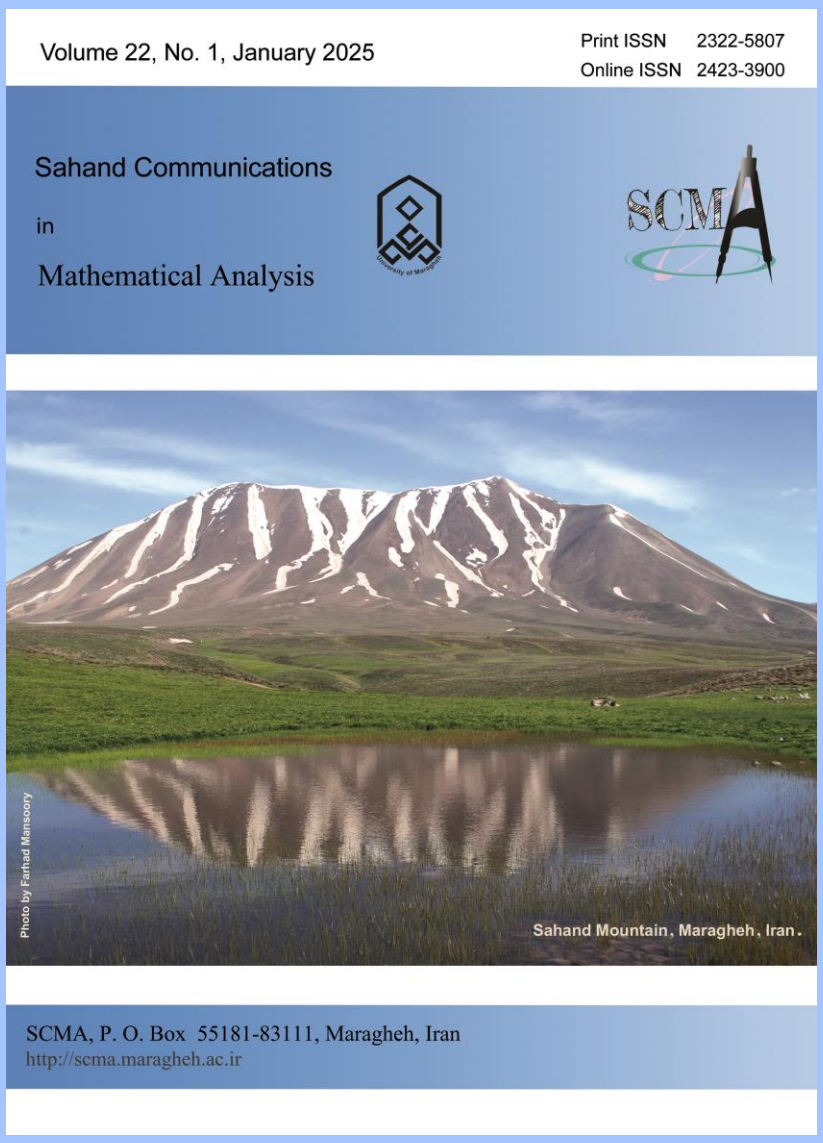
The Results on the Existence of Fixed Points in Hilbert Spaces via Functions of Class *C*

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The Results on the Existence of Fixed Points in Hilbert Spaces via Functions of Class C

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Abstract. In this paper, we obtain a result on the existence of fixed points in Hilbert space using functions of class C. Recently, Chouhan and Desai published a paper [Some Common Fixed-Point Theorems in Hilbert Space, J. Eemerg. Technol. Innov. Res., 2022, 9(10), 1-12]. We observed an error in the proof of Theorem 3.2 on page d316 of their paper. Using the C-class of functions, we have corrected this error and improved their result.

1. Introduction

The existence and uniqueness of a fixed point were established by Banach in 1922, known as the Banach contraction principle [9]. This principle has played a significant role in the development of numerous results in fixed point theory and approximation theory. Many authors have extended and generalized this principle in various directions (see [4, 14, 19] and the references therein). It is a very popular and powerful tool in solving the existence problems in pure and applied sciences. Fixed point theory is commonly used to study the existence and uniqueness of solutions to fractional differential equations. Its wide range of applications has made it highly valued and a subject of ongoing research [5, 17]. This theory has interesting application in medicine [6–8, 16, 18]. In 2014, Ansari [1] introduced the concept of C-class functions and established unique fixed point theorems for specific contractive mappings concerning these functions, see for example [2, 3]. This is a crucial class

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of functions that includes nearly all known types of contractions, beginning with Banach contractions as established in [9].

Chouhan and Desai recently obtained the following result in their paper [13] (Theorem 3.2).

Theorem 1.1 ([13]). *Let $(X, \|\cdot\|)$ be a Hilbert space. Let $T : X \to X$ be a mapping such that*

$$\|Tx - Ty\|^2 \leq a \max\left\{\|Tx - x\|^2, \|Ty - y\|^2, \|x - y\|^2\right\} + b\left(\|Ty - x\|^2 + \|Tx - y\|^2\right), \tag{1.1}$$

for all $x, y \in X$, $a, b \geq 0$ with $a + 2b \leq 1$. Then T has a unique fixed point.

This result is incorrect, as we demonstrate with a counterexample.

The aim of this paper is to improve the results of Chouhan and Desai by utilizing functions of class C. We also introduce a contraction principle by altering distance and C-class functions, and establish a fixed point theorem for this type of contraction within Hilbert spaces.

2. Preliminaries

In this section, we recall some definitions and statements that we will use through the paper.

Definition 2.1 ([2]). A mapping $G : [0, +\infty)^2 \to \mathbb{R}$ is called a C-class function if G is continuous and satisfies the following conditions

(i) $G(s, t) \leq s$,
(ii) $G(s, t) = s$ implies that either $s = 0$ or $t = 0$, for all $s, t \in [0, +\infty)$.

Note, that $G(0, 0) = 0$. The set of $C-$class functions is denoted by $\mathcal{C}$.

Example 2.2 ([2]). The following functions $G : [0, +\infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$, for all $s, t \in [0, +\infty)$:

(1) $G(s, t) = s - t$;
(2) $G(s, t) = ms$ for some $m \in (0, 1)$;
(3) $G(s, t) = \frac{s}{(1+t)^r}$ for some $r \in (0, +\infty)$;
(4) $G(s, t) = \log(t + a^s)/(1 + t)$ for some $a > 1$;
(5) $G(s, t) = \ln(1 + a^s)/2$ for $1 < a < e$;
(6) $G(s, t) = (s + l)^{(1/(1+t)^r)} - l$, $l > 1$ for $r \in (0, +\infty)$;
(7) $G(s, t) = s \log_{t+a} a$, $a > 1$. Indeed, $G(s, t) = s \Rightarrow s = 0$ or $t = 0$;
(8) $G(s, t) = s - \left(\frac{1+s}{2+s}\right)\left(\frac{t}{1+t}\right)$;

(9) $G(s,t) = s\beta(s)$, where $\beta : [0,+\infty) \to (0,1)$ is a continuous function;
(10) $G(s,t) = s - \frac{t}{k+t}$;
(11) $G(s,t) = s - \varphi(s)$ where $\varphi : [0,+\infty) \to [0,+\infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$;
(12) $G(s,t) = sh(s,t)$ where $h : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is a continuous function such that $h(t,s) < 1$ for all $t, s > 0$;
(13) $G(s,t) = s - \left(\frac{2+t}{1+t}\right) t$;
(14) $G(s,t) = \sqrt[n]{\ln(1+s^n)}$, $G(s,t) = s \Rightarrow s = 0$;
(15) $G(s,t) = \phi(s)$ where $\phi : [0,+\infty) \to [0,+\infty)$ is an upper semi-continuous function such that $\phi(0) = 0$, and $\phi(t) < t$ for $t > 0$;
(16) $G(s,t) = \frac{s}{(1+s)^r}$ for some $r \in (0,+\infty)$.

For more examples, see [2, 3].

Definition 2.3 ([15])**.** A function $\psi : [0,+\infty) \to [0,+\infty)$ is called an altering distance function if the following properties are satisfied

(i) ψ is a non-decreasing and continuous,
(ii) $\psi(t) = 0$ if and only if $t = 0$.

The set of altering distance functions is denoted by Φ.

Definition 2.4 ([2, 3])**.** By Φ_u is denoted the class of the functions $\varphi : [0,+\infty) \to [0,+\infty)$ which satisfy the following conditions:

(i) φ is continuous,
(ii) $\varphi(t) > 0$, for all $t > 0$ and $\varphi(0) \geq 0$.

Definition 2.5 ([2, 3])**.** A tripled (ψ, φ, G) where $\psi \in \Psi$, $\varphi \in \Phi_u$ and $G \in \mathcal{C}$ is monotone if for any $x, y \in [0,+\infty)$,

$$x \leqslant y \quad \Longrightarrow \quad G(\psi(x), \varphi(x)) \leqslant G(\psi(y), \varphi(y)).$$

Example 2.6 ([2, 3])**.** (i) Let $G(s,t) = s - t, \phi(x) = \sqrt{x}$,

$$\psi(x) = \begin{cases} \sqrt{x}, & \text{if } 0 \leq x \leq 1, \\ x^2, & \text{if } x > 1, \end{cases}$$

then (ψ, ϕ, G) is monotone.

(ii) Let $G(s,t) = s - t, \phi(x) = x^2$,

$$\psi(x) = \begin{cases} \sqrt{x}, & \text{if } 0 \leq x \leq 1, \\ x^2, & \text{if } x > 1, \end{cases}$$

then (ψ, ϕ, G) is not monotone.

Lemma 2.7 ([12]). *Let E be a Hilbert space and $\{x_n\}$ be a sequence in E such that $\|x_n - x_{n+1}\| \to 0$ as $n \to +\infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\varepsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that:*

(i) $\lim_{k\to+\infty} \left\|x_{m(k)-1} - x_{n(k)+1}\right\| = \varepsilon$;
(ii) $\lim_{k\to+\infty} \left\|x_{m(k)} - x_{n(k)}\right\| = \varepsilon$;
(iii) $\lim_{k\to+\infty} \left\|x_{m(k)-1} - x_{n(k)}\right\| = \varepsilon$;
(iv) $\lim_{k\to+\infty} \left\|x_{m(k)+1} - x_{n(k)+1}\right\| = \varepsilon$;
(v) $\lim_{k\to+\infty} \left\|x_{m(k)} - x_{n(k)-1}\right\| = \varepsilon$.

3. Main Results

Now, we present our main result.

Theorem 3.1. *Let $(X, \|\cdot\|)$ be a Hilbert space. Let $T : X \to X$ be a mapping and $\psi \in \Phi, \varphi \in \Phi_u$, $G \in \mathcal{C}$, where tripled (ψ, φ, G) is monotone. If*

$$\psi\left(\|Tx - Ty\|^2\right) \leq G\left(\psi\left(M(x,y)\right), \varphi\left(M(x,y)\right)\right), \tag{3.1}$$

for all $x, y \in X$, where

$$M(x,y) = a\max\left\{\|Tx - x\|^2, \|Ty - y\|^2, \|x - y\|^2\right\} + b\left(\|Ty - x\|^2 + \|Tx - y\|^2\right), \tag{3.2}$$

and $a, b \geq 0$ such that $a + 4b \leq 1$, then T has a unique fixed point.

Proof. Let $x_0 \in H$. We define a sequence $\{x_n\}$ as $x_{n+1} = Tx_n$, for all $n \geq 0$. From (3.1), we have

$$\begin{aligned}\psi\left(\|x_{n+1} - x_n\|^2\right) &= \psi\left(\|Tx_n - Tx_{n-1}\|^2\right) \\ &\leq G\left(\psi(M(x_n, x_{n-1})), \varphi\left(M(x_n, x_{n-1})\right)\right).\end{aligned} \tag{3.3}$$

From Definition 2.1 (condition (i)), we obtain

$$\psi\left(\|x_{n+1} - x_n\|^2\right) \leq \psi(M(x_n, x_{n-1})). \tag{3.4}$$

Now, since ψ is a non-decreasing, from (3.4) we have

$$\|x_{n+1} - x_n\|^2 \leq M(x_n, x_{n-1}), \tag{3.5}$$

where

$$M\left(x_n, x_{n-1}\right) = a\max\left\{\|Tx_n - x_n\|^2, \|Tx_{n-1} - x_{n-1}\|^2, \|x_n - x_{n-1}\|^2\right\} + b\left(\|Tx_{n-1} - x_n\|^2 + \|Tx_n - x_{n-1}\|^2\right)$$

$$
\begin{aligned}
&= a \max\left\{\|x_{n+1}-x_n\|^2, \|x_n-x_{n-1}\|^2, \|x_n-x_{n-1}\|^2\right\} \\
&\quad + b\left(\|x_n-x_n\|^2 + \|x_{n+1}-x_{n-1}\|^2\right) \\
&= a \max\left\{\|x_{n+1}-x_n\|^2, \|x_n-x_{n-1}\|^2\right\} \\
&\quad + b\left(\|x_{n+1}-x_{n-1}\|^2\right).
\end{aligned}
$$

Since,

$$\|x_{n+1}-x_{n-1}\|^2 \leq 2\left(\|x_{n+1}-x_n\|^2 + \|x_n-x_{n-1}\|^2\right), \tag{3.6}$$

we obtain

$$
\begin{aligned}
\|x_{n+1}-x_n\|^2 \leq a \max\left\{\|x_{n+1}-x_n\|^2, \|x_n-x_{n-1}\|^2\right\} \\
+ 2b\left(\|x_{n+1}-x_n\|^2 + \|x_n-x_{n-1}\|^2\right).
\end{aligned} \tag{3.7}
$$

Now, we consider two cases.

(i) If

$$\max\left\{\|x_{n+1}-x_n\|^2, \|x_n-x_{n-1}\|^2\right\} = \|x_{n+1}-x_n\|^2,$$

then, from (3.7) we have

$$\|x_{n+1}-x_n\|^2 \leq \frac{2b}{1-a-2b}\|x_n-x_{n-1}\|^2. \tag{3.8}$$

(ii) If

$$\max\left\{\|x_{n+1}-x_n\|^2, \|x_n-x_{n-1}\|^2\right\} = \|x_n-x_{n-1}\|^2,$$

then, from (3.7) we have

$$\|x_{n+1}-x_n\|^2 \leq \frac{a+2b}{1-2b}\|x_n-x_{n-1}\|^2. \tag{3.9}$$

Since, $a+4b \leq 1$, from (3.8) and (3.9) we conclude that

$$\|x_{n+1}-x_n\| \leq \|x_n-x_{n-1}\|. \tag{3.10}$$

The inequality (3.10) implies that $\{\|x_{n+1}-x_n\|\}$ is a monotone decreasing sequence. Consequently, there exists $r \geq 0$ such that

$$\|x_{n+1}-x_n\| \to r \quad \text{as} \quad n \to +\infty.$$

Letting $n \to +\infty$ in (3.3), we obtain that

$$
\begin{aligned}
\psi\left(r^2\right) &\leq G\left(\psi\left((a+4b)r^2\right), \varphi\left((a+4b)r^2\right)\right) \\
&\leq \psi((a+4b)r^2).
\end{aligned}
$$

Since ψ is non-decreasing, we obtain

$$\psi\left((a+4b)r^2\right) \leq \psi\left(r^2\right).$$

So, we conclude that

$$G\left(\psi\left((a+4b)r^2\right),\varphi\left((a+4b)r^2\right)\right)=\psi\left((a+4b)r^2\right).$$

Now, from Definition 2.1 (condition (ii)), we get $r=0$. Hence,

$$\lim_{n\to+\infty}\|x_{n+1}-x_n\|=0. \tag{3.11}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. By Lemma 2.7, we conclude that there exists $\epsilon>0$, for which we can find subsequences $\left\{x_{m(k)}\right\}$ and $\left\{x_{n(k)}\right\}$ of $\{Tx_n\}$ with $n(k)>m(k)>k$ such that

$$\begin{aligned}\lim_{k\to+\infty}\left\|x_{m(k)-1}-x_{n(k)+1}\right\| &= \lim_{k\to+\infty}\left\|x_{m(k)}-x_{n(k)}\right\| \\ &= \lim_{k\to+\infty}\left\|x_{m(k)-1}-x_{n(k)}\right\| \\ &= \epsilon.\end{aligned} \tag{3.12}$$

Now, consider the (3.12) and (3.1) as $k\to+\infty$. We have

$$\begin{aligned}&M\left(x_{m(k)},x_{n(k)}\right)\\ &=a\max\left\{\left\|Tx_{m(k)}-x_{m(k)}\right\|^2,\left\|Tx_{n(k)}-x_{n(k)}\right\|^2,\left\|x_{m(k)}-x_{n(k)}\right\|^2\right\}\\ &\quad+b\left(\left\|Tx_{n(k)}-x_{m(k)}\right\|^2+\left\|Tx_{m(k)}-x_{n(k)}\right\|^2\right)\\ &=a\max\left\{\left\|x_{m(k)+1}-x_{m(k)}\right\|^2,\left\|x_{n(k)+1}-x_{n(k)}\right\|^2,\left\|x_{m(k)}-x_{n(k)}\right\|^2\right\}\\ &\quad+b\left(\left\|x_{n(k)+1}-x_{m(k)}\right\|^2+\left\|x_{m(k)+1}-x_{n(k)}\right\|^2\right)\\ &\to a\varepsilon^2+2b\varepsilon^2=(a+2b)\varepsilon^2\\ &\leq\varepsilon^2.\end{aligned}$$

So,

$$\psi\left(\varepsilon^2\right)\leq G\left(\psi\left(\varepsilon^2\right),\varphi\left(\varepsilon^2\right)\right). \tag{3.13}$$

Again, from Definition 2.1, we obtain $\varepsilon=0$. Hence $\{x_n\}$ is a Cauchy sequence. Since H is a Hilbert space the sequence $\{x_n\}$ converges to a point $z\in X$.

Now, we show that z is fixed point of T. Suppose, for the sake of contradiction, that $Tz\neq z$. From (3.1) we obtain

$$\begin{aligned}\psi\left(\|z-Tz\|^2\right) &= \lim_{n\to\infty}\psi\left(\|Tx_n-Tz\|^2\right)\\ &\leq \lim_{n\to\infty}G(\psi\left(M(x_n,z)\right),\varphi\left(M(x_n,z)\right))\\ &\leq \psi(M(x_n,z)),\end{aligned}$$

where with taking limit as $n \to +\infty$,

$$\begin{aligned} M(x_n, z) &= a \max\left\{\|Tx_n - x_n\|^2, \|Tz - z\|^2, \|x_n - z\|^2\right\} \\ &\quad + b\left(\|Tz - x_n\|^2 + \|Tx_n - z\|^2\right) \\ &= a \max\left\{\|x_{n+1} - x_n\|^2, \|Tz - z\|^2, \|x_n - z\|^2\right\} \\ &\quad + b\left(\|Tz - x_n\|^2 + \|x_{n+1} - z\|^2\right) \\ &\to a\|Tz - z\|^2 + b\|Tz - z\|^2 = (a+b)\|Tz - z\|^2 \\ &\leq \|Tz - z\|^2. \end{aligned}$$

So,

$$\begin{aligned} \psi\left(\|Tz - z\|^2\right) &\leq G\left(\psi((a+b)\|Tz - z\|^2), \varphi((a+b)\|Tz - z\|^2)\right) \\ &\leq \psi\left((a+b)\|Tz - z\|^2\right) \\ &\leq \psi\left(\|Tz - z\|^2\right). \end{aligned}$$

From Definition 2.1, we obtain $\|Tz - z\|^2 = 0$. Thus $z = Tz$.

Now we prove that T has a unique fixed point. Suppose that z and w are two distinct fixed points of T. From (3.1), we have

$$\begin{aligned} \psi\left(\|Tw - Tz\|^2\right) &\leq G\left(\psi\left(M(w,z)\right), \varphi\left(M(w,z)\right)\right) \\ &\leq \psi(M(w,z)), \end{aligned}$$

where

$$\begin{aligned} M(w, z) &= a \max\left\{\|Tw - w\|^2, \|Tz - z\|^2, \|w - z\|^2\right\} \\ &\quad + b\left(\|Tz - w\|^2 + \|Tw - z\|^2\right) \\ &= a \max\left\{\|w - w\|^2, \|z - z\|^2, \|w - z\|^2\right\} \\ &\quad + b\left(\|w - z\|^2 + \|z - w\|^2\right) \\ &= (a + 2b)\|z - w\|^2 \leq \|z - w\|^2, \end{aligned}$$

so,

$$\begin{aligned} \psi\left(\|z - w\|^2\right) &= \psi\left(\|Tw - Tz\|^2\right) \\ &\leq G\left(\psi\left(\|z - w\|^2\right), \varphi\left(\|z - w\|^2\right)\right) \\ &\leq \psi\left(\|z - w\|^2\right). \end{aligned}$$

Again, from Definition 2.1, we obtain $\|z-w\|^2 = 0$. Thus $z = w$.

□

Example 3.2. Let $X = \mathbb{R}$, $\|x\| = |x|$, for all $x \in X$ and a mapping $T : X \to X$ be defined by $Tx = \frac{x}{2}$, for all $x \in X$. Also, let

$$\psi(t) = t, \varphi(t) = t, G(s,t) = \frac{2s}{3},$$

for all $s, t \in [0, +\infty)$. Then we have that $\psi \in \Phi, \varphi \in \Phi_u$ and $G \in \mathcal{C}$. In this case, all the conditions of Theorem 3.1 are fulfilled and the mapping T has a fixed point $x = 0$.

Taking $G(s,t) = ms$, $0 < m < 1$ and $am + 4bm < 1$ in Theorem 3.1, we have the following corollary.

Corollary 3.3. *Let $(X, \|\cdot\|)$ be a Hilbert space. Let T be a mapping $T : X \to X$ such that*

$$\|Tx - Ty\|^2 \leq a \max\left\{\|Tx - x\|^2, \|Ty - y\|^2, \|x - y\|^2\right\} + b\left(\|Ty - x\|^2 + \|Tx - y\|^2\right),$$

for all $x, y \in X$ and $a, b \geq 0$ with $a + 4b \leq 1$. Then T has a unique fixed point.

Taking $G(s,t) = ms$, $0 < m < 1$, $a = 0$ and $4bm < 1$ in Theorem 3.1, we have the following corollary.

Corollary 3.4. *Let $(X, \|\cdot\|)$ be a Hilbert space. Let T be a mapping $T : X \to X$ such that*

$$\|Tx - Ty\|^2 \leq b\left(\|Ty - x\|^2 + \|Tx - y\|^2\right), \tag{3.14}$$

for all $x, y \in X$ and $b \geq 0$ with $b < \frac{1}{4}$. Then T has a unique fixed point.

Taking $a = 0$ in Theorem 3.1, we have the following corollary.

Corollary 3.5. *Let $(X, \|\cdot\|)$ be a Hilbert space. Let T be a mapping $T : X \to X$ such that*

$$\psi\left(\|Tx - Ty\|^2\right) \leq G\left(\psi\left(\frac{\|Ty - x\|^2 + \|Tx - y\|^2}{4}\right), \varphi\left(\frac{\|Ty - x\|^2 + \|Tx - y\|^2}{4}\right)\right)$$

for all $x, y \in X$, $\psi \in \Phi, \varphi \in \Phi_u$, and $G \in \mathcal{C}$. Then T has a unique fixed point.

Taking $b = 0$ in Theorem 3.1, we have the following corollary.

Corollary 3.6. *Let $(X, \|\cdot\|)$ be a Hilbert space. Let T be a mapping $T : X \to X$ such that*

$$\psi\left(\|Tx - Ty\|^2\right) \leq G(\psi\left(M(x,y)\right), \varphi\left(M(x,y)\right)) \tag{3.15}$$

where

$$M(x,y) = \max\left\{\|Tx - x\|^2, \|Ty - y\|^2, \|x - y\|^2\right\} \tag{3.16}$$

for all $x, y \in X$, $\psi \in \Phi, \varphi \in \Phi_u$, and $G \in \mathcal{C}$. Then T has a unique fixed point.

In the following Corollaries 2.6. 2.7. and 2.8 we consider that

$$M(x,y) = a\max\left\{\|Tx - x\|^2, \|Ty - y\|^2, \|x - y\|^2\right\} + b\left(\|Ty - x\|^2 + \|Tx - y\|^2\right).$$

Taking $G(s,t) = s - t$, in Theorem 3.1, we have the following corollary.

Corollary 3.7. *Let $(X, \|\cdot\|)$ be a Hilbert space. Let T be a mapping $T : X \to X$ such that*

$$\psi\left(\|Tx - Ty\|^2\right) \leq \psi(M(x,y)) - \varphi\left(M(x,y)\right),$$

for all $x, y \in X$ and $a, b \geq 0$ with $a + 4b \leq 1$ and $\psi \in \Phi, \varphi \in \Phi_u$, $G \in \mathcal{C}$, tripled $(\psi, \varphi, G(s,t) = s - t)$ is monotone. Then T has a unique fixed point.

Taking $G(s,t) = \frac{s}{1+t}$ in Theorem 3.1, we have the following corollary.

Corollary 3.8. *Let $(X, \|\cdot\|)$ be a Hilbert space. Let T be a mapping $T : X \to X$ such that*

$$\psi\left(\|Tx - Ty\|^2\right) \leq \frac{\psi(M(x,y))}{1 + \varphi\left(M(x,y)\right)},$$

for all $x, y \in X$ and $a, b \geq 0$ with $a + 4b \leq 1$ and $\psi \in \Phi, \varphi \in \Phi_u$, $G \in \mathcal{C}$, tripled $\left(\psi, \varphi, G(s,t) = \frac{s}{1+t}\right)$ is monotone. Then T has a unique fixed point.

Taking $G(s,t) = s\log_{t+p} p$, $p > 1$, in Theorem 3.1, we have the following corollary.

Corollary 3.9. *Let $(X, \|\cdot\|)$ be a Hilbert space. Let T be a mapping $T : X \to X$ such that*

$$\psi\left(\|Tx - Ty\|^2\right) \leq \psi(M(x,y)) \cdot \log_{p+\varphi(M(x,y))} p,$$

for all $x, y \in X$ *and* $a, b \geq 0$ *with* $a + 4b \leq 1$ *and* $\psi \in \Phi, \varphi \in \Phi_u$, $G \in \mathcal{C}$, *tripled* $\left(\psi, \varphi, G\left(s,t\right) = s \log_{t+p} p\right), p > 1$ *is monotone. Then* T *has a unique fixed point.*

Remark 3.10. From Theorem 3.2 in [13], for $a = 0$, instead of condition (1.1) we obtain condition

$$\|Tx - Ty\|^2 \leq b\left(\|Ty - x\|^2 + \|Tx - y\|^2\right), \tag{3.17}$$

for all $x, y \in X$ and $b \geq 0$ with $b \leq \frac{1}{2}$. So,

$$\begin{aligned}\|Tx - Ty\| &\leq \sqrt{b\left(\|Ty - x\|^2 + \|Tx - y\|^2\right)} \\ &\leq \sqrt{b}(\|Ty - x\| + \|Tx - y\|),\end{aligned}$$

for all $x, y \in X$ and $b \geq 0$, with $\sqrt{b} \leq \frac{1}{\sqrt{2}}, \left(\frac{1}{2} < \frac{1}{\sqrt{2}}\right)$. That is, we can choice

$$\frac{1}{2} < \sqrt{b} \leq \frac{1}{\sqrt{2}},$$

and this is in contradiction with Chatterjea type contraction [11].

$$\|Tx - Ty\| \leq k(\|Ty - x\| + \|Tx - y\|), \tag{3.18}$$

where, $k \in (0, \frac{1}{2})$.

Note that Theorem 3.2 of Chouhan and Desai [13] is not valid.

Example 3.11. Let $X = \mathbb{R}$, $\|x\| = |x|$, for all $x \in X$ and a mapping $T : X \to X$ be defined by $Tx = x + 1$, for all $x \in X$. Namely, we have

$$\|Tx - Ty\|^2 = |x - y|^2. \tag{3.19}$$

On the other side,

$$\begin{aligned}\frac{1}{2}\left(\|Ty - x\|^2 + \|Tx - y\|^2\right) &= \frac{1}{2}\left(|y + 1 - x|^2 + |x + 1 - y|^2\right) \\ &= \frac{1}{2}\left(|1 + y - x|^2 + |1 - (y - x)|^2\right) \\ &= \frac{1}{2}\left(2 + 2|y - x|^2\right) \\ &= 1 + |y - x|^2 > |y - x|^2.\end{aligned} \tag{3.20}$$

So, we have that a map T fulfills condition (1.1) but T has no fixed point.

Also, we give the following example.

Example 3.12. Also let $S = C([0,1],\mathbb{R})$ with $\langle f,g\rangle = \int_0^1 f(x)g(x)dx$, $\|f\|^2 = \int_0^1 f^2(x)dx$ for all $x \in X$ and a mapping $T : X \to X$ be defined by $T(f(x)) = f(x) + 1$, for all $f \in S$. Now use (3.20) we have

$$\begin{aligned}
&\|T(f(x)) - T(g(x))\|^2 \\
&\quad = \|f(x) - g(x)\|^2 \\
&\quad = \int_0^1 (f(x) - g(x))^2 dx \\
&\quad < \int_0^1 \frac{(f(x) + 1 - g(x))^2 + (g(x) + 1 - f(x))^2}{2} dx \\
&\quad = \frac{1}{2}\int_0^1 (f(x) + 1 - g(x))^2\, dx + \int_0^1 (g(x) + 1 - f(x))^2\, dx \\
&\quad = \frac{1}{2}\left(\|f(x) + 1 - g(x)\|^2 + \|g(x) + 1 - f(x)\|^2\right) \\
&\quad = \frac{\|T(f(x)) - g(x)\|^2 + \|T(g(x)) - f(x)\|^2}{2}.
\end{aligned}$$

Thus T fulfills condition (1.1) but T has no fixed point.

4. Conclusion

The concept of C-class functions was introduced by A. H. Ansari in [2, 3]. By employing C-class functions, we derive new results concerning fixed points in Hilbert spaces. Our findings also lead to some established results from the literature. Additionally, we identify an error in the recent paper [13] by Chouhan and Desai, providing an appropriate example to illustrate the mistake. We have corrected their results and believe that the techniques used in our paper could be applied to a broader range of spaces beyond Hilbert spaces.

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