(t-h) f\left(h, x(h)\right) dh. \tag{3.2}$$

where $U_p(t)$ and $V_p(t)$ are defined in (2.1) *and* (2.2).

Proof. See lemma 4.1 in [21]. □

Definition 3.2. A function $x \in C_a^{\mathcal{T}}$ is considered a fuzzy mild solution of type 1 to the equation (1.1) on the interval $[0,a]$ if it achieves the above equation:

$$x(t) = U_p(t)x_0 + \int_0^t (t-h)^{p-1} \odot V_p(t-h) f\left(h, x(h)\right) \, dh. \tag{3.3}$$

A function $x \in C_a^{\mathcal{T}}$ is considered a fuzzy mild solution of type 2 to the equation (1.1) on the interval $[0,a]$, if it satisfies the above equation:

$$x(t) = U_p(t)x_0 \ominus_H (-1) \odot \int_0^t (t-h)^{p-1} \odot V_p(t-h) f\left(h, x(h)\right) \, dh. \tag{3.4}$$

The following assumptions will be required in the sequel.

(H1) For every $t \in [0,a]$ and $z \in C_a^{\mathcal{T}}$,

- $f(t,.) : C_a^{\mathcal{T}} \to \mathcal{T}$ is fuzzy continuous.
- $f(.,z(.)) : [0,a] \to \mathcal{T}$ is fuzzy strongly measurable.

(H2) For $r > 0$, let $K_r := \left\{ x \in \mathcal{T}, D_\infty\left(x, \widehat{0}\right) \leq r \right\} \subset \mathcal{T}$,

- There exist $p_2 \in [0,p)$ and $\rho(.) \in L^{\frac{1}{p_2}}\left([0,a], \mathbb{R}^+\right)$ such that for all $u, v \in C\left([0,a], K_r\right)$ we have

$$D_\infty\left(f(t,u(t)), f(t,v(t))\right) \leq \rho(t) D_\infty\left(u(t), v(t)\right), \quad t \in [0,a].$$

(H3) For every $t \in [0,a]$ and $z \in C_a^{\mathcal{T}}$,

- There exist $p_1 \in [0,p)$ and $\beta \in L^{\frac{1}{p_1}}\left([0,a], \mathbb{R}^+\right)$ such that

$$D_\infty\left(f(t,z(t)), \widehat{0}\right) \leq \beta(t).$$

(H4) $T(t)$ is a compact operator for all $t > 0$.

(H5) For every $x, y \in \mathcal{K}_{c_0}$ defined by (3.6),

- There is $L > 0$ such that

$$D_\infty\left(x_0, y_0\right) \leq L \; D_a(x,y).$$

Theorem 3.3. *If the previous assumptions* $(H1) - (H5)$ *have been satisfied, then the fuzzy fractional evolution equation* (1.1) *has a fuzzy mild solution of type 1 in* $C_a^{\mathcal{T}}$*, if*

$$L < \frac{1}{C}.$$

Proof. Let's consider the operator $Q : C_a^{\mathcal{T}} \to C_a^{\mathcal{T}}$ defined as follows :

$$(Qx)(t) = U_p(t)x_0 + \int_0^t (t-h)^{p-1} \odot V_p(t-h) f\left(h, x(h)\right) dh. \tag{3.5}$$

It is simple to confirm that the operator Q is well-defined by direct calculating. The fuzzy mild solution that gives (1.1) corresponds to the fixed point of the operator Q given by (3.5), as defined by Definition 3.2.

We will demonstrate in the following that the space $C_a^{\mathcal{T}}$ holds a fixed point for the operator Q.

Let $x \in C_a^{\mathcal{T}}$. Since x is continuous in connection to t and presuming **(H1)**, $f(h, x(h))$ is a measurable function on $[0, a]$. Let

$$d = \frac{p-1}{1-p_1}, \qquad N = \|\beta\|_{L^{\frac{1}{p_1}}[0,a]}.$$

For $t \in [0, a]$, using Hôlder's inequality and **(H3)**, we have

$$\begin{aligned} D_\infty\left(\int_0^t (t-h)^{p-1} \odot f(h, x(h))dh, \widehat{0}\right) &\leq \int_0^t (t-h)^{p-1} \odot D_\infty\left(f(h, x(h)), \widehat{0}\right) dh \\ &\leq \left(\int_0^t (t-h)^{\frac{p-1}{1-p_1}} dh\right)^{1-p_1} \|\beta\|_{L^{\frac{1}{p_1}}[0,t]} \\ &\leq \frac{Na^{(1+d)(1-p_1)}}{(1+d)^{1-p_1}}. \end{aligned}$$

According to the lemma 2.11, and assumption **(H3)**, we obtain

$$\begin{aligned} &D_\infty\left(\int_0^t (t-h)^{p-1} \odot V_p(t-h)\ f(h, x(h))dh, \widehat{0}\right) \\ &\leq \int_0^t (t-h)^{p-1} D_\infty\left(V_p(t-h)f(h, x(h)), \widehat{0}\right) dh \\ &\leq \int_0^t (t-h)^{p-1} \frac{pC}{\Gamma(p+1)} D_\infty\left(f(h, x(h)), \widehat{0}\right) dh \\ &\leq \frac{qNCa^{(1+d)(1-p_1)}}{\Gamma(1+p)(1+d)^{1-p_1}} \end{aligned}$$

for all $t \in [0, a]$. Thus, $\int_0^t (t-h)^{p-1} \odot V_p(t-h)f(h, x(h))dh$ is bounded for all $t \in [0, a]$.

For $x \in C_a^{\mathcal{T}}$, we define

$$(Q_1 x)(t) = U_p(t)x_0, \quad t \in [0, a]$$

and

$$(Q_2 x)(t) = \int_0^t (t-h)^{p-1} \odot V_p(t-h)f(h, x(h))dh, \quad t \in [0, a].$$

Let

$$c_0 = \min\left\{\|x_0\|; C\|x_0\| + \frac{pNCa^{(1+d)(1-p_1)}}{\Gamma(1+p)(1+d)^{1-p_1}}\right\}$$

and

$$\mathcal{K}_{c_0} := \left\{x(.) \in C_a^{\mathcal{T}} : D_\infty\left(x(t), \widehat{0}\right) \leq c_0 \quad \text{for all} \quad t \in [0, a]\right\}. \tag{3.6}$$

It is straightforward to show that $\mathcal{K}_{c_0}$ is a non-empty, convex, and closed subset of $C_a^{\mathcal{T}}$.

Using the Krasnosel'skii fixed point extension, we will show that Q_1x+Q_2x obtains a fixed point in $\mathcal{K}_{c0}$.

Step 1. We show that for all $x, y \in \mathcal{K}_{c_0}, Q_1x + Q_2y \in \mathcal{K}_{c_0}$.

Indeed, with $0 \leq t_1 \leq t_2 \leq a$, we have

$$D_\infty \left((Q_2x)(t_2), (Q_2x)(t_1) \right) = D_\infty \left(\int_0^{t_2} (t_2-h)^{p-1} \odot V_p(t_2-h) f(h,x(h))dh, \right.$$
$$\left. \int_0^{t_1} (t_1-h)^{p-1} \odot V_p(t_1-h) f(h,x(h))dh \right).$$

Then

$$\begin{aligned} D_\infty \left((Q_2x)(t_2), (Q_2x)(t_1) \right) &\leq D_\infty \left(\int_{t_1}^{t_2} (t_2-h)^{p-1} \odot V_p(t_2-h) f(h,x(h))dh, \widehat{0} \right) \\ &+ D_\infty \left(\int_0^{t_1} (t_2-h)^{p-1} \odot V_p(t_2-h) f(h,x(h))dh, \right. \\ &\left. \int_0^{t_1} (t_1-h)^{p-1} \odot V_p(t_2-h) f(h,x(h))dh \right) \\ &+ D_\infty \left(\int_0^{t_1} (t_1-h)^{p-1} \odot V_p(t_2-h) f(h,x(h))dh, \right. \\ &\left. \int_0^{t_1} (t_1-h)^{p-1} \odot V_p(t_1-h) f(h,x(h))dh \right) \\ &= J_1 + J_2 + J_3. \end{aligned}$$

It is evident that J_1, J_2, and J_3 tend to 0 as $t'' - t_1 \to 0$. Thus, $(Q_2x)(t)$ is continuous for $t \in [0,a]$.

Furthermore, we have

$$(Q_1x)(t) = U_p(t)x_0, \quad t \in [0,a].$$

As $U_p(t)$ is continuous for $t \in [0,a]$, then $(Q_1x)(t)$ is also continuous for $t \in [0,a]$.

Then, $\forall x, y \in \mathcal{K}_{c_0}$, $(Q_1x)(t) + (Q_2y)(t)$is continuous for t in $[0,a]$.

Given $x, y \in \mathcal{K}_{c_0}$ and $t \in [0,a]$, we obtain

$$\begin{aligned} D_\infty \left((Q_1x)(t) + (Q_2y)(t), \widehat{0} \right) &\leq D_\infty \left((Q_1x)(t)), \widehat{0} \right) + D_\infty \left((Q_2y)(t)), \widehat{0} \right) \\ &\leq D_\infty \left(U_p(t)x_0, \widehat{0} \right) + D_\infty \left((Q_2y)(t)), \widehat{0} \right) \\ &\leq C \|x_0\| + \frac{pNCa^{(1+d)(1-p_1)}}{\Gamma(1+p)(1+d)^{1-p_1}} \\ &\leq c_0. \end{aligned}$$

Therefore, $Q_1x + Q_2y \in \mathcal{K}_{c_0}$.

Then $Q_1 + Q_2$ is an operator from $\mathcal{K}_{c_0}$ to $\mathcal{K}_{c_0}$.

Step 2. We'll prove that Q_1 is a contraction of $\mathcal{K}_{c_0}$.

For $x, y \in \mathcal{K}_{c_0}$ and $t \in [0,a]$, using hypothesis **(H5)** we have

$$D_\infty \left((Q_1x)(t), (Q_1y)(t) \right) = D_\infty \left(U_p(t)x_0, U_p(t) \odot y_0 \right)$$

$$\leq CD_\infty(x_0, y_0)$$
$$\leq CL\ D_a(x, y)$$

Thus, $\sup_{0\leq t\leq a} D_\infty((Q_1x)(t), (Q_1y)(t)) \leq CL\ D_a(x, y)$, then

$$D_a(Q_1x, Q_1y) \leq K\ D_a(x, y),$$

with $0 < K = CL < 1$.

Hence, Q_1 is a contraction on $\mathcal{K}_{c_0}$.

Step 3. We will prove that Q_2 is a continuous and compact operator on $\mathcal{K}_{c_0}$.

At first, we will show that Q_2 is continuous on $\mathcal{K}_{c_0}$.

Let $\{x_n\} \subseteq \mathcal{K}_{c_0}$ with $x_n \to x$ on $\mathcal{K}_{c_0}$. Using hypothesis **(H2)**, we obtain

$$f(h, x_n(h)) \to f(h, x(h)) \text{ when } \quad n \to \infty \quad \text{ almost every } t \in [0, a].$$

By hypothesis **(H3)**, we also have $D_\infty(f(h, x_n(h)), f(h, x(h))) \leq 2m(s)$. Using the theorem of dominated convergence and the lemma 2.11, we obtain

$$\begin{aligned}
&D_\infty((Q_2x_n)(t), (Q_2x)(t)) \\
&\leq \int_0^t (t-h)^{p-1} D_\infty(V_p(t-h)f(h, x_n(h)), V_p(t-h)f(h, x(h)))\, dh \\
&\leq \int_0^t (t-h)^{p-1} \frac{pC}{\Gamma(1+p)} D_\infty(f(h, x_n(h)), f(h, x(h)))\, dh \\
&\to 0
\end{aligned}$$

as $n \to \infty$, implying that Q_2 is continuous.

Then, we demonstrate the relative compactness of $Q_2(\mathcal{K}_{c_0})$, which is equivalent to the relative compactness of $\{(Q_2x)(t) : x \in \mathcal{K}_{c_0}\}$, where $t \in [0, a]$, and the uniform equicontinuity of the family of functions $\{Q_2x : x \in \mathcal{K}_{c_0}\}$.

We have shown that for any $x \in \mathcal{K}_{c_0}$ and $0 \leq t_1 \leq t_2 \leq a$,

$$D_\infty((Q_2x)(t_2), (Q_2x)(t_1)) \leq J_1 + J_2 + J_3.$$

We now have

$$\begin{aligned}
J_1 &\leq \frac{qNC(t_2 - t_1)^{(1+d)(1-p_1)}}{\Gamma(1+p)(1+d)^{1-p_1}}. \\
J_2 &\leq \frac{pC}{\Gamma(1+p)} \left(\int_0^{t_1} \left((t_1 - h)^d - (t_2 - h)^d \right) dh \right)^{1-p_1} \|\beta\|_{L^{\frac{1}{p_1}}[0,t_1]} \\
&\leq \frac{pNC}{\Gamma(1+p)(1+d)^{1-p_1}} \left((t_1)^{1+d} - (t_2)^{1+d} + (t_2 - t_1)^{1+d} \right)^{1-p_1} \\
&\leq \frac{pNC}{\Gamma(1+p)(1+d)^{1-p_1}} (t_2 - t_1)^{(1+d)(1-p_1)}.
\end{aligned}$$

For $t_1 = 0$, $0 \leq t_2 \leq a$, it is clear that $J_3 = 0$. For $t_1 > 0$, we have

$$\begin{aligned}
J_3 &\leq \int_0^{t_1} (t_1 - h)^{p-1} D_\infty \left(V_p (t_2 - h) f(h, x(h)), V_p (t_1 - h) f(h, x(h))\right) dh \\
&= p \int_0^{t_1} (t_1 - h)^{p-1} D_\infty \left(\int_0^\infty \varphi \phi_p(\varphi) \odot T \left((t_2 - h)^p \varphi\right) f(h, x(h)) d\varphi, \right. \\
&\quad \left. \int_0^\infty \varphi \phi_p(\varphi) \odot T \left((t_1 - h)^p \varphi\right) f(h, x(h)) d\varphi \right) dh \\
&\leq p \int_0^{t_1} \int_0^\infty \varphi \phi_p(\varphi) (t_1 - h)^{p-1} D_\infty \left(T \left((t_2 - h)^p \varphi\right) f(h, x(h)), \right. \\
&\quad \left. T \left((t_1 - h)^p \varphi\right) f(h, x(h))\right) d\varphi dh.
\end{aligned}$$

In $\mathcal{T}$, the difference $u \ominus_{gH} v$, $u, v \in \mathcal{T}$ always exists. Thus, for all $s \in [0, t_1]$, there exists

$$T \left((t_2 - h)^p \varphi\right) f(h, x(h)) \ominus_{gH} T \left((t_1 - h)^p \varphi\right) f(h, x(h)).$$

Then

$$\begin{aligned}
&D_\infty \left(T \left((t_2 - h)^p \varphi\right) f(h, x(h)), T \left((t_1 - h)^p \varphi\right) f(h, x(h))\right) \\
&\quad = D_\infty \left(\left(T \left((t_2 - h)^p \varphi\right) \ominus_{gH} T \left((t_1 - h)^p \varphi\right)\right) f(h, x(h)), \widehat{0} \right) \\
&\quad \leq \left\| \left| T \left((t_2 - h)^p \varphi\right) \ominus_{gH} T \left((t_1 - h)^p \varphi\right) \right| \right\|_{L(\mathcal{T})} D_\infty (f(h, x(h)),
\end{aligned}$$

when $(t_2 - h)^p \varphi - (t_1 - h)^p \varphi \to 0$ as $t_2 \to t_1$ uniformly with relation to $x \in \mathcal{T}$. In that case, $\{Q_2 x : x \in \mathcal{K}_{c_0}\}$ is equicontinuous.

It is sufficient to demonstrate that $S(t) = \{(Q_2 x)(t) : x \in \mathcal{K}_{c_0}\}$ is relatively compact for any $t \in [0, a]$. We define an operator $Q_{\epsilon,\delta}$ for any fixed value of $0 < t \leq a, \forall \epsilon \in (0, t)$ and $\forall \delta > 0$ by

$$\begin{aligned}
(Q_{\epsilon,\delta} x)(t) &= p \odot \int_0^{t-\epsilon} \int_\delta^\infty \varphi \phi_p(\varphi) (t_1 - h)^{p-1} \odot T \left((t - h)^p \varphi\right) f(h, x(h)) d\varphi dh \\
&= p \odot \int_0^{t-\epsilon} \int_\delta^\infty \varphi \phi_p(\varphi) (t_1 - h)^{p-1} \odot \left[T (\epsilon^p \delta) T \left((t - h)^p \varphi - \epsilon^p \delta\right)\right] \\
&\quad f(h, x(h)) d\varphi dh \\
&= T (\epsilon^p \delta) p \int_0^{t-\epsilon} \int_\delta^\infty \varphi \phi_p(\varphi) (t_1 - h)^{p-1} \odot T \left((t - h)^p \varphi - \epsilon^p \delta\right) \\
&\quad f(h, x(h)) d\varphi dh,
\end{aligned}$$

where $x \in \mathcal{K}_{c_0}$. By assumption **(H4)**, $T(\epsilon^p \delta)$ is a compact operator, and then $S_{\epsilon,\delta}(t) = \{(Q_{\epsilon,\delta} x)(t) : x \in \mathcal{K}_{c_0}\}$ is relatively compact. Moreover, $\forall x \in \mathcal{K}_{c_0}$, we have

$$\begin{aligned}
&D_\infty \left((Q_2 x)(t), (Q_{\epsilon,\delta} x)(t)\right) \\
&\quad = p D_\infty \left(\int_0^t \int_0^\delta \varphi (t - h)^{p-1} \phi_p(\varphi) \odot T \left((t - h)^p \varphi\right) f(h, x(h)) d\varphi dh, \widehat{0} \right) \\
&\quad + p D_\infty \left(\int_0^t \int_\delta^\infty \varphi (t - h)^{p-1} \phi_p(\varphi) \odot T \left((t - h)^p \varphi\right) f(h, x(h)) d\varphi dh, \widehat{0} \right) \\
&\quad + p D_\infty \left(\int_0^{t-\epsilon} \int_\delta^\infty \varphi (t - h)^{p-1} \phi_p(\varphi) \odot T \left((t - h)^p \varphi\right) f(h, x(h)) d\varphi dh, \widehat{0} \right)
\end{aligned}$$

$$\leq pD_\infty\left(\int_0^t\int_0^\delta \varphi(t-h)^{p-1}\phi_p(\varphi)\odot T\left((t-h)^p\varphi\right)f(h,x(h))d\varphi dh,\widehat{0}\right)$$
$$+pD_\infty\left(\int_{t-\epsilon}^t\int_\delta^\infty \varphi(t-h)^{p-1}\phi_p(\varphi)\odot T\left((t-h)^p\varphi\right)f(h,x(h))d\varphi dh,\widehat{0}\right).$$

Consequently,

$$D_\infty\left(\left(Q_2x\right)(t),\left(Q_{\epsilon,\delta}x\right)(t)\right)\leq pC\left(\int_0^t(t-h)^{\frac{p-1}{1-p_1}}dh\right)^{1-p_1}\|\beta\|_{L^{\frac{1}{p_1}}[0,t]}\int_0^\delta\varphi\phi_p(\varphi)d\varphi$$
$$+pC\left(\int_{t-\epsilon}^t(t-h)^{\frac{p-1}{1-p_1}}dh\right)^{1-p_1}\|\beta\|_{L^{\frac{1}{p_1}}[0,t]}\int_0^\delta\varphi\phi_p(\varphi)d\varphi$$
$$\leq\frac{pNCa^{(1+d)(1-p_1)}}{(1+d)^{1-p_1}}\int_0^\delta\varphi\phi_p(\varphi)d\varphi+\frac{pNC\epsilon^{(1+d)(1-p_1)}}{\Gamma(1+p)(1+d)^{1-p_1}}.$$

Then, there exist arbitrarily close reasonably compact sets to $S(t),t>0$, suggesting that $S(t),t>0$, is also relatively compact.

We get that $Q_2\left(\mathcal{K}_{c_0}\right)$ is relatively compact by using the theorem of Arzela-Ascoli . In that case, Q_2 is a compact operator.

Finally, by Krasnosel'skii's fixed point extension theorem 2.13, Q_1+Q_2 has a fixed point on $\mathcal{K}_{c_0}$. Therefore, the fuzzy fractional evolution equation (1.1) admits a fuzzy mild solution of type 1 in $C_a^{\mathcal{T}}$. □

Let

$$(\bar{Z}x)(t):=U_p(t)x_0\ominus_H(-1)\odot\int_0^t(t-h)^{p-1}\odot V_p(t-h)f\left(h,x(h)\right)dh,\quad t\in[0,a],$$

and

$$\bar{C}_a^{\mathcal{T}}:=\bar{C}([0,a],\mathcal{T})=\left\{x\in C_a^{\mathcal{T}}:(\bar{Z}x)(t)\quad\text{exists for every}\quad t\in[0,a]\right\}.$$

The following consequence demonstrates that there is a type 2 fuzzy mild solution in $\bar{C}_a^{\mathcal{T}}$.

Theorem 3.4. *Suppose that the previous assumptions are satisfied and that:*

- $\bar{C}_a^{\mathcal{T}}\neq\emptyset$,
- *If* $x\in\bar{C}_a^{\mathcal{T}}$, *then* $\bar{Z}x\in\bar{C}_a^{\mathcal{T}}$.

Then the fuzzy fractional evolution equation (1.1) *has a fuzzy mild solution of type 2 in* $C_a^{\mathcal{T}}$.

Proof. For $x\in\bar{C}_a^{\mathcal{T}}$, $(\bar{Z}x)(t)=\left(Q_1x\right)(t)\ominus_H(-1)\odot\left(Q_2x\right)(t)$.

Let

$$c_0=\min\left\{\|x_0\|;C\|x_0\|+\frac{pNCa^{(1+d)(1-p_1)}}{\Gamma(1+p)(1+d)^{1-p_1}}\right\}$$

and

$$\mathcal{K}_{c_0}:=\left\{x(.)\in\bar{C}_a^{\mathcal{T}}:D_\infty\left(x(t),\widehat{0}\right)\leq c_0\quad\text{pour tout}\quad t\in[0,a]\right\}$$

. Using a method similar to that used for Caputo's [(i)-gH] derivatives, we get that $Q_1 x \ominus_H (-1) \odot Q_2 y \in \mathcal{K}_{c_0}$ for all $x, y \in \mathcal{K}_{c_0} \subset \bar{C}_a^{\mathcal{T}}$, where $(Q_1 x)(t)$ and $(Q_2 x)(t)$ are continuous in $t \in [0, a]$.

Now, for all $x, y \in \mathcal{K}_{c_0}$ we have

$$\begin{aligned} D_\infty \left((Q_1 x)(t) \ominus_H (-1) \odot (Q_2 y)(t), \widehat{0} \right) &\leq D_\infty \left((Q_1 x)(t), \widehat{0} \right) + D_\infty \left((Q_2 y)(t), \widehat{0} \right) \\ &\leq C \|x_0\| + \frac{pNCa^{(1+d)(1-p_1)}}{\Gamma(1+p)(1+d)^{1-p_1}} \\ &\leq c_0, \end{aligned}$$

which means that $Q_1 \ominus_H (-1) \odot Q_2$ is an operator from $\mathcal{K}_{c_0}$ to $\mathcal{K}_{c_0}$.

Then, since Q_1 is a contraction on $\mathcal{K}_{c_0}$ and Q_2 is a continuous and compact operator on $\mathcal{K}_{c_0}$, by Krasnosel'skii's fixed point extension theorem 2.13 we obtain that $Q_1 \ominus_H (-1) \odot Q_2$ has a fixed point on $\mathcal{K}_{c_0}$, which implies that the fuzzy fractional evolution equation (1.1) has a fuzzy mild solution of type 2 in $C_a^{\mathcal{T}}$. □

4. Conclusion

In this current manuscript, we studied the existence of two types of fuzzy mild solutions for the Cauchy problem of the fractional fuzzy evolution equation. Fuzzy semigroup theory, fuzzy fractional calculus, and an application of the Krasnosel'skii fixed point theorem for fuzzy metric spaces are used to prove the existence theorems.

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