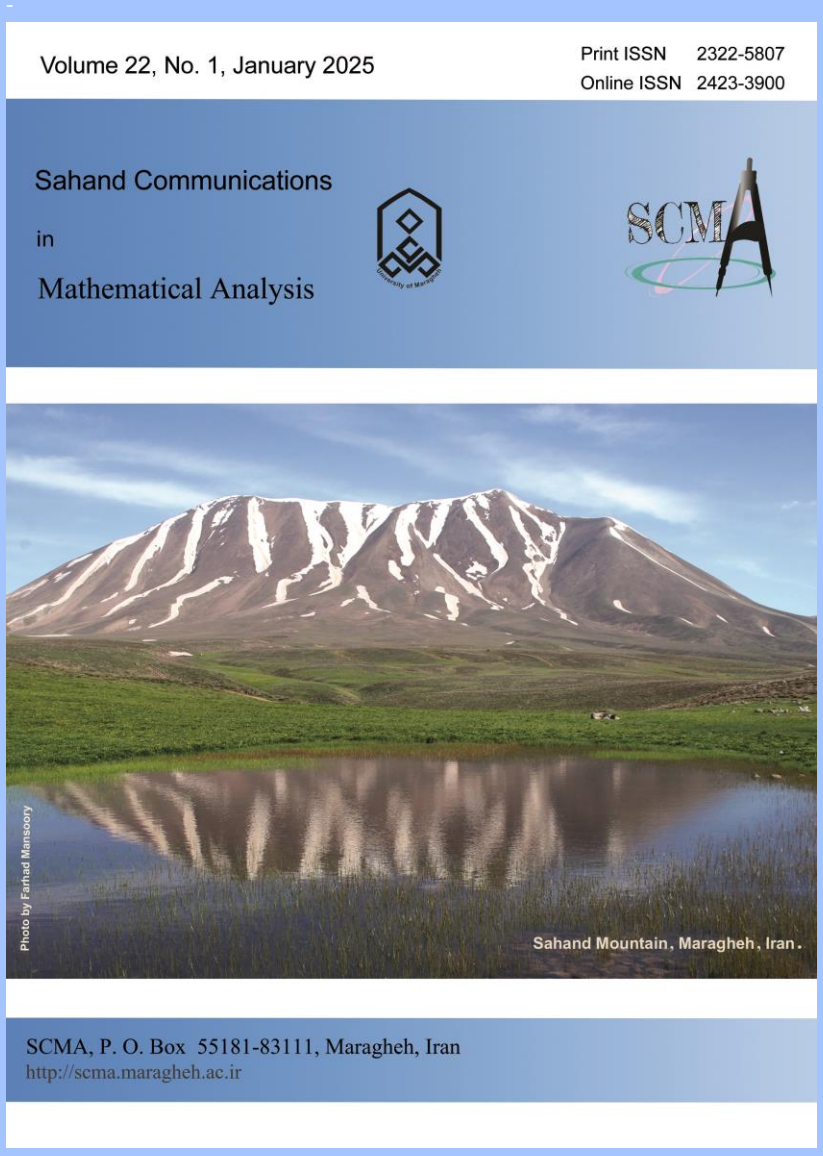
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New Forms of the Open Newton-Cotes-Type Inequalities for a Family of the Quantum Differentiable Convex Functions

Jarunee Soontharanon[1,2], Muhammad Aamir Ali[3], Shahram Rezapour[4,5*], Muhammad Toseef[3] and Thanin Sitthiwirattham[2,6]

Abstract. The main objective of this paper is to establish some new inequalities related to the open Newton-Cotes formulas in the setting of q-calculus. We establish a quantum integral identity first and then prove the desired inequalities for q-differentiable convex functions. These inequalities are useful for determining error bounds for the open Newton-Cotes formulas in both classical and q-calculus. This work distinguishes itself from existing studies by employing quantum operators, leading to sharper and more precise error estimates. These results extend the applicability of Newton-Cotes methods to quantum calculus, offering a novel contribution to the numerical analysis of convex functions. Finally, we provide mathematical examples and computational analysis to validate the newly established inequalities.

1. Introduction

Inequalities play a crucial role in various fields of mathematics. Applications of inequalities are found in diverse fields such as function spaces, fractional differential equations, optimal control theory, and functional analysis [1–5]. In this direction, the Hermite-Hadamard inequality is one of the initial inequalities related to the convex functions and integrals. Hermite [6] introduced this inequality in 1883, and it was later proven by Hadamard [7] in 1893. The mathematical expression of this

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inequality is as follows:

$$g\left(\frac{s+t}{2}\right) \leq \frac{1}{t-s}\int_s^t g(x)\,dx \leq \frac{g(s)+g(t)}{2}. \tag{1.1}$$

If g is a convex function, the above inequality holds. Conversely, the inequality holds for concave functions in the reverse direction.

This particular inequality has proven to be quite advantageous, particularly in the field of approximation theory, where it is frequently applied. With its numerous applications, mathematicians have invested time and energy into studying it and discovering new results. For instance, Dragomir and Agarwal [8] were able to determine the trapezoidal formula's boundaries by calculating the difference between the middle and right sections of the inequality and utilizing differentiable convexity throughout the process. Kirmaci [9] later determined the boundaries of the midpoint formula, which were derived from the same inequality. He calculated the difference between the middle and left portions and also utilized differentiable convexity. Meanwhile, Qi and Xi [10] determined a new inequality known as Bullen's inequality by calculating the difference between the middle section of this inequality and the average of both the left and right portions.

Alp et al. [11], utilized quantum calculus to establish the following new version of the Hermite-Hadamard inequality (1.1).

Theorem 1.1. *For a convex function $g : [s,t] \to \mathbb{R}$, the following inequalities hold for $q \in (0,1)$:*

$$g\left(\frac{qs+t}{1+q}\right) \leq \frac{1}{t-s}\int_s^t g(x)\ {}_sd_qx \leq \frac{qg(s)+g(t)}{1+q}. \tag{1.2}$$

In their research, Bermudo et al. [12] adopted a unique approach to demonstrate various versions of the Hermite-Hadamard inequality (1.1) within the realm of quantum calculus.

Theorem 1.2. *For a convex function $g : [s,t] \to \mathbb{R}$, the following inequalities hold for $q \in (0,1)$:*

$$g\left(\frac{s+qt}{1+q}\right) \leq \frac{1}{t-s}\int_s^t g(x)\ {}^td_qx \leq \frac{g(s)+qg(t)}{1+q} \tag{1.3}$$

and

$$g\left(\frac{s+t}{2}\right) \leq \frac{1}{2(t-s)}\left[\int_s^t g(x)\ {}_sd_qx + \int_s^t g(x)\ {}^td_qx\right] \leq \frac{g(s)+g(t)}{2}. \tag{1.4}$$

In the field of quantum calculus, Alp et al. [11] discovered the bounds of the midpoint formula by subtracting the right portion of inequality (1.2) from its middle section. They utilized q-differentiable convexity

throughout the process. Noor et al. [13] later determined the bounds of the trapezoidal formula in quantum calculus by taking the difference between the left and middle sections of the same inequality (1.2). They also employed differentiable convexity to arrive at their findings. Budak [14] used similar techniques as Alp et al. and Noor et al. to establish the bounds of both midpoint and trapezoidal formulas in quantum calculus using inequality (1.3). Additionally, the authors of [15] established further bounds for both midpoint and trapezoidal formulas in quantum calculus by using inequality (1.4).

Ali et al. [16] conducted a study on quantum integral inequalities and introduced the following quantum calculus version of the Hermite-Hadamard inequality (1.1).

Theorem 1.3. *If $g : [s,t] \to \mathbb{R}$ be a convex function, then we have*
(1.5)
$$g\left(\frac{s+t}{2}\right) \leq \frac{1}{t-s}\left[\int_s^{\frac{s+t}{2}} g(x)\ {}_s d_q x + \int_{\frac{s+t}{2}}^{t} g(x)\ {}^t d_q x\right] \leq \frac{g(s)+g(t)}{2}.$$

Sitthiwirattham et al. established the following new version of Hermite-Hadamard inequality (1.1) in quantum calculus.

Theorem 1.4. *If $g : [s,t] \to \mathbb{R}$ be a convex function, then we have*
(1.6)
$$g\left(\frac{s+t}{2}\right) \leq \frac{1}{t-s}\left[\int_s^{\frac{s+t}{2}} g(x)\ {}^{\frac{s+t}{2}} d_q x + \int_{\frac{s+t}{2}}^{t} g(x)\ {}_{\frac{s+t}{2}} d_q x\right] \leq \frac{g(s)+g(t)}{2}.$$

Ali et al. [16] discovered bounds for the midpoint and trapezoidal formulas in quantum calculus employing inequality (1.5), using the same methods already discussed. To explore more recent inequalities in quantum calculus of Simpson's and Newton's types, see [17, 18] and the references therein.

We have been inspired by ongoing studies to establish a new version of open Newton-Cotes-type inequalities in quantum calculus. To accomplish this, we first establish a new quantum integral identity and then apply it to demonstrate the desired inequalities for q-differentiable convex functions within the framework of quantum calculus. The results presented here have significant role in numerical integration formulas because these results are not proved so far in the literature with the help of quantum calculus. The novelty of the results is that for the first time, the error bounds of open Newton-Cotes formulas are established.

2. Preliminaries

In this section, we briefly review the fundamental concepts of quantum calculus relevant to this study in this part. In 2013, Tariboon and

Ntouyas [19] introduced the concept of left or q_s-derivative and integral, along with their respective properties. To aid in understanding, we will recall some of the definitions presented in their work.

Definition 2.1 ([19])**.** Let $g : [s,t] \to R$ be a continuous function. Then the q_s-derivative of g at $x \in (s,t]$ is defined by

$$ {}_s\mathrm{D}_q g(x) = \frac{g(x) - g(s + q(x-s))}{(1-q)(x-s)}.$$

Also, the q_s-integral is defined by

$$\int_s^x g(\tau)\ {}_s d_q\tau = (1-q)(x-s)\sum_{n=0}^{\infty} q^n g(s + q^n(x-s)).$$

In 2020, Bermudo et al. [12] presented new definitions for quantum derivative and integral using a unique method. These definitions were referred to as right or q^t-derivative and integral. The authors also discussed the fundamental properties of the operators. To clarify, we will provide the definitions from their work below:

Definition 2.2 ([12])**.** Let $g : [s,t] \to \mathbb{R}$ be a continuous function. Then the q^t-derivative of g at $x \in [s,t]$ is defined by

$$ {}^t D_q g(x) = \frac{g(t + q(x-t)) - g(x)}{(1-q)(t-x)}.$$

The q^t-integral is defined by

$$\int_x^t g(\tau)\ {}^t d_q\tau = (1-q)(t-x)\sum_{n=0}^{\infty} q^n g(t + q^n(x-t)).$$

The authors in citations [17] and [18] have presented the formulas for q-integration by parts as follows:

Lemma 2.3. *For continuous functions $h, g : [s,t] \to \mathbb{R}$, the following equality holds:*

$$\begin{aligned}(2.1)\quad &\int_0^c h(\tau)\,{}_s D_q g(s + \tau(t-s))\ {}_0 d_q\tau \\ &= \frac{h(\tau)g(s+\tau(t-s))}{t-s}\bigg|_0^c - \int_0^c g(s + q\tau(t-s))\ {}_0 D_q h(\tau)\ {}_0 d_q\tau.\end{aligned}$$

Lemma 2.4. *For continuous functions $h, g : [s,t] \to \mathbb{R}$, the following equality holds:*

$$\begin{aligned}(2.2)\quad &\int_0^c h(\tau)\ {}^t D_q g(t + \tau(s-t))\ {}_0 d_q\tau \\ &= \int_0^c g(t + q\tau(s-t))\ {}_0 D_q h(\tau)\ {}_0 d_q\tau - \frac{h(\tau)g(t+\tau(s-t))}{t-s}\bigg|_0^c.\end{aligned}$$

3. Main Results

In this section, we establish and demonstrate important integral inequalities for differentiable convex functions. To achieve this, we will first prove the following lemma.

Lemma 3.1. *Let $g : [s,t] \to \mathbb{R}$ be a q-differentiable function over. If ${}_sD_q g$ is integrable over $[s,t]$, then following inequality holds:*

$$\begin{aligned}(3.1)\qquad &\frac{1}{2}\left[g\left(\frac{2s+t}{3}\right)+g\left(\frac{s+2t}{3}\right)\right]-\frac{1}{t-s}\int_s^t g(x)\ {}_sd_qx\\ &=(t-s)\left[\int_0^{\frac{1}{3}} q\tau\ {}_sD_qg\left(s+\tau(t-s)\right)d_q\tau\right.\\ &\quad+\int_{\frac{1}{3}}^{\frac{2}{3}}\left(q\tau-\frac{1}{2}\right)\ {}_sD_qg\left(s+\tau(t-s)\right)d_q\tau\\ &\quad\left.+\int_{\frac{2}{3}}^{1}(q\tau-1)\ {}_sD_qg\left(s+\tau(t-s)\right)d_q\tau\right].\end{aligned}$$

Proof. From right hand side of (3.1) and q-integrals, we have

$$\begin{aligned}(3.2)\qquad &(t-s)\left[\int_0^{\frac{1}{3}} q\tau\ {}_sD_qg\left(s+\tau(t-s)\right)d_q\tau\right.\\ &\quad+\int_{\frac{1}{3}}^{\frac{2}{3}}\left(q\tau-\frac{1}{2}\right)\ {}_sD_qg\left(s+\tau(t-s)\right)d_q\tau\\ &\quad\left.+\int_{\frac{2}{3}}^{1}(q\tau-1)\ {}_sD_qg\left(s+\tau(t-s)\right)d_q\tau\right]\\ &=(t-s)\left[\int_0^{\frac{1}{3}} q\tau\ {}_sD_qg\left(s+\tau(t-s)\right)d_q\tau\right.\\ &\quad+\int_0^{\frac{2}{3}}\left(q\tau-\frac{1}{2}\right)\ {}_sD_qg\left(s+\tau(t-s)\right)d_q\tau\\ &\quad-\int_0^{\frac{1}{3}}\left(q\tau-\frac{1}{2}\right)\ {}_sD_qg\left(s+\tau(t-s)\right)d_q\tau\\ &\quad-\int_0^{\frac{2}{3}}(q\tau-1)\ {}_sD_qg\left(s+\tau(t-s)\right)d_q\tau\\ &\quad\left.+\int_0^{1}(q\tau-1)\ {}_sD_qg\left(s+\tau(t-s)\right)d_q\tau\right]\end{aligned}$$

$$= (t-s)\left[\frac{1}{2}\int_0^{\frac{1}{3}} {}_sD_q g\left(s+\tau\left(t-s\right)\right)d_q\tau + \frac{1}{2}\int_0^{\frac{2}{3}} {}_sD_q g\left(s+\tau\left(t-s\right)\right)d_q\tau + \int_0^1 \left(q\tau-1\right) {}_sD_q g\left(s+\tau\left(t-s\right)\right)d_q\tau\right].$$

With the help of Lemma 2.3, we have

$$\int_0^{\frac{1}{3}} {}_sD_q g\left(s+\tau\left(t-s\right)\right)d_q\tau = \frac{1}{t-s}\left[g\left(\frac{2s+t}{3}\right)-g\left(s\right)\right], \tag{3.3}$$

$$\int_0^{\frac{2}{3}} {}_sD_q g\left(s+\tau\left(t-s\right)\right)d_q\tau = \frac{1}{t-s}\left[g\left(\frac{s+2t}{3}\right)-g\left(s\right)\right], \tag{3.4}$$

$$\begin{aligned}\int_0^1 \left(q\tau-1\right) {}_sD_q g\left(s+\tau\left(t-s\right)\right)d_q\tau &= \frac{(q-1)}{t-s}g\left(t\right)+\frac{g\left(s\right)}{t-s}-\frac{q}{t-s}\int_0^1 g\left(s+q\tau\left(t-s\right)\right)d_q\tau\\ &= \frac{(q-1)}{t-s}g\left(t\right)+\frac{g\left(s\right)}{t-s}-\frac{q}{t-s}\\ &\quad\times\left[\frac{(1-q)}{q}\sum_{n=0}^{\infty}q^n g\left(s+q^n\left(t-s\right)\right)-\frac{(1-q)}{q}g\left(t\right)\right]\\ &= \frac{g\left(s\right)}{t-s}-\frac{q}{\left(t-s\right)^2}\int_s^t g\left(x\right) {}_sd_qx.\end{aligned} \tag{3.5}$$

Thus, we obtain the required equality by using (3.3)-(3.5) in (3.2). □

Theorem 3.2. *Let all conditions of Lemma 3.1 hold. If $|{}_sD_qg|$ is convex function, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{2}\left[g\left(\frac{2s+t}{3}\right)+g\left(\frac{s+2t}{3}\right)\right]-\frac{1}{t-s}\int_s^t g\left(x\right) {}_sd_qx\right|\\ &\quad\le (t-s)\left[\left(\frac{q}{27\left(1+q+q^2\right)}+s_1\left(q\right)+\frac{15-4q-4q^2}{27\left(1+q+q^2\right)\left(1+q\right)}\right)\left|{}_sD_qg\left(t\right)\right|\right.\\ &\qquad\left.+\left(\frac{2q}{27\left(1+q\right)}+s_3\left(q\right)+\frac{\left(1+q+q^2\right)-9q^3+8}{27\left(1+q+q^2\right)\left(1+q\right)}\right)\left|{}_sD_qg\left(s\right)\right|\right],\end{aligned}$$

where

$$s_1\left(q\right)=\int_{\frac{1}{3}}^{\frac{2}{3}}\tau\left|q\tau-\frac{1}{2}\right|d_q\tau=\begin{cases}\frac{9-5q-5q^2}{54[2]_q[3]_q}, & q\in\left(0,\frac{3}{4}\right)\\ \frac{6q^2+6q-3}{108[2]_q[3]_q}, & q\in\left[\frac{3}{4},1\right)\end{cases}$$

$$s_2(q)=\int_{\frac{1}{3}}^{\frac{2}{3}}\left|q\tau-\frac{1}{2}\right|d_q\tau=\begin{cases}\frac{3-3q}{18[2]_q}, & q\in\left(0,\frac{3}{4}\right)\\ \\ \frac{q}{18[2]_q}, & q\in\left[\frac{3}{4},1\right)\end{cases}$$

$$s_3(q)=\int_{\frac{1}{3}}^{\frac{2}{3}}(1-\tau)\left|q\tau-\frac{1}{2}\right|d_q\tau=s_2(q)-s_1(q).$$

Proof. By taking modulus in (3.1) and using convexity of $|{}_sD_qg|$, we have

$$\begin{aligned}&\left|\frac{1}{2}\left[g\left(\frac{2s+t}{3}\right)+g\left(\frac{s+2t}{3}\right)\right]-\frac{1}{t-s}\int_s^t g(x)\ {}_sd_qx\right|\\&\leq(t-s)\left[\int_0^{\frac{1}{3}}q\tau\left|{}_sD_qg(s+\tau(t-s))\right|d_q\tau\right.\\&\quad+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|q\tau-\frac{1}{2}\right|\left|{}_sD_qg(s+\tau(t-s))\right|d_q\tau\\&\quad\left.+\int_{\frac{2}{3}}^{1}(1-q\tau)\left|{}_sD_qg(s+\tau(t-s))\right|d_q\tau\right]\\&\leq(t-s)\left[\int_0^{\frac{1}{3}}q\tau\left[\tau\left|{}_sD_qg(t)\right|+(1-\tau)\left|{}_sD_qg(s)\right|\right]d_q\tau\right.\\&\quad+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|q\tau-\frac{1}{2}\right|\left[\tau\left|{}_sD_qg(t)\right|+(1-\tau)\left|{}_sD_qg(s)\right|\right]d_q\tau\\&\quad\left.+\int_{\frac{2}{3}}^{1}(1-q\tau)\left[\tau\left|{}_sD_qg(t)\right|+(1-\tau)\left|{}_sD_qg(s)\right|\right]d_q\tau\right]\\&=(t-s)\left[\left(\frac{q}{27[3]_q}+s_1(q)+\frac{15-4q-4q^2}{27[2]_q[3]_q}\right)\left|{}_sD_qg(t)\right|\right.\\&\quad\left.+\left(\frac{q\left(3q^2+2q+2\right)}{[4]_q+q[2]_q}+s_3(q)+\frac{[3]_q-9q^3+8}{27[2]_q[3]_q}\right)\left|{}_sD_qg(s)\right|\right].\end{aligned}$$

Thus, the proof is completed. □

Remark 3.3. We obtain the following inequality after taking the limit as $q\to1^-$ in Theorem 3.2:

$$\begin{aligned}&\left|\frac{1}{2}\left[g\left(\frac{2s+t}{3}\right)+g\left(\frac{s+2t}{3}\right)\right]-\frac{1}{t-s}\int_s^t g(x)\,dx\right|\\&\leq\frac{5(t-s)}{72}\left[\left|g'(s)\right|+\left|g'(t)\right|\right].\end{aligned}$$

Theorem 3.4. *Let all conditions of Lemma 3.1 hold. If* $|{}_sD_qg|^p$, $p \geq 1$ *is convex function, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{2}\left[g\left(\frac{2s+t}{3}\right)+g\left(\frac{s+2t}{3}\right)\right]-\frac{1}{t-s}\int_s^t g(x)\ {}_sd_qx\right| \\
&\quad\leq (t-s)\left[\left(\frac{q}{9\,[2]_q}\right)^{1-\frac{1}{p}}\left(\frac{q\,|{}_sD_qg(t)|^p}{27\,[3]_q}+\frac{q\left(3q^2+2q+2\right)}{[4]_q+q\,[2]_q}\,|{}_sD_qg(s)|^p\right)^{\frac{1}{p}}\right. \\
&\qquad+\left(s_2(q)\right)^{1-\frac{1}{p}}\left(s_1(q)\,|{}_sD_qg(t)|^p+s_3(q)\,|{}_sD_qg(s)|^p\right)^{\frac{1}{p}} \\
&\qquad+\left(\frac{9-6\,[2]_q+4q}{9\,[2]_q}\right)^{1-\frac{1}{p}}\left(\frac{15-4q\,[2]_q}{27\left([4]_q+q\,[2]_q\right)}\,|{}_sD_qg(t)|^p\right. \\
&\qquad\left.\left.+\frac{13q-6q^2-6}{27\,[3]_q}\,|{}_sD_qg(s)|^p\right)^{\frac{1}{p}}\right],
\end{aligned}$$

where $s_1(q)$*-*$s_3(q)$ *are defined in Theorem 3.2.*

Proof. Taking modulus of equality (3.1) and using power mean inequality, we have

$$\begin{aligned}
&\left|\frac{1}{2}\left[g\left(\frac{2s+t}{3}\right)+g\left(\frac{s+2t}{3}\right)\right]-\frac{1}{t-s}\int_s^t g(x)\ {}_sd_qx\right| \\
&\quad\leq (t-s)\left[\left(\int_0^{\frac{1}{3}} q\tau d_q\tau\right)^{1-\frac{1}{p}}\left(\int_0^{\frac{1}{3}}\tau\,|{}_sD_qg(s+\tau(t-s))|^p\,d_q\tau\right)^{\frac{1}{p}}\right. \\
&\qquad+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|q\tau-\frac{1}{2}\right|d_q\tau\right)^{1-\frac{1}{p}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|q\tau-\frac{1}{2}\right|\,|{}_sD_qg(s+\tau(t-s))|^p\,d_q\tau\right)^{\frac{1}{p}} \\
&\qquad\left.+\left(\int_{\frac{2}{3}}^1(1-q\tau)\,d_q\tau\right)^{1-\frac{1}{p}}\left(\int_{\frac{2}{3}}^1(1-q\tau)\,|{}_sD_qg(s+\tau(t-s))|^p\,d_q\tau\right)^{\frac{1}{p}}\right].
\end{aligned}$$

We have the following relation by using the convexity of $|{}_sD_qg|^p$

$$\begin{aligned}
&\left|\frac{1}{2}\left[g\left(\frac{2s+t}{3}\right)+g\left(\frac{s+2t}{3}\right)\right]-\frac{1}{t-s}\int_s^t g(x)\ {}_sd_qx\right| \\
&\quad\leq (t-s)\left[\left(\int_0^{\frac{1}{3}} q\tau d_q\tau\right)^{1-\frac{1}{p}}\right. \\
&\qquad\left.\times\left(\int_0^{\frac{1}{3}} q\tau\left[\tau\,|{}_sD_qg(t)|^p+(1-\tau)\,|{}_sD_qg(s)|^p\right]d_q\tau\right)^{\frac{1}{p}}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|q\tau-\frac{1}{2}\right|d_q\tau\right)^{1-\frac{1}{p}}\\
&\times\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|q\tau-\frac{1}{2}\right|\left[\tau\left|{}_sD_qg\left(t\right)\right|^p+\left(1-\tau\right)\left|{}_sD_qg\left(s\right)\right|^p\right]d_q\tau\right)^{\frac{1}{p}}\\
&+\left(\int_{\frac{2}{3}}^{1}\left(1-q\tau\right)d_q\tau\right)^{1-\frac{1}{p}}\\
&\times\left(\int_{\frac{2}{3}}^{1}\left(1-q\tau\right)\left[\tau\left|{}_sD_qg\left(t\right)\right|^p+\left(1-\tau\right)\left|{}_sD_qg\left(s\right)\right|^p\right]d_q\tau\right)^{\frac{1}{p}}\Bigg]\\
&=\left(t-s\right)\Bigg[\left(\frac{q}{9\left[2\right]_q}\right)^{1-\frac{1}{p}}\left(\frac{q\left|{}_sD_qg\left(t\right)\right|^p}{27\left[3\right]_q}+\frac{q\left(3q^2+2q+2\right)}{\left[4\right]_q+q\left[2\right]_q}\left|{}_sD_qg\left(s\right)\right|^p\right)^{\frac{1}{p}}\\
&+\left(s_2\left(q\right)\right)^{1-\frac{1}{p}}\left(s_1\left(q\right)\left|{}_sD_qg\left(t\right)\right|^p+s_3\left(q\right)\left|{}_sD_qg\left(s\right)\right|^p\right)^{\frac{1}{p}}\\
&+\left(\frac{9-6\left[2\right]_q+4q}{9\left[2\right]_q}\right)^{1-\frac{1}{p}}\\
&\times\left(\frac{15-4q\left[2\right]_q}{27\left(\left[4\right]_q+q\left[2\right]_q\right)}\left|{}_sD_qg\left(t\right)\right|^p+\frac{13q-6q^2-6}{27\left[3\right]_q}\left|{}_sD_qg\left(s\right)\right|^p\right)^{\frac{1}{p}}\Bigg].
\end{aligned}
$$

Thus, the proof is completed. □

Remark 3.5. We obtain the following inequality after taking the limit as $q\to1^-$ in Theorem 3.2:

$$
\begin{aligned}
&\left|\frac{1}{2}\left[g\left(\frac{2s+t}{3}\right)+g\left(\frac{s+2t}{3}\right)\right]-\frac{1}{t-s}\int_s^t g\left(x\right)dx\right|\\
&\quad\leq\frac{t-s}{18}\left[\left(\frac{2\left|g'\left(s\right)\right|^p+7\left|g'\left(t\right)\right|^p}{9}\right)^{\frac{1}{p}}+\left(\frac{7\left|g'\left(s\right)\right|^p+2\left|g'\left(t\right)\right|^p}{9}\right)^{\frac{1}{p}}\right.\\
&\qquad\left.+\frac{1}{2}\left(\frac{\left|g'\left(s\right)\right|^p+\left|g'\left(t\right)\right|^p}{2}\right)^{\frac{1}{p}}\right].
\end{aligned}
$$

Theorem 3.6. *Let all conditions of Lemma 3.1 hold. If $|{}_sD_qg|^p$, $p>1$ is convex function, then the following inequality holds:*

$$
\left|\frac{1}{2}\left[g\left(\frac{2s+t}{3}\right)+g\left(\frac{s+2t}{3}\right)\right]-\frac{1}{t-s}\int_s^t g\left(x\right)\ {}_sd_qx\right|
$$

$$\leq (t-s)\left[\frac{q}{\left(3^{r+1}\left[r+1\right]_q\right)^{\frac{1}{r}}}\left(\frac{\left|{}_sD_qg\left(t\right)\right|^p+\left(3\left[2\right]_q-1\right)\left|{}_sD_qg\left(s\right)\right|^p}{9\left[2\right]_q}\right)^{\frac{1}{p}}\right.$$
$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|q\tau-\frac{1}{2}\right|^r d_q\tau\right)^{\frac{1}{r}}\left(\frac{\left|{}_sD_qg\left(t\right)\right|^p+q\left|{}_sD_qg\left(s\right)\right|^p}{3\left[2\right]_q}\right)^{\frac{1}{p}}$$
$$\left.+\left(\int_{\frac{2}{3}}^{1}\left(1-q\tau\right)^r d_q\tau\right)^{\frac{1}{r}}\left(\frac{5\left|{}_sD_qg\left(t\right)\right|^p+\left(3\left[2\right]_q-5\right)\left|{}_sD_qg\left(s\right)\right|^p}{9\left[2\right]_q}\right)^{\frac{1}{p}}\right],$$

where $p+r=rp$.

Proof. Taking modulus of equality (3.1) and using Hölder inequality, we have

$$\left|\frac{1}{2}\left[g\left(\frac{2s+t}{3}\right)+g\left(\frac{s+2t}{3}\right)\right]-\frac{1}{t-s}\int_s^t g\left(x\right)\ {}_sd_qx\right|$$
$$\leq (t-s)\left[\left(\int_0^{\frac{1}{3}}\left(q\tau\right)^r d_q\tau\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{3}}\left|{}_sD_qg\left(s+\tau\left(t-s\right)\right)\right|^p d_q\tau\right)^{\frac{1}{p}}\right.$$
$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|q\tau-\frac{1}{2}\right|^r d_q\tau\right)^{\frac{1}{r}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|{}_sD_qg\left(s+\tau\left(t-s\right)\right)\right|^p d_q\tau\right)^{\frac{1}{p}}$$
$$\left.+\left(\int_{\frac{2}{3}}^{1}\left(1-q\tau\right)^r d_q\tau\right)^{\frac{1}{r}}\left(\int_{\frac{2}{3}}^{1}\left|{}_sD_qg\left(s+\tau\left(t-s\right)\right)\right|^p d_q\tau\right)^{\frac{1}{p}}\right].$$

We have the following relation by using the convexity of $\left|{}_sD_qg\right|^p$

$$\left|\frac{1}{2}\left[g\left(\frac{2s+t}{3}\right)+g\left(\frac{s+2t}{3}\right)\right]-\frac{1}{t-s}\int_s^t g\left(x\right)\ {}_sd_qx\right|$$
$$\leq (t-s)\left[\left(\int_0^{\frac{1}{3}}\left(q\tau\right)^r d_q\tau\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{3}}\left[\tau\left|{}_sD_qg\left(t\right)\right|^p+\left(1-\tau\right)\left|{}_sD_qg\left(s\right)\right|^p\right]d_q\tau\right)^{\frac{1}{p}}\right.$$
$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|q\tau-\frac{1}{2}\right|^r d_q\tau\right)^{\frac{1}{r}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left[\tau\left|{}_sD_qg\left(t\right)\right|^p+\left(1-\tau\right)\left|{}_sD_qg\left(s\right)\right|^p\right]d_q\tau\right)^{\frac{1}{p}}$$
$$\left.+\left(\int_{\frac{2}{3}}^{1}\left(1-q\tau\right)^r d_q\tau\right)^{\frac{1}{r}}\left(\int_{\frac{2}{3}}^{1}\left[\tau\left|{}_sD_qg\left(t\right)\right|^p+\left(1-\tau\right)\left|{}_sD_qg\left(s\right)\right|^p\right]d_q\tau\right)^{\frac{1}{p}}\right]$$
$$=(t-s)\left[\frac{q}{\left(3^{r+1}\left[r+1\right]_q\right)^{\frac{1}{r}}}\left(\frac{\left|{}_sD_qg\left(t\right)\right|^p+\left(3\left[2\right]_q-1\right)\left|{}_sD_qg\left(s\right)\right|^p}{9\left[2\right]_q}\right)^{\frac{1}{p}}\right.$$

$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|q\tau-\frac{1}{2}\right|^{r}d_q\tau\right)^{\frac{1}{r}}\left(\frac{|{}_sD_qg(t)|^p+q\,|{}_sD_qg(s)|^p}{3\,[2]_q}\right)^{\frac{1}{p}}$$
$$+\left(\int_{\frac{2}{3}}^{1}(1-q\tau)^{r}d_q\tau\right)^{\frac{1}{r}}\left(\frac{5\,|{}_sD_qg(t)|^p+\left(3\,[2]_q-5\right)|{}_sD_qg(s)|^p}{9\,[2]_q}\right)^{\frac{1}{p}}\Bigg].$$

Thus, the proof is completed. □

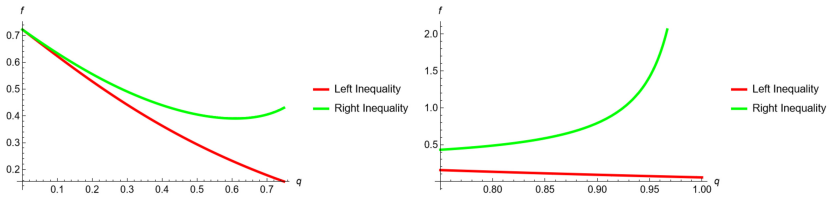
Remark 3.7. We obtain the following inequality after taking the limit as $q\to 1^-$ in Theorem 3.2:

$$\left|\frac{1}{2}\left[g\left(\frac{2s+t}{3}\right)+g\left(\frac{s+2t}{3}\right)\right]-\frac{1}{t-s}\int_s^t g(x)\,dx\right|$$
$$\leq t-s\left[\left(\frac{1}{3^{r+1}(r+1)}\right)^{\frac{1}{r}}\left(\frac{|g'(s)|^p+5\,|g'(t)|^p}{18}\right)^{\frac{1}{p}}\right.$$
$$+\left(\frac{1}{3^{r+1}(r+1)}\right)^{\frac{1}{r}}\left(\frac{5\,|g'(s)|^p+|g'(t)|^p}{18}\right)^{\frac{1}{p}}$$
$$\left.+\left(\frac{2}{6^{r+1}(r+1)}\right)^{\frac{1}{r}}\left(\frac{|g'(s)|^p+|g'(t)|^p}{6}\right)^{\frac{1}{p}}\right].$$

4. Numerical Examples

In this section, we provide numerical examples that demonstrate the validity of the new inequalities in q-calculus for convex functions where q is within the range of 0 to 1.

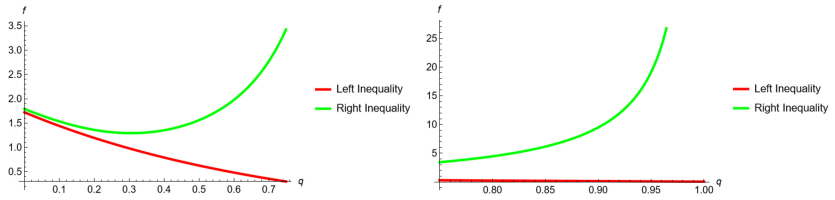
Example 4.1. Let $f:[s,t]=[0,1]\to\mathbb{R}$ be a function such that $f(x)=x^2$ and q-derivative is convex on the $[s,t]$. Then, from the Figure 1 it is clear that the inequality given in Theorem 3.2 is valid.



(a) (b)

Figure 1. Comparative analysis of inequalities of Theorem 3.2 for the discretization of q, when q lies between 0 and 1.

Example 4.2. Let $f : [s,t] = [1,2] \to \mathbb{R}$ be a function such that $f(x) = x^2$ and q-derivative is convex on the $[s,t]$. Then, from the Figure 2 it is clear that the inequality given in Theorem 3.4 is valid.



(a) (b)

FIGURE 2. Comparative analysis of inequalities of Theorem 3.4 for the discretization of q, when q lies between 0 and 1.

Example 4.3. Let $f : [s,t] = [1,2] \to \mathbb{R}$ be a function such that $f(x) = x^2$ and q-derivative is convex on the $[s,t]$. Then, from the Figure 3 it is clear that the inequality given in Theorem 3.6 is valid.

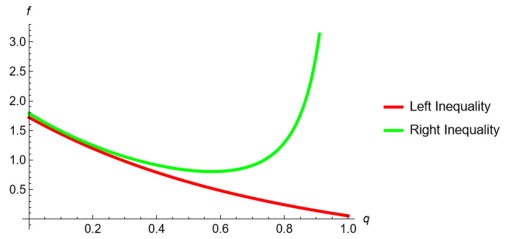


FIGURE 3. Comparative analysis of inequalities of Theorem 3.6 for the discretization of q, when q lies between 0 and 1.

5. Conclusion

This study used the framework of q-calculus to derive new inequalities for convex functions, which are crucial for establishing sharper error bounds in the Open Newton-Cotes formula for numerical integration. These inequalities not only refine classical results but also extend previously established ones, bridging gaps in the literature. The computational analysis conducted in this study confirmed the accuracy and practical relevance of these bounds, demonstrating their effectiveness in reducing numerical errors within the quantum calculus framework.

Furthermore, the enhanced precision offers potential improvements in various numerical methods, especially for applications requiring high accuracy in the approximation of convex integrals. This work paves the way for future research, including extending these inequalities to functions of two variables or exploring other generalized forms of convexity Such extensions could lead to further advancements and address intriguing challenges in the field of numerical analysis, with possible applications in multidimensional integration and optimization algorithms.

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