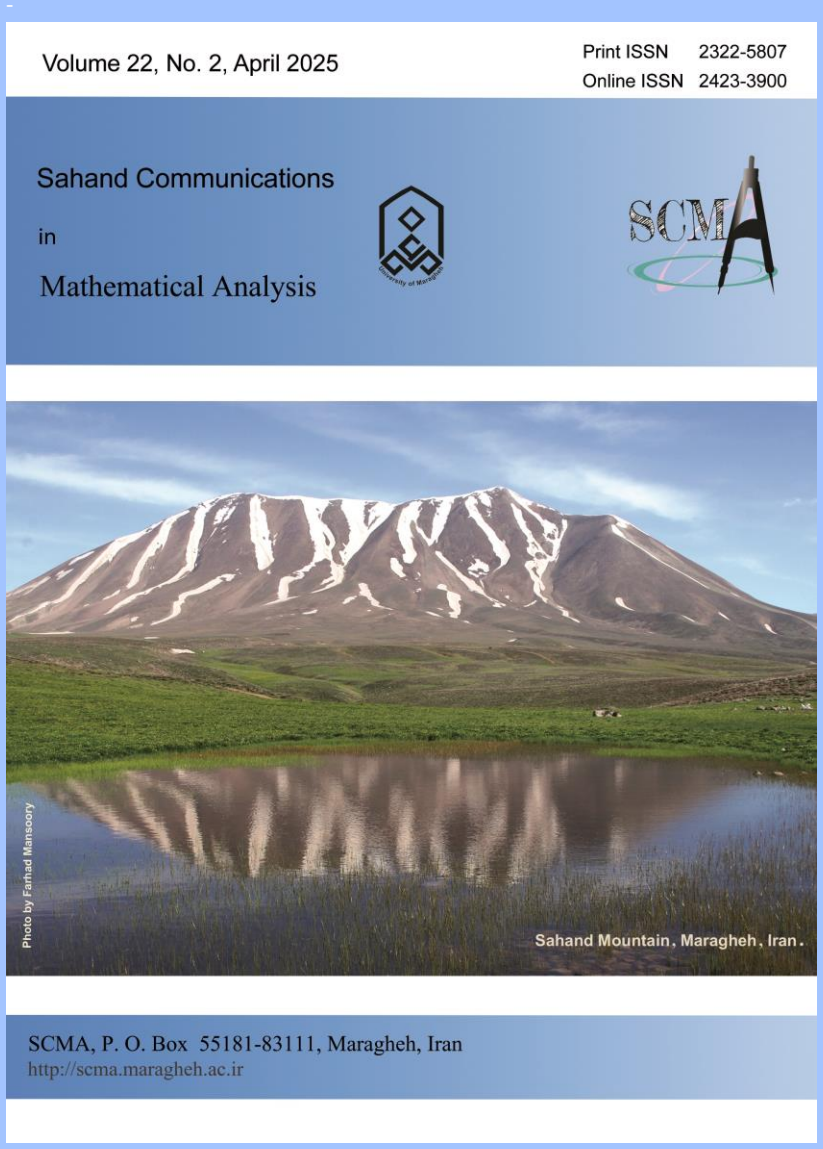
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On Bullen-Type Inequalities for Fractional Integrals with Exponential Kernels

Romaissa Sassane[1], Abdelghani Lakhdari[2*] and Badreddine Meftah[3]

Abstract. In this paper, we investigate the Bullen inequality in the context of fractional integrals with exponential kernels. Building upon the foundational works in the field, we first introduce a new integral identity. From this identity, we derive several novel Bullen-type inequalities for differentiable convex functions. To validate our theoretical findings, we provide a numerical example along with graphical representations, demonstrating the accuracy and applicability of our results. The results obtained are not only new for the fractional integrals considered in our study, but as α approaches 1, we also derive additional novel results for the classical integral.

1. Introduction and Preliminaries

Let $\mathcal{D}$ be an interval of real numbers.

Definition 1.1. A function $\mathfrak{h} : \mathcal{D} \to \mathbb{R}$ is said to be convex, if

$$\mathfrak{h}\left(\mu y_1 + (1-\mu)\, y_2\right) \leq \mu \mathfrak{h}\left(y_1\right) + (1-\mu)\, \mathfrak{h}(y_2)$$

holds for all $y_1, y_2 \in \mathcal{D}$ and all $\mu \in [0,1]$.

The concept of convexity is crucial in both pure and applied mathematics. It is fundamental in optimization, functional analysis, and geometry, providing insights into the behavior of functions and sets. One of the most interesting results related to convexity is the Hermite-Hadamard inequality [6, 8], which bounds the value of a convex function using its values at the endpoints and the midpoint of an interval. This

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inequality can be expressed as follows: For a convex function $\mathfrak{h}$ on $[\sigma, \rho]$, we have

$$\mathfrak{h}\left(\tfrac{\sigma+\rho}{2}\right) \leq \tfrac{1}{\rho-\sigma}\int_{\sigma}^{\rho}\mathfrak{h}(y)dy \leq \tfrac{\mathfrak{h}(\sigma)+(\rho)}{2}. \tag{1.1}$$

In 2010, Tseng et al. [15] provided an improved version of inequality (1.1) as follows:

$$\begin{aligned}\mathfrak{h}\left(\tfrac{\sigma+\rho}{2}\right) &\leq \tfrac{1}{2}\left[\mathfrak{h}\left(\tfrac{3\sigma+\rho}{4}\right)+\mathfrak{h}\left(\tfrac{\sigma+3\rho}{4}\right)\right]\\ &\leq \tfrac{1}{\rho-\sigma}\int_{\sigma}^{\rho}\mathfrak{h}(y)dy\\ &\leq \tfrac{1}{2}\left[\tfrac{\mathfrak{h}(\sigma)+(\rho)}{2}+\mathfrak{h}\left(\tfrac{\sigma+\rho}{2}\right)\right]\\ &\leq \tfrac{\mathfrak{h}(\sigma)+(\rho)}{2}.\end{aligned} \tag{1.2}$$

The third inequality in (1.2) was previously established in [3] and is known as the Bullen inequality; see also [4]:

In [11], Hwang et al. established the following Bullen-type inequality.

$$\left|\tfrac{1}{2}\left(\tfrac{\mathfrak{h}(\sigma)+\mathfrak{h}(\rho)}{2}+\mathfrak{h}\left(\tfrac{\sigma+\rho}{2}\right)\right)-\tfrac{1}{\rho-\sigma}\int_{\sigma}^{\rho}\mathfrak{h}\left(y\right)dy\right| \leq \tfrac{\rho-\sigma}{16}\left(\left|\mathfrak{h}'\left(\sigma\right)\right|+\left|\mathfrak{h}'\left(\rho\right)\right|\right),$$

where $|\mathfrak{h}'|$ is convex on $[\sigma, \rho]$.

The study of Bullen-type inequalities has garnered significant attention in recent years, leading to various advancements in the field. Hamida and Meftah [7] have explored Bullen-type inequalities through the lens of Riemann-Liouville fractional integrals, demonstrating their broad applications. Du et al. [5] have focused on these inequalities using generalized fractional integrals, expanding the scope of their applicability.

Hezenci and Budak [9] investigated Bullen-type inequalities for differentiable convex functions using conformable fractional integrals. Hezenci et al. [10] further contributed by conducting a comprehensive study on the conformable fractional version of Bullen-type inequalities for twice-differentiable convex mappings. Furthermore, Yu et al. [19] have contributed to the field by establishing certain error bounds on Bullen-type integral inequalities within the framework of fractal spaces. For more These collective efforts highlight the diverse approaches and significant advancements in understanding and applying Bullen-type inequalities. For additional works on Bullen-type inequalities, we refer the readers to [14, 17] and the references therein.

Fractional calculus extends traditional calculus to non-integer orders, offering tools to model processes with memory and hereditary properties, common in complex systems. It has become essential in fields like

physics and engineering for its ability to describe anomalous diffusion, viscoelastic materials, and more. Several researchers have shown significant interest in establishing new fractional integral inequalities due to their broad applicability and potential to advance the understanding of fractional calculus in various fields [2, 12, 18, 23].

In [1], Ahmad et al. introduced the fractional integral operators with exponential kernels in the following manner:

Definition 1.2 ([1])**.** Let $\mathfrak{h} \in L[\sigma, \rho]$. The fractional integrals with exponential kernels $\mathcal{I}^{\alpha}_{\sigma^+}\mathfrak{h}(\mu)$ and $\mathcal{I}^{\alpha}_{\rho^-}\mathfrak{h}(\mu)$ of order $\alpha \in (0,1)$ are defined by

$$\mathcal{I}^{\alpha}_{\sigma^+}\mathfrak{h}(\mu) = \frac{1}{\alpha}\int_{\sigma}^{\mu} \exp\left(-\tfrac{1-\alpha}{\alpha}(\mu - y)\right)\mathfrak{h}(y)\,dy, \quad \mu > \sigma$$

and

$$\mathcal{I}^{\alpha}_{\rho^-}\mathfrak{h}(\mu) = \frac{1}{\alpha}\int_{\mu}^{\rho} \exp\left(-\tfrac{1-\alpha}{\alpha}(y - \mu)\right)\mathfrak{h}(y)\,dy, \quad \rho > \mu,$$

respectively.

For the sake of simplicity, we will use the following notation throughout the rest of the article: $A = \frac{1-\alpha}{\alpha}(\sigma - \rho)$.

Remark 1.3. Based on Definition 1.2, the following statements can be derived:

(i) $\lim_{\alpha\to 1}\mathcal{I}^{\alpha}_{\sigma^+}\mathfrak{h}(\varkappa) = \int_{\sigma}^{\varkappa}\mathfrak{h}(y)\,dy$, and $\lim_{\alpha\to 1}\mathcal{I}^{\alpha}_{\rho^-}\mathfrak{h}(\mu) = \int_{\varkappa}^{\rho}\mathfrak{h}(y)\,dy$.
(ii) $\lim_{\alpha\to 0}\mathcal{I}^{\alpha}_{\sigma^+}\mathfrak{h}(\mu) = \lim_{\alpha\to 0}\mathcal{I}^{\alpha}_{\rho^-}\mathfrak{h}(\mu) = \mathfrak{h}(\mu)$.

In the same paper, the authors derived the Hermite-Hadamard inequality and trapezoid-type inequalities using the newly defined fractional integrals.

Theorem 1.4 ([1])**.** *Let $\mathfrak{h} : [\sigma, \rho] \to \mathbb{R}$ be a positive function such that $\mathfrak{h} \in L[\sigma, \rho]$ with $0 \le \sigma < \rho$. If $\mathfrak{h}$ is convex on $[\sigma, \rho]$, then for $\alpha \in (0,1)$, the following inequalities are valid:*

$$\mathfrak{h}\left(\frac{\sigma+\rho}{2}\right) \le \frac{1-\alpha}{2\left(1-\exp\left(-A\right)\right)}\left[\mathcal{I}^{\alpha}_{\sigma^+}\mathfrak{h}(\rho) + \mathcal{I}^{\alpha}_{\rho^-}\mathfrak{h}(\sigma)\right] \le \frac{\mathfrak{h}(\sigma)+\mathfrak{h}(\rho)}{2}.$$

The study of integral inequalities for fractional integrals with exponential kernels has seen significant advancements. The trapezoid-type inequality was provided by Ahmad et al. in [1]. In their publication [16], Wu et al. introduced midpoint-type inequalities, broadening the application of classical inequalities. Using convexity, Li et al. provided some new Hermite-Hadamard-type inequalities in [13]. In [21], the authors examined a variation of the Hermite-Hadamard inequality and developed Hermite-Hadamard-type inequalities for left-sided fractional integrals with exponential kernels, thereby expanding the theoretical

framework. Notably, Yuan et al. provided an in-depth analysis of these operators in [22]. They performed a parametric study on the same operator previously investigated by Ahmad et al. [1] and Wu et al. [16], and their findings included significant results that enabled the determination of error bounds for various quadrature rules. This work has substantially enhanced the understanding and application of these fractional integral operators.

Inspired by the aforementioned works, this paper investigates the Bullen inequality within the framework of fractional integrals with exponential kernels. We start by presenting a new integral identity, from which we derive several new Bullen-type inequalities for differentiable convex functions. The study concludes with a numerical example and graphical representations that confirm the accuracy of our results.

2. Main Results

In order to prove our results, we need the following lemma.

Lemma 2.1. *Let $\mathfrak{h} : \mathcal{D} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\mathcal{D}^\circ$, $\sigma, \rho \in \mathcal{D}^\circ$ with $\sigma < \rho$, and $\mathfrak{h}' \in L^1[\sigma, \rho]$, then the following equality holds*

$$\begin{aligned}
&\frac{\mathfrak{h}(\sigma)+\mathfrak{h}(\rho)+2\mathfrak{h}\left(\frac{\sigma+\rho}{2}\right)}{4} - \frac{1-\alpha}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left(\mathcal{I}^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)^-}\mathfrak{h}(\sigma) + \mathcal{I}^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)^+}\mathfrak{h}(\rho)\right) \\
&\quad = \frac{\rho-\sigma}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)} \\
&\qquad \times \left(\int_0^{\frac{1}{2}} \left(\frac{1+\exp\left(-\frac{1}{2}A\right)}{2} - \exp(-A\mu)\right)\mathfrak{h}'((1-\mu)\sigma+\mu\rho)\,d\mu\right. \\
&\qquad \left. - \int_{\frac{1}{2}}^1 \left(\frac{1+\exp\left(-\frac{1}{2}A\right)}{2} - \exp(-A(1-\mu))\right)\mathfrak{h}'((1-\mu)\sigma+\mu\rho)\,d\mu\right),
\end{aligned}$$

where $A = \frac{1-\alpha}{\alpha}(\rho-\sigma)$.

Proof. Let

$$\mathcal{H}_1 = \int_0^{\frac{1}{2}} \left(\tfrac{1}{2} - \exp(-A\mu) + \tfrac{1}{2}\exp\left(-\tfrac{1}{2}A\right)\right)\mathfrak{h}'((1-\mu)\sigma+\mu\rho)\,d\mu$$

and

$$\mathcal{H}_2 = \int_{\frac{1}{2}}^1 \left(\tfrac{1}{2} - \exp(-A(1-\mu)) + \tfrac{1}{2}\exp\left(-\tfrac{1}{2}A\right)\right)\mathfrak{h}'((1-\mu)\sigma+\mu\rho)\,d\mu.$$

Integrating by parts $\mathcal{H}_1$, we get

(2.1)

$$\mathcal{H}_1 = \int_0^{\frac{1}{2}} \left(\tfrac{1}{2} - \exp(-A\mu) + \tfrac{1}{2}\exp\left(-\tfrac{1}{2}A\right)\right)\mathfrak{h}'((1-\mu)\sigma+\mu\rho)\,d\mu$$

$$
\begin{aligned}
&= \tfrac{1}{\rho-\sigma}\left(\tfrac{1}{2}-\exp\left(-A\mu\right)+\tfrac{1}{2}\exp\left(-\tfrac{1}{2}A\right)\right)\mathfrak{h}\left(\left(1-\mu\right)\sigma+\mu\rho\right)\Big|_0^{\frac{1}{2}} \\
&\quad -\tfrac{A}{\rho-\sigma}\int_0^{\frac{1}{2}}\left(\exp\left(-A\mu\right)\right)\mathfrak{h}\left(\left(1-\mu\right)\sigma+\mu\rho\right)d\mu \\
&= \tfrac{1}{2(\rho-\sigma)}\left(1-\exp\left(-\tfrac{1}{2}A\right)\right)\left(\mathfrak{h}\left(\tfrac{\sigma+\rho}{2}\right)+\mathfrak{h}\left(\sigma\right)\right) \\
&\quad -\tfrac{1-\alpha}{\alpha(\rho-\sigma)}\int_\sigma^{\frac{\sigma+\rho}{2}}\exp\left(-\tfrac{1-\alpha}{\alpha}\left(\nu-\sigma\right)\right)\mathfrak{h}\left(\nu\right)d\nu \\
&= \tfrac{1}{2(\rho-\sigma)}\left(1-\exp\left(-\tfrac{1}{2}A\right)\right)\left(\mathfrak{h}\left(\tfrac{\sigma+\rho}{2}\right)+\mathfrak{h}\left(\sigma\right)\right)-\tfrac{1-\alpha}{\rho-\sigma}\mathcal{I}^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)^-}\mathfrak{h}\left(\sigma\right).
\end{aligned}
$$

Similarly, we get

(2.2)

$$
\begin{aligned}
\mathcal{H}_2 &= \int_{\frac{1}{2}}^1\left(\tfrac{1}{2}-\exp\left(-A\left(1-\mu\right)\right)+\tfrac{1}{2}\exp\left(-\tfrac{1}{2}A\right)\right)\mathfrak{h}'\left(\left(1-\mu\right)\sigma+\mu\rho\right)d\mu \\
&= \tfrac{1}{\rho-\sigma}\left(\tfrac{1}{2}-\exp\left(-A\left(1-\mu\right)\right)+\tfrac{1}{2}\exp\left(-\tfrac{1}{2}A\right)\right)\mathfrak{h}\left(\left(1-\mu\right)\sigma+\mu\rho\right)\Big|_{\frac{1}{2}}^1 \\
&\quad +\tfrac{A}{\rho-\sigma}\int_{\frac{1}{2}}^1\left(\exp\left(-A\left(1-\mu\right)\right)\right)\mathfrak{h}\left(\left(1-\mu\right)\sigma+\mu\rho\right)d\mu \\
&= -\tfrac{1}{2(\rho-\sigma)}\left(1-\exp\left(-\tfrac{1}{2}A\right)\right)\left(\mathfrak{h}\left(\rho\right)+\mathfrak{h}\left(\tfrac{\sigma+\rho}{2}\right)\right) \\
&\quad +\tfrac{1-\alpha}{\alpha(\rho-\sigma)}\int_{\frac{\sigma+\rho}{2}}^\rho\exp\left(-\frac{1-\alpha}{\alpha}\left(\rho-\nu\right)\right)\mathfrak{h}\left(\nu\right)d\nu \\
&= -\tfrac{1}{2(\rho-\sigma)}\left(1-\exp\left(-\tfrac{1}{2}A\right)\right)\left(\mathfrak{h}\left(\rho\right)+\mathfrak{h}\left(\tfrac{\sigma+\rho}{2}\right)\right)+\tfrac{1-\alpha}{\rho-\sigma}\mathcal{I}^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)^+}\mathfrak{h}\left(\rho\right).
\end{aligned}
$$

By taking the difference between equations (2.1) and (2.2), and then multiplying the resulting expression by $\frac{\rho-\sigma}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}$, we get the desired result. □

Theorem 2.2. *Let $\mathfrak{h}:[\sigma,\rho]\to\mathbb{R}$ be a differentiable function on $[\sigma,\rho]$ such that $\mathfrak{h}'\in L^1[\sigma,\rho]$ with $0\le\sigma<\rho$. If $|\mathfrak{h}'|$ is convex, then we have*

$$
\begin{aligned}
&\left|\tfrac{\mathfrak{h}(\sigma)+\mathfrak{h}(\rho)+2\mathfrak{h}\left(\frac{\sigma+\rho}{2}\right)}{4}-\tfrac{1-\alpha}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left(I^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)^-}\mathfrak{h}\left(\sigma\right)+I^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)^+}\mathfrak{h}\left(\rho\right)\right)\right| \\
&\quad\le\tfrac{\rho-\sigma}{4\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\Omega_1\left(A\right)\left(\left|\mathfrak{h}'\left(\sigma\right)\right|+\left|\mathfrak{h}'\left(\rho\right)\right|\right)+2\Omega_2\left(A\right)\left|\mathfrak{h}'\left(\tfrac{\sigma+\rho}{2}\right)\right|\right],
\end{aligned}
$$

where $A=\frac{1-\alpha}{\alpha}\left(\rho-\sigma\right)$, and $\Omega_1\left(A\right)$ and $\Omega_2\left(A\right)$ are defined as in (2.3) *and* (2.4)*, respectively.*

Proof. From Lemma 2.1, properties of modulus, and the convexity of $|\mathfrak{h}'|$, we have

$$
\begin{aligned}
&\left|\frac{\mathfrak{h}(\sigma)+\mathfrak{h}(\rho)+2\mathfrak{h}\left(\frac{\sigma+\rho}{2}\right)}{4}-\frac{1-\alpha}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left(I^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)^{-}}\mathfrak{h}(\sigma)+I^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)^{+}}\mathfrak{h}(\rho)\right)\right| \\
&\quad\leq \frac{\rho-\sigma}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)} \\
&\qquad\times\left[\int_0^{\frac{1}{2}}\left|\frac{1+\exp\left(-\frac{1}{2}A\right)}{2}-\exp(-A\mu)\right|\left|\mathfrak{h}'\left((1-\mu)\sigma+\mu\rho\right)\right|d\mu\right. \\
&\qquad\left.+\int_{\frac{1}{2}}^1\left|\frac{1+\exp\left(-\frac{1}{2}A\right)}{2}-\exp(-A(1-\mu))\right|\left|\mathfrak{h}'\left((1-\mu)\sigma+\mu\rho\right)\right|d\mu\right] \\
&\quad= \frac{\rho-\sigma}{4\left(1-\exp\left(-\frac{1}{2}A\right)\right)} \\
&\qquad\times\left[\int_0^1\left|\frac{1+\exp\left(-\frac{1}{2}A\right)}{2}-\exp\left(-\frac{\mu}{2}A\right)\right|\left|\mathfrak{h}'\left((1-\mu)\sigma+\mu\frac{\sigma+\rho}{2}\right)\right|d\mu\right. \\
&\qquad\left.+\int_0^1\left|\frac{1+\exp\left(-\frac{1}{2}A\right)}{2}-\exp\left(-\frac{\mu}{2}A\right)\right|\left|\mathfrak{h}'\left(\mu\frac{\sigma+\rho}{2}+(1-\mu)\rho\right)\right|d\mu\right] \\
&\quad\leq \frac{\rho-\sigma}{4\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\left(\int_0^1\left|\frac{1}{2}-\exp\left(-\frac{\mu}{2}A\right)+\frac{1}{2}\exp\left(-\frac{1}{2}A\right)\right|\right.\right. \\
&\qquad\left.\times\left((1-\mu)\left|\mathfrak{h}'(\sigma)\right|+\mu\left|\mathfrak{h}'\left(\frac{\sigma+\rho}{2}\right)\right|\right)d\mu\right) \\
&\qquad+\left(\int_0^1\left|\frac{1}{2}-\exp\left(-\frac{\mu}{2}A\right)+\frac{1}{2}\exp\left(-\frac{1}{2}A\right)\right|\right. \\
&\qquad\left.\left.\times\left(\mu\left|\mathfrak{h}'\left(\frac{\sigma+\rho}{2}\right)\right|+(1-\mu)\left|\mathfrak{h}'(\rho)\right|\right)d\mu\right)\right] \\
&\quad= \frac{\rho-\sigma}{4\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\left(\int_0^1\left|\frac{1}{2}-\exp\left(-\frac{\mu}{2}A\right)+\frac{1}{2}\exp\left(-\frac{1}{2}A\right)\right|(1-\mu)\,d\mu\right)\right. \\
&\qquad\times\left(\left|\mathfrak{h}'(\sigma)\right|+\left|\mathfrak{h}'(\rho)\right|\right) \\
&\qquad\left.+2\left(\int_0^1\left|\frac{1}{2}-\exp\left(-\frac{\mu}{2}A\right)+\frac{1}{2}\exp\left(-\frac{1}{2}A\right)\right|\mu d\mu\right)\left|\mathfrak{h}'\left(\frac{\sigma+\rho}{2}\right)\right|\right] \\
&\quad= \frac{\rho-\sigma}{4\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\Omega_1(A)\left(\left|\mathfrak{h}'(\sigma)\right|+\left|\mathfrak{h}'(\rho)\right|\right)+2\Omega_2(A)\left|\mathfrak{h}'\left(\frac{\sigma+\rho}{2}\right)\right|\right],
\end{aligned}
$$

where we have used

(2.3)

$$
\Omega_1(A)=\int_0^1\left|\frac{1}{2}-\exp\left(-\frac{\mu}{2}A\right)+\frac{1}{2}\exp\left(-\frac{1}{2}A\right)\right|(1-\mu)\,d\mu
$$

$$= 2\left[\frac{A+(A-8)\exp\left(-\frac{1}{2}A\right)}{8A} + \frac{(A-2)\left(\exp\left(-\frac{1}{2}A\right)\right)}{A^2}\ln\left(\frac{1}{2}+\frac{1}{2}\exp\left(-\frac{1}{2}A\right)\right)\right.$$
$$\left. + \frac{1+\exp\left(-\frac{1}{2}A\right)}{A^2}\left(\ln\left(\frac{1}{2}+\frac{1}{2}\exp\left(-\frac{1}{2}A\right)\right)\right)^2\right]$$

and

(2.4)

$$\Omega_2(A) = \int_0^1 \left|\frac{1}{2} - \exp\left(-\frac{\mu}{2}A\right) + \frac{1}{2}\exp\left(-\frac{1}{2}A\right)\right| \mu d\mu$$
$$= 4\left[\frac{A+(A+8)\exp\left(-\frac{1}{2}A\right)}{16A} + \frac{1+\exp\left(-\frac{1}{2}A\right)}{A^2}\ln\left(\frac{1}{2}+\frac{1}{2}\exp\left(-\frac{1}{2}A\right)\right)\right.$$
$$\left. - \frac{1+\exp\left(-\frac{1}{2}A\right)}{2A^2}\left(\ln\left(\frac{1}{2}+\frac{1}{2}\exp\left(-\frac{1}{2}A\right)\right)\right)^2\right].$$

The proof is completed. □

Corollary 2.3. *In Theorem 2.2, using the convexity of* $|\mathfrak{h}'|$*, we get*

$$\left|\frac{\mathfrak{h}(\sigma)+\mathfrak{h}(\rho)+2\mathfrak{h}\left(\frac{\sigma+\rho}{2}\right)}{4} - \frac{1-\alpha}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left(I^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)^-}\mathfrak{h}(\sigma) + I^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)^+}\mathfrak{h}(\rho)\right)\right|$$
$$\leq \frac{\rho-\sigma}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left(\frac{1+\exp\left(-\frac{1}{2}A\right)}{4} + \frac{1+\exp\left(-\frac{1}{2}A\right)}{A}\ln\left(\frac{1+\exp\left(-\frac{1}{2}A\right)}{2}\right)\right)$$
$$\times\left(\left|\mathfrak{h}'(\sigma)\right| + \left|\mathfrak{h}'(\rho)\right|\right).$$

Corollary 2.4. *In Theorem 2.2, as* α *aproaches* 1*, then we obtain*

$$\left|\frac{\mathfrak{h}(\sigma)+\mathfrak{h}(\rho)+2\mathfrak{h}\left(\frac{\sigma+\rho}{2}\right)}{4} - \frac{1}{\rho-\sigma}\int_\sigma^\rho \mathfrak{h}(y)\,dy\right|$$
$$\leq \frac{\rho-\sigma}{32}\left(\left|\mathfrak{h}'(\sigma)\right| + 2\left|\mathfrak{h}'\left(\frac{\sigma+\rho}{2}\right)\right| + \left|\mathfrak{h}'(\rho)\right|\right).$$

Corollary 2.5. *In Corollary 2.3, as* α *aproaches* 1*, then we obtain*

$$\left|\frac{\mathfrak{h}(\sigma)+\mathfrak{h}(\rho)+2\mathfrak{h}\left(\frac{\sigma+\rho}{2}\right)}{4} - \frac{1}{\rho-\sigma}\int_\sigma^\rho \mathfrak{h}(y)\,dy\right| \leq \frac{\rho-\sigma}{16}\left(\left|\mathfrak{h}'(\sigma)\right| + \left|\mathfrak{h}'(\rho)\right|\right),$$

which was offered by Hwang et al. in [11].

Theorem 2.6. *Let* $\mathfrak{h}:[\sigma,\rho]\to\mathbb{R}$ *be a differentiable function on* (σ,ρ) *such that* $\mathfrak{h}'\in L^1[\sigma,\rho]$ *with* $0\leq\sigma<\rho$*. If* $|\mathfrak{h}'|^q$ *is convex where* $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$*, then we have*

$$\left|\frac{\mathfrak{h}(\sigma)+\mathfrak{h}(\rho)+2\mathfrak{h}\left(\frac{\sigma+\rho}{2}\right)}{4} - \frac{1-\alpha}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left(I^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)^-}\mathfrak{h}(\sigma) + I^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)^+}\mathfrak{h}(\rho)\right)\right|$$
$$\leq \frac{\rho-\sigma}{4\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left(\int_0^1 \left|\frac{1}{2} - \exp\left(-\frac{\mu}{2}A\right) + \frac{1}{2}\exp\left(-\frac{1}{2}A\right)\right|^p d\mu\right)^{\frac{1}{p}}$$

$$\times \left[\left(\frac{|\mathfrak{h}'(\sigma)|^q + \left|\mathfrak{h}'\left(\frac{\sigma+\rho}{2}\right)\right|^q}{2} \right)^{\frac{1}{q}} + \left(\frac{\left|\mathfrak{h}'\left(\frac{\sigma+\rho}{2}\right)\right|^q + |\mathfrak{h}'(\rho)|^q}{2} \right)^{\frac{1}{q}} \right],$$

where $A = \frac{1-\alpha}{\alpha}(\rho - \sigma)$.

Proof. From Lemma 2.1, properties of modulus, Hölder's inequality and the convexity of $|\mathfrak{h}'|^q$, we have

$$\begin{aligned}
&\left| \frac{\mathfrak{h}(\sigma)+\mathfrak{h}(\rho)+2\mathfrak{h}\left(\frac{\sigma+\rho}{2}\right)}{4} - \frac{1-\alpha}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)} \left(I^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)^-}\mathfrak{h}(\sigma) + I^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)^+}\mathfrak{h}(\rho) \right) \right| \\
&\leq \frac{\rho-\sigma}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)} \left[\left(\int_0^{\frac{1}{2}} \left| \tfrac{1}{2} - \exp(-A\mu) + \tfrac{1}{2}\exp\left(-\tfrac{1}{2}A\right) \right|^p d\mu \right)^{\frac{1}{p}} \right. \\
&\quad \times \left(\int_0^{\frac{1}{2}} \left|\mathfrak{h}'\left((1-\mu)\sigma + \mu\rho\right)\right|^q d\mu \right)^{\frac{1}{q}} \\
&\quad + \left(\int_{\frac{1}{2}}^1 \left| \tfrac{1}{2} - \exp(-A(1-\mu)) + \tfrac{1}{2}\exp\left(-\tfrac{1}{2}A\right) \right|^p d\mu \right)^{\frac{1}{p}} \\
&\quad \left. \times \left(\int_{\frac{1}{2}}^1 \left|\mathfrak{h}'\left((1-\mu)\sigma + \mu\rho\right)\right|^q d\mu \right)^{\frac{1}{q}} \right] \\
&= \frac{\rho-\sigma}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)} \left[\left(\frac{1}{2}\int_0^1 \left| \tfrac{1}{2} - \exp\left(-\tfrac{\mu}{2}A\right) + \tfrac{1}{2}\exp\left(-\tfrac{1}{2}A\right) \right|^p d\mu \right)^{\frac{1}{p}} \right. \\
&\quad \times \left(\frac{1}{2}\int_0^1 \left|\mathfrak{h}'\left((1-\mu)\sigma + \mu\tfrac{\sigma+\rho}{2}\right)\right|^q d\mu \right)^{\frac{1}{q}} \\
&\quad + \left(\frac{1}{2}\int_0^1 \left| \tfrac{1}{2} - \exp\left(-\tfrac{\mu}{2}A\right) + \tfrac{1}{2}\exp\left(-\tfrac{1}{2}A\right) \right|^p d\mu \right)^{\frac{1}{p}} \\
&\quad \left. \times \left(\frac{1}{2}\int_0^1 \left|\mathfrak{h}'\left(\mu\tfrac{\sigma+\rho}{2} + (1-\mu)\rho\right)\right|^q d\mu \right)^{\frac{1}{q}} \right] \\
&\leq \frac{\rho-\sigma}{4\left(1-\exp\left(-\frac{1}{2}A\right)\right)} \left(\int_0^1 \left| \tfrac{1}{2} - \exp\left(-\tfrac{\mu}{2}A\right) + \tfrac{1}{2}\exp\left(-\tfrac{1}{2}A\right) \right|^p d\mu \right)^{\frac{1}{p}} \\
&\quad \times \left[\left(\int_0^1 \left((1-\mu)\left|\mathfrak{h}'(\sigma)\right|^q + \mu\left|\mathfrak{h}'\left(\tfrac{\sigma+\rho}{2}\right)\right|^q \right) d\mu \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(\int_0^1 \left(\mu\left|\mathfrak{h}'\left(\tfrac{\sigma+\rho}{2}\right)\right|^q + (1-\mu)\left|\mathfrak{h}'(\rho)\right|^q \right) d\mu \right)^{\frac{1}{q}} \right]
\end{aligned}$$

$$= \frac{\rho-\sigma}{4\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left(\int_0^1 \left|\frac{1}{2}-\exp\left(-\frac{\mu}{2}A\right)+\frac{1}{2}\exp\left(-\frac{1}{2}A\right)\right|^p d\mu\right)^{\frac{1}{p}}$$
$$\times\left[\left(\frac{|\mathfrak{h}'(\sigma)|^q+\left|\mathfrak{h}'\left(\frac{\sigma+\rho}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathfrak{h}'\left(\frac{\sigma+\rho}{2}\right)\right|^q+|\mathfrak{h}'(\rho)|^q}{2}\right)^{\frac{1}{q}}\right].$$

The proof is completed. □

Corollary 2.7. *In Theorem 2.6, using the convexity of* $|\mathfrak{h}'|^q$*, we get*

$$\left|\frac{\mathfrak{h}(\sigma)+\mathfrak{h}(\rho)+2\mathfrak{h}\left(\frac{\sigma+\rho}{2}\right)}{4}-\frac{1-\alpha}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left(I^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)^-}\mathfrak{h}(\sigma)+I^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)^+}\mathfrak{h}(\rho)\right)\right|$$
$$\leq \frac{\rho-\sigma}{4\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left(\int_0^1 \left|\frac{1}{2}-\exp\left(-\frac{\mu}{2}A\right)+\frac{1}{2}\exp\left(-\frac{1}{2}A\right)\right|^p d\mu\right)^{\frac{1}{p}}$$
$$\times\left[\left(\frac{3|\mathfrak{h}'(\sigma)|^q+|\mathfrak{h}'(\rho)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\mathfrak{h}'(\sigma)|^q+3|\mathfrak{h}'(\rho)|^q}{4}\right)^{\frac{1}{q}}\right].$$

Corollary 2.8. *In Theorem 2.6, as* α *aproaches* 1*, then we obtain*

$$\left|\frac{\mathfrak{h}(\sigma)+\mathfrak{h}(\rho)+2\mathfrak{h}\left(\frac{\sigma+\rho}{2}\right)}{4}-\frac{1}{\rho-\sigma}\int_\sigma^\rho \mathfrak{h}(y)\,dy\right|$$
$$\leq \frac{\rho-\sigma}{8}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|\mathfrak{h}'(\sigma)|^q+\left|\mathfrak{h}'\left(\frac{\sigma+\rho}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathfrak{h}'\left(\frac{\sigma+\rho}{2}\right)\right|^q+|\mathfrak{h}'(\rho)|^q}{2}\right)^{\frac{1}{q}}\right],$$

where we have used the fact that

$$\lim_{\alpha\to 1}\frac{1-2\exp\left(-\frac{\mu}{2}A\right)+\exp\left(-\frac{1}{2}A\right)}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}=\mu-\frac{1}{2}.$$

Corollary 2.9. *In Corollary 2.7, as* α *aproaches* 1*, then we obtain*

$$\left|\frac{\mathfrak{h}(\sigma)+\mathfrak{h}(\rho)+2\mathfrak{h}\left(\frac{\sigma+\rho}{2}\right)}{4}-\frac{1}{\rho-\sigma}\int_\sigma^\rho \mathfrak{h}(y)\,dy\right|$$
$$\leq \frac{\rho-\sigma}{8}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{3|\mathfrak{h}'(\sigma)|^q+|\mathfrak{h}'(\rho)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\mathfrak{h}'(\sigma)|^q+3|\mathfrak{h}'(\rho)|^q}{4}\right)^{\frac{1}{q}}\right],$$

which is the same result provided by Hamida and Meftah in Corollary 3.4 from [7].

Theorem 2.10. *Let* $\mathfrak{h}:[\sigma,\rho]\to\mathbb{R}$ *be a differentiable function on* $[\sigma,\rho]$ *such that* $\mathfrak{h}'\in L^1[\sigma,\rho]$ *with* $0\leq\sigma<\rho$*. If* $|\mathfrak{h}'|^q$ *is convex where* $q\geq 1$*, then we have*

$$\left|\frac{\mathfrak{h}(\sigma)+\mathfrak{h}(\rho)+2\mathfrak{h}\left(\frac{\sigma+\rho}{2}\right)}{4}-\frac{1-\alpha}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left(I^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)^-}\mathfrak{h}(\sigma)+I^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)^+}\mathfrak{h}(\rho)\right)\right|$$
$$\leq \frac{\rho-\sigma}{4\left(1-\exp\left(-\frac{1}{2}A\right)\right)}$$

$$\times \left(\frac{1+\exp\left(-\frac{1}{2}A\right)}{2} + \frac{2+2\exp\left(-\frac{1}{2}A\right)}{A} \ln\left(\frac{1}{2} + \frac{1}{2}\exp\left(-\frac{1}{2}A\right)\right) \right)^{1-\frac{1}{q}}$$
$$\times \left[\left(\Omega_1(A) \left|\mathfrak{h}'(\sigma)\right|^q + \Omega_2(A) \left|\mathfrak{h}'\left(\tfrac{\sigma+\rho}{2}\right)\right|^q \right)^{\frac{1}{q}} \right.$$
$$\left. + \left(\Omega_2(A) \left|\mathfrak{h}'\left(\tfrac{\sigma+\rho}{2}\right)\right|^q + \Omega_1(A) \left|\mathfrak{h}'(\rho)\right|^q \right)^{\frac{1}{q}} \right],$$

where $\Omega_1(A)$ *and* $\Omega_2(A)$ *are defined as in* (2.3) *and* (2.4)*, respectively.*

Proof. From Lemma 2.1, properties of modulus, power mean inequality and the convexity of $|\mathfrak{h}'|^q$, we have

$$\left| \frac{\mathfrak{h}(\sigma)+\mathfrak{h}(\rho)+2\mathfrak{h}\left(\frac{\sigma+\rho}{2}\right)}{4} - \frac{1-\alpha}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)} \left(I^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)^-}\mathfrak{h}(\sigma) + I^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)^+}\mathfrak{h}(\rho) \right) \right|$$
$$\leq \frac{\rho-\sigma}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)} \left[\left(\int_0^{\frac{1}{2}} \left| \frac{1+\exp\left(-\frac{1}{2}A\right)}{2} - \exp(-A\mu) \right| d\mu \right)^{1-\frac{1}{q}} \right.$$
$$\times \left(\int_0^{\frac{1}{2}} \left| \frac{1+\exp\left(-\frac{1}{2}A\right)}{2} - \exp(-A\mu) \right| \left|\mathfrak{h}'((1-\mu)\sigma+\mu\rho)\right|^q d\mu \right)^{\frac{1}{q}}$$
$$+ \left(\int_{\frac{1}{2}}^1 \left| \frac{1+\exp\left(-\frac{1}{2}A\right)}{2} - \exp(-A(1-\mu)) \right| d\mu \right)^{1-\frac{1}{q}}$$
$$\times \left(\int_{\frac{1}{2}}^1 \left| \frac{1+\exp\left(-\frac{1}{2}A\right)}{2} - \exp(-A(1-\mu)) \right| \right.$$
$$\left. \left. \times \left|\mathfrak{h}'((1-\mu)\sigma+\mu\rho)\right|^q d\mu \right)^{\frac{1}{q}} \right]$$
$$= \frac{\rho-\sigma}{4\left(1-\exp\left(-\frac{1}{2}A\right)\right)} \left[\left(\int_0^1 \left| \frac{1}{2} - \exp\left(-\frac{\mu}{2}A\right) + \frac{1}{2}\exp\left(-\frac{1}{2}A\right) \right| d\mu \right)^{1-\frac{1}{q}} \right.$$
$$\times \left(\int_0^1 \left| \frac{1+\exp\left(-\frac{1}{2}A\right)}{2} - \exp\left(-\frac{\mu}{2}A\right) \right| \left|\mathfrak{h}'\left((1-\mu)\sigma+\mu\tfrac{\sigma+\rho}{2}\right)\right|^q d\mu \right)^{\frac{1}{q}}$$
$$+ \left(\int_0^1 \left| \frac{1+\exp\left(-\frac{1}{2}A\right)}{2} - \exp\left(-\frac{\mu}{2}A\right) \right| d\mu \right)^{1-\frac{1}{q}}$$
$$\left. \times \left(\int_0^1 \left| \frac{1+\exp\left(-\frac{1}{2}A\right)}{2} - \exp\left(-\frac{\mu}{2}A\right) \right| \left|\mathfrak{h}'\left(\mu\tfrac{\sigma+\rho}{2}+(1-\mu)\rho\right)\right|^q d\mu \right)^{\frac{1}{q}} \right]$$
$$\leq \frac{\rho-\sigma}{4\left(1-\exp\left(-\frac{1}{2}A\right)\right)} \left(\int_0^1 \left| \frac{1}{2} - \exp\left(-\frac{\mu}{2}A\right) + \frac{1}{2}\exp\left(-\frac{1}{2}A\right) \right| d\mu \right)^{1-\frac{1}{q}}$$

$$
\begin{aligned}
&\times\left[\left(\int_0^1\left|\tfrac{1}{2}-\exp\left(-\tfrac{\mu}{2}A\right)+\tfrac{1}{2}\exp\left(-\tfrac{1}{2}A\right)\right|\right.\right.\\
&\times\left.\left.\left((1-\mu)\left|\mathfrak{h}'(\sigma)\right|^q+\mu\left|\mathfrak{h}'\left(\tfrac{\sigma+\rho}{2}\right)\right|^q\right)d\mu\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\left|\tfrac{1}{2}-\exp\left(-\tfrac{\mu}{2}A\right)+\tfrac{1}{2}\exp\left(-\tfrac{1}{2}A\right)\right|\right.\\
&\times\left.\left(\mu\left|\mathfrak{h}'\left(\tfrac{\sigma+\rho}{2}\right)\right|^q+(1-\mu)\left|\mathfrak{h}'(\rho)\right|^q\right)d\mu\right)^{\frac{1}{q}}\Bigg]\\
=&\ \tfrac{\rho-\sigma}{4\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left(\int_0^1\left|\tfrac{1}{2}-\exp\left(-\tfrac{\mu}{2}A\right)+\tfrac{1}{2}\exp\left(-\tfrac{1}{2}A\right)\right|d\mu\right)^{1-\frac{1}{q}}\\
&\times\left[\left(\left(\int_0^1\left|\tfrac{1+\exp\left(-\frac{1}{2}A\right)}{2}-\exp\left(-\tfrac{\mu}{2}A\right)\right|(1-\mu)\,d\mu\right)\left|\mathfrak{h}'(\sigma)\right|^q\right.\right.\\
&+\left.\left(\int_0^1\left|\tfrac{1+\exp\left(-\frac{1}{2}A\right)}{2}-\exp\left(-\tfrac{\mu}{2}A\right)\right|\mu d\mu\right)\left|\mathfrak{h}'\left(\tfrac{\sigma+\rho}{2}\right)\right|^q\right)^{\frac{1}{q}}\\
&+\left(\left(\int_0^1\left|\tfrac{1+\exp\left(-\frac{1}{2}A\right)}{2}-\exp\left(-\tfrac{\mu}{2}A\right)\right|\mu d\mu\right)\left|\mathfrak{h}'\left(\tfrac{\sigma+\rho}{2}\right)\right|^q\right.\\
&+\left.\left.\left(\int_0^1\left|\tfrac{1+\exp\left(-\frac{1}{2}A\right)}{2}-\exp\left(-\tfrac{\mu}{2}A\right)\right|(1-\mu)\,d\mu\right)\left|\mathfrak{h}'(\rho)\right|^q\right)^{\frac{1}{q}}\right]\\
=&\ \tfrac{\rho-\sigma}{4\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\\
&\times\left(\tfrac{1+\exp\left(-\frac{1}{2}A\right)}{2}+\tfrac{2+2\exp\left(-\frac{1}{2}A\right)}{A}\ln\left(\tfrac{1}{2}+\tfrac{1}{2}\exp\left(-\tfrac{1}{2}A\right)\right)\right)^{1-\frac{1}{q}}\\
&\times\left[\left(\Omega_1(A)\left|\mathfrak{h}'(\sigma)\right|^q+\Omega_2(A)\left|\mathfrak{h}'\left(\tfrac{\sigma+\rho}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.\\
&+\left.\left(\Omega_2(A)\left|\mathfrak{h}'\left(\tfrac{\sigma+\rho}{2}\right)\right|^q+\Omega_1(A)\left|\mathfrak{h}'(\rho)\right|^q\right)^{\frac{1}{q}}\right],
\end{aligned}
$$

where we have used (2.3) and (2.4). The proof is completed. □

Corollary 2.11. *In Theorem 2.10, using the convexity of $|\mathfrak{h}'|^q$, we get*

$$
\begin{aligned}
&\left|\tfrac{\mathfrak{h}(\sigma)+\mathfrak{h}(\rho)+2\mathfrak{h}\left(\frac{\sigma+\rho}{2}\right)}{4}-\tfrac{1-\alpha}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left(I^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)^-}\mathfrak{h}(\sigma)+I^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)^+}\mathfrak{h}(\rho)\right)\right|\\
&\quad\leq\tfrac{\rho-\sigma}{4\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\\
&\qquad\times\left(\tfrac{1+\exp\left(-\frac{1}{2}A\right)}{2}+\tfrac{2+2\exp\left(-\frac{1}{2}A\right)}{A}\ln\left(\tfrac{1}{2}+\tfrac{1}{2}\exp\left(-\tfrac{1}{2}A\right)\right)\right)^{1-\frac{1}{q}}
\end{aligned}
$$

$$\times \left[\left(\Omega_3(A) \left| \mathfrak{h}'(\sigma) \right|^q + \frac{\Omega_2(A)}{2} \left| \mathfrak{h}'(\rho) \right|^q \right)^{\frac{1}{q}} + \left(\frac{\Omega_2(A)}{2} \left| \mathfrak{h}'(\sigma) \right|^q + \Omega_3(A) \left| \mathfrak{h}'(\rho) \right|^q \right)^{\frac{1}{q}} \right],$$

where $\Omega_3(A) = \Omega_1(A) + \frac{1}{2}\Omega_2(A)$.

Corollary 2.12. *In Theorem 2.10, as* α *aproaches* 1*, then we obtain*

$$\left| \frac{\mathfrak{h}(\sigma)+\mathfrak{h}(\rho)+2\mathfrak{h}\left(\frac{\sigma+\rho}{2}\right)}{4} - \frac{1}{\rho-\sigma}\int_\sigma^\rho \mathfrak{h}(y)\,dy \right| \leq \frac{\rho-\sigma}{16}\left[\left(\frac{|\mathfrak{h}'(\sigma)|^q + \left|\mathfrak{h}'\left(\frac{\sigma+\rho}{2}\right)\right|^q}{2} \right)^{\frac{1}{q}} + \left(\frac{\left|\mathfrak{h}'\left(\frac{\sigma+\rho}{2}\right)\right|^q + |\mathfrak{h}'(\rho)|^q}{2} \right)^{\frac{1}{q}} \right].$$

Corollary 2.13. *In Corollary 2.11, as* α *aproaches* 1*, then we obtain*

$$\left| \frac{\mathfrak{h}(\sigma)+\mathfrak{h}(\rho)+2\mathfrak{h}\left(\frac{\sigma+\rho}{2}\right)}{4} - \frac{1}{\rho-\sigma}\int_\sigma^\rho \mathfrak{h}(y)\,dy \right| \leq \frac{\rho-\sigma}{16}\left[\left(\frac{3|\mathfrak{h}'(\sigma)|^q + |\mathfrak{h}'(\rho)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{|\mathfrak{h}'(\sigma)|^q + 3|\mathfrak{h}'(\rho)|^q}{4} \right)^{\frac{1}{q}} \right],$$

which was offered by Hamida and Meftah in Corollary 3.6 from [7].

3. Illustrative Example

In this section, to validate the accuracy of our results, we present a numerical example accompanied by graphical illustrations. Note that the figures are created using the MATLAB software.

Example 3.1. Consider the function $\mathfrak{h} : [0,1] \to \mathbb{R}$ defined by $\mathfrak{h}(y) = e^y$. This function satisfies the conditions of our theorems, as its derivative, $\mathfrak{h}'(y) = e^y$, is convex over the interval $[0,1]$.

For $A = \frac{1-\alpha}{\alpha}$, the right-hand side of the inequality provided in Theorem 2.2 is given by:

$$\begin{aligned} RHS := \frac{1}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)} \Bigg[& (1+\exp(1)) \left(\frac{A+(A-8)\exp\left(-\frac{1}{2}A\right)}{8A} \right. \\ & + \frac{(A-2)\left(\exp\left(-\frac{1}{2}A\right)\right)}{A^2} \ln\left(\tfrac{1}{2} + \tfrac{1}{2}\exp\left(-\tfrac{1}{2}A\right)\right) \\ & \left. + \frac{1+\exp\left(-\frac{1}{2}A\right)}{A^2} \left(\ln\left(\tfrac{1}{2} + \tfrac{1}{2}\exp\left(-\tfrac{1}{2}A\right)\right)\right)^2 \right) \\ & + 4\left(\frac{A+(A+8)\exp\left(-\frac{1}{2}A\right)}{16A} + \frac{1+\exp\left(-\frac{1}{2}A\right)}{A^2} \ln\left(\tfrac{1}{2} + \tfrac{1}{2}\exp\left(-\tfrac{1}{2}A\right)\right) \right. \\ & \left. - \frac{1+\exp\left(-\frac{1}{2}A\right)}{2A^2} \left(\ln\left(\tfrac{1}{2} + \tfrac{1}{2}\exp\left(-\tfrac{1}{2}A\right)\right)\right)^2 \right) \exp\left(\tfrac{1}{2}\right) \Bigg], \end{aligned}$$

and the left-hand side of the same inequality is given by:

(i) For $\alpha \neq \frac{1}{2}$

$$LHS := \left| \frac{1+\exp(1)+2\exp\left(\frac{1}{2}\right)}{4} - \frac{(1-\alpha)\left[(2\alpha-1)\exp(1)\left(1-\exp\left(\frac{1}{2\alpha}\right)\right)+\exp\left(\frac{2\alpha-1}{2\alpha}\right)-1\right]}{2(2\alpha-1)\left(1-\exp\left(-\frac{1}{2}\right)\right)} \right|.$$

(ii) For $\alpha = \frac{1}{2}$

$$LHS := \left| \frac{1+\exp(1)+2\exp\left(\frac{1}{2}\right)}{4} - \frac{\exp(1)(1-\exp(1))+1}{4\left(1-\exp\left(-\frac{1}{2}\right)\right)} \right|.$$

The values RHS and LHS are depicted in Figure 1 for $\alpha \in (0,1)$.

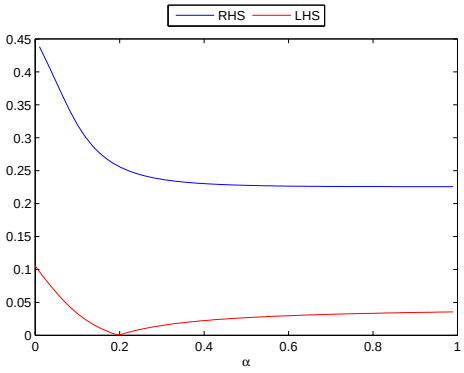


FIGURE 1. Graphical representation of Theorem 2.2

This representation shows that the left-hand side consistently lies below the right-hand side, validating the accuracy of the result established in Theorem 2.2.

4. CONCLUSION

In conclusion, this study has extended the understanding of Bullen-type inequalities by incorporating fractional integrals with exponential kernels. The introduction of a new integral identity allowed us to derive significant new inequalities for differentiable convex functions. The numerical example provided, along with the accompanying graphical representations, confirms the theoretical results and illustrates their practical relevance. These findings open up new avenues for further research in the area of fractional calculus and its applications to integral inequalities.

Conflict of interest. The authors state that they do not have any conflicts of interest.

Data availability. Not applicable

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