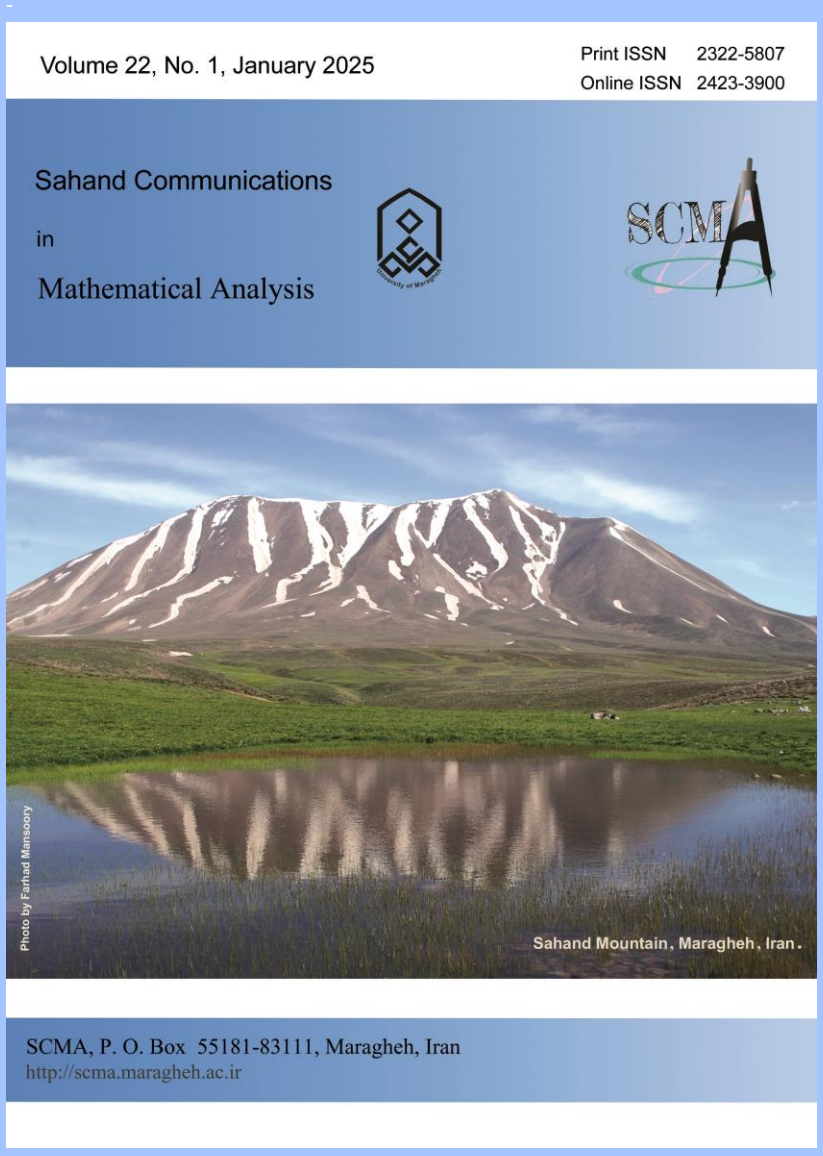
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Euler Operational Matrix of Integration and Collocation Method for Solving Functional Integral Equations

Sohrab Bazm[1,*] and Fatemeh Pahlevani[2]

Abstract. In this paper, the functional Volterra integral equations of the Hammerstein type are studied. First, some conditions that ensure the existence and uniqueness of the solutions to these equations within the space of square-integrable functions are established and then the Euler operational matrix of integration is constructed and applied within the collocation method for approximating the solutions. This approach transforms the integral equation into a set of nonlinear algebraic equations, which can be efficiently solved by employing standard numerical methods like Newton's method or Picard iteration. One significant advantage of this method lies in its ability to avoid the need for direct integration to discretize the integral operator. Error estimates are provided and two illustrative examples are included to demonstrate the method's effectiveness and practical applicability.

1. Introduction

The literature contains an extensive range of studies focused on the numerical solutions of various types of Volterra and Fredholm integral equations (see, for example, [3, 13, 16, 18, 21, 26, 31, 40, 45]). In recent decades, the study of functional integral equations has attracted the attention of many researchers because of their importance in modeling real-world problems. Such equations have applications in various scientific fields such as physics, biology, engineering, optimal control theory, economics and models of swelling porous media [2, 24, 32, 41, 47, 51, 60]. These equations are also present in the reformulation of Hammerstein

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integral equations, as well as in a variety of problems involving both ordinary and partial differential equations[5, 25, 29, 39, 44, 48].

There are several studies in the field of investigating the existence and uniqueness of solutions to functional integral equations (see, for example, [7, 8, 15, 17, 27, 28, 30, 53, 55]). However, the development of computational techniques for solving such equations is still in its early stages and has only recently gained attention. For instance, the authors in [17] utilized Euler and trapezoidal discretization methods to approximate the solution of the functional Volterra-Urysohn integral equation:

$$u(x) = f(x) + g\left(x, \int_0^x k(x, y, u(y))dy\right), \quad x \in [0, \bar{X}]. \tag{1.1}$$

An iterative method, combining the quasilinearization technique and the Jacobi-Galerkin method, was introduced in [61] to solve functional Volterra integral equations of the form:

$$u(t) = f\left(t, u(t), \int_0^t k(t, s, u(s))ds\right), \quad t \in [0, T].$$

In [45], a fixed-point approach based on the Krasnoselskij iteration was proposed to create an efficient algorithm for solving (1.1). The Fredholm version of (1.1) was numerically approximated using the Nyström method with Gauss quadrature rules in [15] and through the collocation method with sigmoidal basis functions in [4]. In addition, [52] and [6] developed distinct methodologies: the former utilized a collocation approach with piecewise continuous basis functions, while the latter employed a spectral element method featuring Gauss-Lobatto-Legendre collocation points, aimed at solving functional integral equations of the following type

$$u(x) = f(x) + g\left(x, \int_a^b k(x, y)u(y)dy\right), \quad x \in [a, b].$$

In this paper, we examine a functional integral equation of the Hammerstein type, given by

$$u(x) = f(x) + g\left(x, \int_0^x k(x, y)H(y, u(y))dy\right), \quad x \in I := [0, 1], \tag{1.2}$$

where the functions f, k, g and H are known and u denotes the unknown solution. When $g(x, y) = y$, the integral equation (1.2) has been analyzed extensively in the relevant literature (see, for example, [11, 12, 14, 19, 20, 22, 31]).

Discretization techniques are essential for analyzing integral equations. Consequently, a range of projection methods, like the Galerkin and collocation approaches, have been employed (see, for instance, [3,

38, 43]), usually producing a system of algebraic equations solvable by either iterative or direct methods.

Recently, the operational matrix method has emerged as an effective approach for discretizing problems involving integral operators. The core concept behind this method is the integral approximation

$$\int_0^x \Theta(t)dt \approx P\Theta(x),$$

where $\Theta(x) = [\theta_1(x), \ldots, \theta_N(x)]^T$ represents a column vector of basis functions and P is an $N \times N$ square matrix, referred to as the operational matrix of integration. To date, the operational matrices of integration P have been derived for various basis functions, including Bernoulli [9], Chelyshkov [14], block-pulse [50], Legendre wavelets [42], Haar wavelets [49], Bernstein [59] and barycentric rational cardinal basis functions [54].

Euler polynomials play a pivotal role in diverse disciplines, including engineering, physics and computer science, by streamlining complex computations. It enables the conversion of differential equations into more manageable algebraic forms and facilitates the representation of infinite series. Moreover, it bridges the gap between trigonometric functions and exponential expressions, thereby enhancing analytical capabilities across these fields. A novel application of Euler polynomials in the examination and prediction of air quality pollutants, as well as in mechanical engineering, can be found in references [57] and [34], respectively.

In this paper, we derive the operational matrix of integration of Euler polynomials and apply it with the collocation method to discretize the functional Volterra-Hammerstein integral equation (1.2), resulting in a system of nonlinear algebraic equations. A key feature that distinguishes this method from other discretization approaches is its ability to discretize the integral operator without requiring any actual integration, making it a more computationally efficient alternative.

The reminder of this paper is organized as follows. Section 2 applies the fixed-point method to establish the existence and uniqueness of the solution to equation (1.2). Sections 3.1 and 3.2 introduce Euler polynomials along with their properties, while Section 3.3 is dedicated to deriving the operational matrix of integration and multiplication for these polynomials. In Section 4, we apply the Euler operational matrices in conjunction with the collocation method to reduce equation (1.2) into a set of nonlinear algebraic equations. Section 5 provides an error bound for the proposed method. Two numerical examples are included in Section 6 to illustrate the effectiveness and accuracy of the method, along with a comparison to results found in other studies. Finally, Section 7 concludes the paper.

2. Existence and Uniqueness of the Solution

In this section, we examine the conditions for the existence and uniqueness of solutions related to equation (1.2). The functional framework for our analysis is the Hilbert space $\mathfrak{L}^2([0,1])$, which is equipped with the standard inner product and the corresponding norm defined as

$$\langle u, v\rangle = \int_0^1 u(x)v(x)dx, \qquad \|u\|_2 = \langle u, u\rangle^{\frac{1}{2}}.$$

Definition 2.1. A function $f : [0,1] \to \mathbb{R}$ is said to satisfy the Carathéodory's conditions if the following properties hold:

(i) $f(x,y)$ is continuous in x for each fixed y;
(ii) $f(x,y)$ is continuous in y for each fixed x;
(iii) there exists a non-negative Lebesgue-integrable function $m : [0,1] \to \mathbb{R}$ such that $|f(x,y)| \le m(y)$, for all $(x,y) \in [0,1] \times \mathbb{R}$.

We will consider the following conditions on the functions $f(x)$, $g(x,y)$, $k(x,y)$ and $H(x,y)$:

($\mathbf{C_1}$) $f \in \mathfrak{L}^2([0,1])$;

($\mathbf{C_2}$) The kernel $k \in \mathfrak{L}^2([0,1]\times[0,1])$, i.e.,

$$\|k\|_2 = \left(\int_0^1 \int_0^1 |k(x,y)|^2\, dxdy\right)^{1/2} < \infty.$$

($\mathbf{C_3}$) $g, H : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfiy the Carathéodory's conditions and there exist two positive constants α, β and two nonnegative functions $\theta, \tau \in \mathfrak{L}^2([0,1])$ such that

$$|g(x,y)| \le \theta(x) + \alpha|y|, \qquad |H(x,y)| \le \tau(x) + \beta|y|, \quad x \in [0,1],\ y \in \mathbb{R}.$$

($\mathbf{C_4}$) g and H fulfill the Lipschitz condition with respect to their second arguments with the Lipschitz constants μ, δ, respectively, i.e.,

$$\begin{aligned} &|g(x,u) - g(x,v)| \le \mu|u-v|, \\ &|H(x,u) - H(x,v)| \le \delta|u-v|, \quad x \in [0,1],\ u,v \in \mathbb{R}. \end{aligned} \tag{2.1}$$

($\mathbf{C_5}$) The kernel k and constants μ, δ satisfy $\mu\delta\|k\|_2 < 1$.

The functional integral equation (1.2) can be transformed into an operator form as shown below:

$$u(x) = f(x) + (Fu)(x),$$

where

$$(Fu)(x) = (GKu)(x) = g\left(x, (Ku)(x)\right),$$

and

$$(Ku)(x) = \int_0^x k(x,y)H\left(y, u(y)\right) dy,$$

defines the Hammerstein integral operator, while $(Gu)(x) = g(x, u(x))$ is known as the Nemytskii operator [33, Definition 2]. So, the functional equation (1.2) can be reformulated into a fixed-point equation of the type $u = Au$, where $Au = f + Fu.$. The following result guarantees the existence and uniqueness of the solution to (1.2) in $\mathfrak{L}^2([0,1])$.

Lemma 2.2. *Under the conditions given in* $(\mathbf{C_1})$-$(\mathbf{C_5})$*, the equation* (1.2) *admits a unique solution within the space* $\mathfrak{L}^2([0,1])$.

Proof. For any arbitrary $u \in \mathfrak{L}^2([0,1])$, it follows from $(\mathbf{C_3})$ and Minkowski's inequality that

$$\begin{aligned} \|Fu\|_2 &= \left(\int_0^1 |g(x,(Ku)(x))|^2 \, dx \right)^{1/2} \\ &\leq \left(\int_0^1 |\theta(x) + \alpha \, |(Ku)(x)||^2 \, dx \right)^{1/2} \\ &= \|\theta + \alpha \, |Ku|\|_2 \\ &\leq \|\theta\|_2 + \alpha \|Ku\|_2. \end{aligned} \tag{2.2}$$

It also follows from Hölder's inequality and $(\mathbf{C_3})$ that

$$\begin{aligned} |(Ku)(x)| &= \left| \int_0^x k(x,y) H(y,u(y)) dy \right| \\ &\leq \int_0^x |k(x,y) H(y,u(y))| \, dy \\ &\leq \int_0^1 |k(x,y) H(y,u(y))| \, dy \\ &\leq \left(\int_0^1 |k(x,y)|^2 dy \right)^{1/2} \left(\int_0^1 |H(y,u(y))|^2 \, dy \right)^{1/2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|Ku\|_2 &= \left\{ \int_0^1 |(Ku)(x)|^2 \, dx \right\}^{1/2} \\ &\leq \left\{ \int_0^1 \left(\int_0^1 |k(x,y)|^2 \, dy \right) \left(\int_0^1 |\tau(y) + \beta |u(y)||^2 \, dy \right) dx \right\}^{1/2} \\ &= \|k\|_2 \ \ \|\tau + \beta |u|\|_2 \\ &\leq \|k\|_2 \ \left(\|\tau\|_2 + \beta \|u\|_2 \right). \end{aligned} \tag{2.3}$$

Now, (2.2) and (2.3) yield

$$\|Fu\|_2 \leq \|\theta\|_2 + \alpha \|k\|_2 \ \big(\|\tau\|_2 + \beta \|u\|_2 \big),$$

which together with $(\mathbf{C_2})$ implies that $Fu \in \mathfrak{L}^2([0,1])$. Therefore, from $(\mathbf{C_1})$ it follows that A maps $\mathfrak{L}^2([0,1])$ on itself. In order to apply the Banach fixed-point theorem, it is crucial to show that the operator A is a contraction mapping. Let $u, v \in \mathfrak{L}^2([0,1])$. By applying the inequalities given in $(\mathbf{C_4})$ and Hölder's inequality, we derive

$$\begin{aligned}
|g(x,(Ku)(x)) - g(x,(Kv)(x))| &\leq \mu\,|(Ku)(x) - (Kv)(x)| \\
&= \mu \left| \int_0^x k(x,y)\,(H(y,u(y)) - H(y,v(y)))\,dy \right| \\
&\leq \mu \int_0^x |k(x,y)|\,|H(y,u(y)) - H(y,v(y))|\,dy \\
&\leq \mu \int_0^1 |k(x,y)|\,|H(y,u(y)) - H(y,v(y))|\,dy \\
&\leq \mu\delta \int_0^1 |k(x,y)|\,|u(y) - v(y)|\,dy \\
&\leq \mu\delta \left(\int_0^1 |k(x,y)|^2\,dy \right)^{1/2} \left(\int_0^1 |u(y) - v(y)|^2\,dy \right)^{1/2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|Au - Av\|_2 &= \|GKu - GKv\|_2 \\
&= \left\{ \int_0^1 |g(x,(Ku)(x)) - g(x,(Kv)(x))|^2\,dx \right\}^{1/2} \\
&\leq \mu\delta \left\{ \int_0^1 \left(\int_0^1 |k(x,y)|^2 dy \right) \left(\int_0^1 |u(y) - v(y)|^2 dy \right) dx \right\}^{1/2} \\
&= \mu\delta \|k\|_2 \|u - v\|_2,
\end{aligned}$$

which together with $(\mathbf{C_5})$ gives the result. □

3. Euler Polynomials and their Operational Matrix of Integration

3.1. **Definitions.** Euler polynomials and numbers (introduced by Euler in 1740) have various applications in number theory, contributing to a rich literature on the subject [58]. Euler polynomials $E_n(x)$ are characterized by the generating formula ([46, page 588]):

$$\sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = \frac{2e^{tx}}{1+e^t}, \quad |t| < \pi.$$

For nonnegative integers n, Euler polynomials $E_n(x)$ can also be generated from the following relations:

(3.1)

$$(i)\quad E_n(x)=\tfrac{1}{2^n}\sum_{k=1}^{n+1}\sum_{j=0}^{k-1}(-1)^j\binom{n+1}{k}(x+j)^n,\quad [46,\text{ Chapter 24}]$$

$$(ii)\quad \sum_{k=0}^{n}\binom{n}{k}E_k(x)+E_n(x)=2x^n,\quad n\geq 0,[23]$$

$$(iii)\quad E_n(x+h)=\sum_{k=0}^{n}\binom{n}{k}E_k(x)h^{n-k},\quad [46,\text{ Chapter 24}]$$

$$(iv)\quad E_n(x)=\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)x^k,\quad n\geq 0,[35].$$

The first four Euler polynomials are

$$\begin{aligned}E_0(x)&=1,\\ E_1(x)&=x-\tfrac{1}{2},\\ E_2(x)&=x^2-x,\\ E_3(x)&=x^3-\tfrac{3}{2}x^2+\tfrac{1}{4}.\end{aligned}$$

Based on the mentioned relations in (3.1) concerning Euler polynomials, the following expressions can be derived [46, Chapter 24]:

$$\begin{aligned}&(i)\quad E_n(x+1)+E_n(x)=2x^n,,\\ &(ii)\quad E_n'(x)=nE_{n-1}(x),\\ &(iii)\quad E_n(1-x)=(-1)^nE_n(x),\\ &(iv)\quad (-1)^{n+1}E_n(-x)=E_n(x)-2x^n.\end{aligned}\tag{3.2}$$

In number theory, Euler numbers E_n represent a sequence of integers that can be expressed through the following Taylor series expansion [36]:

$$\begin{aligned}&\sum_{n=0}^{\infty}E_n\frac{t^n}{n!}=\frac{2}{e^t+e^{-t}}=\frac{1}{\cosh(t)}=\operatorname{sech}(\mathrm{t}),\quad |t|<\frac{\pi}{2},\\ &\sum_{n=0}^{\infty}(-1)^nE_{2n}\frac{t^{2n}}{2n!}=\frac{1}{\cos(t)}=\sec(t),\quad |t|<\frac{\pi}{2}.\end{aligned}\tag{3.3}$$

Notably, all Euler numbers with odd indices are zero, while the even-indexed ones alternate in sign. These numbers, as outlined in equation (3.3), appear as coefficients in the Taylor series expansions of both the secant and hyperbolic secant functions. It is mentionable that $E_n\neq E_n(0)$ for $n\geq 1$. Euler numbers are related to Euler polynomials by

the following formula [1]:

$$E_n = 2^n E_n\left(\frac{1}{2}\right).$$

Let N be a fixed nonnegative integer and define

$$T(x) = \left[1, x, x^2, \ldots, x^N\right]^T, \tag{3.4}$$

and

$$E(x) = \left[E_0(x), E_1(x), E_2(x), \ldots, E_N(x)\right]^T. \tag{3.5}$$

Using part (ii) of equation (3.1), we obtain

$$DE(x) = T(x),$$

where D is a lower triangular matrix of order $N+1$ defined as

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ \frac{1}{2}\binom{1}{0} & 1 & 0 & 0 & \ldots & 0 \\ \frac{1}{2}\binom{2}{0} & \frac{1}{2}\binom{2}{1} & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}\binom{N}{0} & \frac{1}{2}\binom{N}{1} & \frac{1}{2}\binom{N}{2} & \cdots & \frac{1}{2}\binom{N}{N-1} & 1 \end{pmatrix}.$$

Since D is an invertible matrix, it follows that

$$E(x) = D^{-1}T(x). \tag{3.6}$$

Additionally, by varying n from 0 to N in part (iv) of (3.1) and then expressing the results in matrix form, we obtain

$$E(x) = \widehat{D}T(x), \tag{3.7}$$

where $\widehat{D}$ is a lower triangular matrix of order $N+1$ defined as

(3.8)

$$\widehat{D} = \begin{pmatrix} \binom{0}{0}E_0(0) & 0 & 0 & \ldots & 0 \\ \binom{1}{0}E_1(0) & \binom{1}{1}E_0(0) & 0 & \ldots & 0 \\ \binom{2}{0}E_2(0) & \binom{2}{1}E_1(0) & \binom{2}{2}E_0(0) & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{N}{0}E_N(0) & \binom{N}{1}E_{N-1}(0) & \binom{N}{2}E_{N-2}(0) & \ldots & \binom{N}{N}E_0(0) \end{pmatrix},$$

Equations (3.6) and (3.7) directly indicate that $\widehat{D} = D^{-1}$. Moreover, based on equation (3.7), we define the dual matrix corresponding to $E(x)$ as

$$J = \int_0^1 E(x)E^T(x)dx = \widehat{D}G(1)\widehat{D}^T, \tag{3.9}$$

where $\widehat{D}$ refers to the matrix defined in (3.8) and $G(x)$ denotes the Gram matrix, which is defined as follows

$$G(x) = \int_0^x T(y)T^T(y)dy = [G_{ij}(x)]_{i,j=0}^N, \qquad G_{ij}(x) = \frac{x^{i+j+1}}{i+j+1}. \tag{3.10}$$

3.2. **Function approximation.** Let $\mathcal{H}_N = \text{span}\{E_0(x), \ldots, E_N(x)\}$. Since $E_i(x)$ for $i = 0, 1, \ldots, N$ are linearly independent, it follows that $\mathcal{H}_N$ is an $(N+1)$-dimensional subspace of the Hilbert space $\mathfrak{L}^2([0,1])$ and is closed ([37, Theorem 2.4-3]). Moreover, for any function $u \in \mathcal{H}$, a unique optimal approximation $\bar{u}_N \in \mathcal{H}_N$ exists, as stated in [37, Theorem 6.2-5], such that

$$\|u - \bar{u}_N\|_2 \leq \|u - v\|_2, \quad \forall v \in \mathcal{H}_N.$$

Since $\bar{u}_N \in \mathcal{H}_N$, there exist unique coefficients $u_0, u_1, \ldots, u_N$ such that:

$$u(x) \simeq \bar{u}_N(x) = \sum_{k=0}^N u_k E_k(x) = E^T(x)U, \tag{3.11}$$

where $E(x)$ denotes the vector defined in (3.5) and $U = \big[u_0, u_1, \ldots, u_N\big]^T$. We refer to U as the Euler coefficient vector corresponding to u.
Now, define

$$E_{ij}(x,t) = E_i(x)E_j(t), \quad i,j = 0, 1, \ldots, N.$$

Following the same reasoning applied to univariate functions, any function $k(x,t) \in \mathfrak{L}^2([0,1] \times [0,1])$ can be approximated by its unique best approximation in the space $\mathcal{H}_N^2 = \text{span}\{E_{ij}(x,t), i,j = 0, 1, \ldots, N\}$ expressed as follows:

$$\begin{aligned} k(x,t) \simeq \bar{k}_N(x,t) &= \sum_{m=0}^N \sum_{n=0}^N k_{m,n} E_m(x)E_n(t) \\ &= E^T(x)KE(t), \end{aligned} \tag{3.12}$$

where $K = [k_{m,n}]_{m,n=0}^N$ denotes the Euler coefficient matrix corresponding to k.

In the following, we aim to derive formulas for calculating the Euler coefficient vector U and the Euler coefficient matrix K, as defined in equations (3.11) and (3.12), respectively. To achieve this, we will

first introduce Bernoulli polynomials and highlight some of their key properties, which play a crucial role in the subsequent derivations.

Definition 3.1. The classical Bernoulli polynomials $B_n(x)$ of degree n are constructed from the following relation (see [9, 10, 12])

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k(x) = (n+1)x^n, \quad n = 0, 1, \ldots.$$

The Bernoulli polynomials have many notable properties. For instance, they satisfy the following relations [9, 10, 12]:

$$\begin{aligned} &(i) \quad B_n'(x) = nB_{n-1}(x), \quad n \geq 1, \\ &(ii) \quad \int_0^1 B_n(x)dx = \begin{cases} 1, & n = 0, \\ 0, & n \geq 1. \end{cases} \end{aligned} \tag{3.13}$$

Euler polynomials can be computed using the Bernoulli polynomials $B_n(x)$ as described by the following relation [56]:

$$E_n(x) = \frac{2}{n+1}\left(B_{n+1}(x) - 2^{n+1}B_{n+1}\left(\frac{x}{2}\right)\right).$$

An additional key relationship between the Euler and Bernoulli polynomials, which serves as the foundation for proving the two theorems in this section, is provided below [35, Section 1]:

$$E_n(x) = \sum_{k=0}^{n} \frac{-2}{k+1}\binom{n}{k} E_{k+1}(0)B_{n-k}(x), \quad n \geq 0. \tag{3.14}$$

In the following, we assume that the summation $\sum_{n_1}^{n_2}$ equals zero whenever $n_1 > n_2$.

Theorem 3.2. *Let $u \in \mathfrak{L}^2([0,1])$ be an arbitrary function and be approximated in terms of Euler polynomials by $\bar{u}_N(x) = \sum_{k=0}^{N} u_k E_k(x) = E^T(x)U$. Then, the Euler coefficients u_k, for all $k = 0, 1, 2, \ldots, N$, can be derived using the following expression*

$$u_k = \frac{1}{k!}\int_0^1 u^{(k)}(x)dx - \sum_{r=1}^{N-k}\binom{k+r}{r} u_{k+r}W_r, \quad k = N, \ldots, 0, \tag{3.15}$$

where

$$W_r = \frac{-2}{r+1}E_{r+1}(0), \quad r = 0, 1, \ldots, N. \tag{3.16}$$

Proof. Integrating (3.14) over the interval $[0,1]$ and utilizing part (ii) in (3.13) yields:

$$\int_0^1 E_n(x)dx = W_n, \quad n \geq 0, \tag{3.17}$$

where W_n is defined in (3.16) for $n \geq 0$. By repeatedly differentiating (3.11) k times and applying part (ii) in (3.2) after each differentiation, we arrive at the following approximation:

$$\frac{1}{k!}u^{(k)}(x) \simeq \sum_{r=0}^{N-k} \binom{k+r}{r} u_{k+r}E_r(x). \tag{3.18}$$

Subsequently, by integrating (3.18) from 0 to 1 and employing (3.16), while noting that $W_0 = 1$, We derive the desired result, i.e., equation (3.15). □

Theorem 3.3. *Let $k \in \mathfrak{L}^2([0,1]\times[0,1])$ be an arbitrary function, which is approximated using Euler polynomials by the expression $\bar{k}_N(x,t) = \sum_{m=0}^{N}\sum_{n=0}^{N} k_{m,n}E_m(x)E_n(t) = E^T(x)KE(t)$. The coefficients k_{mn} in this expansion, for all $m,n = 0,1,\ldots,N$, can be computed via the following relation:*

$$\begin{aligned} k_{m,n} = {} & \frac{1}{m!n!}\int_0^1\int_0^1 \frac{\partial^{m+n}k(x,t)}{\partial x^m \partial t^n}dxdt \\ & - \sum_{j=1}^{N-n}\binom{n+j}{j}k_{m,n+j}W_j \\ & - \sum_{i=1}^{N-m}\binom{m+i}{i}W_i\sum_{j=0}^{N-n}\binom{n+j}{j}k_{m+i,n+j}W_j, \\ & m,n = N,N-1,\ldots,0. \end{aligned} \tag{3.19}$$

Proof. By successively differentiating (3.12) m times with respect to x and n times with respect to t and applying (ii) in (3.2) after each differentiation, we arrive at the following expression:

$$\begin{aligned} \frac{1}{m!n!}\frac{\partial^{m+n}k(x,t)}{\partial x^m\partial t^n} \simeq {} & \sum_{i=0}^{N-m}\binom{m+i}{i}E_i(x)\sum_{j=0}^{N-n}\binom{n+j}{j}k_{m+i,n+j}E_j(t), \\ & m,n = N,N-1,\ldots,0. \end{aligned} \tag{3.20}$$

Now, by integrating (3.20) with respect to x and t both over $[0,1]$ and using (3.17), we get

$$\begin{aligned}(3.21)\qquad &\frac{1}{m!n!}\int_0^1\int_0^1 \frac{\partial^{m+n}k(x,t)}{\partial x^m \partial t^n}dxdt \\ &= W_0^2 k_{m,n} + W_0\sum_{j=1}^{N-n} k_{m,n+j}\binom{n+j}{j}W_j \\ &\quad + \sum_{i=1}^{N-m}\binom{m+i}{i}W_i\sum_{j=0}^{N-n} k_{m+i,n+j}\binom{n+j}{j}W_j, \\ &\quad m,n = N, N-1,\ldots,0.\end{aligned}$$

Noting that $W_0 = 1$, (3.21) gives the result, i.e., (3.19). □

3.3. **Euler operational matrix of integration.** For a fixed $x \in [0,1]$, integrating both sides of part (ii) in (3.2) over interval $[0,x]$ yields:

$$\int_0^x E_{n-1}(t)dt = \frac{1}{n}\left(E_n(x) - E_n(0)\right), \quad n \geq 1.$$

Thus, the integral of Euler vector $E(x)$ over interval $[0,x]$ can be formulated as:

$$(3.22)\qquad \int_0^x E(t)dt = PE(x) + \frac{1}{N+1}E_{N+1}(x)e_{N+1},$$

where

$$(3.23)\qquad P = \begin{pmatrix} -E_1(0) & 1 & 0 & \ldots & 0 \\ -\frac{E_2(0)}{2} & 0 & \frac{1}{2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ -\frac{E_N(0)}{N} & 0 & 0 & \ldots & \frac{1}{N} \\ -\frac{E_{N+1}(0)}{N+1} & 0 & 0 & \ldots & 0 \end{pmatrix},$$

and e_{N+1} represents the last column of the identity matrix I of dimension $N+1$. We refer to P as the Euler operational matrix of integration of order $N+1$.

By omitting the second term in the right-hand side of (3.22), the integral of Euler vector $E(x)$ over interval $[0,x]$ can be represented as:

$$(3.24)\qquad \int_0^x E(t)dt \simeq PE(x).$$

Based on equations (3.11) and (3.24), the integral of any function $u \in \mathcal{H}$ over interval $[0, x]$ can be approximated in terms of Euler polynomials, providing an effective approach for approximating integrals in this context. Consequently, we can establish the following relation.

$$\int_0^x u(t)dt \simeq \int_0^x U^T E(t)dt = U^T \int_0^x E(t)dt \simeq U^T P E(x).$$

Now, define

$$\Pi(x) = E(x)E^T(x),$$

which we referred to as the product matrix of Euler polynomials. Let $C = [c_0, c_1, \ldots, c_N]^T$ be an arbitrary vector in $\mathbb{R}^{N+1}$. Then, we can deduce from equation (3.7) that

$$\begin{aligned} \Pi(x)C &= \left(\hat{D}T(x)\right)E^T(x)C \\ &= \hat{D}\left[E^T(x)C, xE^T(x)C, \ldots, x^N E^T(x)C\right]^T \\ &= \hat{D}\left[\sum_{i=0}^N c_i E_i(x), \sum_{i=0}^N c_i x E_i(x), \ldots, \sum_{i=0}^N c_i x^N E_i(x)\right]^T, \end{aligned} \tag{3.25}$$

where $T(x)$ and $\hat{D}$ are defined by (3.4) and (3.8), respectively. Utilizing equation (3.11), we can approximate $x^k E_i(x)$, for $i, k = 0, 1, \ldots, N$, using Euler polynomials as:

$$x^k E_i(x) \simeq E^T(x)\delta_{k,i}, \tag{3.26}$$

where $\delta_{k,i}$ is the Euler coefficient vector defined as

$$\delta_{k,i} = [\delta_0^{k,i}, \delta_1^{k,i}, \ldots, \delta_N^{k,i}]^T. \tag{3.27}$$

From (3.11) and 3.26, we obtain

$$\begin{aligned} \int_0^1 x^k E_i(x)E(x)dx &\simeq \left(\int_0^1 E(x)E^T(x)dx\right)\delta_{k,i} \\ &= J\delta_{k,i}, \quad i, k = 0, 1, \ldots, N, \end{aligned} \tag{3.28}$$

where J is the matrix defined by (3.9). Thus, for $i, k = 0, 1, \ldots, N$, it follows from (3.28) that

$$\begin{aligned} \delta_{k,i} &\simeq J^{-1}\int_0^1 x^k E_i(x)E(x)dx \\ &= J^{-1}\left[\int_0^1 x^k E_i(x)E_0(x)dx, \ldots, \int_0^1 x^k E_i(x)E_N(x)dx\right]^T. \end{aligned}$$

Also, by using (3.26) and (3.27), we have

$$\sum_{i=0}^N c_i x^k E_i(x) \simeq \sum_{i=0}^N c_i \left(E^T(x)\delta_{k,i}\right) \tag{3.29}$$

$$
\begin{aligned}
&= \sum_{i=0}^{N} c_i \sum_{j=0}^{N} \delta_j^{k,i} E_j(x) \\
&= \sum_{j=0}^{N} E_j(x) \sum_{i=0}^{N} c_i \delta_j^{k,i} \\
&= E^T(x) \left[\sum_{i=0}^{N} c_i \delta_0^{k,i}, \sum_{i=0}^{N} c_i \delta_1^{k,i}, \ldots, \sum_{i=0}^{N} c_i \delta_N^{k,i} \right]^T \\
&= E^T(x)[\delta_{k,0}, \delta_{k,1}, \ldots, \delta_{k,N}]C \\
&= E^T(x)\tilde{\Delta}_k,
\end{aligned}
$$

where

$$\tilde{\Delta}_k = \Delta_k C, \quad , k = 0, 1, \ldots, N,$$

in which Δ_k is an $(N+1) \times (N+1)$ matrix whose columns are given by $\delta_{k,i}$, $i = 0, 1, \ldots, N$. Now, applying (3.29) into (3.25) results in the following approximation:

$$\Pi(x)C \simeq \widehat{C}E(x), \tag{3.30}$$

where $\widehat{C} = \widehat{D}\tilde{C}^T$ in which $\tilde{C} = [\tilde{\Delta}_0, \tilde{\Delta}_1, \ldots, \tilde{\Delta}_N]$.

4. Numerical Solution of Functional Integral Equations

In this section, we investigate functional integral equations of the form (1.2), assuming that the functions f, g, k and H satisfy conditions ($\mathbf{C_1}$)-($\mathbf{C_5}$) outlined in Section 2. Under these assumptions, Lemma 2.2 ensures the existence and uniqueness of a solution $u \in \mathfrak{L}^2([0,1])$ to equation (1.2). Our objective here is to numerically approximate the unknown solution u by employing the Euler operational matrix of integration with collocation method. To this end, we define

$$z(x) = H(x, u(x)). \tag{4.1}$$

By substituting (4.1) into (1.2), we immediately obtain

$$u(x) = f(x) + g\left(x, \int_0^x k(x,y)z(y)dy\right), \quad x \in [0,1]. \tag{4.2}$$

Thus, from (4.1), it follows that z satisfies the following nonlinear integral equation:

$$z(x) = H\left(x, f(x) + g\left(x, \int_0^x k(x,y)z(y)dy\right)\right). \tag{4.3}$$

Now, we use Euler polynomials to approximate both the driving term $f(x)$ and the kernel $k(x,y)$, as outlined in equations (3.11) and (3.12).

Specifically, we express them in the following forms:

$$f(x) \approx f_N(x) = E^T(x)F, \tag{4.4}$$

$$k(x,y) \approx k_N(x,y) = E^T(x)KE(y), \tag{4.5}$$

where the Euler coefficient vector $F = \left[f_0, f_1, \ldots, f_N\right]^T$ and the Euler coefficient matrix K are determined using Theorems 3.2 and 3.3, respectively. Our approach aims to find a function $z_N \in \mathcal{H}_{\mathcal{N}}$ that satisfies the functional integral equation:

$$z_N(x) = H\left(x, f_N(x) + g\left(x, \int_0^x k_N(x,y) z_N(y) dy\right)\right). \tag{4.6}$$

Let

$$z_N(x) = \sum_{i=0}^{N} z_i E_i(x) = E^T(x)Z, \tag{4.7}$$

where $Z = \left[z_0, z_1, \ldots, z_N\right]^T$. By substituting equations (4.4), (4.5) and (4.7) into the functional integral equation (4.6) and applying equations (3.30) and (3.24), we get:

(4.8)

$$\begin{aligned} E^T(x)Z &= H\left(x, E^T(x)F + g\left(x, \int_0^x E^T(x)KE(y)E^T(y)Z dy\right)\right) \\ &= H\left(x, E^T(x)F + g\left(x, E^T(x)K\int_0^x \Pi(y)Z dy\right)\right) \\ &\simeq H\left(x, E^T(x)F + g\left(x, E^T(x)K\widehat{Z}\int_0^x E(y) dy\right)\right) \\ &\simeq H\left(x, E^T(x)F + g\left(x, E^T(x)K\widehat{Z}PE(x)\right)\right). \end{aligned}$$

In (4.8), P denotes the Euler operational matrix of integration defined by (3.23).

Let us now define Newton-Cotes nodes x_l within the interval $[0,1]$ as follows:

$$x_l = \frac{2l+1}{2(N+1)}, \quad l = 0, 1, \ldots, N. \tag{4.9}$$

By collocating (4.8) at Newton-Cotes nodes (4.9), we obtain

(4.10)

$$E^T(x_l)Z = H\left(x_l, E^T(x_l)F + g\left(x_l, E^T(x_l)K\widehat{Z}PE(x_l)\right)\right),\ l = 0, \ldots, N.$$

The equation (4.10) represents a system of $N+1$ nonlinear algebraic equations for unknowns z_0, z_1, $\ldots$, z_N. The nonlinear system (4.10) can be solved through numerical approaches, such as Newton's method

or Picard iteration method. Once the unknowns are determined, the approximate solution to equation (4.3) can be obtained using equation (4.7). In the last step, the desired approximation u to the equation (1.2) will be calculated as follows:

$$u_N(x) = f_N(x) + g\left(x, \int_0^x k_N(x,y) z_N(y) dy\right) \tag{4.11}$$

$$= E^T(x)F + g\left(x, \int_0^x E^T(x) K E(y) E^T(y) Z dy\right)$$

$$= E^T(x)F + g\left(x, E^T(x) K \widehat{D} G(x) \widehat{D}^T Z\right), \quad x \in [0,1], \tag{4.12}$$

where $\widehat{D}$ and $G(x)$ refer to the matrices defined by equations (3.8) and (3.10), respectively.

Remark 4.1. The solvability of system (4.10) is a challenging problem and we were unable to prove whether a unique solution exists. However, our numerical tests (see Section 6) indicate that system (4.10) can be solved in many cases and its solution can be efficiently approximated using standard iterative methods. In our work, we used the *FindRoot* function in *Mathematica* to solve this system, which applies Newton's method by default. In all the examples we studied, this function successfully found an accurate approximate solution, even when starting with an initial guess of zero.

Remark 4.2. One key advantage of this method compared to other discretization techniques for solving integral equations is that it avoids the need for explicitly evaluating integrals. This greatly simplifies the computational process. Instead of direct integration, the method utilizes an operational matrix of integration, resulting in improved efficiency and numerical stability.

5. Error Estimate

For $f \in \mathfrak{L}^2([0,1])$ (or $f \in \mathfrak{L}^2([0,1]\times[0,1])$), we define

$$e_N(f) = \|f - \bar{f}_N\|_2, \tag{5.1}$$

where $\bar{f}_N$ denotes the Euler polynomial approximation of f, as defined in (3.11) (or (3.12)). The following theorem holds.

Theorem 5.1. *Let $u(x)$ be the exact solution of* (1.2) *and let $u_N(x)$ denote its approximation using Euler polynomials, as obtained in* (4.12) *through the proposed method. Furthermore, assume that*

(i) *H is a bounded function on $[0,1]\times\mathbb{R}$; that is, there exists a positive number ρ such that $|H(x,y)| \le \rho$ for all $(x,y) \in [0,1]\times\mathbb{R}$;*

(ii) $k \in \mathfrak{L}^2\big([0,1]\times[0,1]\big)$ *and* $\tilde{k} = \|k\|_2$*;*
(iii) g *and* H *satisfy the Lipshitz conditions* (2.1) *given in* $(\mathbf{C_4})$ *in Section 2.*
(iv) $\mu\delta(\tilde{k} + e_N(k)) < 1$.

Then, the error estimate below holds:

$$\|u - u_N\|_2 \leq \frac{e_N(f) + \mu\rho e_N(k)}{1 - \mu\delta(\tilde{k} + e_N(k))}.$$

Proof. We define the linear operators $\mathcal{K}, \mathcal{K}_N : \mathfrak{L}^2\big([0,1]\big) \to \mathfrak{L}^2\big([0,1]\big)$ as:

$$(\mathcal{K}v)(x) = \int_0^x k(x,t)v(t)dt, \qquad (\mathcal{K}_N v)(t) = \int_0^x k_N(x,t)v(t)dt,$$

where k_N is the Euler polynomial approximation of the kernel function $k(x,t)$. Then, for any $v \in \mathfrak{L}^2\big([0,1]\big)$, it follows from Hölder's inequality that

$$\begin{aligned}
\|\mathcal{K}v\|_2 &= \left(\int_0^1 |(\mathcal{K}v)(x)|^2 dx\right)^{\frac{1}{2}} \\
&= \left(\int_0^1 \left|\int_0^x k(x,t)v(t)dt\right|^2 dx\right)^{\frac{1}{2}} \\
&\leq \left\{\int_0^1 \left(\int_0^x |k(x,t)|^2 dt\right)\left(\int_0^x |v(t)|^2 dt\right) dx\right\}^{\frac{1}{2}} \\
&\leq \left\{\int_0^1 \left(\int_0^1 |k(x,t)|^2 dt\right)\left(\int_0^1 |v(t)|^2 dt\right) dx\right\}^{\frac{1}{2}} \\
&= \|k\|_2\|v\|_2.
\end{aligned} \tag{5.2}$$

Following a similar approach as outlined in (5.2), we derive

$$\|(\mathcal{K} - \mathcal{K}_N)v\|_2 \leq \|k - k_N\|_2\|v\|_2, \quad \forall v \in \mathfrak{L}^2\left([0,1]\right). \tag{5.3}$$

From (4.11) and (4.2) and applying assumption (iii), we obtain:

$$\begin{aligned}
\|u - u_N\|_2 &\leq \|f - f_N\|_2 + \|g(.,(\mathcal{K}z)(.)) - g(.,(\mathcal{K}_N z_N)(.))\|_2 \\
&\leq e_N(f) + \mu\|\mathcal{K}z - \mathcal{K}_N z_N\|_2.
\end{aligned} \tag{5.4}$$

Also from (5.2), (5.3) and assumption (i), we obtain

$$\begin{aligned}
\|\mathcal{K}z - \mathcal{K}_N z_N\|_2 &\leq \|\mathcal{K}(z - z_N)\|_2 + \|(\mathcal{K} - \mathcal{K}_N)z_N\|_2 \\
&\leq \|k\|_2\|z - z_N\|_2 + \|k - k_N\|_2\|z_N\|_2 \\
&\leq \tilde{k}\|z - z_N\|_2 + e_N(k)(\|z - z_N\|_2 + \|z\|_2) \\
&= \Big(\tilde{k} + e_N(k)\Big)\|z - z_N\|_2 + e_N(k)\|H(.,u(.))\|_2
\end{aligned} \tag{5.5}$$

$$\leq \left(\tilde{k} + e_N(k)\right) \|z - z_N\|_2 + \rho e_N(k).$$

By employing (4.1), (4.6) and assumption (iii), we conclude that

$$\begin{aligned} |z(x) - z_N(x)| &= |H(x, u(x)) - H(x, u_N(x))| \\ &\leq \delta |u(x) - u_N(x)|, \quad \forall x \in [0, 1], \end{aligned}$$

which leads to

$$\|z - z_N\|_2 \leq \delta \|u - u_N\|_2. \tag{5.6}$$

Consequently, by using relations (5.4)-(5.6), we obtain

$$\|u - u_N\|_2 \leq e_N(f) + \mu\delta\big(\tilde{k} + e_N(k)\big)\|u - u_N\|_2 + \mu\rho e_N(k). \tag{5.7}$$

Now, the result follows directly from (5.7) and assumption (iv). □

6. Numerical Illustrations

The proposed method was implemented on two functional equations of the form (1.2). The global error,

$$\varepsilon_N = \frac{1}{|u|_{\max}} \sqrt{\frac{1}{N+1} \sum_{l=0}^{N} [e_N(x_l)]^2}$$

is computed and reported in Tables 1 and 2 for selected values of N. Here, $u(x)$ denotes the exact solution of the integral equation, $u_N(x)$ is the approximate solution obtained via (4.12) and $|u|_{\max}$ denotes the maximum absolute value of u over the interval $[0, 1]$, with $x_i = i/N$ for $i = 0, 1, \ldots, N$. The results are further compared with those from the Euler and trapezoidal methods presented in [17].

The computations were performed on a personal computer, with the codes were written in Mathematica 12. Our approach utilized Mathematica's *FindRoot[f, x, x0]* function to solve the nonlinear system (4.10), which applies Newton's method by default to find a numerical root of f, starting from the initial guess $x = x_0$. In the example presented, the initial point was chosen as the zero vector. To calculate $|u|_{\max}$ in ε_N, we used Mathematica's *Maximize[f, x]* function to compute the maximum value of f with respect to x. For the given examples, the CPU time required to execute the proposed method was calculated for various numbers of Euler basis functions (denoted as N) and reported in the tables. These results were compared with the CPU times needed to execute the Euler and trapezoidal methods outlined in [17] for the same N values. The *Timing[expr]* command was utilized to measure the CPU times.

Example 6.1. Let us consider the following functional integral equation

$$u(x) = f(x) + g\left(x, \int_0^x \frac{1}{10} e^{x-y} u(y) e^{-u^2(y)} dy\right), \quad x \in [0,1], \tag{6.1}$$

where

$$g(x,y) = \frac{\sin(x)}{20(1+y^2)}, \qquad f(x) = e^{-x} - \frac{\sin(x)}{20 + \frac{1}{20} e^{2x} \left(\frac{1}{e} - e^{-e^{-2x}}\right)^2}.$$

The exact solution of the problem is given by $u(x) = e^{-x}$. Equation (6.1) represents a functional integral equation of the form (1.2) with $k(x,y) = \frac{1}{10} e^{x-y}$ and $H(x,y) = y e^{-y^2}$. It can be easily checked that equation (6.1) satisfies conditions $(\mathbf{C_1})$-$(\mathbf{C_4})$, with the following parameters:

$$\tau(x) \equiv 0, \quad \theta(x) \equiv 1/20, \quad \alpha \geq 1, \quad \beta \geq 1, \quad \|k\|_2 \simeq 0.118,$$
$$\mu \geq \max\left\{\left|\frac{\partial g(x,y)}{\partial y}\right| : x \in [0,1], y \in \mathbb{R}\right\} \simeq 0.027, \quad \delta \geq 1.$$

Therefore, by Lemma 2.2, $u(x) = e^{-x}$ is the only solution to (6.1) in the space $\mathfrak{L}^2([0,1])$. Moreover, it can be shown that conditions $(\mathbf{A_1}) - (\mathbf{A_3})$ in [17, Section 2] hold. So, it follows from [17, Theorem 2.1] that $u(x) = e^{-x}$ is the unique solution of (6.1) within $L^\infty([0,1])$. Additionally, by [17, Theorems 3.3 and 3.4], we find that both the Euler and trapezoidal methods proposed in [17] converge to the exact solution $u(x) = e^{-x}$.

For this example, Table 1 shows the global error ε_N and the CPU time spent for our proposed method and the methods described in [17]. The results clearly demonstrate that the proposed method is computationally more efficient and converges faster to the exact solution $u(x) = e^{-x}$ compared to the methods in [17].

Example 6.2. Consider the following functional integral equation

$$u(x) = x - \frac{1}{100} x \cos(x - \sin(x)) + \frac{1}{100} x \cos\left(\int_0^x \sin(x-y) u(y) dy\right), \tag{6.2}$$
$$x \in [0,1].$$

Equation (6.2) is a functional integral equation of the form (1.2) with

$$f(x) = x - \frac{1}{100} x \cos(x - \sin(x)), \qquad g(x,y) = \frac{1}{100} x \cos(y),$$
$$k(x,y) = \sin(x-y), \qquad H(x,y) = y.$$

The exact solution is $u(x) = x$. It can be easily verified that (6.2) satisfies $(\mathbf{C_1})$-$(\mathbf{C_5})$ with

$$\theta(x) \equiv 1/100, \quad \alpha \geq 1, \quad \tau(x) \equiv 0, \quad \beta \geq 1, \quad \|k\|_2 \simeq 0.3821,$$

$$\mu \geq \max\left\{\left|\frac{\partial g(x,y)}{\partial y}\right| : x \in [0,1], y \in \mathbb{R}\right\} = 0.01, \quad \delta \geq 1.$$

TABLE 1. The global error ε_N and CPU time (in seconds) corresponding to Example 6.1.

Euler's method [17]			Trapezoidal method [17]			Present method		
N	ε_N	CPU time	N	ε_N	CPU time	N	ε_N	CPU time
1	1.55×10^{-4}	0.063	1	1.05×10^{-5}	0.078	1	8.62×10^{-6}	0.078
3	3.39×10^{-5}	0.063	3	1.56×10^{-6}	0.078	3	1.44×10^{-7}	0.453
5	1.76×10^{-5}	0.078	5	5.35×10^{-7}	0.078	5	1.11×10^{-8}	2.734
7	1.16×10^{-5}	0.078	7	2.66×10^{-7}	0.094	7	1.77×10^{-9}	15.47
9	8.66×10^{-6}	0.094	9	1.58×10^{-7}	0.094	9	5.91×10^{-10}	75.80
⋮	⋮	⋮	⋮	⋮	⋮			
20	3.55×10^{-6}	0.141	20	3.08×10^{-8}	0.313			
40	1.70×10^{-6}	0.438	40	7.57×10^{-9}	0.469			
80	8.33×10^{-7}	2.688	80	1.88×10^{-9}	2.781			
125	5.29×10^{-7}	9.266	125	7.66×10^{-10}	9.828			
250	2.63×10^{-7}	101.0	250	1.91×10^{-10}	101.8			
500	1.31×10^{-7}	810.3	500	4.76×10^{-11}	816.9			

So, by Lemma 2.2, $u(x) = x$ is the unique solution of (6.2) in $\mathfrak{L}^2([0,1])$. Similarly, it can be demonstrated that conditions $(\mathbf{C_1})$-$(\mathbf{C_3})$ in [17, Section 2] are satisfied and according to [17, Theorem 2.1], $u(x) = x$ is the unique solution of (6.2) in $L^\infty([0,1])$ and according to [17, Theorems 3.3 and 3.4], the Euler and trapezoidal methods presented in [17] converge to the exact solution $u(x) = x$.

For this example, Table 2 presents the global error ε_N and the CPU time required for our proposed method as well as the methods outlined in [17]. The results clearly illustrate that the proposed method is more computationally efficient and achieves faster convergence to the exact solution $u(x) = x$ compared to the methods in [17].

7. Conclusions

In this research, we studied functional integral equations of the form (1.2). Using a fixed-point approach under specific smoothness conditions, we demonstrated the existence and uniqueness of the solution within the space of square-integrable functions. Following this theoretical analysis, we developed a collocation method to obtain a numerical approximation of the solution. This method employs Euler operational matrices, transforming the original problem (1.2) into a system of nonlinear algebraic equations (4.10). One key advantage of this method over other discretization techniques for solving integral equations is its

ability to bypass the explicit computation of integrals, thereby greatly simplifying the computational process. Instead of directly evaluating integrals, the method utilizes an operational matrix of integration, making it both more efficient and numerically stable.

TABLE 2. The global error ε_N and CPU time (in seconds) corresponding to Example 6.2.

Euler's method [17]			Trapezoidal method [17]			Present method		
N	ε_N	CPU time	N	ε_N	CPU time	N	ε_N	CPU time
1	8.87×10^{-5}	0.047	1	8.87×10^{-5}	0.047	1	5.82×10^{-6}	5.719
3	1.29×10^{-5}	0.047	3	1.29×10^{-5}	0.047	3	6.44×10^{-8}	22.73
5	4.12×10^{-6}	0.063	5	4.12×10^{-6}	0.063	5	1.40×10^{-9}	52.70
7	1.95×10^{-6}	0.063	7	1.95×10^{-6}	0.078	7	1.01×10^{-10}	98.66
9	1.13×10^{-6}	0.063	9	1.13×10^{-6}	0.094	9	2.18×10^{-12}	173.5
⋮	⋮	⋮	⋮	⋮	⋮			
20	2.06×10^{-6}	0.078	20	2.06×10^{-7}	0.078			
40	4.91×10^{-8}	0.156	40	4.91×10^{-8}	0.156			
80	1.20×10^{-8}	0.766	80	1.20×10^{-8}	0.906			
125	4.86×10^{-9}	2.563	125	4.86×10^{-9}	2.906			
250	1.21×10^{-9}	18.67	250	1.21×10^{-9}	23.11			
500	3.00×10^{-10}	206.4	500	3.00×10^{-10}	258.8			

In Section 6, the numerical results obtained with the proposed method are compared to those derived from the Euler and trapezoidal methods described in [17]. For a given N, all these methods require solving a nonlinear system of $N + 1$ algebraic equations with an equal number of unknowns. The numerical results in the tables show that the proposed method is not only highly accurate but also cost-effective for solving (1.2). Furthermore, the results highlight that the proposed approach delivers superior performance and achieves significantly faster convergence to the exact solution.

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