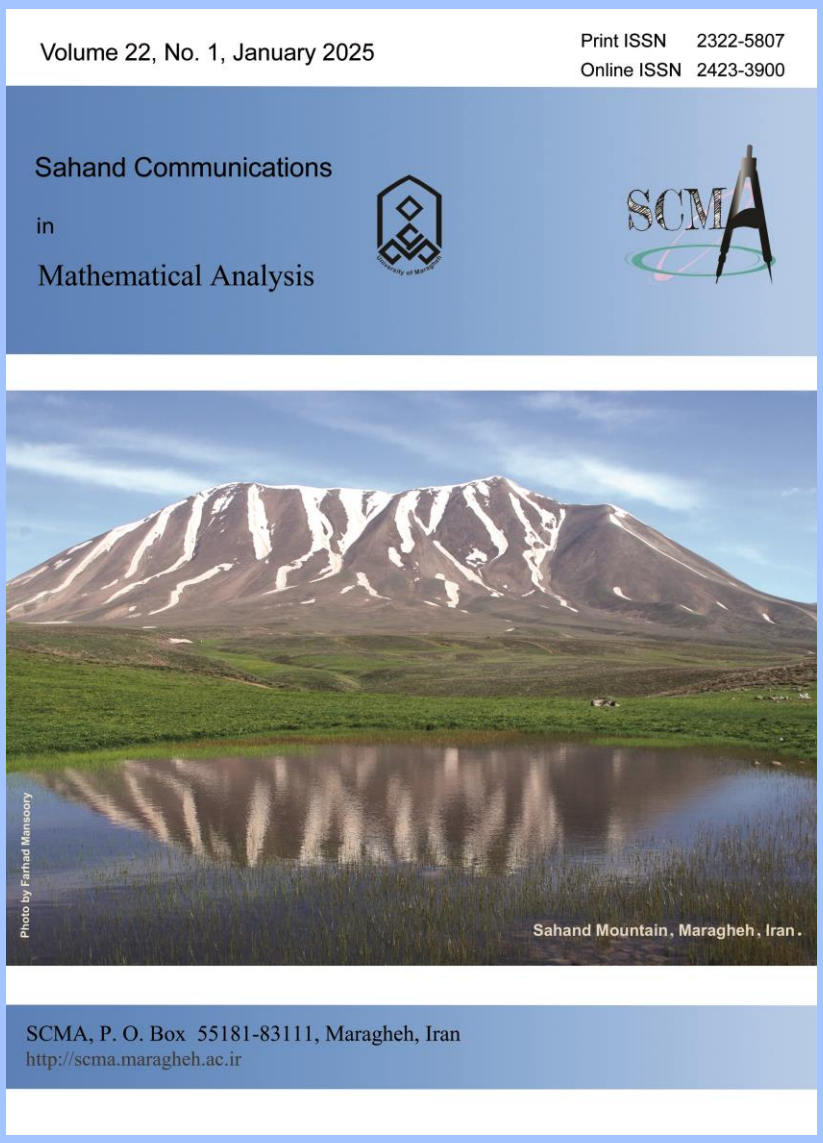
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Sharp Estimates of Hermitian Toeplitz Determinants for Some Subclasses of Sakaguchi Type Function Related to Sine Function

Sangarambadi Padmanabhan Vijayalakshmi[1], Sibel Yalçın[2*] and Tirumalai Vinjimur Sudharsan[3]

Abstract. Hermitian Toeplitz determinants are utilized across various fields, such as functional analysis, applied mathematics, physics, and technical sciences. This paper establishes a link with specific subclasses of analytic functions. Extensive research exists regarding estimating second and third Hankel determinants for normalized analytic functions within this domain. The current research seeks to establish precise upper and lower bounds for the second and third-order Hermitian Toeplitz determinants associated with specific novel subclasses of Sakaguchi-type functions, $S_s^*(\sin z)$, $S_c^*(\sin z)$ and $S_p^q(\sin z)$ related to the sine function. Further, the sharp estimates of Zalcman functional $|a_{n+m-1} - a_n a_m|$ for $n = 2$ and $n = 2$, $m = 3$ are considered.

1. Introduction

Let $\mathcal{A}$ represent the class of analytic functions characterized by the specified form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots ; \quad (z \in \Delta := \{z \in \mathbb{C} : |z| < 1\}). \tag{1.1}$$

Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ that comprises univalent functions.

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* Corresponding author.

Let $\mathbb{P}$ denote the class of analytic functions characterized by a positive real part and expressed in the form

$$p(z) = 1 + \sum_{k=1}^{\infty} b_k z^k, \quad z \in \Delta. \tag{1.2}$$

For two functions $f, g \in \mathcal{A}$, the function f is subordinate to g in Δ denoted as $f \prec g$, if there exists a Schwarz function such that $\omega(0) = 0$ and $|\omega(z)| \leq |z|$ satisfying the relation $f(z) = g(\omega(z)), z \in \Delta$. Furthermore, if the function g is univalent in Δ, then the condition $f \prec g$ holds if and only if $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$.

Research on the Hankel determinant has attracted significant interest in recent years, with numerous researchers establishing sharp bounds for the second and third Hankel determinants. In the domain of Geometric Function Theory, a notable conjecture introduced by Zalcman in 1960 posits that the coefficients belonging to class $\mathcal{S}$ adhere to the inequality

$$|a_k^2 - a_{2k-1}| \leq (k-1)^2.$$

The equality will hold only for the Koebe Function

$$k(z) = \frac{z}{(1-z)^2}$$

and its rotation. When $k = 2$, equality is held for the famous Fekete-Szegö inequality. Many researchers [5, 20] studied the Zalcman functional. Hermitian Toeplitz matrices play a crucial role in signal processing and are extensively applied in fields such as functional analysis, applied mathematics, physics, and various engineering disciplines. [23].

In recent years, researchers have extensively studied Hermitian Toeplitz determinants whose elements are coefficients of functions in subclasses of $\mathcal{A}$ [8, 9, 14, 16, 17, 26]. Hermitian Toeplitz determinant represents a specific instance of a Hankel determinant. In 1954, Hartman and Winter [13] defined the Hermitian Toeplitz determinant $\mathbb{HT}_{q,k}$ $q \geq 1$ $k \geq 0$ as follows:

$$\mathbb{HT}_{q,k} = \begin{vmatrix} a_k & a_{k+1} & \cdots & a_{k+q-1} \\ \overline{a}_{k+1} & a_k & \cdots & a_{k+q-2} \\ \vdots & & & \\ \overline{a}_{k+q-1} & \overline{a}_{k+q-2} & \cdots & a_k \end{vmatrix}.$$

In particular,

$$\mathbb{HT}_{2,1}(f) = 1 - |a_2|^2,$$

$$\mathbb{HT}_{3,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ \overline{a}_2 & 1 & a_2 \\ \overline{a}_3 & \overline{a}_2 & 1 \end{vmatrix}$$

$$= 2\mathcal{R}e\left(a_2^2\overline{a_3}\right) - 2|a_2|^2 - |a_3|^2 + 1.$$

In [21], Sakaguchi introduced the class S_s^*, the class of functions that are the class of starlike functions concerning symmetric points. In 1991, Chakrabarti and Jagannathan presented the concept of (p,q) calculus to generalize or unify various types of q-oscillator algebras. Numerous researchers [3, 4, 6, 12, 22], driven by the potential of these applications, have advanced the theory of quantum calculus grounded in the two-parameter (p,q)-integer. This theory has proven highly effective across various domains, including difference equations, Lie groups, hypergeometric series, and the physical sciences.

The $(p,q)-$ bracket or twin basic number $[k]_{p,q}$ is defined by

$$[k]_{p,q} = \begin{cases} \frac{p^k - q^k}{p-q}, & \text{if } q \neq p \\ kp^{k-1}, & \text{if } q = p \end{cases}$$

where $0 < q \leq p \leq 1$.

For $0 < q < 1$, q-bracket $[k]_q$ for $k = 0, 1, 2, \ldots$ is given by $[k]_q = [k]_{1,q}$. The (p,q)- derivative of a function f is defined by

$$\mathcal{D}_{p,q}f(z) = \begin{cases} \frac{f(pz)-f(qz)}{(p-q)z}, & \text{if } q \neq p, z \neq 0 \\ 1, & \text{if } p \neq q, z = 0 \\ f'(z), & \text{if } p = q. \end{cases}$$

In particular, $\mathcal{D}_{p,q}z^k = [k]_{p,q}z^{k-1}$. For a function $f \in A$, the (p,q)-derivative operator is given by

$$\mathcal{D}_{p,q}f(z) = 1 + \sum_{k=2}^{\infty}[k]_{p,q}a_k z^{k-1}. \tag{1.3}$$

In this study, we derive sharp estimates for the second and third-order Hermitian Toeplitz determinants pertaining to specific subclasses of Sakaguchi-type functions $S_s^*(\sin z), S_c^*(\sin z)$ and $S_p^q(\sin z)$ related to sine functions. Further, the sharp estimates of Zalcman Functional $|a_{k+m-1} - a_k a_m|$ for $k = 2$ and $k = 2$, $m = 3$ are obtained.

2. Priliminary Results

Lemma 2.1 ([10])**.** *Let $p \in \mathbb{P}$ characterized by the form* (1.2)*. Then $|b_k| \leq 2, k = 1, 2, 3, \ldots$ and the equality holds for the function $p(z) = \frac{1+z}{1-z}$.*

Lemma 2.2 ([19]). *Let $p \in \mathbb{P}$ characterized by the form (1.2). Then, For any real numbers λ and σ, the following precise estimates are valid.*

$$|b_3 + \lambda b_1 b_2 + \sigma b_1^3| \leq 1, \quad (\lambda, \sigma) \in D_1 \cup D_2,$$

where,

$$D_1 = \left\{ (\lambda, \sigma) \in \mathbb{R}^2 : |\lambda| \leq \frac{1}{2}, -1 \leq \sigma \leq 1 \right\},$$

$$D_2 = \left\{ (\lambda, \sigma) \in \mathbb{R}^2 : \frac{1}{2} \leq |\lambda| \leq 2, \ \frac{4}{27}(|\lambda| + 1)^3 - (|\lambda| + 1) \leq \sigma \leq 1 \right\}.$$

The extremal function is expressed as $p(z) = z^3$.

Lemma 2.3 ([11]). *Let $p \in \mathbb{P}$ characterized by the form (1.2). Then, for all $\lambda \in \mathbb{C}$, we have $|b_2 + \lambda b_1^2| \leq \max\{1, |\lambda|\}$.*

Lemma 2.4 ([18]). *Let $p \in \mathbb{P}$ characterized by the form (1.2). Then there exist some ξ, $\zeta \in \mathbb{C}$ with $|\xi| \leq 1$, $|\zeta| \leq 1$, such that*

$$2b_2 = b_1^2 + \left(4 - b_1^2\right) \xi,$$

$$4b_3 = b_1^3 + 2b_1\xi \left(4 - b_1^2\right) - \left(4 - b_1^2\right) b_1 \xi^2 + 2 \left(4 - b_1^2\right) (1 - |\xi|^2)\zeta.$$

Definition 2.5 ([15]). Let $S_s^*(\sin z)$ represent the subclass of $\mathcal{S}$ comprising functions f defined by the form (1.1), which satisfies the following condition

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec 1 + \sin(z), \quad \text{for } z \in \Delta,$$

This indicates that the value of $\frac{zf'(z)}{f(z)-f(-z)}$ is situated within an eight-shaped area in the right half of the complex plane.

Definition 2.6. Denote $S_c^*(\sin z)$ the subclass of $\mathcal{S}$ consisting of convex functions with respect to symmetric points, which satisfies the following condition

$$\frac{(2zf'(z))'}{(f(z) - f(-z))'} \prec 1 + \sin(z), \quad \text{for } z \in \Delta.$$

Definition 2.7. Let $S_p^q(\sin z)$ represent the subclass of $\mathcal{S}$ comprising functions f defined by the form (1.1), which satisfies the following condition

$$\frac{2z\mathcal{D}_{p,q}f(z)}{f(z) - f(-z)} \prec 1 + \sin(z), \quad \text{for } z \in \Delta, \ 0 < q \leq p \leq 1.$$

We observed that when $p = q$ this class simplifies to the class $S_s^*(\sin z)$ [15].

Remark 2.8. We shall illustrate with an example that the aforementioned classes $S_s^*(\sin z)$, $S_c^*(\sin z)$, $S_p^q(\sin z)$ are non empty.

(i) Taking $f_1(z) = z + az^2$, $a \in \mathbb{C}$, then

$$\phi_*(z) = \frac{2zf_1'(z)}{f_1(z) - f_1(-z)} = 1 + 2za, \quad z \in \Delta.$$

For the values of $z = 0.9e^{i\theta}$, $a = 0.1$, it is clear from the Figure 1 made with GNU octave computer software, we have $\phi_*(\Delta) \subset 1 + \sin(z)$ and hence the class $S_s^*(\sin z)$ is nonempty.

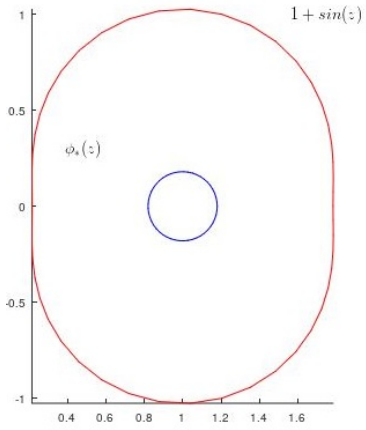


FIGURE 1. The image of $\phi_*(z)$, (blue colour) is contained in $1 + \sin(z)$(red colour)

(ii) Taking $f_1(z) = z + az^2$, $a \in \mathbb{C}$, then

$$\begin{aligned}\psi_*(z) &= \frac{(2zf_1'(z))'}{(f_1(z) - f_1(-z))'} \\ &= 1 + 4za, z \in \Delta.\end{aligned}$$

For the values of $z = 0.9e^{i\theta}$, $a = 0.1$, it is clear from the Figure 2 made with GNU octave computer software, we have $\psi_*(\Delta) \subset 1 + \sin(z)$. Hence the class $S_c^*(\sin z)$ is nonempty.

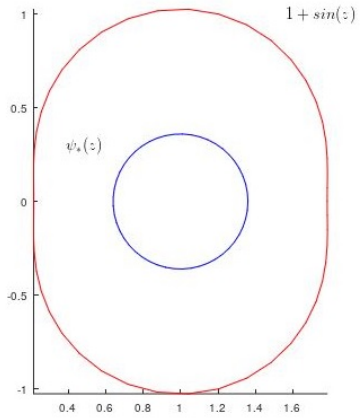


FIGURE 2. The image of $\psi_*(z)$ (blue colour) is contained in $1+\sin(z)$(red colour)

(iii) Taking $f_1(z) = z + az^2$, $a \in \mathbb{C}$, then

$$\begin{aligned}\chi_*(z) &= \frac{2z\mathcal{D}_{p,q}f(z)}{f(z)-f(-z)} \\ &= 1+(p+q)az, \quad z \in \Delta.\end{aligned}$$

For the values of $z = 0.9e^{i\theta}$, $p = 0.5$, $q = 0.3$, and $a = 0.1$, it is clear from the Figure 3, made with GNU octave computer software, we have $\chi_*(\Delta) \subset 1+\sin(z)$. Hence, the class $S_p^q(\sin z)$ is nonempty.

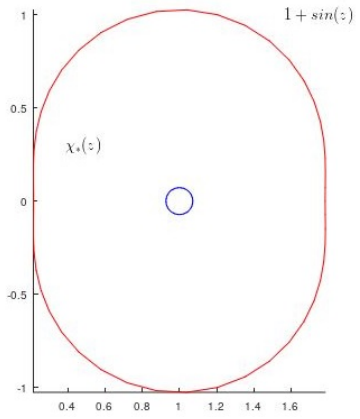


FIGURE 3. The image of $\chi_*(z)$ (blue colour) is contained in $1+\sin(z)$(red colour)

3. Main Results

Theorem 3.1. *Let the function $f(z)$ characterized by the form* (1.1) *belongs to the class $S_s^*(\sin z)$. Then,*

$$\frac{3}{4} \leq \mathbb{HT}_{2,1}(f) \leq 1 \tag{3.1}$$

and

$$\frac{1}{2} \leq \mathbb{HT}_{3,1}(f) \leq 1. \tag{3.2}$$

The inequalities are sharp.

Proof. Let $f(z) \in S_s^*(\sin z)$, then there exists a Schwarz function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, satisfying

$$\frac{2zf'(z)}{f(z) - f(-z)} = 1 + \sin(\omega(z)), \quad \text{for } z \in \Delta. \tag{3.3}$$

specify the function

$$p(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + b_1 z + b_2 z^2 + \cdots \tag{3.4}$$

where $p \in \mathbb{P}$. This implies

$$\begin{aligned} \omega(z) &= \frac{p(z) - 1}{1 + p(z)} \\ &= \frac{b_1}{2} z + \left(\frac{b_2}{2} - \frac{b_1^2}{4} \right) z^2 + \left(\frac{b_3}{2} - \frac{b_1 b_2}{2} + \frac{b_1^3}{8} \right) z^3 + \cdots . \end{aligned}$$

After some simple computations, we obtain

(3.5)

$$\begin{aligned} 1 + \sin(\omega(z)) &= 1 + \omega(z) - \frac{\omega(z)^3}{3!} + \frac{\omega(z)^5}{5!} + \cdots \\ &= 1 + \frac{1}{2} b_1 z + \left(\frac{b_2}{2} - \frac{b_1^2}{4} \right) z^2 + \left(\frac{5b_1^3}{48} + \frac{b_3 - b_1 b_2}{2} \right) z^3 + \cdots \end{aligned}$$

and

$$\frac{2zf'(z)}{f(z) - f(-z)} = 1 + 2a_2 z + 2a_3 z^2 + (4a_4 - 2a_2 a_3) z^3 + \cdots . \tag{3.6}$$

From (3.3) and (3.4), we infer that

$$a_2 = \frac{b_1}{4}, \tag{3.7}$$

$$a_3 = \frac{1}{8} \left(2b_2 - b_1^2 \right). \tag{3.8}$$

Since $|b_1| \leq 2$, it follows that

$$\mathbb{HT}_{2,1}(f) = 1 - |a_2|^2 = 1 - \left|\frac{b_1}{4}\right|^2 \leq 1$$

and

$$\mathbb{HT}_{2,1}(f) \geq \frac{3}{4}.$$

The equality for upper and lower estimation is true when $f(z) = z$ and $f(z) = z + \frac{z^2}{2}$ respectively. Since the class $S_s^*(\sin z)$ and $\mathbb{HT}_{3,1}(f)$ are rotationally invariant, we may assume that $b_1 \in [0, 2]$. Using (3.5), (3.6) and Lemma2.4 for some $\xi \in \overline{\Delta}$, we have

$$\begin{aligned}(3.9)\quad \mathbb{HT}_{3,1}(f) =& 1 + \frac{1}{64}[b_1^2\left(4 - b_1^2\right)\mathcal{R}e(\bar{\xi})] - \frac{b_1^2}{8} - \frac{1}{64}\left(2b_2 - b_1^2\right)^2 \\ =& 1 + \frac{1}{64}\left[b_1^2\left(4 - b_1^2\right)\mathcal{R}e(\bar{\xi})\right] - \frac{b_1^2}{8} - \frac{1}{64}\left[\left(4 - b_1^2\right)^2|\xi|^2\right] \\ =& \mathbb{F}(b_1^2, |\xi|, \mathcal{R}e\left(\bar{\xi}\right)).\end{aligned}$$

It is apparent that $\mathbb{HT}_{3.1}(f) \leq \mathbb{F}(b_1^2, |\xi|, |\xi|)$ and $\mathbb{HT}_{3.1}(f) \geq \mathbb{F}(b_1^2, |\xi|, -|\xi|)$. Setting $b_1^2 = x \in [0, 4]$ and $|\xi| = y \in [0, 1]$, from (3.7), we have

$$\begin{aligned}\mathbb{HT}_{3,1}(f) &\leq 1 + \frac{1}{64}\left[xy(4 - x) - 8x - (4 - x)^2y^2\right] \\ &= \mathbb{F}(x, y)\end{aligned}$$

and

$$\begin{aligned}\mathbb{HT}_{3,1}(f) &\geq 1 - \frac{1}{64}\left[xy(4 - x) + 8x + (4 - x)^2y^2\right] \\ &= G(x, y).\end{aligned}$$

On the boundary points $(0, 4) \times (0, 1)$, we find

$$\begin{aligned}\mathbb{F}(0, y) &= 1 - \frac{y^2}{4} \leq 1, \\ \mathbb{F}(4, y) &= \frac{1}{2}, \\ \mathbb{F}(x, 0) &= 1 - \frac{x}{2} \leq 1, \\ \mathbb{F}(x, 1) &= \frac{3}{4} - \frac{1}{32}\left[x^2 - 2x\right] \leq \frac{3}{4}.\end{aligned}$$

The second derivative test reveals that no critical points are present within the domain $(0, 4) \times (0, 1)$. Consequently, based on the preceding analysis, it can be concluded that the maximum value of $\mathbb{F}(x, y)$ is 1. Similarly,

$$G(0, y) \leq 1, \qquad G(4, y) = \frac{1}{2}, \qquad G(x, 0) \leq 1, \qquad G(x, 1) \leq \frac{3}{4}.$$

Hence,

$$\frac{1}{2} \leq \mathbb{HT}_{3,1}(f) \leq 1.$$

The equality for upper and lower estimation is true when $f(z) = z$ and $f(z) = z + \frac{z^2}{2}$ respectively. □

Theorem 3.2. *Let the function $f(z)$ characterized by the form* (1.1) *belongs to the class $S_c^*(\sin z)$. Then,*

$$\frac{15}{16} \leq \mathbb{HT}_{2,1}(f) \leq 1$$

and

$$\frac{7}{8} \leq \mathbb{HT}_{3,1}(f) \leq 1.$$

The inequalities are sharp.

Proof. Let $f(z) \in S_c^*(\sin z)$. In a similar manner, as indicated by the Theorem 3.1 we get

$$\frac{(2zf'(z))'}{(f(z) - f(-z))'} = 1 + \sin(\omega(z)), \quad \text{for } z \in \Delta. \tag{3.10}$$

After simple computations, we obtain

$$\frac{(2zf'(z))'}{(f(z) - f(-z))'} = 1 + 4a_2 z + 6a_3 z^2 + (16a_4 - 12a_2 a_3)z^3 + \cdots . \tag{3.11}$$

From (3.3) and (3.9), we infer that

$$a_2 = \frac{b_1}{8}, \tag{3.12}$$

$$a_3 = \frac{1}{24}\left(2b_2 - b_1^2\right), \tag{3.13}$$

$$a_4 = \frac{1}{16}\left[\frac{2b_1^3}{48} - \frac{3b_1 b_2}{8} + \frac{b_3}{2}\right]. \tag{3.14}$$

Since $|b_1| \leq 2$, it follows that

$$\mathbb{HT}_{2,1}(f) = 1 - |a_2|^2 = 1 - \left|\frac{b_1}{8}\right|^2 \leq 1$$

and

$$\mathbb{HT}_{2,1}(f) \geq \frac{15}{16}.$$

The equality for upper and lower estimation is true when $f(z) = z$ and $f(z) = z + \frac{z^2}{4}$ respectively. Using (3.10), (3.11) and Lemma 2.4 for some $\xi \in \overline{\Delta}$, we have

(3.15)

$$\mathbb{HT}_{3,1}(f) = 1 + \frac{1}{768}\left[b_1^2\left(4 - b_1^2\right)\mathcal{R}e(\overline{\xi})\right] - \frac{b_1^2}{32} - \frac{1}{576}\left[\left(4 - b_1^2\right)^2 |\xi|^2\right]$$

$$= \mathbb{F}\left(b_1^2, |\xi|, \mathcal{R}e(\bar{\xi})\right).$$

It is apparant that

$$\mathbb{HT}_{3.1}(f) \leq \mathbb{F}(b_1^2, |\xi|, |\xi|), \qquad \mathbb{HT}_{3.1}(f) \geq \mathbb{F}(b_1^2, |\xi|, -|\xi|).$$

Setting $b_1^2 = x \in [0,4]$ *and* $|\xi| = y \in [0,1]$, from (3.13), we have

$$\begin{aligned}\mathbb{HT}_{3,1}(f) &\leq 1 + \frac{1}{768}[x(4-x)y] - \frac{x}{32} - \frac{1}{576}\left[(4-x)^2y^2\right] \\ &= \mathbb{F}(x,y)\end{aligned}$$

and

$$\begin{aligned}\mathbb{HT}_{3,1}(f) &\geq 1 + \frac{1}{768}[-x(4-x)y] - \frac{x}{32} - \frac{1}{576}\left[(4-x)^2y^2\right] \\ &= G(x,y).\end{aligned}$$

On the boundary points $(0,4)\times(0,1)$, we find

$$\mathbb{F}(0,y) \leq 1, \qquad \mathbb{F}(4,y) = \frac{7}{8}, \qquad \mathbb{F}(x,0) \leq 1, \qquad \mathbb{F}(x,1) \leq \frac{35}{36}.$$

Thus, no critical points are present within the domain $(0,4)\times(0,1)$. Consequently, based on the preceding analysis, it can be concluded that the maximum value of $\mathbb{F}(x,y)$ is 1. Similarly,

$$G(0,y) \leq 1, \qquad G(4,y) = \frac{7}{8}, \qquad G(x,0) \leq 1, \qquad G(x,1) \leq \frac{35}{36}.$$

Hence,

$$\frac{7}{8} \leq \mathbb{HT}_{3,1}(f) \leq 1.$$

The equality for upper and lower estimation is true when $f(z) = z$ and $f(z) = z + \frac{z^2}{4}$ respectively. □

Theorem 3.3. *Let the function $f(z)$ characterized by the form* (1.1) *belongs to the class $S_p^q(\sin z)$, $0 < q \leq p \leq 1$. Then,*

$$\frac{t_2^2 - 1}{t_2^2} \leq \mathbb{HT}_{2,1}(f) \leq 1$$

and

$$\frac{t_2^2 - 2}{t_2^2} \leq \mathbb{HT}_{3,1}(f) \leq 1$$

where, $t_k = [k]_{p,q} = \frac{p^k - q^k}{p-q}$. The inequalities are sharp.

Proof. Let $f(z) \in S_p^q(\sin z)$. In a similar manner, as indicated by the Theorem 3.1, we get

$$\frac{2z\mathcal{D}_{p,q}f(z)}{f(z) - f(-z)} = 1 + \sin(\omega(z)), \quad \text{for } z \in \Delta. \tag{3.16}$$

From (1.3), we have

$$\frac{2z\mathcal{D}_{p,q}f(z)}{f(z)-f(-z)} = 1 + [2]_{p,q}a_2z + ([3]_{p,q}-1)a_3z^2 + \cdots . \tag{3.17}$$

From (3.15) and (3.3),

$$a_2 = \frac{b_1}{2t_2}, \tag{3.18}$$

$$a_3 = \frac{1}{t_3-1}\left(\frac{2b_2-b_1^2}{4}\right), \tag{3.19}$$

$$a_4 = \frac{1}{t_4}\left[\frac{5b_1^3}{48} - \frac{b_1^3}{8(t_3-1)} - \frac{b_1b_2}{2} + \frac{b_3}{2} + \frac{b_1b_2}{4(t_3-1)}\right], \tag{3.20}$$

where $t_k = [k]_{p,q} = \frac{p^k-q^k}{p-q}$.

Using (3.16)-(3.18), we get

$$\begin{aligned}\mathbb{HT}_{2,1}(f) &= 1 - |a_2^2| \\ &= 1 - \left|\frac{b_1^2}{4t_2^2}\right| \\ &\leq 1\end{aligned} \tag{3.21}$$

and using Lemma 2.1, we get

$$\mathbb{HT}_{2,1}(f) \geq \frac{t_2^2-1}{t_2^2}. \tag{3.22}$$

The upper and lower bound of the inequality is best possible when $f(z) = z$ and $f(z) = z + \frac{z^2}{t_2}$ respectively. Using (3.16), (3.17) and Lemma 2.4 for some $\xi \in \overline{\Delta}$, we have

$$\begin{aligned}\mathbb{HT}_{3,1}(f) = 1 &+ \frac{1}{8t_2^2(t_3-1)}\left[b_1^2\left(4-b_1^2\right)\mathcal{R}e(\overline{\xi})\right] \\ &- \frac{b_1^2}{2t_2^2} - \frac{1}{16(t_3-1)^2}\left[\left(4-b_1^2\right)^2|\xi|^2\right] \\ &= \mathbb{F}(b_1^2, |\xi|, \mathcal{R}e(\overline{\xi})).\end{aligned} \tag{3.23}$$

It is evident that $\mathbb{HT}_{3.1}(f) \leq \mathbb{F}(b_1^2, |\xi|, |\xi|)$ and $\mathbb{HT}_{3.1}(f) \geq \mathbb{F}(b_1^2, |\xi|, -|\xi|)$. Setting $b_1^2 = x \in [0,4]$ and $|\xi| = y \in [0,1]$, from (3.21), we have

$$\begin{aligned}\mathbb{HT}_{3,1}(f) &\leq 1 + \frac{1}{8t_2^2(t_3-1)}[xy(4-x)] - \frac{x}{2t_2^2} - \frac{1}{16(t_3-1)^2}\left[(4-x)^2y^2\right] \\ &= \mathbb{F}(x,y)\end{aligned}$$

and

$$\mathbb{HT}_{3,1}(f) \geq 1 + \frac{1}{8t_2^2(t_3-1)}[-xy(4-x)] - \frac{x}{2t_2^2} - \frac{1}{16(t_3-1)^2}\left[(4-x)^2y^2\right]$$

$$= G(x, y).$$

On the boundary points $(0, 4) \times (0, 1)$, we find

$$\begin{aligned}
\mathbb{F}(0, y) &= 1 - \frac{y^2}{(t_3 - 1)^2} \leq 1, \\
\mathbb{F}(4, y) &= 1 - \frac{2}{t_2^2} = \frac{t_2^2 - 2}{t_2^2}, \\
\mathbb{F}(x, 0) &\leq 1 - \frac{x}{t_2^2} \leq 1, \\
\mathbb{F}(x, 1) &\leq \frac{t_3}{(t_3 - 1)^2}(t_3 - 2).
\end{aligned}$$

From the preceding analysis, we can deduce that the maximum value of $\mathbb{F}(x, y)$ is 1 and similarly minimum value of

$$G(x, y) = \frac{t_2^2 - 2}{t_2^2}.$$

Hence,

$$\frac{t_2^2 - 2}{t_2^2} \leq \mathbb{HT}_{3,1}(f) \leq 1.$$

The upper and the lower bound of the inequality is best possible when $f(z) = z$ and $f(z) = z + \frac{z^2}{t_2}$ respectively. □

Corollary 3.4. *When $p = q$, we get the bounds of $\mathbb{HT}_{2,1}(f)$ and $\mathbb{HT}_{3,1}(f)$ given by (3.1) and (3.2).*

4. Sharp Upper Bounds of Generalized Zalcman Functional

In [15], Katarzyna derived the generalized Zalcman functional for functions belonging to the class $S_s^*(\sin z)$. We will examine specific instances of the generalized Zalcman functional, particularly $|a_{k+m-1} - a_k a_m|$, for the cases where $k = 2$ and $k = 2, m = 3$, focusing on functions from the classes $S_c^*(\sin z)$ and $S_p^q(\sin z)$.

Theorem 4.1. *Let the function $f(z)$ characterized by the form* (1.1) *belongs to the class $S_c^*(\sin z)$. Then,*

(i) $|a_3 - a_2^2| \leq \frac{1}{12}$.

(ii) $|a_2 a_3 - a_4| \leq \frac{1}{32}$.

The inequalities are sharp.

Proof. Using (3.10) and (3.11), we find

$$\left|a_3 - a_2^2\right| = \left|\frac{[2b_2 - b_1^2]}{24} - \frac{b_1^2}{64}\right|$$

$$= \frac{1}{12}\left|b_2 - \frac{11}{16}b_1^2\right|.$$

Using Lemma 2.3 with $\lambda = -\frac{11}{16}$, we get

$$\left|a_3 - a_2^2\right| \leq \frac{1}{12}.$$

From (3.10), (3.11) and (3.12), we find

$$|a_4 - a_2a_3| = \frac{1}{384}\left|3b_1^3 - 13b_1b_2 + 12b_3\right|$$
$$= \frac{1}{32}\left|b_3 - \frac{13}{12}b_1b_2 + \frac{1}{4}b_1^3\right|.$$

Applying Lemma 2.2 with $\lambda = -\frac{13}{12}$ and $\sigma = \frac{1}{4}$, we get

$$|a_4 - a_2a_3| \leq \frac{1}{32}.$$

These inequalities are sharp for the functions given by (3.10) with $\omega(z) = \frac{z^2}{2}$ and $\omega(z) = \frac{z^3}{2}$. □

Theorem 4.2. *Let the function $f(z)$ characterized by the form (1.1) belongs to the class $S_p^q(\sin z), 0 < q \leq p \leq 1$. Then,*

(i) $|a_3 - a_2^2| \leq \frac{1}{|1-t_3|}$.

(ii) $|a_2a_3 - a_4| \leq \frac{1}{t_4}$.

The inequalities are sharp.

Proof. Using (3.16) -(3.18), we find

$$|a_3 - a_2^2| = \left|\frac{1}{1-t_3}\left[\frac{b_1^2}{4} - \frac{b_2}{2}\right] - \frac{b_1^2}{4t_2^2}\right|.$$

Applying Lemma 2.4, we get

$$|a_3 - a_2^2| = \left|\frac{1}{4(1-t_3)}\left(4 - b_1^2\right)\xi - \frac{1}{4t_2^2}b_1^2\right|.$$

Let $b_1 = b$. Since $|b_1| = |b| \leq 2$, By employing Lemma 2.4, we derive that b lies within the interval $[0, 2]$. Furthermore, by utilizing the triangle inequality with $\rho = |\xi| < 1$, we arrive at the following result.

$$|a_3 - a_2^2| \leq \frac{1}{4|1-t_3|}\left(4 - b^2\right)\rho + \frac{1}{4|t_2^2|}b^2 = \mathbb{F}(\rho)$$

$\mathbb{F}'(\rho) > 0$. Hence the maximum value is attained when $\rho = 1$.

Thus, $\mathbb{F}(\rho) \leq \frac{1}{4|1-t_3|}\left(4 - b^2\right) + \frac{1}{4t_2^2}b^2 = \mathcal{G}(b)$. After simple computations, $\mathcal{G}'(b) = 0$ which gives, $b = 0$. At $b = 0$, $\mathcal{G}''(b) < 0$. Thus, the upper limit of $\mathbb{F}(\rho)$ is associated with the condition where $\rho = 1$ and $b = 0$.

Hence $|a_3 - a_2^2| \leq \frac{1}{|1-t_3|}$. This inequality is sharp for the functions given by (3.16) with $\omega(z) = z^2$.

Further, again using (3.16) to (3.18), we obtain

$$\begin{aligned}|a_4 - a_2a_3| &= \left|\frac{b_1^3}{48t_4(t_3-1)}[5(t_3-1)-6]\right.\\ &\quad - \frac{1}{4t_4(t_3-1)}[2b_1b_2(t_3-1)-b_1b_2]\\ &\quad \left.+\frac{b_3}{2t_4} - \frac{b_1b_2}{4t_2(t_3-1)} + \frac{b_1^3}{8t_2(t_3-1)}\right|\\ &= \left|\alpha b_1^3 - \beta b_1b_2 + \gamma b_3\right|,\end{aligned}$$

where,

$$\begin{aligned}\alpha &= \frac{5t_2|t_3-1| - 6t_2 + 6t_4}{48t_2t_4|t_3-1|},\\ \beta &= \frac{1}{4t_2t_4|t_3-1|}[t_2(2|t_3-1|-1) - t_4],\\ \gamma &= \frac{1}{2t_4}.\end{aligned}$$

Substituting the values of b_2 and b_3 from Lemma 2.4, we have

$$\begin{aligned}&|a_4 - a_2a_3|\\ &= \left|\alpha b_1^3 - \frac{\beta b_1}{2}[b_1^2 + \xi\left(4-b_1^2\right)]\right.\\ &\quad \left.+\frac{\gamma}{4}\left[b_1^3 + 2\xi b_1\left(4-b_1^2\right) - \xi^2 b_1\left(4-b_1^2\right) + 2\zeta(1-|\xi|^2)\left(4-b_1^2\right)\right]\right|.\end{aligned}$$

applying Lemma 2.4 , proceeding in a similar fashion, we obtain

$$\begin{aligned}|a_4 - a_2a_3| &\leq \frac{b^3}{4}[4\alpha + 2\beta + \gamma] + \frac{1}{2}\rho b\left(4-b^2\right)(\beta+\gamma)\\ &\quad + \frac{1}{4}\rho^2\left(4-b^2\right)\gamma(b-2) + \frac{1}{2}\left(4-b^2\right)\gamma\\ &= \mathbb{F}(\rho).\end{aligned}$$

Next we maximize the function $\mathbb{F}(\rho)$.

$\mathbb{F}'(\rho) = 0$ gives $\rho = \frac{b(\beta+\gamma)}{\gamma(2-b)}$. Set $\rho^* = \frac{b(\beta+\gamma)}{\gamma(2-b)}$. Now $0 \leq \rho^* \leq 1$. Also, we have $\mathbb{F}''(\rho) < 0$. Thus ρ^* is the only value in the interval $[0,1]$ at which $\mathbb{F}(\rho)$ attains maximum. Hence $\mathbb{F}(\rho) \leq \mathbb{F}(\rho^*)$. Thus

$$\begin{aligned}\mathbb{F}(\rho) &\leq \frac{b^3}{4}[4\alpha + 2\beta + \gamma] + \frac{(\beta+\gamma)^2b^2\left(4-b^2\right)}{\gamma(2-b)}\\ &\quad + \frac{1}{4\gamma^2(2-b)^2}b^2(\beta+\gamma)^2\left(4-b^2\right)\gamma(b-2) + \frac{1}{2}\left(4-b^2\right)\gamma\end{aligned}$$

$$= \mathcal{G}(b).$$

After simple computations, $\mathcal{G}'(b) = 0$ implies $b = 0$. At $b = 0$, $\mathcal{G}''(b) < 0$. Consequently, the upper limit of ρ is represented by $\rho = \rho*$ with $b = 0$. Therefore, it follows that $|a_4 - a_2a_3| \leq 2\gamma$.

That is $|a_4 - a_2a_3| \leq \frac{1}{t_4}$. This inequality is sharp for the functions given by (3.16) with $\omega(z) = z^3$. □

5. Conclusion

Hermitian Toeplitz determinants are utilized across various disciplines, such as functional analysis, applied mathematics, physics, and technical sciences. Quantum calculus is crucial in multiple fields, including mathematics and physics, particularly in ordinary fractional calculus. This paper establishes a link with specific subclasses of analytic functions. The current study focuses on Hermitian Toeplitz determinants associated with certain subclasses of Sakaguchi-type functions, namely $S_s^*(\sin z)$, $S_c^*(\sin z)$, and $S_p^q(\sin z)$, which are connected to sine functions. Additionally, we derived estimates for the Zalcman functional $|a_{k+m-1} - a_k a_m|$ for the cases where $k = 2$ and $k = 2$, $m = 3$. These findings may hold significance in various mathematics, engineering, science, and technology domains. We motivate researchers to delve deeper into the precise estimates concerning logarithmic coefficients and the fourth-order Hermitian Toeplitz determinants associated with these categories, including convex, starlike, and other pertinent Subcategories of analytic and univalent functions within the realm of geometric function theory.

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[1] Department of Mathematics, Dwaraka Doss Goverdhan Doss Vaishnav College, Chennai, India.
Email address: vijishreekanth@gmail.com

[2] Department of Mathematics, Faculty of Arts and Sciences, Bursa Uludag University, Bursa, TÜRKİYE.
Email address: syalcin@uludag.edu.tr

[3] Department of Mathematics, S.I.V.E.T College, Gowrivakkam, Chennai, India.
Email address: tvsudharsan@gmail.com