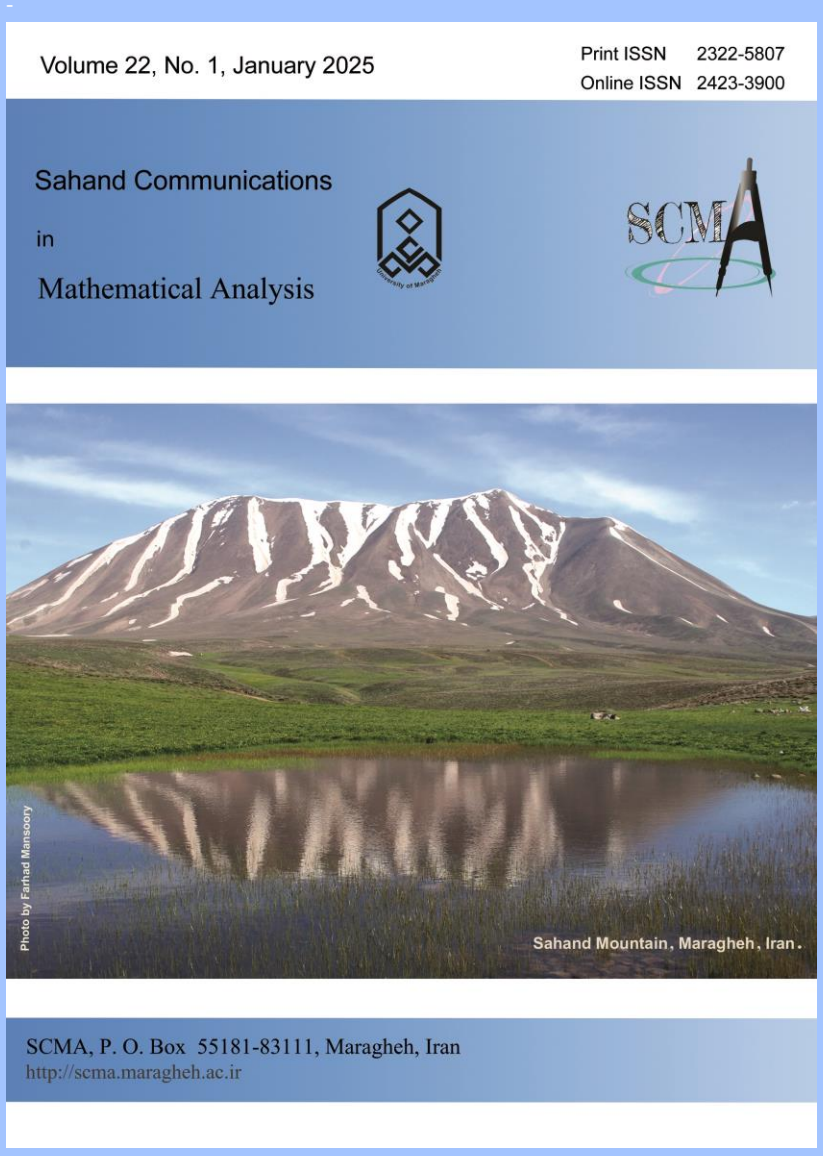
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A Comprehensive Study on the Zeros of the Two-Parameter Mittag-Leffler Function

Farnoosh Abooali[1*] and Aliasghar Jodayree Akbarfam[2]

Abstract. The Mittag-Leffler function appears as an analytical solution of some fractional differential equations. The behavior of the zeros of the Mittag-Leffler function, especially their asymptotic distribution, plays a fundamental role in the study of eigenvalue problems. The goal here is to derive a different form of the higher-order asymptotic distribution of the zeros of the Mittag-Leffler function.

1. Introduction

The Mittag-Leffler function

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta+\alpha k)},\quad \alpha>0,\quad \beta\in\mathbf{C},\quad z\in\mathbf{C} \tag{1.1}$$

was introduced by Humbert [26], Agarwal [29], Humbert and Agarwal [27], Cleota and Hughes [8], Wiman [6, 7], for the solution of fractional differential equations and integral equations of Abel type.

The close connection of these functions to differential equations of fractional order has become more and more popular in modelling natural and technical processes and situations (Babenko [42], Bagley and Torvik [28], Blair [12, 13], Caputo and Mainardi [23], El-Sayed [2], Gorenflo et al. [30], Gorenflo and Mainardi [31, 35], Gorenflo and Rutman [32], Gorenflo and Vessella [33], Luchko [22], Luchko and Srivastava [43], Mainardi [11], Schneider and Wyss [41], Slonimski [14], Westerlund and Ekstam [40]), Lang [20, 21], Hilfer [37, 38], Aydi [18] and Saxena [39].

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In the computation of eigenvalues of fractional order problems and corresponding inverse problems, their asymptotic form plays an important role, especially in the appearance of the potential function in different forms, which gives a deeper understanding of the study of inverse problems. For example, while studying non-classical Sturm-Liouville problems, the higher-order distribution of their eigenvalues is closely connected to the form of the potential function. For more details, see [4, 9, 15–17] for more details.

2. On the Number of Zeros of $E_{\alpha,\beta}(z)$

The Mittag-Leffler function given by (1.1) is an entire function of order $\frac{1}{\alpha}$ [35]. Consequently, all Mittag-Leffler functions $E_{\alpha,\beta}(z)$ will have an infinite number of zeros with the possible exception where $\frac{1}{\alpha}$ is an integer. The previous statement says nothing about the number of zeros when $\frac{1}{\alpha}$ is an integer. In these cases, there may be no zeros, a finite number of zeros, or an infinite number of zeros. From [19] we see that in case $2 < \alpha < 3$, the Mittag-Leffler function has an infinite number of real zeros and may have a finite number of non-real zeros. This case is identified by the phase $[\infty/f]$ or $[\infty/0]$ in [19], where the first symbol of the pair indicates that there is an infinite number of real zeros, and the second symbol of the pair refers to the number of non-real zeros of the Mittag-Leffler. For more details see [19]. Mittag-Leffler functions have the same zeros and show the same functional behaviors to which the alpha and beta phase diagrams are applied, as in Figure 1.

We note that the asymptotic form of the zeros of the Mittag-Leffler function has already been studied; however, using the method proposed below, we can obtain more precise statements regarding their asymptotic distribution. Thus, we improve Sedletskii's results [5], by showing that the higher order asymptotic distribution of the zeros of the Mittag-Leffler function with two parameters are given by:

Theorem 2.1. *Let the values of parameters α, β satisfy either one of the following relations:*

(1) $0 < \alpha < 2$, $\beta \in \mathbf{C}$, *and* $\beta \neq 0, -1, -2, \ldots$ *when* $\alpha = 1$
(2) $\alpha = 2$, $\mathbf{Re}\,\beta > 3$.

Then all the zeros z_n of the Mittag-Leffler function $E_{\alpha,\beta}(z)$ with sufficiently large modulus are simple and the asymptotic relation

$$(z_n)^{\frac{1}{\alpha}} = 2\pi ni - \tau_\beta \alpha \log(2\pi ni) + \frac{\frac{d_\beta}{c_\beta}}{(2\pi ni)^\alpha} + (\tau_\beta \alpha)^2 \frac{\log(2\pi ni)}{2\pi ni} - (\tau_\beta \alpha)^2 \frac{\log c_\beta}{2\pi ni} + \alpha_n,$$

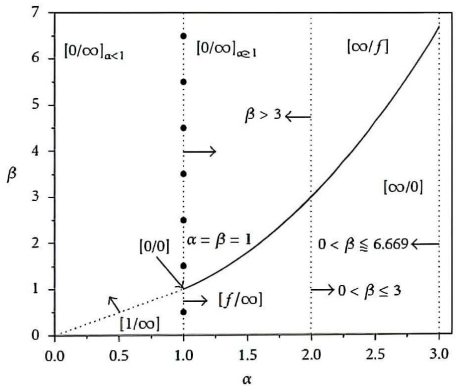


FIGURE 1. The case $\alpha-\beta$ phase diagram for $E_{\alpha,\beta}(z)$. The arrows point to the particular phase into which a Mittag-Leffler function whose parameters lie on a phase boundary belongs. Dotted lines are used to indicate phase boundaries where α and β are known precisely and solid lines for boundaries where α and/or β may be determined as accurately as desired but not known precisely. In opposition to the remainder of the points on the line $\alpha = 1$, the dark points at $\beta = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \ldots$ belong to phase $[0/\infty]_{\alpha<1}$ (i.e., where $0 < \alpha < 1$) and $\beta = \frac{1}{2}$ for the phase $[1/\infty]$.

holds for $|n| \to \infty$ *where, for* $\alpha < 2$,

$$\alpha_n = O\left(\frac{\log|n|}{|n|^{1+\alpha}}\right) + O\left(\frac{1}{|n|^{2\alpha}}\right) + O\left(\frac{\log^2|n|}{|n|^2}\right),$$

while for $\alpha = 2$,

$$\alpha_n = \frac{e^{\pm i\pi\beta}}{c_\beta^2(2\pi i n)^{-4\tau_\beta}} + O\left(\frac{1}{|n|^{-8\tau_\beta}}\right) + O\left(\frac{\log|n|}{|n|^{1-4\tau_\beta}}\right) + O\left(\frac{\log^2|n|}{|n|^2}\right)$$

and

$$c_\beta = \tfrac{\alpha}{\Gamma(\beta-\alpha)}, \qquad d_\beta = \tfrac{\alpha}{\Gamma(\beta-2\alpha)},$$

$$\tau_\beta = 1 + \tfrac{1-\beta}{\alpha}, \qquad \beta \neq \alpha - l, l \in \mathbb{Z}_+,$$

$$c_\beta = \tfrac{\alpha}{\Gamma(\beta-2\alpha)}, \qquad d_\beta = \tfrac{\alpha}{\Gamma(\beta-3\alpha)},$$

$$\tau_\beta = 2 + \tfrac{1-\beta}{\alpha}, \qquad \beta = \alpha - l, l \in \mathbb{Z}_+,$$

The proof of the theorem is based on the integral representation of the Mittag-Leffler function $E_{\alpha,\beta}(z)$ and on the following lemma.

Lemma 2.2 ([5]). *Let $A \in \mathbf{C}$ and $\delta \in (0, \frac{\pi}{2})$ be fixed numbers and let the sets Z, W be defined for $R > 0$ by the relations*

$$Z = Z_{\delta,R} = \{z : |\arg z| < \pi - \delta,\ |z| > R\},$$
$$W = W_{\delta,R} = \{z : |\arg z| < \pi - 2\delta,\ |z| > 2R\}.$$

Then for sufficiently large $R > 0$ the equation $z - A \log z = w$, $w \in W$, has a unique zero $z = z(w) \in Z$. This zero is simple and it is given asymptotically by

$$z = w + A \log w + A^2 \frac{\log w}{w} + O\left(\frac{\log^2 w}{w^2}\right), \quad \text{as } w \to \infty.$$

2.1. **Integral Representations of the Mittag-Leffler Function.** For the Mittag-Leffler functions, the integral representation along the Hankel loop with arbitrary β has been treated by Erdélyi et al. [3] and Dzherbashyan [24],[25]. They considered

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i\alpha} \int_{\gamma(\epsilon;\delta)} \frac{e^{\zeta^{1/\alpha}} \zeta^{(1-\beta)/\alpha}}{\zeta - z} d\zeta, \quad z \in G^{(-)}(\epsilon;\delta),$$

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} e^{z^{1/\alpha}} + \frac{1}{2\pi i\alpha} \int_{\gamma(\epsilon;\delta)} \frac{e^{\zeta^{1/\alpha}} \zeta^{(1-\beta)/\alpha}}{\zeta - z} d\zeta, \quad z \in G^{(+)}(\epsilon;\delta),$$

under the conditions

$$0 < \alpha < 2, \qquad \frac{\pi\alpha}{2} < \delta \le \min\{\pi, \pi\alpha\}. \tag{2.1}$$

The contour $\gamma(\epsilon;\delta)$ can be seen in Figure 2.

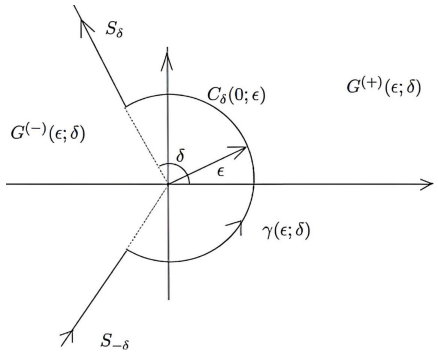


FIGURE 2. Hankel contour

It consists of two rays $S_{-\delta}$ and S_δ ($\arg\zeta = -\delta$, $|\zeta| \ge \epsilon$ and $\arg\zeta = \delta$, $|\zeta| \ge \epsilon$) and a circular are $C_\delta(0;\epsilon)$ ($|\zeta| = \epsilon, -\delta \le \arg\zeta \le \delta$). According

to the calculations that were done [34]:

(2.2)

$$E_{\alpha,\beta}(z) = \int_{\epsilon}^{+\infty} K[\alpha,\beta,\delta,r,z]dr + \int_{-\delta}^{\delta} P[\alpha,\beta,\epsilon,\varphi,z]d\varphi, \quad z \in G^{(-)}(\epsilon;\delta),$$

and

$$E_{\alpha,\beta}(z) = \int_{\epsilon}^{+\infty} K[\alpha,\beta,\delta,r,z]dr + \int_{-\delta}^{\delta} P[\alpha,\beta,\epsilon,\varphi,z]d\varphi + \frac{1}{\alpha} z^{(1-\beta)/\alpha} e^{z^{1/\alpha}}, \quad z \in G^{(+)}(\epsilon;\delta), \tag{2.3}$$

with

$$\tilde{K}[\alpha,\beta,r,z] = \frac{1}{\pi\alpha} r^{(1-\beta)/\alpha} e^{-r^{1/\alpha}} \frac{\sin(\pi(1-\beta)) - z\sin(\pi(1-\beta+\alpha))}{r^2 - 2rz\cos(\pi\alpha) + z^2},$$

$$P[\alpha,\beta,\epsilon,\varphi,z] = \frac{\epsilon^{1+(1-\beta)/\alpha}}{2\pi\alpha} \frac{e^{\epsilon^{1/\alpha}\cos(\varphi/\alpha)}(\cos(\omega) + i\sin(\omega))}{\epsilon e^{\imath\varphi} - z},$$

$$\omega = \epsilon^{1/\alpha}\sin(\varphi/\alpha) + \varphi(1 + (1-\beta)/\alpha).$$

2.2. **The Case $0 < \alpha \leq 1$.** Let us now concider the case $0 < \alpha \leq 1$, $z \neq 0$. By condition (2.1) we can choose $\delta = \min\{\pi, \pi\alpha\} = \pi\alpha$. Then the kernel $K[\alpha,\beta,\delta,r,z]$ looks simpler [34]:

(2.4)

$$K[\alpha,\beta,\delta,r,z] = \tilde{K}[\alpha,\beta,r,z] = \frac{1}{\pi\alpha} r^{(1-\beta)/\alpha} e^{-r^{1/\alpha}} \frac{r\sin(\pi(1-\beta)) - z\sin(\pi(1-\beta+\alpha))}{r^2 - 2rz\cos(\pi\alpha) + z^2}.$$

In order to calculate the function $E_{\alpha,\beta}(z)$ at a point $z \in \mathbf{C}$, $z \neq 0$, in formulas (2.2)-(2.4), we will specify three states for $\arg z$, namely

$$A) \quad |\arg z| > \pi\alpha, \qquad B) \quad |\arg z| = \pi\alpha, \qquad C) \quad |\arg z| < \pi\alpha.$$

First we discuss on a case A): $|\arg z| > \pi\alpha$.

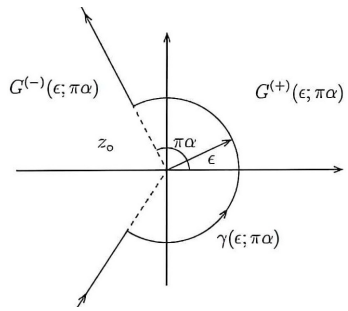


FIGURE 3. The case $|\arg z| > \pi\alpha$.

In this case, z is always (for arbitrary ϵ) in the negative region, that is, $z \in G^{(-)}(\epsilon;\delta)$.

Note that the Mittag-Leffler function has no real zeros for $z > 0$, because in the case $\beta > 1+\alpha$, the Gamma function is positive.

Lemma 2.3 ([34]). *Under the conditions*

$$0 < \alpha \leq 1, \quad \beta \in \mathbf{R}, \qquad |\arg z| > \pi\alpha, \quad z \neq 0$$

the function $E_{\alpha,\beta}(z)$ has the representations

(2.5)

$$E_{\alpha,\beta}(z) = \int_{\epsilon}^{\infty} \tilde{K}[\alpha,\beta,r,z]dr + \int_{-\pi\alpha}^{\pi\alpha} P[\alpha,\beta,\epsilon,\varphi,z]d\varphi, \quad \epsilon > 0, \quad \beta \in \mathbf{R},$$

(2.6)

$$E_{\alpha,\beta}(z) = \int_{0}^{\infty} \tilde{K}[\alpha,\beta,r,z]dr, \quad if \quad \beta < 1+\alpha,$$

(2.7)

$$E_{\alpha,\beta}(z) = -\frac{\sin(\pi\alpha)}{\pi\alpha}\int_{0}^{\infty} \frac{e^{-r^{1/\alpha}}}{r^2 - 2rz\cos(\pi\alpha) + z^2}dr - \frac{1}{z}, \quad if \quad \beta = 1+\alpha.$$

Lemma 2.4. *Let $0 < \alpha < 1$ and k be an integer number. Then*

$$A := \int_{0}^{\infty} r^{k+\frac{1}{2\alpha}} e^{-r^{\frac{1}{2\alpha}}} dr = 2\alpha\Gamma(2\alpha k + 2\alpha + 1),$$
$$B := \int_{0}^{\infty} r^{k} e^{-r^{\frac{1}{2\alpha}}} dr = 2\alpha\Gamma(2\alpha k + 2\alpha).$$

Proof. In integral A, let $x = r^{\frac{1}{2\alpha}}$. Then,

$$A := 2\alpha\mathcal{L}\left[x^{2\alpha k+2\alpha}\right]\Big|_{p=1} = 2\alpha\Gamma(2\alpha k + 2\alpha + 1)$$

where p is the Laplace transform parameter. By proceeding analogously as in the proof of integral A, the integral B above can be derived. □

3. Higher Order Asymptotics

. In this part we begin by rewriting (2.4) and separating into partial fractions so as to find:

(3.1)
$$\begin{aligned} &\frac{1}{r^2 - 2rz\cos(\pi\alpha) + z^2} \\ &\quad = \frac{1}{(r - ze^{i\pi\alpha})(r - ze^{-i\pi\alpha})} \\ &\quad = \frac{1}{2iz\sin(\pi\alpha)}\left\{\frac{1}{r - ze^{i\pi\alpha}} - \frac{1}{r - ze^{-i\pi\alpha}}\right\} \end{aligned}$$

$$= \frac{1}{-2iz^2\sin(\pi\alpha)} \sum_{k=0}^{\infty} \left(\frac{r}{z}\right)^k \left\{ \frac{1}{e^{i\pi\alpha(k+1)}} - \frac{1}{e^{-i\pi\alpha(k+1)}} \right\}$$

$$= \frac{1}{-2iz^2\sin(\pi\alpha)} \sum_{k=0}^{\infty} \left(\frac{r}{z}\right)^k \left\{ e^{-i\pi\alpha(k+1)} - e^{i\pi\alpha(k+1)} \right\}$$

$$= \frac{1}{z^2\sin(\pi\alpha)} \sum_{k=0}^{\infty} \left(\frac{r}{z}\right)^k \left\{ \frac{e^{-i\pi\alpha(k+1)} - e^{i\pi\alpha(k+1)}}{2i} \right\}$$

$$= \frac{1}{z^2\sin(\pi\alpha)} \sum_{k=0}^{\infty} \left(\frac{r}{z}\right)^k \sin\left(\pi\alpha(k+1)\right).$$

Now, using Lemma 2.3, the proof of which can also be found in [34], we obtain the asymptotic form of the zeros of the Mittag-Leffler function in this case: We consider (2.5) for the state ($\beta < 1 + \alpha$) that is (2.6). Consequently,

$$\int_{-\pi\alpha}^{\pi\alpha} P[\alpha, \beta, \epsilon, \varphi, z] d\varphi \to 0 \quad \text{if } \epsilon \to 0.$$

Furthermore, we have

$$\tilde{K}[\alpha, \beta, r, z] = O\left(r^{\frac{1-\beta}{\alpha}}\right), \quad r \to 0.$$

Reinserting the values of K we get,

(3.2)

$$\begin{aligned}
E_{\alpha,\beta}(z) &= \int_0^{\infty} \tilde{K}[\alpha, \beta, r, z] dr \\
&= \int_0^{\infty} \frac{1}{\pi\alpha} r^{\frac{1-\beta}{\alpha}} e^{-r^{\frac{1}{\alpha}}} \left\{ \frac{r\sin\left(\pi(1-\beta)\right)}{z^2\sin(\pi\alpha)} \sum_{k=0}^{\infty} \left(\frac{r}{z}\right)^k \sin\left(\pi\alpha(k+1)\right) \right. \\
&\quad \left. - \frac{z\sin\left(\pi(1-\beta+\alpha)\right)}{z^2\sin(\pi\alpha)} \sum_{k=0}^{\infty} \left(\frac{r}{z}\right)^k \sin\left(\pi\alpha(k+1)\right) \right\} dr \\
&= \int_0^{\infty} \left\{ \frac{1}{\pi\alpha} r^{\frac{1-\beta}{\alpha}} e^{-r^{\frac{1}{\alpha}}} r \frac{\sin\left(\pi(1-\beta)\right)}{z^2\sin(\pi\alpha)} \sum_{k=0}^{\infty} \frac{r^k}{z^k} \sin\left(\pi\alpha(k+1)\right) \right\} dr \\
&\quad - \int_0^{\infty} \left\{ \frac{1}{\pi\alpha} r^{\frac{1-\beta}{\alpha}} e^{-r^{\frac{1}{\alpha}}} \frac{\sin\left(\pi(1-\beta+\alpha)\right)}{z\sin(\pi\alpha)} \sum_{k=0}^{\infty} \frac{r^k}{z^k} \sin\left(\pi\alpha(k+1)\right) \right\} dr \\
&= \frac{1}{\pi\alpha z^2} \frac{\sin\left(\pi(1-\beta)\right)}{\sin(\pi\alpha)} \sum_{k=0}^{\infty} \frac{1}{z^k} \sin\left(\pi\alpha(k+1)\right) \int_0^{\infty} r^{\frac{1-\beta}{\alpha}+1+k} e^{-r^{\frac{1}{\alpha}}} dr \\
&\quad - \frac{1}{\pi\alpha z} \frac{\sin\left(\pi(1-\beta+\alpha)\right)}{\sin(\pi\alpha)} \sum_{k=0}^{\infty} \frac{1}{z^k} \sin\left(\pi\alpha(k+1)\right) \int_0^{\infty} r^{\frac{1-\beta}{\alpha}+k} e^{-r^{\frac{1}{\alpha}}} dr
\end{aligned}$$

$$= \frac{1}{\pi\alpha z^2}\frac{\sin\left(\pi(1-\beta)\right)}{\sin(\pi\alpha)}\sum_{k=0}^{\infty}\frac{1}{z^k}\sin\left(\pi\alpha(k+1)\right)\alpha\Gamma\left(\alpha(k+1)+\alpha-\beta+1\right)$$
$$- \frac{1}{\pi\alpha z}\frac{\sin\left(\pi(1-\beta+\alpha)\right)}{\sin(\pi\alpha)}\sum_{k=0}^{\infty}\frac{1}{z^k}\sin\left(\pi\alpha(k+1)\right)\alpha\Gamma\left(\alpha k+\alpha-\beta+1\right).$$

If $E_{\alpha,\beta}(z)=0$, then

$$\frac{1}{\pi\alpha z^2}\frac{\sin\left(\pi(1-\beta)\right)}{\sin(\pi\alpha)}\sum_{k=0}^{\infty}\frac{1}{z^k}\sin\left(\pi\alpha(k+1)\right)\alpha\Gamma\left(\alpha(k+1)+\alpha-\beta+1\right)$$
$$= \frac{1}{\pi\alpha z}\frac{\sin\left(\pi(1-\beta+\alpha)\right)}{\sin(\pi\alpha)}\sum_{k=0}^{\infty}\frac{1}{z^k}\sin\left(\pi\alpha(k+1)\right)\alpha\Gamma\left(\alpha k+\alpha-\beta+1\right).$$

And now to get z:

$$\frac{\frac{1}{z}}{\frac{1}{z^2}} = \frac{\frac{1}{\pi\alpha}\frac{\sin(\pi(1-\beta))}{\sin(\pi\alpha)}\sum_{k=0}^{\infty}\frac{1}{z^k}\sin\left(\pi\alpha(k+1)\right)\alpha\Gamma\left(\alpha(k+1)+\alpha-\beta+1\right)}{\frac{1}{\pi\alpha}\frac{\sin(\pi(1-\beta+\alpha))}{\sin(\pi\alpha)}\sum_{k=0}^{\infty}\frac{1}{z^k}\sin\left(\pi\alpha(k+1)\right)\alpha\Gamma\left(\alpha(k+1)-\beta+1\right)},$$

so,

$$z = \frac{\sin\left(\pi(1-\beta)\right)\sum_{k=0}^{\infty}\frac{1}{z^k}\sin\left(\pi\alpha(k+1)\right)\Gamma\left(\alpha(k+1)+\alpha-\beta+1\right)}{\sin\left(\pi(1-\beta+\alpha)\right)\sum_{k=0}^{\infty}\frac{1}{z^k}\sin\left(\pi\alpha(k+1)\right)\Gamma\left(\alpha(k+1)-\beta+1\right)}$$
$$= \frac{\Gamma(1-\beta+\alpha)\Gamma(\beta-\alpha)}{\Gamma(1-\beta)\Gamma(\beta)}\frac{\sum_{k=0}^{\infty}\frac{1}{z^k}\sin\left(\pi\alpha(k+1)\right)\Gamma\left(\alpha(k+1)+\alpha-\beta+1\right)}{\sum_{k=0}^{\infty}\frac{1}{z^k}\sin\left(\pi\alpha(k+1)\right)\Gamma\left(\alpha(k+1)-\beta+1\right)}.$$

Since, $\sin(\pi z)=\frac{\pi}{\Gamma(z)\Gamma(1-z)}$, we get:

$$z = \frac{\Gamma(1-\beta+\alpha)\Gamma(\beta-\alpha)}{\Gamma(1-\beta)\Gamma(\beta)}\frac{\sum_{k=0}^{\infty}\frac{1}{z^k}\frac{\pi}{\Gamma(\alpha(k+1))\Gamma(1-\alpha(k+1))}}{\sum_{k=0}^{\infty}\frac{1}{z^k}\frac{\pi}{\Gamma(\alpha(k+1))\Gamma(1-\alpha(k+1))}}$$
$$\times\frac{\Gamma\left(\alpha(k+1)+\alpha-\beta+1\right)}{\Gamma\left(\alpha(k+1)-\beta+1\right)}.$$

We have with $\left(f(z)=\dfrac{1}{\Gamma(z)}\right)$,

$$z=\frac{f(\beta)f(1-\beta)}{f(\beta-\alpha)f(1-\beta+\alpha)}\frac{\sum\limits_{k=0}^{\infty}\frac{1}{z^k}f\left(\alpha(k+1)\right)f\left(1-\alpha(k+1)\right)}{\sum\limits_{k=0}^{\infty}\frac{1}{z^k}f\left(\alpha(k+1)\right)f\left(1-\alpha(k+1)\right)}\times\frac{\Gamma\left(\alpha k+2\alpha-\beta+1\right)}{\Gamma\left(\alpha k+\alpha-\beta+1\right)}. \tag{3.3}$$

Now we write,

$$\begin{aligned}A_k&=f\left(\alpha(k+1)\right)f\left(1-\alpha(k+1)\right),\\ B_k&=\Gamma\left(\alpha k+2\alpha-\beta+1\right),\\ C_k&=\Gamma\left(\alpha k+\alpha-\beta+1\right).\end{aligned}$$

in (3.3) to find that:

$$z=\frac{f(\beta)f(1-\beta)}{f(\beta-\alpha)f(1-\beta+\alpha)}\frac{\sum\limits_{k=0}^{\infty}\frac{1}{z^k}A_kB_k}{\sum\limits_{k=0}^{\infty}\frac{1}{z^k}A_kC_k}. \tag{3.4}$$

Recalling the formula we have torsional multiplication

$$\left(\sum_{k=0}^{\infty}a_kz^k\right)\left(\sum_{k=0}^{\infty}b_kz^k\right)=\sum_{k=0}^{\infty}c_kz^k,\qquad c_k=\sum_{n=0}^{k}a_nb_{k-n}.$$

and using it here, we get,

$$\frac{\sum\limits_{k=0}^{\infty}\frac{1}{z^k}A_kB_k}{\sum\limits_{k=0}^{\infty}\frac{1}{z^k}A_kC_k}=\sum_{k=0}^{\infty}\frac{1}{z^k}D_k.$$

Thus,

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{1}{z^k}A_kB_k&=\left(\sum_{k=0}^{\infty}\frac{1}{z^k}A_kC_k\right)\left(\sum_{k=0}^{\infty}\frac{1}{z^k}D_k\right)\\ &=\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{1}{z^m}A_mC_mD_{k-m}\frac{1}{z^{(k-m)}}\\ &=\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{1}{z^k}A_mC_mD_{k-m},\end{aligned}$$

after which we find,

$$A_k B_k = \sum_{m=0}^{k} A_m C_m D_{k-m}. \tag{3.5}$$

This gives the sequence D_n where,

$$\begin{aligned}
k=0 \quad &\Longrightarrow \quad D_0 = \frac{B_0}{C_0},\\
k=1 \quad &\Longrightarrow \quad D_1 = \frac{A_1B_1 - A_1C_1D_0}{A_0C_0},\\
k=2 \quad &\Longrightarrow \quad D_2 = \frac{A_2B_2 - A_1C_1D_1 - A_2C_2D_0}{A_0C_0},\\
k=3 \quad &\Longrightarrow \quad D_3 = \frac{A_3B_3 - A_1C_1D_2 - A_2C_2D_1 - A_3C_3D_0}{A_0C_0},\\
&\vdots
\end{aligned}$$

Substituting these in (3.4), the asymptotic forms of the zeros of the Mittag-Leffler functions are can be obtained, that is,

$$z = \frac{f(\beta)f(1-\beta)}{f(\beta-\alpha)f(1-\beta+\alpha)}\left(D_0 + \frac{1}{z}D_1 + \frac{1}{z^2}D^2 + \cdots\right),$$

or

$$z = \left(\frac{f(\beta)f(1-\beta)}{f(\beta-\alpha)f(1-\beta+\alpha)}\right)D_0 + O\left(\frac{1}{z}\right). \tag{3.6}$$

Defining μ by,

$$\mu = \frac{f(\beta)f(1-\beta)}{f(\beta-\alpha)f(1-\beta+\alpha)},$$

using lemma 2.2 and writing $z = e^w$, $w \in \mathbf{C}$, we obtain

$$z = e^w = \mu D_0 + \frac{\mu D_1}{z} + O\left(\frac{1}{z^2}\right).$$

In particular, we find,

$$w_n = 2\pi n i + \ln(\mu D_0) + \epsilon_n, \quad \text{where } \epsilon_n \to 0, \quad \text{as } |n| \to \infty,$$

so that $z_n = e^{w_n}$. This method, therefore, allows us to find more terms in the asymptotic expansion of the zeros of the Mittag-Leffler function. For more details, refer to [5]. Now, we consider the case where $\beta = 1+\alpha$. From

$$E_{\alpha,\beta}(z) = -\frac{\sin(\pi\alpha)}{\pi\alpha}\int_0^\infty \frac{e^{-r^{1/\alpha}}}{r^2 - 2rz\cos(\pi\alpha) + z^2}dr - \frac{1}{z},$$

we have the formulas

$$\lim_{\epsilon\to 0}\int_{-\pi\alpha}^{\pi\alpha} P[\alpha,\beta,\epsilon,\varphi,z]d\varphi = -\frac{1}{z}.$$

Using the proof of (3.1), we get

$$\begin{aligned}\tilde{K}[\alpha,\beta,r,z] &= -\frac{\sin(\pi\alpha)}{\pi\alpha}\frac{e^{-r^{\frac{1}{\alpha}}}}{r^2-2rz\cos(\pi\alpha)+z^2} \\ &= -\frac{\sin(\pi\alpha)}{\pi\alpha}\frac{e^{-r^{\frac{1}{\alpha}}}}{z^2\sin(\pi\alpha)}\sum_{k=0}^{\infty}\left(\frac{r}{z}\right)^k \sin\left(\pi\alpha(k+1)\right) \\ &= -\frac{1}{\pi\alpha}e^{-r^{\frac{1}{\alpha}}}\frac{1}{z^2}\sum_{k=0}^{\infty}\left(\frac{r}{z}\right)^k \sin\left(\pi\alpha(k+1)\right).\end{aligned} \tag{3.7}$$

Calculating the integral,

(3.8)

$$\begin{aligned}\int_0^{\infty}\tilde{K}[\alpha,\beta,r,z]dr &= \int_0^{\infty} -\frac{1}{\pi\alpha}e^{-r^{\frac{1}{\alpha}}z^2}\sum_{k=0}^{\infty}\frac{r^k}{z^k}\sin\left(\pi\alpha(k+1)\right)dr \\ &= -\frac{1}{\pi\alpha z^2}\sum_{k=0}^{\infty}\frac{1}{z^k}\sin\left(\pi\alpha(k+1)\right)\int_0^{\infty} r^k e^{-r^{\frac{1}{\alpha}}}dr.\end{aligned}$$

and substituting this into the previous displays, we find,

$$E_{\alpha,\beta}(z) = -\frac{1}{\pi\alpha z^2}\sum_{k=0}^{\infty}\frac{1}{z^k}\sin\left(\pi\alpha(k+1)\right)\alpha\Gamma(\alpha k+\alpha) - \frac{1}{z} = 0, \tag{3.9}$$

from which we obtain,

$$\begin{aligned}z &= -\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{1}{z^k}\sin\left(\pi\alpha(k+1)\right)\Gamma(\alpha k+\alpha) \\ &= -\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{1}{z^k}\frac{\pi}{\Gamma\left(\alpha(k+1)\right)\Gamma\left(1-\alpha(k+1)\right)}\Gamma(\alpha k+\alpha).\end{aligned}$$

Writing $f(z) = \frac{1}{\Gamma(z)}$, this becomes,

$$\begin{aligned}z &= -\sum_{k=0}^{\infty}\frac{1}{z^k}f(\alpha k+\alpha)f\left(1-\alpha(k+1)\right)\frac{1}{f(\alpha k+\alpha)} \\ &= -\sum_{k=0}^{\infty}\frac{1}{z^k}f\left(1-\alpha(k+1)\right),\end{aligned} \tag{3.10}$$

i.e.,

$$z = -\left(f(1-\alpha) + \frac{1}{z} f(1-2\alpha) + \frac{1}{z^2} f(1-3\alpha) + \frac{1}{z^3} f(1-4\alpha) + \cdots \right).$$

From this, the asymptotic form in this case reads as follows:

$$z = -f(1-\alpha) + O\left(\frac{1}{z}\right). \tag{3.11}$$

Writing $f(1-\alpha) = \eta$ and using Lemma 2.2 with $z = e^w$, $w \in \mathbf{C}$, we obtain

$$e^w = -\eta + O\left(\frac{1}{z}\right),$$

or, in particular,

$$w_n = 2\pi n i - \ln \eta + \epsilon_n, \quad \text{where } \epsilon_n \to 0, \quad \text{as } |n| \to \infty.$$

Thus, $z_n = e^{w_n}$. So, in this case, we obtained higher-order terms in the asymptotic form of the zeros of the Mittag-Leffler function [5]. The following case to be considered is case B) where, $|\arg z| = \pi\alpha$. In this case, we are not allowed to make ϵ arbitrarily small in our contour $\gamma(\epsilon;\delta)$, because z is directly lying on this contour if $\epsilon < |z|$.

Lemma 3.1 ([34])**.** *Under the conditions*

$$0 < \alpha \leq 1, \quad \beta \in \mathbf{R}, \qquad |\arg z| = \pi\alpha, \quad z \neq 0,$$

the function $E_{\alpha,\beta}(z)$ has the representation

$$E_{\alpha,\beta}(z) = \int_{\epsilon}^{\infty} \tilde{K}[\alpha,\beta,r,z]dr + \int_{-\pi\alpha}^{\pi\alpha} P[\alpha,\beta,\epsilon,\varphi,z]d\varphi, \quad \epsilon > |z|, \tag{3.12}$$

with the same $\tilde{K}$, P as defined in Lemma 2.3 -[34]. In this case, the asymptotic form of the roots becomes the same as in case A). Finally, we discuss case C) where, $|\arg z| < \pi\alpha$.
In this case, if $0 < \epsilon < |z|$, then z is in the region $G^{(+)}(\epsilon;\delta)$.

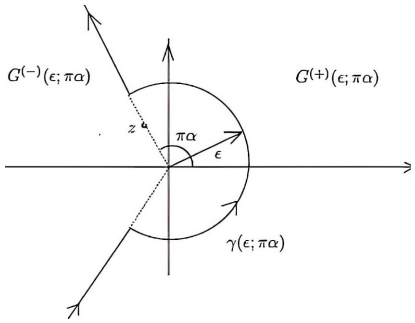


FIGURE 4. The case $|\arg z| = \pi\alpha$.

Lemma 3.2. *Under the conditions*

$$0 < \alpha \leq 1, \quad \beta \in \mathbf{R}, \qquad |\arg z| < \pi\alpha, \quad z \neq 0,$$

the function $E_{\alpha,\beta}(z)$ *has the representations*

$$E_{\alpha,\beta}(z) = \int_{\epsilon}^{\infty} \tilde{K}[\alpha,\beta,r,z]dr + \int_{-\pi\alpha}^{\pi\alpha} P[\alpha,\beta,\epsilon,\varphi,z]d\varphi + \frac{1}{\alpha} z^{\frac{1-\beta}{\alpha}} e^{z^{\frac{1}{\alpha}}}, \quad 0 < \epsilon < |z|, \tag{3.13}$$

$$E_{\alpha,\beta}(z) = \int_{\epsilon}^{\infty} \tilde{K}[\alpha,\beta,r,z]dr + \frac{1}{\alpha} z^{\frac{1-\beta}{\alpha}} e^{z^{\frac{1}{\alpha}}}, \quad \textit{if } \beta < 1+\alpha, \tag{3.14}$$

$$E_{\alpha,\beta}(z) = -\frac{\sin(\pi\alpha)}{\pi\alpha} \int_0^{\infty} \frac{e^{-r^{1/\alpha}}}{r^2 - 2rz\cos(\pi\alpha) + z^2} dr - \frac{1}{z} + \frac{1}{\alpha z} e^{z^{\frac{1}{\alpha}}}, \quad \textit{if} \quad \beta = 1+\alpha. \tag{3.15}$$

Proof. This is similar to the proof of [34, Lemma 2.3] and so is omitted. □

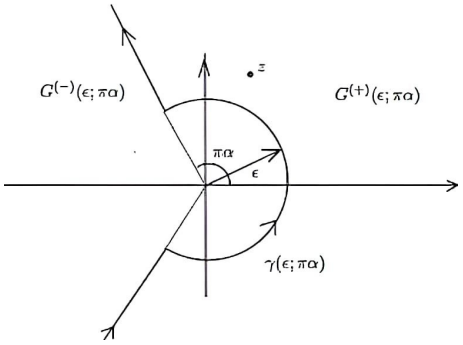


FIGURE 5. The case $|\arg z| < \pi\alpha$.

Now we write for (3.14)

$$zE_{\alpha,\beta}(z) = \int_{\epsilon}^{\infty} z\tilde{K}\left[\alpha,\beta,r,z\right] dr + \frac{1}{\alpha} z^{\frac{1-\beta+\alpha}{\alpha}} e^{z^{\frac{1}{\alpha}}},$$

since the $1-\beta+\alpha > 0$, we see that

$$z^{\frac{1-\beta+\alpha}{\alpha}} e^{z^{\frac{1}{\alpha}}} \longrightarrow \infty \quad \text{as } |z| \to \infty,$$

and

$$\int_{\epsilon}^{\infty} z\tilde{K}\left[\alpha,\beta,r,z\right] dr \longrightarrow 0 \quad \text{as } |z| \to \infty,$$

consequently therefore $zE_{\alpha,\beta}(z)$ have no zeros, when $|z| \to \infty$.

For more details see the proof of [34, Theorem 3.2].

Now we write the asymptotic expansion for (3.15). Using (3.8) and (3.9):

(3.16)
$$E_{\alpha,\beta}(z) = -\frac{1}{\pi\alpha z^2}\sum_{k=0}^{\infty}\frac{1}{z^k}\sin\left(\pi\alpha(k+1)\right)\alpha\Gamma(\alpha k+\alpha) - \frac{1}{z} + \frac{1}{\alpha z}e^{z^{\frac{1}{\alpha}}} = 0,$$

i.e.,

$$\frac{1}{\alpha z}e^{z^{\frac{1}{\alpha}}} = \frac{1}{\pi\alpha z^2}\sum_{k=0}^{\infty}\frac{\sin\left(\pi\alpha(k+1)\right)\alpha\Gamma(\alpha k+\alpha)}{z^k} + \frac{1}{z},$$

from which,

$$\begin{aligned} e^{z^{\frac{1}{\alpha}}} &= \frac{1}{\pi z}\sum_{k=0}^{\infty}\frac{\sin\left(\pi\alpha(k+1)\right)\alpha\Gamma(\alpha k+\alpha)}{z^k} + \alpha \\ &= \alpha\left[\frac{1}{\pi\alpha z}\sum_{k=0}^{\infty}\frac{\sin\left(\pi\alpha(k+1)\right)\alpha\Gamma(\alpha k+\alpha)}{z^k} + 1\right]. \end{aligned}$$

Let $\hat{G}_k := \sin\left(\pi\alpha(k+1)\right)\Gamma(\alpha k+\alpha)z^k$. Then

$$e^{z^{\frac{1}{\alpha}}} = \alpha\left[\frac{1}{\pi z}\sum_{k=0}^{\infty}\frac{\hat{G}_k}{z^k} + 1\right].$$

Finally,

$$\begin{aligned} z_r^{\frac{1}{\alpha}} &= 2\pi ri + \ln[\alpha] + \ln\left[\sum_{k=0}^{\infty}\frac{\hat{G}_k}{z^k}\frac{1}{\pi z} + 1\right] \\ &= 2\pi ri + \ln[\alpha] + \sum_{k=0}^{\infty}\frac{\hat{G}_k}{\pi z^{k+1}} - \frac{1}{2}\left(\sum_{k=0}^{\infty}\frac{\hat{G}_k}{\pi z^{k+1}}\right)^2 + \frac{1}{3}\left(\sum_{k=0}^{\infty}\frac{\hat{G}_k}{\pi z^{k+1}}\right)^3 - \cdots, \end{aligned}$$

where $r \in \mathbb{N}$.

The Case $1 < \alpha < 3$. According to Figure 1, we see that there are an infinite number of complex roots in the interval $1 < \alpha < 2$ and an infinite number of real roots in the interval $2 < \alpha < 3$; by using the following reciprocal relation, the higher-order asymptotic form of the zeros of the Mittag-Leffler function can be obtained in this region.

The Recursion Relation, [36]

Let α and β are real numbers with $\alpha > 0$. Calculating $E_{\alpha,\beta}(z)$ for the case $\alpha > 1$ can be reduced to the case $0 < \alpha \leq 1$ by virtue of the recursion relation

$$E_{\alpha,\beta}(z) = \frac{1}{2m+1} \sum_{h=-m}^{m} E_{\frac{\alpha}{2m+1},\beta}\left(z^{\frac{1}{2m+1}} e^{\frac{i2\pi h}{2m+1}}\right), \tag{3.17}$$

which is valid for all $\beta \in \mathbf{R}$, $z \in \mathbf{C}$, where $m = [\frac{\alpha-1}{2}] + 1$.

Using the definition (1.1) of the Mittag-Leffler function and replacing $m = 1$ in (3.17) for this region, we get

$$\begin{aligned} E_{\alpha,\beta}(z) &= \frac{1}{3} \sum_{h=-1}^{1} E_{\frac{\alpha}{3},\beta}\left(z^{\frac{1}{3}} e^{\frac{i2\pi h}{3}}\right) \\ &= \frac{1}{3}\left[E_{\frac{\alpha}{3},\beta}\left(z^{\frac{1}{3}} e^{\frac{-2\pi i}{3}}\right) + E_{\frac{\alpha}{3},\beta}\left(z^{\frac{1}{3}}\right) + E_{\frac{\alpha}{3},\beta}\left(z^{\frac{1}{3}} e^{\frac{2\pi i}{3}}\right)\right]. \end{aligned} \tag{3.18}$$

To prove the series expansion for the Mittag-Leffler function $E_{\alpha,\beta}(z)$ as given, we will begin by breaking down the series over the arithmetic progression $\{3r, 3r+1, 3r+2\}$. This approach allows us to systematically sum the terms and obtain the desired expression. Following the proof, we will demonstrate how the transcendental equation for the zeros of the Mittag-Leffler function can be written in a more compact form.

$$\begin{aligned} \frac{1}{3} E_{\frac{\alpha}{3},\beta}\left(z^{\frac{1}{3}} e^{\frac{-2\pi i}{3}}\right) &= \frac{1}{3} \sum_{k=0}^{\infty} \frac{\left(z^{\frac{1}{3}} e^{\frac{-2\pi i}{3}}\right)^k}{\Gamma(\frac{\alpha}{3}k + \beta)} \\ &= \frac{1}{3} \sum_{k=0}^{\infty} \frac{z^{\frac{k}{3}} e^{\frac{-2\pi i k}{3}}}{\Gamma(\frac{\alpha}{3}k + \beta)} \\ &= \frac{1}{3} \sum_{r=0}^{\infty} \frac{z^r e^{\frac{-2\pi i(3r)}{3}}}{\Gamma\left(\frac{3r\alpha}{3} + \beta\right)} + \frac{1}{3} \sum_{r=0}^{\infty} \frac{z^{\frac{3r+1}{3}} e^{\frac{-2\pi i(3r+1)}{3}}}{\Gamma(\frac{(3r+1)\alpha}{3} + \beta)} \\ &\quad + \frac{1}{3} \sum_{r=0}^{\infty} \frac{z^{\frac{3r+2}{3}} e^{\frac{-2\pi i(3r+2)}{3}}}{\Gamma\left(\frac{(3r+2)\alpha}{3} + \beta\right)} \\ &= \frac{1}{3} \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(r\alpha + \beta)} + \frac{e^{\frac{-2\pi i}{3}}}{3} \sum_{r=0}^{\infty} \frac{z^{r+\frac{1}{3}}}{\Gamma(r\alpha + \beta + \frac{\alpha}{3})} \\ &\quad + \frac{e^{\frac{-4\pi i}{3}}}{3} \sum_{r=0}^{\infty} \frac{z^{r+\frac{2}{3}}}{\Gamma(r\alpha + \beta + \frac{2\alpha}{3})} \end{aligned}$$

And now, for the second equation, we have:

$$\frac{1}{3}E_{\frac{\alpha}{3},\beta}\left(z^{\frac{1}{3}}\right)=\sum_{r=0}^{\infty}\frac{z^{\frac{1}{3}}}{\Gamma(\frac{\alpha}{3}k+\beta)}$$

Next, for the third equation, we have:

$$\begin{aligned}\frac{1}{3}E_{\frac{\alpha}{3},\beta}\left(z^{\frac{1}{3}}e^{\frac{2\pi i}{3}}\right)&=\frac{1}{3}\sum_{r=0}^{\infty}\frac{z^{r}}{\Gamma(r\alpha+\beta)}+\frac{e^{\frac{2\pi i}{3}}}{3}\sum_{r=0}^{\infty}\frac{z^{r+\frac{1}{3}}}{\Gamma(r\alpha+\beta+\frac{\alpha}{3})}\\&\quad+\frac{e^{\frac{4\pi i}{3}}}{3}\sum_{r=0}^{\infty}\frac{z^{r+\frac{2}{3}}}{\Gamma\left(r\alpha+\beta+\frac{2\alpha}{3}\right)}\end{aligned}$$

By summing the three equations, we obtain the following result:

$$\begin{aligned}E_{\alpha,\beta}(z)&=\frac{2}{3}\sum_{r=0}^{\infty}\frac{z^{r}}{\Gamma(\alpha r+\beta)}+\frac{2}{3}\cos\left(\frac{2\pi}{3}\right)\sum_{r=0}^{\infty}\frac{z^{r+\frac{1}{3}}}{\Gamma(\alpha r+\frac{\alpha}{3}+\beta)}\\&\quad+\frac{2}{3}\cos\left(\frac{4\pi}{3}\right)\sum_{r=0}^{\infty}\frac{z^{r+\frac{2}{3}}}{\Gamma(\alpha r+\frac{2\alpha}{3}+\beta)}+\frac{1}{3}\sum_{r=0}^{\infty}\frac{z^{\frac{r}{3}}}{\Gamma\left(\frac{\alpha}{3}r+\beta\right)}.\end{aligned}$$

The transcendental equation for zeros of the Mittag-Leffler function may be written more compactly as,

$$E_{\alpha,\beta}(z)=-z^{\frac{1}{3}}E_{\alpha,\frac{\alpha}{3}+\beta}(z)-z^{\frac{2}{3}}E_{\alpha,\frac{2\alpha}{3}+\beta}(z)+E_{\frac{\alpha}{3},\beta}\left(z^{\frac{1}{3}}\right),\tag{3.19}$$

for any $p\in N$ and $|z|\to\infty$.

We know that whenever $\delta\leq|\arg z|\leq\pi$,

$$E_{\alpha,\beta}(z)=-\sum_{k=1}^{p}\frac{z^{-k}}{\Gamma(\beta-\alpha k)}+O(|z|^{-1-p}).$$

Substituting the previous asymptotic form of the Mittag-Leffler functions in this case in (3.19) with $k=1$ we obtain,

(3.20)

$$\begin{aligned}&-\frac{1}{z\Gamma(\beta-\alpha)}-\frac{1}{z^{\frac{2}{3}}\Gamma\left(\frac{\alpha}{3}+\beta-\alpha\right)}-\frac{1}{z^{\frac{1}{3}}\Gamma\left(\frac{2\alpha}{3}+\beta-\alpha\right)}+\frac{1}{z^{\frac{1}{3}}\Gamma\left(\beta-\frac{\alpha}{3}\right)}\\&\quad+\sum_{k=2}^{p}\left\{-\frac{z^{-k}}{\Gamma(\beta-\alpha k)}-\frac{z^{\frac{1}{3}-k}}{\Gamma(\frac{\alpha}{3}+\beta-\alpha k)}-\frac{z^{\frac{2}{3}-k}}{\Gamma(\frac{2\alpha}{3}+\beta-\alpha k)}+\frac{z^{\frac{-k}{3}}}{\Gamma(\beta-\frac{\alpha}{3}k)}\right\}\\&\quad+O\left(|z|^{-1-p}\right)+O\left(|z|^{-\frac{2}{3}-p}\right)+O\left(|z|^{-\frac{1}{3}-p}\right)+O\left(|z|^{-1-p}\right)+O\left(|z|^{\frac{-1-p}{3}}\right)\\&=0.\end{aligned}$$

Since $p \geq 2$ we can combine the above error terms and rewrite (3.20) as,

(3.21)

$$-\frac{1}{z\Gamma(\beta-\alpha)}-\frac{1}{z^{\frac{2}{3}}\Gamma\left(\frac{\alpha}{3}+\beta-\alpha\right)}-\frac{1}{z^{\frac{1}{3}}\Gamma\left(\frac{2\alpha}{3}+\beta-\alpha\right)}+\frac{1}{z^{\frac{1}{3}}\Gamma\left(\beta-\frac{\alpha}{3}\right)}$$
$$+\sum_{k=2}^{p}\varpi_k(z)+O\left(|z|^{-(1+p)/3}\right)=0,$$

as $|z|\to\infty$ where,

$$\varpi_k(z):=-\frac{z^{-k}}{\Gamma(\beta-\alpha k)}-\frac{z^{\frac{1}{3}-k}}{\Gamma\left(\frac{\alpha}{3}+\beta-\alpha k\right)}-\frac{z^{\frac{2}{3}-k}}{\Gamma\left(\frac{2\alpha}{3}+\beta-\alpha k\right)}+\frac{z^{\frac{-k}{3}}}{\Gamma\left(\beta-\frac{\alpha}{3}k\right)}.$$

Multiplying both sides of (3.21) by z and simplifying, we get,

$$\frac{z^{\frac{1}{3}}}{\Gamma(\beta-\frac{2\alpha}{3})}=-\frac{1}{\Gamma(\beta-\alpha)}+\sum_{k=2}^{p}z\varpi_k(z)+O\left(|z|^{(2-p)/3}\right),$$

or,

$$z^{\frac{1}{3}}=-\frac{\Gamma\left(\beta-\frac{2\alpha}{3}\right)}{\Gamma(\beta-\alpha)}+\Gamma\left(\beta-\frac{2\alpha}{3}\right)\sum_{k=2}^{p}z\varpi_k(z)+O\left(|z|^{(2-p)/3}\right). \tag{3.22}$$

Recall that $f(x)=\dfrac{1}{\Gamma(x)}$ (see the definition above (3.3)). In terms of f this asymptotic form now becomes,

$$z^{\frac{1}{3}}=-\frac{f(\beta-\alpha)}{f\left(\beta-\frac{2\alpha}{3}\right)}+\frac{1}{f\left(\beta-\frac{2\alpha}{3}\right)}\sum_{k=2}^{p}z\varpi_k(z)+O\left(|z|^{(2-p)/3}\right) \tag{3.23}$$

Writing $\Delta=\frac{f(\beta-\alpha)}{f\left(\beta-\frac{2\alpha}{3}\right)}$, $\Delta'=\frac{1}{f\left(\beta-\frac{2\alpha}{3}\right)}\sum\limits_{k=2}^{p}z\varpi_k(z)$, using lemma 2.2 and $z^{\frac{1}{3}}=e^w$.

We then have

$$z^{\frac{1}{3}}=e^w=-\Delta+\Delta'+O\left(|z|^{(2-p)/3}\right),$$

from which

$$w_n=2\pi ni-\ln\Delta+\ln\Delta'+\epsilon_n,\quad \epsilon_n\to 0,\quad n\to\pm\infty,$$

and now $z_n^{\frac{1}{3}}=e^{w_n}$. So, in this case, we successfully derived the higher-order terms in the asymptotic expansion of the zeros of the Mittag-Leffler function [5].

In this case, $2<\alpha<3$, the Mittag-Leffler function has an infinite

number of real zeros. Similar to the previous case by using the recursion formula (3.17) and the integral representation (3.15) one can obtain the higher-order asymptotic distribution of zeros.
We see that

$$\arg\left(z^{\frac{1}{3}}e^{\frac{2i\pi h}{3}}\right) = \arg\left(z^{\frac{1}{3}}\right) + \frac{2\pi h}{3}$$
$$= \begin{cases} \frac{\pi}{3} + \frac{2\pi h}{3}, & \text{for} \quad z < 0, \\ \frac{2\pi h}{3}, & \text{for} \quad z > 0. \end{cases}$$

Noting the following table,

h	$z > 0$	$z < 0$
-1	$\frac{-2\pi}{3}$	$\frac{-\pi}{3}$
0	0	$\frac{\pi}{3}$
1	$\frac{2\pi}{3}$	π

In any cases, $\arg\left(z^{\frac{1}{3}}e^{\frac{2i\pi h}{3}}\right) \leq 2\pi < \pi\alpha$, therefore we have to use the integral representation (3.15). Note that $\beta = 1 + \alpha$.

(3.24)

$$E_{\frac{\alpha}{3},\beta}\left(z^{\frac{1}{3}}e^{\frac{2i\pi h}{3}}\right) = \frac{\sin\left(\frac{\pi\alpha}{3}\right)}{\frac{\pi\alpha}{3}} \int_0^\infty \frac{e^{-r^{1/3\alpha}}}{r^2 - 2rz^{\frac{1}{3}}e^{\frac{2i\pi h}{3}}\cos\left(\frac{\pi\alpha}{3}\right) + \left(z^{\frac{1}{3}}e^{\frac{2i\pi h}{3}}\right)^2} dr$$
$$- \frac{1}{z^{\frac{1}{3}}e^{\frac{2i\pi h}{3}}} + \frac{1}{\frac{\alpha}{3}z^{\frac{1}{3}}e^{\frac{2i\pi h}{3}}} e^{\left(z^{\frac{1}{3}}e^{\frac{2i\pi h}{3}}\right)^{\frac{1}{3\alpha}}}$$
$$= \frac{-3ie^{\frac{-2\pi ih}{3}}}{2\pi\alpha z^{\frac{1}{3}}} \int_0^\infty \left[\frac{e^{-r^{\frac{1}{3\alpha}}}}{r - z^{\frac{1}{3}}e^{\frac{i\pi(2h+\alpha)}{3}}} - \frac{e^{-r^{\frac{1}{3\alpha}}}}{r - z^{\frac{1}{3}}e^{\frac{i\pi(2h-\alpha)}{3}}}\right] dr$$
$$- \frac{1}{z^{\frac{1}{3}}e^{\frac{2i\pi h}{3}}} + \frac{3}{\alpha z^{\frac{1}{3}}e^{\frac{2i\pi h}{3}}} e^{z^{\frac{1}{9\alpha}}e^{\frac{2i\pi h}{9\alpha}}}$$

$$= \frac{-3ie^{\frac{-2\pi ih}{3}}}{2\pi\alpha z^{\frac{1}{3}}} \left(\frac{-1}{z^{\frac{1}{3}}e^{\frac{i\pi(2h+\alpha)}{3}}}\right) \int_0^\infty e^{-r^{\frac{1}{3\alpha}}} \sum_{k=0}^\infty \left(\frac{r}{z^{\frac{1}{3}}e^{\frac{i\pi(2h+\alpha)}{3}}}\right)^k dr$$
$$+ \frac{-3ie^{\frac{-2\pi ih}{3}}}{2\pi\alpha z^{\frac{1}{3}}} \left(\frac{1}{z^{\frac{1}{3}}e^{\frac{i\pi(2h-\alpha)}{3}}}\right) \int_0^\infty e^{-r^{\frac{1}{3\alpha}}} \sum_{k=0}^\infty \left(\frac{r}{z^{\frac{1}{3}}e^{\frac{i\pi(2h-\alpha)}{3}}}\right)^k dr$$

$$- \frac{1}{z^{\frac{1}{3}} e^{\frac{2i\pi h}{3}}} + \frac{3}{\alpha z^{\frac{1}{3}} e^{\frac{2i\pi h}{3}}} e^{z^{\frac{1}{9\alpha}} e^{\frac{2i\pi h}{9\alpha}}}$$

$$= \frac{3ie^{\frac{i\pi(-4h-\alpha)}{3}}}{2\pi\alpha z^{\frac{2}{3}}} \sum_{k=0}^{\infty} \frac{z^{\frac{-1}{3k}}}{e^{\frac{ik\pi(2h+\alpha)}{3}}} \int_0^{\infty} r^k e^{-r^{\frac{1}{3\alpha}}} dr$$
$$- \frac{3ie^{\frac{i\pi(-4h+\alpha)}{3}}}{2\pi\alpha z^{\frac{2}{3}}} \sum_{k=0}^{\infty} \frac{z^{\frac{-1}{3k}}}{e^{\frac{ik\pi(2h-\alpha)}{3}}} \int_0^{\infty} r^k e^{-r^{\frac{1}{3\alpha}}} dr - \frac{1}{z^{\frac{1}{3}} e^{\frac{2i\pi h}{3}}}$$
$$+ \frac{3}{\alpha z^{\frac{1}{3}} e^{\frac{2i\pi h}{3}}} e^{z^{\frac{1}{9\alpha}} e^{\frac{2i\pi h}{9\alpha}}}$$

By applying Lemma 2.4 we see that

$$E_{\frac{\alpha}{3},\beta}\left(z^{\frac{1}{3}} e^{\frac{2i\pi h}{3}}\right) = \frac{3ie^{\frac{i\pi(-4h-\alpha)}{3}}}{2\pi\alpha z^{\frac{2}{3}}} \sum_{k=0}^{\infty} \frac{z^{\frac{-k}{3}}}{e^{\frac{ik\pi(2h+\alpha)}{3}}} 3\alpha\Gamma(3\alpha k + 3\alpha)$$
$$- \frac{3ie^{\frac{i\pi(-4h+\alpha)}{3}}}{2\pi\alpha z^{\frac{2}{3}}} \sum_{k=0}^{\infty} \frac{z^{\frac{-k}{3}}}{e^{\frac{ik\pi(2h-\alpha)}{3}}} 3\alpha\Gamma(3\alpha k + 3\alpha)$$
$$- \frac{1}{z^{\frac{1}{3}} e^{\frac{2i\pi h}{3}}} + \frac{3}{\alpha z^{\frac{1}{3}} e^{\frac{2i\pi h}{3}}} e^{z^{\frac{1}{9\alpha}} e^{\frac{2i\pi h}{9\alpha}}}$$

Then,

(3.25)

$$E_{\frac{\alpha}{3},\beta}\left(z^{\frac{1}{3}} e^{\frac{2i\pi h}{3}}\right) = \frac{9e^{\frac{-4i\pi h}{3}}}{\pi z^{\frac{2}{3}}} \sum_{k=0}^{\infty} e^{\frac{-2i\pi hk}{3}} z^{\frac{-k}{3}} \sin\frac{\pi\alpha(k+1)}{3} \Gamma\left(3\alpha k + 3\alpha\right)$$
$$- \frac{1}{z^{\frac{1}{3}} e^{\frac{2i\pi h}{3}}} + \frac{3}{\alpha z^{\frac{1}{3}} e^{\frac{2i\pi h}{3}}} e^{z^{\frac{1}{9\alpha}} e^{\frac{2i\pi h}{9\alpha}}}$$

For $2 < \alpha < 3$ and $\beta = \alpha + 1$, using recursion relation we can write

(3.26)

$$E_{\alpha,\beta}(z) = \frac{1}{3} \sum_{k=-1}^{1} E_{\frac{\alpha}{3},\beta}\left(z^{\frac{1}{3}} e^{\frac{2i\pi h}{3}}\right)$$
$$= \frac{3e^{\frac{-4i\pi}{3}}}{\pi z^{\frac{2}{3}}} \sum_{k=0}^{\infty} e^{\frac{-2i\pi k}{3}} z^{\frac{-k}{3}} \sin\frac{\pi\alpha(k+1)}{3} \Gamma(3\alpha k + 3\alpha) - \frac{1}{z^{\frac{1}{3}} e^{\frac{2i\pi}{3}}}$$
$$+ \frac{3}{\alpha z^{\frac{1}{3}} e^{\frac{2i\pi}{3}}} e^{z^{\frac{1}{9\alpha}} e^{\frac{2i\pi}{9\alpha}}} + \frac{3}{\pi z^{\frac{2}{3}}} \sum_{k=0}^{\infty} z^{\frac{-k}{3}} \sin\frac{\pi\alpha(k+1)}{3} \Gamma(3\alpha k + 3\alpha)$$
$$- \frac{1}{z^{\frac{1}{3}}} + \frac{3}{\alpha z^{\frac{1}{3}}} e^{z^{\frac{1}{9\alpha}}} + \frac{3e^{\frac{4i\pi}{3}}}{\pi z^{\frac{2}{3}}} \sum_{k=0}^{\infty} e^{\frac{2i\pi k}{3}} z^{\frac{-k}{3}} \sin\frac{\pi\alpha(k+1)}{3} \Gamma(3\alpha k + 3\alpha)$$
$$- \frac{1}{z^{\frac{1}{3}} e^{\frac{-2i\pi}{3}}} + \frac{3}{\alpha z^{\frac{1}{3}} e^{\frac{-2i\pi}{3}}} e^{z^{\frac{1}{9\alpha}} e^{\frac{-2i\pi}{9\alpha}}}$$

$$
\begin{aligned}
&= \frac{6}{\pi}\sum_{k=0}^{\infty}\cos\left(\frac{2\pi k+4\pi}{4}\right) z^{\frac{-k-2}{3}}\sin\frac{\pi\alpha(k+1)}{3}\Gamma(3\alpha k+3\alpha)\\
&\quad -2z^{\frac{-1}{3}}\cos\frac{2\pi}{3}+\frac{3}{\alpha}z^{\frac{-1}{3}}\left[e^{\cos\frac{2\pi}{3}+2z^{\frac{1}{9\alpha}}\cos\frac{2\pi}{9\alpha}\cos\left(\frac{2\pi}{3}+z^{\frac{1}{9\alpha}}\sin\frac{2\pi}{9\alpha}\right)}\right]\\
&\quad +\frac{3}{\pi z^{\frac{2}{3}}}\sum_{k=0}^{\infty}z^{\frac{-k}{3}}\sin\frac{\pi\alpha(k+1)}{3}\Gamma(3\alpha k+3\alpha)-\frac{1}{z^{\frac{1}{3}}}+\frac{3}{\alpha z^{\frac{1}{3}}}e^{z^{\frac{1}{9\alpha}}}\\
&= \frac{3}{\pi}\sum_{k=0}^{\infty}\left[2\cos\left(\frac{2\pi k+4\pi}{4}\right)+1\right] z^{\frac{-k-2}{3}}\sin\frac{\pi\alpha(k+1)}{3}\Gamma(3\alpha k+3\alpha)\\
&\quad -z^{\frac{-1}{3}}\left(2\cos\frac{2\pi}{3}+1\right)+\frac{3}{\alpha z^{\frac{1}{3}}}e^{z^{\frac{1}{9\alpha}}}\\
&\quad +\frac{3}{\alpha}z^{\frac{-1}{3}}\left[e^{\cos\frac{2\pi}{3}+2z^{\frac{1}{9\alpha}}\cos\frac{2\pi}{9\alpha}\cos\left(\frac{2\pi}{3}+z^{\frac{1}{9\alpha}}\sin\frac{2\pi}{9\alpha}\right)}\right]
\end{aligned}
$$

For finding zeroes of the Mittag-Leffler function, we multiply both sides by $z^{\frac{1}{3}}e^{-z^{\frac{1}{9\alpha}}}$ and letting $|z|\to\infty$, we see that the transcendental equation is

$$e^{\cos\frac{2\pi}{3}+2\cos\frac{2\pi}{9\alpha}\cos\left(\frac{2\pi}{3}+z^{\frac{1}{9\alpha}}\sin\frac{2\pi}{9\alpha}\right)}+1=0$$

or

(3.27)

$$z_m = e^{\frac{9\alpha}{\sin\frac{2\pi}{9\alpha}}}\left[\cos^{-1}\frac{\frac{1}{2}+i(2m+1)\pi}{2\cos\frac{2\pi}{9\alpha}}-\frac{2\pi}{3}\right],\quad m=0,\pm1,\pm2,\ldots$$

Note that in other cases, the Mittag-Leffler functions have, at most, finite numbers of zeros, and thus, these cases are ignored.

4. Conclusion

In this paper, we studied the higher-order asymptotic distribution of the zeros of the generalized Mittag-Leffler function in terms of the reciprocal Gamma function, which plays an important role as a numerical starting point. Recall that the reciprocal Gamma function $\frac{1}{\Gamma(z)}=f(z)$ is an entire function, with zeros at $z=0,-1,-2,-3,\ldots$. Using our method, we derived more terms in the asymptotic distribution of the zeroes of the generalized Mittag-Leffler function. These results will now allow us to look deeper into the inverse problem of fractional differential equations in which the generalized Mittag-Leffler function appears.

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References

1. A. Carpinteri and F. Mainardi, *Fractals and Fractional Calculus in Continuum Mechanics*,CISM Int. Cent. Mech. Sci., Springer, 2014, pp. 378.
2. A. El-Sayed, *Fractional-Order Diffusion-Wave Equations*, Int. J. Theor. Phys., 35 (1996), pp. 311-322.
3. A. Erdélyi, W. Magnus, F. Oberhettinger and F. Tricomi, *Higher Transcendental Functions*, Bateman Project, McGraw-Hill, New York, 3 (1955).
4. A. Jodayree-Akbarfam, A.B. Mingarelli, *Duality for an Indefinite Inverse Sturm-Liouville Problem*, J. Math. Anal. Appl., Elsevier. 312 (2005), pp. 435-463.
5. A.M. Sedletskii, *Asymptotic formulas for zeros of functions of Mittag-Leffler type*, Anal. Math., 20 (1994), pp. 117-132.
6. A. Wiman, *Über den Fundamentalsatz in der Theorie der Funktionen* $E_\alpha(x)$, Acta Math., 29 (1905), pp. 191-201.
7. A. Wiman, *Über die Nullstellen der Funktionen* $E_\alpha(x)$, Acta Math., 29 (1905), pp. 217-234.
8. C.G. Fry, *Asymptotic developments of certain integral functions*, Ph.D. Dissertation, Purdue University, 1939.
9. F. Dastmalchi-Saei and A. Jodayree-Akbarfam, *Higher-Order Asymptotic Formula for the Eigenvalues of Sturm-Liouville Problem with n Turning Points*, J. Math. Anal. Appl., 371 (2010), pp. 1-15.
10. F. Mainardi and R. Gorenflo, *The Mittag-Leffler function in the Riemann-Liouville fractional calculus*, Bound. Value Prob. Spec. Funct. Fract. Calc., Belarusian State Univ., Minsk, 1996, pp. 215-225.
11. F. Mainardi, *Fractional calculus: some basic problems in continuum and statistical mechanics*, arXiv preprint arXiv:1201.0863, 2012.
12. G.S. Blair, *Psychorheology: Links between the Past and the Present*, J. Texture Stud., 5 (1974), pp. 3-12.
13. G.S. Blair, *The Role of Psychophysics in Rheology*, J. Colloid Sci., 2 (1947), pp. 21-32.
14. G.L. Lonimski, *About a Law of Deformation of Polymers*, Dokl. Akad. Nauk SSSR Phys., 140 (1961), pp. 343-346.
15. H.R. Marasi, S. Joujehi and A. Jodayree Akbarfam, *The infinite product representation of solutions of indefinite problem with a finite number of arbitrary turning points*, Math. Commun., 28 (2023), pp. 49-67.

16. H.R. Marasi and A. Jodayree Akbarfam, *On the canonical solution of indefinite problem with m turning points of even order*, J. Math. Anal. Appl., 332 (2007), pp. 1071-1086.
17. H.R. Marasi, *Asymptotic form and infinite product representation of solution of a second order initial value problem with a complex parameter and a finite number of turning points*, J. Contemp. Math. Anal., 46 (2011), pp. 212-226.
18. H. Aydi, H.R. Marasi, H. Piri and A. Talebi, *A solution to the new Caputo–Fabrizio fractional KDV equation via stability*, J. Math. Anal., 8 (2017), pp. 147-155.
19. J.W. Hanneken, B.N. Achar and D.M. Vaught, *An Alpha-Beta Phase Diagram Representation of the Zeros and Properties of the Mittag-Leffler Function*, Adv. Math. Phys., Hindawi, 2013.
20. K.R. Lang, *Astrophysical Formulae*, Radiation, Gas Processes and High Energy Astrophysics, Springer-Verlag, New York, NY, 1 (1999).
21. K.R. Lang, *Astrophysical Formulae in Space, Time, Matter and Cosmology*, Springer, New York, 2 (1999).
22. Y.F. Luchko, *Operational method in fractional calculus*, Fract. Calc. Appl. Anal., Citeseer, 2 (1999), pp. 463-488.
23. M. Caputo and F. Mainardi, *Linear Models of Dissipation in Unelastic Solids*, Riv. Nuovo Cimento, 1 (1971), pp. 161-198.
24. M.M. Dzhrbashyan, *On the Integral Representation of Functions Continuous on Several Rays (Generalization of the Fourier Integral)*, Izv. Rossiiskoi Akad. Nauk Ser. Mat., 18 (1954), pp. 427-448.
25. M.M. Dzhrbashyan, *Integral Transforms and Representations of Functions in the Complex Domain*, Nauka, Moscow, 1966.
26. P. Humbert, *Quelques résultats relatifs à la fonction de Mittag-Leffler*, C. R. Hebd. Séances Acad. Sci., Gauthier Villars, 15 (1953) pp. 1467-1468.
27. P. Humbert and R.P. Agarwal, *Sur la fonction de Mittag-Leffler et quelques-unes de ses généralisations*, Bull. Sci. Math., 77 (1953), pp. 180-185.
28. P. Torvik and R. Bagley, *On the appearance of the fractional derivative in the behavior of real materials*, J. Appl. Mech., 51 (1984), pp. 294-300.
29. R.P. Agarwal, *A propos d'une note de M. Pierre Humbert*, C. R. Math. Acad. Sci., Paris, 236 (1953), pp. 2031-2032.
30. R. Gorenflo, Y. Luchko and P. Zabrejko, *On solvability of linear fractional differential equations in Banach spaces*, Fract. Calc. Appl. Anal., (2) 1999.

31. R. Gorenflo and F. Mainardi, *Fractional Oscillations and Mittag-Leffler Functions*, Int. Workshop Recent Adv. Appl. Math. (RAAM96), 1996.
32. R. Gorenflo and R. Rutman, *On ultraslow and intermediate processes*, Transform Methods Spec. Functions, Sci. Cult. Technol., Singapore, 1995.
33. R. Gorenflo and S. Vessella, *Abel Integral Equations*, Springer, Berlin, 1991.
34. R. Gorenflo, J. Loutchko and Y. Luchko, *Computation of the Mittag-Leffler Function $E_{\alpha,\beta}(z)$ and Its Derivative*, Fract. Calc. Appl. Anal., 2002.
35. R. Gorenflo and F. Mainardi, *Fractional Calculus: Integral and Differential Equations of Fractional Order*, Springer, Vienna, 1997.
36. R. Hilfer and H.J. Seybold, *Computation of the Generalized Mittag-Leffler Function and Its Inverse in the Complex Plane*, Integral Transforms Spec. Funct., 17(2006) pp. 637-652.
37. R. Hilfer, *Fractional Diffusion Based on Riemann-Liouville Fractional Derivatives*, J. Phys. Chem. B, ACS Publications, 104 (2000) pp. 3914-3917.
38. R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.
39. R.K. Saxena, J. Ram and M. Vishnoi, *Certain properties of generalized Mittag-Leffler function*, Proc. 3rd Ann. Conf. Soc. Spec. Funct. Their Appl., Chennai, India, 2002, pp. 77-81.
40. S. Westerlund and L. Ekstam, *Capacitor theory*, IEEE Trans. Dielectr. Electr. Insul., IEEE, 1 (1994), pp. 826-839.
41. W.R. Schneider and W. Wyss, *Fractional diffusion and wave equations*, J. Math. Phys., Am. Inst. Phys., 30 (1989), pp. 134-144.
42. Y.I. Babenko, *Heat and Mass Transfer*, Khimiya, Leningrad, Russia, 1986.
43. Y.F. Luchko and H.M. Srivastava, *The exact solution of certain differential equations of fractional order by using operational calculus*, Comput. Math. Appl., 29 (1995), pp. 73-85.

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