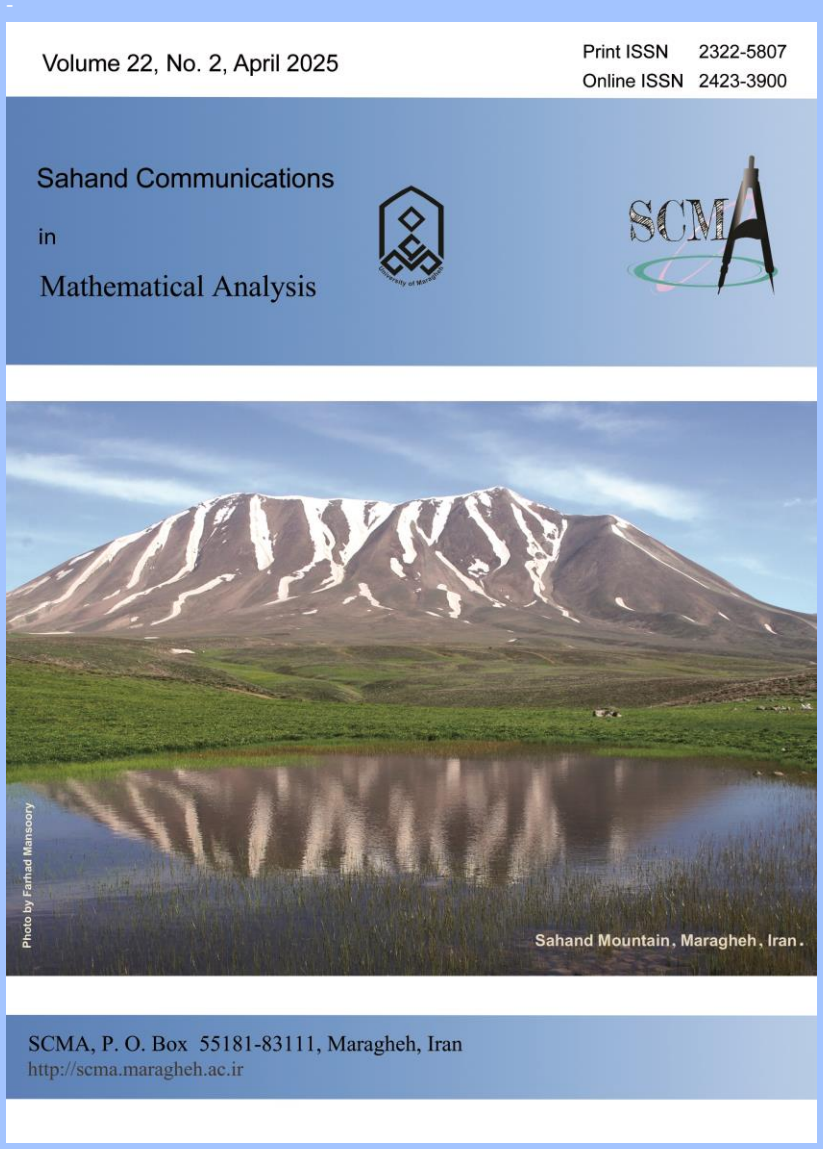
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On Certain Ostrowski Type Integral Inequalities Involving Atangana-Baleanu Katugampola Fractional Integral Operator for Convex Function with Applications

Artion Kashuri[1], Muhammad Sadaqat Talha[2] and Soubhagya Kumar Sahoo[3*]

Abstract. In this paper, new generalized variants of Ostrowski's type identities involving the Atangana-Baleanu-Katugampola fractional integral operator for differentiable convex and twice differentiable convex functions are presented. Using these equalities or lemmas along with several known identities, new inequalities for convex functions and the Atangana-Baleanu-Katugampola, fractional integral operators are proved. By making appropriate choices of parameters, some connections between our results and various other findings are also recognized in the paper. Finally, some applications to unique means for positive real numbers are offered.

1. Introduction

Integral inequalities are one of the major areas of mathematical inequalities. They have been extensively developed over the last few decades due to their significant applications in many areas of scientific research. Over the past few decades, as mathematical models have been established in various of applied sciences, inequalities have proven particularly useful, especially physics and engineering. The extensive involvement of integral inequalities in various branches of mathematics continues to attract mathematicians and researchers to work on them. Since the introduction of Ostrowski's inequality, Hardy's inequality and Hermite-Hadamard's inequality, many researchers have shown interest

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in the subject of integral inequalities. Numerous generalizations of these inequalities have been proved for several functions, such as differentiable convex and s-convex functions (see [12]).

The theory of convex analysis has become one of the most fascinating and useful areas of study in both pure and applied sciences over the last few decades. This is largely due to its wide range of applications. It is also well-known that the theory of convex functions is closely linked to the theory of inequalities, particularly integral inequalities. Various new categories of inequalities related to convex functions have been derived and applied to other fields of study, as evidenced by [6, 8, 18].

The impact of inequalities involving convex functions is as profound and significant in various branches of mathematics, solving numerous problems with remarkable efficiency. Convexity also plays a crucial role in optimizing solutions to mathematical problems. In higher dimensions, many researchers utilize inequalities of higher-order convex functions. Numerous new inequalities involving convex functions have been derived and successfully applied to other fields. One can see in [8] and [18].

In the mathematical study, using integral and fractional differential operators to generalise classical conclusions has gained popularity and drawn the attention of numerous authors and mathematicians. In recent decades, fractional calculus has become increasingly important and popular among scholars since it deals with derivatives and integrals of arbitrary real or complex orders. This is because, as [5, 9, 10, 22] and the references therein show, it has been shown to have applications in various domains, including the social sciences and the natural sciences. Researchers have constructed several inequalities using fractional differential and integral operators; some of these inequalities have been shown to help analyze the solutions of specific fractional differential and integral problems. Let's review some important convex function definitions and findings, as in [12]. Convex function plays a significant role in many different fields of science and is commonly defined as:

Definition 1.1 ([12, Definition 1.1])**.** Suppose $\aleph : I = [\wp_1, \wp_2] \subseteq \mathbb{R} \to \mathbb{R}$, is convex function if

$$\aleph(\wp_1 + (1-\varsigma)\wp_2) \leq \varsigma\aleph(\wp_1) + (1-\varsigma)\aleph(\wp_2), \tag{1.1}$$

holds for all $\wp_1, \wp_2 \in I$ and $\varsigma \in [0,1]$. And $\aleph$ is concave if $(-\aleph)$ is convex.

The Hermite-Hadamard inequalities are among the most significant inequalities that have garnered the attention of many mathematicians in this field. In [8], these inequalities are recognized as well-known results in the literature and are presented as follows:

Theorem 1.2 ([8, Equation 1.1, page 1]). *If $\aleph$ is a convex function on $[\wp_1, \wp_2]$ with $\wp_1 < \wp_2$, then*

$$\aleph\left(\frac{\wp_1+\wp_2}{2}\right) \leq \frac{1}{\wp_2-\wp_1}\int_{\wp_1}^{\wp_2}\aleph(\hbar)d\hbar \leq \frac{\aleph(\wp_1)+\aleph(\wp_2)}{2}, \tag{1.2}$$

holds.

Lemma 1.3 ([20]). *For $0 < \beta \leq 1$ and $0 \leq \wp_1 < \wp_2$, we have*

$$\left|\wp_1^\beta - \wp_2^\beta\right| \leq (\wp_2-\wp_1)^\beta.$$

Now, we look forward to the classical Ostrowski inequality, which is very interesting and useful from the point of view of integral inequality.

Definition 1.4 ([17, Equation 11.1, page 8]). Suppose $\aleph : [\wp_1, \wp_2] \subseteq \mathbb{R} \to \mathbb{R}$ is a differentiable mapping, such that $\aleph \in [\wp_1, \wp_2]$, where $\wp_1, \wp_2 \in I$ with $\wp_1 < \wp_2$. If $\left|\aleph'(\hbar)\right| \leq M$, for all $\hbar \in [\wp_1, \wp_2]$. Then the given inequality holds:

$$\left|\aleph(\hbar) - \frac{1}{\wp_2-\wp_1}\int_{\wp_1}^{\wp_2}\aleph(\hbar)d\hbar\right| \leq M(\wp_2-\wp_1)\left[\frac{1}{4}+\frac{\left(\hbar-\frac{\wp_1+\wp_2}{2}\right)^2}{(\wp_2-\wp_1)^2}\right], \tag{1.3}$$

here the constant $\frac{1}{4}$ is the least possible value.

Some essential mathematical preliminaries and definitions of fractional calculus theory are given below:

Definition 1.5 ([11, Equation 1.1, page 225], [21, Definition 2]). Let $\aleph \in L_1[\wp_1, \wp_2]$. The Riemann-Liouville integrals $J^\beta_{\wp_1^+}\aleph(\varsigma)$ and $J^\beta_{\wp_2^-}\aleph(\varsigma)$ of order $\beta > 0$ are defined by:

$$J^\beta_{\wp_1^+}\aleph(\varsigma) = \frac{1}{\Gamma(\beta)}\int_{\wp_1}^{\varsigma}(\varsigma-\hbar)^{\beta-1}\aleph(\hbar)d\hbar, \quad \varsigma > \wp_1,$$

and

$$J^\beta_{\wp_2^-}\aleph(\varsigma) = \frac{1}{\Gamma(\beta)}\int_{\varsigma}^{\wp_2}(\hbar-\varsigma)^{\beta-1}\aleph(\hbar)d\hbar, \quad \varsigma < \wp_2,$$

respectively, where $\Gamma(\beta) = \int_0^\infty e^{-\varsigma}\varsigma^{\beta-1}d\varsigma$. Here $J^0_{\wp_2^-}\aleph(\varsigma) = J^0_{\wp_1^+}\aleph(\varsigma) = \aleph(\varsigma)$.

Definition 1.6 ([7, Definition 1.3]). The Hadamard's fractional integrals operators $H^\beta_{\wp_1^+}\aleph(\varsigma)$ and $H^\beta_{\wp_2^-}\aleph(\varsigma)$ of order $\beta > 0$ are defined by:

$$H^\beta_{\wp_1^+}\aleph(\varsigma) = \frac{1}{\Gamma(\beta)}\int_{\wp_1}^{\varsigma}(\ln\varsigma - \ln\hbar)^{\beta-1}\frac{\aleph(\hbar)}{\hbar}d\hbar, \quad \varsigma > \wp_1,$$

and

$$H^{\beta}_{\wp_2^-}\aleph(\varsigma)=\frac{1}{\Gamma(\beta)}\int_{\varsigma}^{\wp_2}(\ln\hbar-\ln\varsigma)^{\beta-1}\frac{\aleph(\hbar)}{\hbar}d\hbar,\quad \varsigma<\wp_2,$$

respectively.

Definition 1.7 ([3])**.** Let the space $X_c^p(\wp_1,\wp_2)$ $(c\in\mathbb{R})$ and $1\leq p\leq\infty$, which is the space of all complex valued Lebesgue measurable functions $\aleph$ on $(\wp_1,\wp_2)$, for which $\|\aleph\|_{X_c^p}<\infty$, where the norm $\|.\|_{X_c^p}$ is defined by

$$\|\aleph\|_{X_c^p}=\left(\int_{\wp_1}^{\wp_2}\mid\varsigma^c\aleph(\varsigma)\mid^p\frac{d\varsigma}{\varsigma}\right)^{\frac{1}{p}},\quad (1\leq p<\infty),$$

and for $p=\infty$,

$$\|\varsigma\|_{X_c^{\infty}}=ess\sup_{\wp_1\leq\varsigma\leq\wp_2}\mid\varsigma^c\aleph(\varsigma)\mid.$$

Over the last few decades, fractional calculus has become a powerful and versatile mathematical tool with applications across variuos scientific and engineering fields. Many researchers are increasingly focusing on newly introduced fractional operators. Katugampola introduced a new fractional integral that unifies the Hadamard fractional and Riemann-Liouville (RL) integrals into a single, more general form, as follows:

Definition 1.8 ([15, Equation 3.3 and 3.4, page 4])**.** Let $[\wp_1,\wp_2]\subset\mathbb{R}$ be a finite interval. Then the left-sided and the right-sided fractional integrals of order $(\beta>0)$ of $\aleph\in X_c^p(\wp_1,\wp_2)$, called Katugampola fractional integral is given by:

$${}^{\eth}I^{\beta}_{\wp_1^+}\{\aleph(\varsigma)\}=\frac{\eth^{1-\beta}}{\Gamma(\beta)}\int_{\wp_1}^{\varsigma}\frac{\hbar^{\eth-1}}{(\varsigma^{\eth}-\hbar^{\eth})^{1-\beta}}\aleph(\hbar)d\hbar,$$

and

$${}^{\eth}I^{\beta}_{\wp_2^-}\{\aleph(\varsigma)\}=\frac{\eth^{1-\beta}}{\Gamma(\beta)}\int_{\varsigma}^{\wp_2}\frac{\hbar^{\eth-1}}{(\hbar^{\eth}-\varsigma^{\eth})^{1-\beta}}\aleph(\hbar)d\hbar,$$

with $\wp_1<\varsigma<\wp_2$ and $\eth>0$, if the integrals exists.

Theorem 1.9 ([15, Theorem 4.1])**.** *Let $\beta>0$ and $\eth>0$, then for $\varsigma>\wp_1$, we have*

(i) $\lim_{\eth\to1}\ {}^{\eth}I^{\beta}_{\wp_1^+}\{\aleph(\varsigma)\}=J^{\beta}_{\wp_1^+}\aleph(\varsigma)$,

(ii) $\lim_{\eth\to0^+}\ {}^{\eth}I^{\beta}_{\wp_1^-}\aleph(\varsigma)=H^{\beta}_{\wp_1^+}\aleph(\varsigma)$.

Similar results also hold for right-sided operators.

In their recent work, Atangana and Baleanu introduced a new type of fractional derivatives. These derivatives utilize a generalization of the Mittag-Leffler function as a non-singular and non-local kernel, while

retaining all the key properties of traditional fractional derivatives [16, 19].

Definition 1.10 ([4, Definition 4])**.** The fractional integral called Atangana-Baleanu of the function $\aleph \in H^1(\wp_1, \wp_2)$ is given by:

$$ {}^{AB}_{\wp_1}I^{\beta}_{\varsigma}\{\aleph(\varsigma)\} = \frac{1-\beta}{\mathrm{M}(\beta)}\aleph(\varsigma) + \frac{\beta}{\mathrm{M}(\beta)\Gamma(\beta)}\int_{\wp_1}^{\varsigma}(\varsigma-\hbar)^{\beta-1}\aleph(\hbar)d\hbar, \quad \varsigma > \wp_1, $$

where $\wp_2 > \wp_1, \beta \in [0,1]$ and $\mathrm{M}(\beta) > 0$ satisfies the property $\mathrm{M}(0) = \mathrm{M}(1) = 1$.

In [1, Definition 2.3], the right-side of (AB)-fractional integral operator is given as:

$$ {}^{AB}_{\wp_2}I^{\beta}_{\varsigma}\{\aleph(\varsigma)\} = \frac{1-\beta}{\mathrm{M}(\beta)}\aleph(\varsigma) + \frac{\beta}{\mathrm{M}(\beta)\Gamma(\beta)}\int_{\varsigma}^{\wp_2}(\hbar-\varsigma)^{\beta-1}\aleph(\hbar)d\hbar, \quad \varsigma < \wp_2. $$

Here, $\Gamma(\beta)$ is the Euler Gamma function and $\mathrm{M}(\beta) > 0$ is the positive normalizing function, ensuring that (AB)-integral of the positive function is always positive. And it is noteworthy to mention here that, for order $\beta \to 1$, it recovers the classical integral. Some recent results related to Atangana-Baleanu (AB)-fractional operator were proved in [23].

The ABK-fractional derivative has been defined as:

Definition 1.11 ([14, Definition 2.1])**.** Let $[\wp_1, \wp_2] \subset \mathbb{R}$ be a finite interval. Then the left-sided and right-sided ABK-fractional integrals of order $\beta > 0$ of $\aleph \in X_c^{\varrho}(\wp_1, \wp_2)$ are defined as follows:

$$ {}^{ABK_{\eth}}_{\wp_1^+}I^{\beta}_{\varsigma}\aleph(\varsigma) = \frac{1-\beta}{\mathrm{M}(\beta)}\aleph(\varsigma) + \frac{\eth^{1-\beta}\beta}{\Gamma(\beta)\mathrm{M}(\beta)}\int_{\wp_1}^{\varsigma}\frac{\hbar^{\eth-1}}{(\varsigma^{\eth}-\hbar^{\eth})^{1-\beta}}\aleph(\hbar)d\hbar, $$

for $\varsigma > \wp_1 \geq 0$, and

$$ {}^{ABK_{\eth}}_{\wp_2^-}I^{\beta}_{\varsigma}\aleph(\varsigma) = \frac{1-\beta}{\mathrm{M}(\beta)}\aleph(\varsigma) + \frac{\eth^{1-\beta}\beta}{\Gamma(\beta)\mathrm{M}(\beta)}\int_{\varsigma}^{\wp_2}\frac{\hbar^{\eth-1}}{(\hbar^{\eth}-\varsigma^{\eth})^{1-\beta}}\aleph(\hbar)d\hbar, $$

for $\varsigma < \wp_2$, where $\eth > 0$ and $\mathrm{M}(\beta) > 0$ satisfies the property $\mathrm{M}(0) = \mathrm{M}(1) = 1$.

It follows that ABK-fractional integral of the positive function is always positive.

Remark 1.12. Whenever the parameter $\eth \to 1$, our results will retrieve results for the (AB)-fractional integral operator in [2, 13].

The primary aim of this paper is to present a generalization of various types of integral inequalities for convex functions using Atangana-Baleanu Katugampola fractional integrals. We derive new identities involving first and second-order derivatives. The techniques and ideas

employed to obtain the main results in this paper have the potential to generate new findings in the field of integral inequalities. Finally, we provide some applications to special means for positive real numbers.

2. Ostrowski Type Integral Inequalities for Differentiable Convex Function

This section will prove an identity related to the Atangana-Baleanu-Katugampola, a fractional operator for differentiable functions. Then, we will present our main results by employing this and some fundamental integral inequalities.

Lemma 2.1. *Let $\beta > 0$, $\eth > 0$ and $\aleph : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable on I^0 such that $\aleph \in X_c^p(\wp_1^\eth, \wp_2^\eth)$ for all $\wp_1^\eth, \wp_2^\eth \in I$ and $\wp_1^\eth < \wp_2^\eth$. If $\aleph' \in L_1[\wp_1^\eth, \wp_2^\eth]$, then for all $\varsigma^\eth \in [\wp_1^\eth, \wp_2^\eth]$ and $\beta \in (0,1]$, the following identity*

$$
\begin{aligned}
(2.1)\qquad & \left[\frac{\left(\varsigma^\eth - \wp_1^\eth\right)^\beta + \left(\wp_2^\eth - \varsigma^\eth\right)^\beta}{\eth\left(\wp_2^\eth - \wp_1^\eth\right)}\right]\aleph\left(\varsigma^\eth\right) \\
&\quad + \frac{\Gamma(\beta)\mathrm{M}(\beta)}{\eth^{1-\beta}\left(\wp_2^\eth - \wp_1{}^\eth\right)}\left[\ {}^{ABK_\eth}_{\wp_1^+}I_\varsigma^\beta\aleph(\varsigma^\eth) + \ {}^{ABK_\eth}_{\wp_2^-}I_\varsigma^\beta\aleph(\varsigma^\eth)\right] \\
&\quad - \frac{(1-\beta)\,\Gamma(\beta)}{\eth^{1-\beta}\left(\wp_2^\eth - \wp_1^\eth\right)}\left[\aleph(\wp_1^\eth) + \aleph(\wp_2^\eth)\right] \\
&= \frac{\left(\varsigma^\eth - \wp_1^\eth\right)^{\beta+1}}{\left(\wp_2^\eth - \wp_1^\eth\right)}\int_0^1 \ell^{\eth\beta}.\ell^{\eth-1}\aleph'\left(\ell^\eth\varsigma^\eth + (1-\ell^\eth)\wp_1^\eth\right)d\ell \\
&\quad - \frac{\left(\wp_2{}^\eth - \varsigma^\eth\right)^{\beta+1}}{\left(\wp_2^\eth - \wp_1^\eth\right)}\int_0^1 \ell^{\eth\beta}.\ell^{\eth-1}\aleph'\left(\ell^\eth\varsigma^\eth + (1-\ell^\eth)\wp_2^\eth\right)d\ell,
\end{aligned}
$$

holds true.

Proof. For easier computations, let us write

$$
\begin{aligned}
(2.2)\qquad I &= \frac{\left(\varsigma^\eth - \wp_1^\eth\right)^{\beta+1}}{\left(\wp_2^\eth - \wp_1^\eth\right)}\int_0^1 \ell^{\eth\beta}.\ell^{\eth-1}\aleph'\left(\ell^\eth\varsigma^\eth + (1-\ell^\eth)\wp_1^\eth\right)d\ell \\
&\quad - \frac{\left(\wp_2^\eth - \varsigma^\eth\right)^{\beta+1}}{\left(\wp_2^\eth - \wp_1^\eth\right)}\int_0^1 \ell^{\eth\beta}.\ell^{\eth-1}\aleph'\left(\ell^\eth\varsigma^\eth + (1-\ell^\eth)\wp_2^\eth\right)d\ell \\
&= \frac{\left(\varsigma^\eth - \wp_1^\eth\right)^{\beta+1}}{\left(\wp_2^\eth - \wp_1^\eth\right)}I_1 - \frac{\left(\wp_2{}^\eth - \varsigma^\eth\right)^{\beta+1}}{\left(\wp_2^\eth - \wp_1^\eth\right)}I_2.
\end{aligned}
$$

Employing integration by parts, gives

(2.3)

$$\begin{aligned} I_1 &= \int_0^1 \ell^{\eth\beta}.\ell^{\eth-1}\aleph'\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_1^{\eth}\right)d\ell \\ &= \ell^{\eth\beta}\frac{\aleph\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_1^{\eth}\right)}{\eth\left(\varsigma^{\eth}-\wp_1^{\eth}\right)}\Bigg|_0^1 - \beta\int_0^1\left(\ell^{\eth}\right)^{\beta-1}\ell^{\eth-1}\frac{\aleph\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_1^{\eth}\right)}{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)}d\ell \\ &= \frac{\aleph\left(\varsigma^{\eth}\right)}{\eth\left(\varsigma^{\eth}-\wp_1^{\eth}\right)} - \frac{\beta}{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)}\int_0^1\left(\ell^{\eth}\right)^{\beta-1}\ell^{\eth-1}\aleph\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_1^{\eth}\right)d\ell. \end{aligned}$$

By changing the variable for the right side of (2.3) with $\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_1^{\eth}=\hbar^{\varrho}$, we get

(2.4)

$$\begin{aligned} I_1 &= \frac{\aleph\left(\varsigma^{\eth}\right)}{\eth\left(\varsigma^{\eth}-\wp_1^{\eth}\right)} - \frac{\beta}{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)}\int_{\wp_1^{\eth}}^{\varsigma^{\eth}}\left(\frac{\hbar^{\varrho}-\wp_1^{\eth}}{\varsigma^{\eth}-\wp_1^{\eth}}\right)^{\beta-1}\frac{\hbar^{\eth-1}}{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)}\aleph\left(\hbar^{\varrho}\right)d\hbar \\ &= \frac{\aleph\left(\varsigma^{\eth}\right)}{\eth\left(\varsigma^{\eth}-\wp_1^{\eth}\right)} - \frac{\beta}{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+1}}\int_{\wp_1^{\eth}}^{\varsigma^{\eth}}\frac{\hbar^{\eth-1}}{\left(\hbar^{\varrho}-\wp_1^{\eth}\right)^{1-\beta}}\aleph\left(\hbar^{\varrho}\right)d\hbar. \end{aligned}$$

By multiplying $\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+1}}{\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}$ and adding and subtracting $\frac{1-\beta}{\mathrm{M}(\beta)}\aleph\left(\wp_1^{\eth}\right)$ on the right side of (2.4), we get

$$\begin{aligned} I_1 &= \frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta}}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\aleph\left(\varsigma^{\eth}\right) + \frac{\mathrm{M}(\beta)\Gamma(\beta)}{\varrho^{1-\beta}\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[{}^{ABK}_{\wp_1^+}\!\mathfrak{D}_{\eth}\,I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth}) - \frac{1-\beta}{\mathrm{M}(\beta)}\aleph\left(\wp_1^{\eth}\right)\right] \\ &= \frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta}}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\aleph\left(\varsigma^{\eth}\right) + \frac{\mathrm{M}(\beta)\Gamma(\beta)}{\varrho^{1-\beta}\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[{}^{ABK}_{\wp_1^+}\!\mathfrak{D}_{\eth}\,I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})\right] \\ &\quad - \frac{1-\beta}{\varrho^{1-\beta}\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\Gamma(\beta)\aleph\left(\wp_1^{\eth}\right). \end{aligned}$$

By using similar steps, we have

$$\begin{aligned} I_2 &= \frac{-\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta}}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\aleph\left(\varsigma^{\eth}\right) - \frac{\mathrm{M}(\beta)\Gamma(\beta)}{\varrho^{1-\beta}\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[{}^{ABK}_{\wp_2^-}\!\mathfrak{D}_{\eth}\,I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth}) + \frac{1-\beta}{\mathrm{M}(\beta)}\aleph\left(\wp_2^{\eth}\right)\right] \\ &= -\frac{\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta}}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\aleph\left(\varsigma^{\eth}\right) - \frac{\mathrm{M}(\beta)\Gamma(\beta)}{\varrho^{1-\beta}\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[{}^{ABK}_{\wp_2^-}\!\mathfrak{D}_{\eth}\,I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})\right] \\ &\quad + \frac{1-\beta}{\varrho^{1-\beta}\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\Gamma(\beta)\aleph\left(\wp_2^{\eth}\right). \end{aligned}$$

By putting the values of I_1 and I_2 in (2.2), we get (2.1), which is our final result. □

Theorem 2.2. *Let $\beta > 0$, $\eth > 0$ and $\aleph : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable on I^0, such that $\aleph \in X_c^p(\wp_1^\eth, \wp_2^\eth)$, for all $\wp_1^\eth, \wp_2^\eth \in I$ and $\wp_1^\eth < \wp_2^\eth$. If $\aleph' \in L_1[\wp_1^\eth, \wp_2^\eth]$, and $\left|\aleph'\right|$ is a convex function, then for all $\varsigma^\eth \in \left[\wp_1^\eth, \wp_2^\eth\right]$ and $\beta \in (0,1]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\left[\frac{\left(\varsigma^\eth - \wp_1^\eth\right)^\beta + \left(\wp_2{}^\eth - \varsigma^\eth\right)^\beta}{\eth\left(\wp_2^\eth - \wp_1^\eth\right)}\right]\aleph\left(\varsigma^\eth\right)\right. \\
&\quad + \frac{\Gamma(\beta)\mathrm{M}(\beta)}{\eth^{1-\beta}\left(\wp_2^\eth - \wp_1^\eth\right)}\left[{}^{ABK_\eth}_{\wp_1^+}I_\varsigma^\beta\aleph(\varsigma^\eth) + {}^{ABK_\eth}_{\wp_2^-}I_\varsigma^\beta\aleph(\varsigma^\eth)\right] \\
&\quad \left. - \frac{(1-\beta)\,\Gamma(\beta)}{\eth^{1-\beta}\left(\wp_2^\eth - \wp_1^\eth\right)}\left[\aleph(\wp_1^\eth) + \aleph(\wp_2^\eth)\right]\right| \\
&\leq \frac{\left(\varsigma^\eth - \wp_1^\eth\right)^{\beta+1}}{\eth\,(\beta+2)\left(\wp_2^\eth - \wp_1^\eth\right)}\left[\left|\aleph'\left(\varsigma^\eth\right)\right| + \frac{\left|\aleph'\left(\wp_1^\eth\right)\right|}{(\beta+1)}\right] \\
&\quad + \frac{\left(\wp_2^\eth - \varsigma^\eth\right)^{\beta+1}}{\eth\,(\beta+2)\left(\wp_2^\eth - \wp_1^\eth\right)}\left[\left|\aleph'\left(\varsigma^\eth\right)\right| + \frac{\left|\aleph'\left(\wp_2^\eth\right)\right|}{(\beta+1)}\right].
\end{aligned}$$

Proof. By using Lemma 2.1 and convexity of $\left|\aleph'\right|$, we get

$$\begin{aligned}
&\left|\left[\frac{\left(\varsigma^\eth - \wp_1^\eth\right)^\beta + \left(\wp_2^\eth - \varsigma^\eth\right)^\beta}{\eth\left(\wp_2^\eth - \wp_1^\eth\right)}\right]\aleph\left(\varsigma^\eth\right)\right. \\
&\quad + \frac{\Gamma(\beta)\mathrm{M}(\beta)}{\eth^{1-\beta}\left(\wp_2^\eth - \wp_1^\eth\right)}\left[{}^{ABK_\eth}_{\wp_1^+}I_\varsigma^\beta\aleph(\varsigma^\eth) + {}^{ABK_\eth}_{\wp_2^-}I_\varsigma^\beta\aleph(\varsigma^\eth)\right] \\
&\quad \left. - \frac{(1-\beta)\,\Gamma(\beta)}{\eth^{1-\beta}\left(\wp_2^\eth - \wp_1^\eth\right)}\left[\aleph(\wp_1^\eth) + \aleph(\wp_2^\eth)\right]\right| \\
&\leq \frac{\left(\varsigma^\eth - \wp_1^\eth\right)^{\beta+1}}{\left(\wp_2^\eth - \wp_1^\eth\right)}\int_0^1 \ell^{\eth\beta}.\ell^{\eth-1}\left|\aleph'\left(\ell^\eth\varsigma^\eth + (1-\ell^\eth)\wp_1^\eth\right)\right|d\ell \\
&\quad + \frac{\left(\wp_2^\eth - \varsigma^\eth\right)^{\beta+1}}{\left(\wp_2^\eth - \wp_1^\eth\right)}\int_0^1 \ell^{\eth\beta}.\ell^{\eth-1}\left|\aleph'\left(\ell^\eth\varsigma^\eth + (1-\ell^\eth)\wp_2^\eth\right)\right|d\ell \\
&\leq \frac{\left(\varsigma^\eth - \wp_1^\eth\right)^{\beta+1}}{\left(\wp_2^\eth - \wp_1^\eth\right)}\int_0^1 \ell^{\eth\beta}.\ell^{\eth-1}\left(\ell^\eth\left|\aleph'\left(\varsigma^\eth\right)\right| + (1-\ell^\eth)\left|\aleph'\left(\wp_1^\eth\right)\right|\right)d\ell \\
&\quad + \frac{\left(\wp_2^\eth - \varsigma^\eth\right)^{\beta+1}}{\left(\wp_2^\eth - \wp_1^\eth\right)}\int_0^1 \ell^{\eth\beta}.\ell^{\eth-1}\left(\ell^\eth\left|\aleph'\left(\varsigma^\eth\right)\right| + (1-\ell^\eth)\left|\aleph'\left(\wp_2^\eth\right)\right|\right)d\ell
\end{aligned}$$

$$= \frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+1}}{\eth\left(\beta+2\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\left|\aleph'\left(\varsigma^{\eth}\right)\right|+\frac{\left|\aleph'\left(\wp_1^{\eth}\right)\right|}{(\beta+1)}\right]$$
$$+\frac{\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+1}}{\eth\left(\beta+2\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\left|\aleph'\left(\varsigma^{\eth}\right)\right|+\frac{\left|\aleph'\left(\wp_2^{\eth}\right)\right|}{(\beta+1)}\right].$$

So the proof is completed. □

Corollary 2.3. *If we take* $\left|\aleph'\right| \leq M$ *in Theorem 2.2, we obtain the following inequality:*

$$\left|\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta}+\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta}}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph\left(\varsigma^{\eth}\right)\right.$$
$$+\frac{\Gamma(\beta)\mathrm{M}(\beta)}{\eth^{1-\beta}\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[{}^{ABK_{\eth}}_{\wp_1^{+}}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})+{}^{ABK_{\eth}}_{\wp_2^{-}}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})\right]$$
$$\left.-\frac{(1-\beta)\,\Gamma(\beta)}{\eth^{1-\beta}\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\aleph(\wp_1^{\eth})+\aleph(\wp_2^{\eth})\right]\right|$$
$$\leq \frac{M}{\eth\left(\beta+1\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+1}+\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+1}\right].$$

Theorem 2.4. *Let* $\beta>0$, $\eth>0$ *and* $\aleph: I\subset\mathbb{R}\to\mathbb{R}$ *be differentiable on* I^0, *such that* $\aleph\in X_c^p(\wp_1^{\eth},\wp_2^{\eth})$, *for all* $\wp_1^{\eth},\wp_2^{\eth}\in I$ *and* $\wp_1^{\eth}<\wp_2^{\eth}$. *If* $\aleph'\in L_1[\wp_1^{\eth},\wp_2^{\eth}]$, *and* $\left|\aleph'\right|^q$ *is a convex function, then for all* $\varsigma^{\eth}\in\left[\wp_1^{\eth},\wp_2^{\eth}\right]$ *and* $\beta\in(0,1]$, *then the following inequality holds:*

$$\left|\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta}+\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta}}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph\left(\varsigma^{\eth}\right)\right.$$
$$+\frac{\Gamma(\beta)\mathrm{M}(\beta)}{\eth^{1-\beta}\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[{}^{ABK_{\eth}}_{\wp_1^{+}}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})+{}^{ABK_{\eth}}_{\wp_2^{-}}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})\right]$$
$$\left.-\frac{(1-\beta)\,\Gamma(\beta)}{\eth^{1-\beta}\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\aleph(\wp_1^{\eth})+\aleph(\wp_2^{\eth})\right]\right| \tag{2.5}$$
$$\leq\left[\frac{1}{\eth\left(p\beta+1\right)}\right]^{\frac{1}{p}}\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+1}}{\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\frac{\left|\aleph'\left(\varsigma^{\eth}\right)\right|^q+\left|\aleph'\left(\wp_1^{\eth}\right)\right|^q}{2\eth}\right]^{\frac{1}{q}}\right.$$
$$\left.+\frac{\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+1}}{\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\frac{\left|\aleph'\left(\varsigma^{\eth}\right)\right|^q+\left|\aleph'\left(\wp_2^{\eth}\right)\right|^q}{2\eth}\right]^{\frac{1}{q}}\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and* $q>1$.

Proof. By using Lemma 2.1 and Hölder's inequality, we get

$$
\begin{aligned}
(2.6)\qquad &\left|\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta}+\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta}}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph\left(\varsigma^{\eth}\right)\right.\\
&\quad+\frac{\Gamma(\beta)\mathrm{M}(\beta)}{\eth^{1-\beta}\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[{}^{ABK_{\eth}}_{\wp_1^{+}}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})+{}^{ABK_{\eth}}_{\wp_2^{-}}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})\right]\\
&\quad\left.-\frac{(1-\beta)\,\Gamma(\beta)}{\eth^{1-\beta}\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\aleph(\wp_1^{\eth})+\aleph(\wp_2^{\eth})\right]\right|\\
&\leq\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+1}}{\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\int_0^1\ell^{\eth\beta}\ell^{\eth-1}\left|\aleph'\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_1^{\eth}\right)\right|d\ell\\
&\quad+\frac{\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+1}}{\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\int_0^1\ell^{\eth\beta}\ell^{\eth-1}\left|\aleph'\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_2^{\eth}\right)\right|d\ell\\
&\leq\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+1}}{\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\int_0^1\ell^{p\eth\beta}\ell^{\eth-1}d\ell\right]^{\frac{1}{p}}\\
&\quad\times\left[\int_0^1\ell^{\eth-1}\left|\aleph'\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_1^{\eth}\right)\right|^{q}d\ell\right]^{\frac{1}{q}}\\
&\quad+\frac{\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+1}}{\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\int_0^1\ell^{p\eth\beta}\ell^{\eth-1}d\ell\right]^{\frac{1}{p}}\\
&\quad\times\left[\int_0^1\ell^{\eth-1}\left|\aleph'\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_2^{\eth}\right)\right|^{q}d\ell\right]^{\frac{1}{q}}.
\end{aligned}
$$

By the convexity of $\left|\aleph'\right|^{q}$, we get

$$
\begin{aligned}
(2.7)\qquad &\int_0^1\ell^{\eth-1}\left|\aleph'\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_1^{\eth}\right)\right|^{q}d\ell\\
&\leq\int_0^1\ell^{\eth-1}\left(\ell^{\eth}\left|\aleph'\left(\varsigma^{\eth}\right)\right|^{q}+(1-\ell^{\eth})\left|\aleph'\left(\wp_1^{\eth}\right)\right|^{q}\right)d\ell\\
&=\frac{\left|\aleph'\left(\varsigma^{\eth}\right)\right|^{q}+\left|\aleph'\left(\wp_1^{\eth}\right)\right|^{q}}{2\eth},
\end{aligned}
$$

and

$$
(2.8)\qquad \int_0^1\ell^{\eth-1}\left|\aleph'\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_2^{\eth}\right)\right|^{q}d\ell
$$

$$\leq \int_0^1 \ell^{\eth-1}\left(\ell^{\eth}\left|\aleph'\left(\varsigma^{\eth}\right)\right|^q+\left(1-\ell^{\eth}\right)\left|\aleph'\left(\wp_2^{\eth}\right)\right|^q\right)d\ell$$
$$=\frac{\left|\aleph'\left(\varsigma^{\eth}\right)\right|^q+\left|\aleph'\left(\wp_2^{\eth}\right)\right|^q}{2\eth}.$$

Combining (2.7) and (2.8) with (2.6), we get (2.5). So the proof is completed. □

Corollary 2.5. *If we take* $\left|\aleph'\right| \leq M$ *in Theorem 2.4, we will get the following inequality:*

$$\left|\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta}+\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta}}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph\left(\varsigma^{\eth}\right)\right.$$
$$+\frac{\Gamma(\beta)\mathrm{M}(\beta)}{\eth^{1-\beta}\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[{}^{ABK_{\eth}}_{\wp_1^+}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})+{}^{ABK_{\eth}}_{\wp_2^-}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})\right]$$
$$\left.-\frac{(1-\beta)\,\Gamma(\beta)}{\eth^{1-\beta}\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\aleph(\wp_1^{\eth})+\aleph(\wp_2^{\eth})\right]\right|$$
$$\leq\left[\frac{1}{\eth\left(p\beta+1\right)}\right]^{\frac{1}{p}}\frac{M}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+1}+\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+1}\right].$$

Theorem 2.6. *Let* $\beta>0$ *and* $\eth>0$. *Let* $\aleph: I\subset\mathbb{R}\to\mathbb{R}$ *is differentiable on* I^0, *such that* $\aleph\in X_c^p(\wp_1^{\eth},\wp_2^{\eth})$, *for all* $\wp_1^{\eth},\wp_2^{\eth}\in I$ *and* $\wp_1^{\eth}<\wp_2^{\eth}$. *If* $\aleph'\in L_1[\wp_1^{\eth},\wp_2^{\eth}]$, *and* $\left|\aleph'\right|^q$ *is a convex function, then for all* $\varsigma^{\eth}\in\left[\wp_1^{\eth},\wp_2^{\eth}\right]$ *and* $\beta\in(0,1]$, *then the following inequality holds:*

$$\left|\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta}+\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta}}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph\left(\varsigma^{\eth}\right)\right.$$
$$+\frac{\Gamma(\beta)\mathrm{M}(\beta)}{\eth^{1-\beta}\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[{}^{ABK_{\eth}}_{\wp_1^+}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})+{}^{ABK_{\eth}}_{\wp_2^-}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})\right]$$
$$\left.-\frac{(1-\beta)\,\Gamma(\beta)}{\eth^{1-\beta}\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\aleph(\wp_1^{\eth})+\aleph(\wp_2^{\eth})\right]\right|$$
$$\leq\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+1}}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\frac{1}{p\left(p\beta+1\right)}+\frac{\left|\aleph'\left(\varsigma^{\eth}\right)\right|^q+\left|\aleph'\left(\wp_1^{\eth}\right)\right|^q}{2q}\right]$$
$$+\frac{\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+1}}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\frac{1}{p\left(p\beta+1\right)}+\frac{\left|\aleph'\left(\varsigma^{\eth}\right)\right|^q+\left|\aleph'\left(\wp_2^{\eth}\right)\right|^q}{2q}\right], \tag{2.9}$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and* $q>1$.

Proof. By using Lemma 2.1 and Young's inequality ($xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q$), we get

$$
\begin{aligned}
(2.10)\qquad & \left| \left[\frac{\left(\varsigma^{\eth} - \wp_1^{\eth}\right)^{\beta} + \left(\wp_2^{\eth} - \varsigma^{\eth}\right)^{\beta}}{\eth\left(\wp_2^{\eth} - \wp_1^{\eth}\right)} \right] \aleph\left(\varsigma^{\eth}\right) \right. \\
& + \frac{\Gamma(\beta)\mathrm{M}(\beta)}{\eth^{1-\beta}\left(\wp_2^{\eth} - \wp_1^{\eth}\right)} \left[{}^{ABK_{\eth}}_{\wp_1^{+}} I_{\varsigma}^{\beta} \aleph(\varsigma^{\eth}) + {}^{ABK_{\eth}}_{\wp_2^{-}} I_{\varsigma}^{\beta} \aleph(\varsigma^{\eth}) \right] \\
& \left. - \frac{(1-\beta)\,\Gamma(\beta)}{\eth^{1-\beta}\left(\wp_2^{\eth} - \wp_1^{\eth}\right)} \left[\aleph(\wp_1^{\eth}) + \aleph(\wp_2^{\eth}) \right] \right| \\
& \leq \frac{\left(\varsigma^{\eth} - \wp_1^{\eth}\right)^{\beta+1}}{\left(\wp_2^{\eth} - \wp_1^{\eth}\right)} \int_0^1 \ell^{\eth\beta}.\ell^{\eth-1} \left| \aleph'\left(\ell^{\eth}\varsigma^{\eth} + (1-\ell^{\eth})\wp_1^{\eth}\right) \right| d\ell \\
& + \frac{\left(\wp_2^{\eth} - \varsigma^{\eth}\right)^{\beta+1}}{\left(\wp_2^{\eth} - \wp_1^{\eth}\right)} \int_0^1 \ell^{\eth\beta}.\ell^{\eth-1} \left| \aleph'\left(\ell^{\eth}\varsigma^{\eth} + (1-\ell^{\eth})\wp_2^{\eth}\right) \right| d\ell \\
& \leq \frac{\left(\varsigma^{\eth} - \wp_1^{\eth}\right)^{\beta+1}}{\left(\wp_2^{\eth} - \wp_1^{\eth}\right)} \left[\frac{1}{p} \int_0^1 \ell^{p\eth\beta}\ell^{\eth-1} d\ell \right. \\
& \left. + \frac{1}{q} \int_0^1 \ell^{\eth-1} \left| \aleph'\left(\ell^{\eth}\varsigma^{\eth} + (1-\ell^{\eth})\wp_1^{\eth}\right) \right|^q d\ell \right] \\
& + \frac{\left(\wp_2^{\eth} - \varsigma^{\eth}\right)^{\beta+1}}{\left(\wp_2^{\eth} - \wp_1^{\eth}\right)} \left[\frac{1}{p} \int_0^1 \ell^{p\eth\beta}\ell^{\eth-1} d\ell \right. \\
& \left. + \frac{1}{q} \int_0^1 \ell^{\eth-1} \left| \aleph'\left(\ell^{\eth}\varsigma^{\eth} + (1-\ell^{\eth})\wp_2^{\eth}\right) \right|^q d\ell \right].
\end{aligned}
$$

By the convexity of $\left|\aleph'\right|^q$, we get

$$
\begin{aligned}
(2.11)\qquad & \int_0^1 \ell^{\eth-1} \left| \aleph'\left(\ell^{\eth}\varsigma^{\eth} + (1-\ell^{\eth})\wp_1^{\eth}\right) \right|^q d\ell \\
& \leq \int_0^1 \ell^{\eth-1} \left(\ell^{\eth} \left| \aleph'\left(\varsigma^{\eth}\right) \right|^q + (1-\ell^{\eth}) \left| \aleph'\left(\wp_1^{\eth}\right) \right|^q \right) d\ell \\
& = \frac{\left| \aleph'\left(\varsigma^{\eth}\right) \right|^q + \left| \aleph'\left(\wp_1^{\eth}\right) \right|^q}{2\eth},
\end{aligned}
$$

and

$$
\begin{aligned}
(2.12)\qquad & \int_0^1 \ell^{\eth-1} \left| \aleph'\left(\ell^{\eth}\varsigma^{\eth} + (1-\ell^{\eth})\wp_2^{\eth}\right) \right|^q d\ell \\
& \leq \int_0^1 \ell^{\eth-1} \left(\ell^{\eth} \left| \aleph'\left(\varsigma^{\eth}\right) \right|^q + (1-\ell^{\eth}) \left| \aleph'\left(\wp_2^{\eth}\right) \right|^q \right) d\ell
\end{aligned}
$$

$$= \frac{\left|\aleph'\left(\varsigma^{\eth}\right)\right|^q + \left|\aleph'\left(\wp_2^{\eth}\right)\right|^q}{2\eth}.$$

Combining (2.11) and (2.12) with (2.10), we get (2.9). So the proof is completed. □

Corollary 2.7. *If we choose* $\left|\aleph'\right|^q \leq M$ *in Theorem 2.6, we will obtain the following inequality:*

$$\begin{aligned}
&\left|\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta}+\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta}}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph\left(\varsigma^{\eth}\right)\right. \\
&\quad+\frac{\Gamma(\beta)\mathrm{M}(\beta)}{\eth^{1-\beta}\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[{}^{ABK_{\eth}}_{\wp_1^+}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})+{}^{ABK_{\eth}}_{\wp_2^-}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})\right] \\
&\quad\left.-\frac{(1-\beta)\,\Gamma(\beta)}{\eth^{1-\beta}\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\aleph(\wp_1^{\eth})+\aleph(\wp_2^{\eth})\right]\right| \\
&\leq \frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+1}}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\frac{1}{p\left(p\beta+1\right)}+\frac{M}{q}\right]+\frac{\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+1}}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\frac{1}{p\left(p\beta+1\right)}+\frac{M}{q}\right].
\end{aligned}$$

Theorem 2.8. *Let* $\beta>0$ *and* $\eth>0$. *Let* $\aleph: I\subset\mathbb{R}\to\mathbb{R}$ *is differentiable on* I^0, *such that* $\aleph\in X_c^p(\wp_1^{\eth},\wp_2^{\eth})$, *for all* $\wp_1^{\eth},\wp_2^{\eth}\in I$ *and* $\wp_1^{\eth}<\wp_2^{\eth}$. *If* $\aleph'\in L_1[\wp_1^{\eth},\wp_2^{\eth}]$, *and* $\left|\aleph'\right|^q$ *is a convex function, then for all* $\varsigma^{\eth}\in[\wp_1^{\eth},\wp_2^{\eth}]$ *and* $\beta\in(0,1]$, *then the following inequality holds:*

(2.13)

$$\begin{aligned}
&\left|\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta}+\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta}}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph\left(\varsigma^{\eth}\right)\right. \\
&\quad+\frac{\Gamma(\beta)\mathrm{M}(\beta)}{\eth^{1-\beta}\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[{}^{ABK_{\eth}}_{\wp_1^+}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})+{}^{ABK_{\eth}}_{\wp_2^-}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})\right] \\
&\quad\left.-\frac{(1-\beta)\,\Gamma(\beta)}{\eth^{1-\beta}\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\aleph(\wp_1^{\eth})+\aleph(\wp_2^{\eth})\right]\right| \\
&\leq\left[\frac{1}{\eth\left(\beta+1\right)}\right]^{1-\frac{1}{q}}\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+1}}{\eth(\beta+2)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\left|\aleph'\left(\varsigma^{\eth}\right)\right|^q+\frac{\left|\aleph'\left(\wp_1^{\eth}\right)\right|^q}{\beta+1}\right]^{\frac{1}{q}}\right. \\
&\quad\left.+\frac{\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+1}}{\eth(\beta+2)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\left|\aleph'\left(\varsigma^{\eth}\right)\right|^q+\frac{\left|\aleph'\left(\wp_2^{\eth}\right)\right|^q}{\beta+1}\right]^{\frac{1}{q}}\right],
\end{aligned}$$

where $q\geq 1$.

Proof. By using Lemma 2.1 and power-mean inequality, we get

$$
\begin{aligned}
(2.14)\qquad & \left|\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta}+\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta}}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph\left(\varsigma^{\eth}\right)\right. \\
&\quad +\frac{\Gamma(\beta)\mathrm{M}(\beta)}{\eth^{1-\beta}\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[{}^{ABK_{\eth}}_{\wp_1^{+}}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})+{}^{ABK_{\eth}}_{\wp_2^{-}}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})\right] \\
&\quad \left.-\frac{(1-\beta)\,\Gamma(\beta)}{\eth^{1-\beta}\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\aleph(\wp_1^{\eth})+\aleph(\wp_2^{\eth})\right]\right| \\
&\quad \leq \frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+1}}{\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\int_0^1 \ell^{\eth\beta}\ell^{\eth-1}\left|\aleph'\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_1^{\eth}\right)\right|d\ell \\
&\quad + \frac{\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+1}}{\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\int_0^1 \ell^{\eth\beta}\ell^{\eth-1}\left|\aleph'\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_2^{\eth}\right)\right|d\ell \\
&\quad \leq \frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+1}}{\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\int_0^1 \ell^{\eth\beta}\ell^{\eth-1}d\ell\right]^{1-\frac{1}{q}} \\
&\quad \times\left[\int_0^1 \ell^{\eth\beta}\ell^{\eth-1}\left|\aleph'\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_1^{\eth}\right)\right|^q d\ell\right]^{\frac{1}{q}} \\
&\quad + \frac{\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+1}}{\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\int_0^1 \ell^{\eth\beta}\ell^{\eth-1}d\ell\right]^{1-\frac{1}{q}} \\
&\quad \times\left[\int_0^1 \ell^{\eth\beta}\ell^{\eth-1}\left|\aleph'\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_2^{\eth}\right)\right|^q d\ell\right]^{\frac{1}{q}}.
\end{aligned}
$$

By the convexity of $\left|\aleph'\right|^q$, we get

$$
\begin{aligned}
(2.15)\qquad & \int_0^1 \ell^{\eth\beta}\ell^{\eth-1}\left|\aleph'\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_1^{\eth}\right)\right|^q d\ell \\
&\leq \int_0^1 \ell^{\eth\beta}.\ell^{\eth-1}\left(\ell^{\eth}\left|\aleph'\left(\varsigma^{\eth}\right)\right|^q+(1-\ell^{\eth})\left|\aleph'\left(\wp_1^{\eth}\right)\right|^q\right)d\ell \\
&= \frac{\left|\aleph'\left(\varsigma^{\eth}\right)\right|^q}{\eth(\beta+2)}+\frac{\left|\aleph'\left(\wp_1^{\eth}\right)\right|^q}{\eth(\beta+2)(\beta+1)},
\end{aligned}
$$

and

$$
\begin{aligned}
(2.16)\qquad & \int_0^1 \ell^{\eth\beta}\ell^{\eth-1}\left|\aleph'\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_2^{\eth}\right)\right|^q d\ell \\
&\leq \int_0^1 \ell^{\eth\beta}\ell^{\eth-1}\left(\ell^{\eth}\left|\aleph'\left(\varsigma^{\eth}\right)\right|^q+(1-\ell^{\eth})\left|\aleph'\left(\wp_2^{\eth}\right)\right|^q\right)d\ell
\end{aligned}
$$

$$= \frac{\left|\aleph'\left(\varsigma^{\eth}\right)\right|^q}{\eth(\beta+2)} + \frac{\left|\aleph'\left(\wp_2^{\eth}\right)\right|^q}{\eth(\beta+2)(\beta+1)}.$$

Combining (2.15) and (2.16) with (2.14), we get (2.13). So the proof is completed. □

Corollary 2.9. *If we choose* $\left|\aleph'\right| \leq M$ *in Theorem 2.8, we will obtain the following inequality:*

$$\begin{aligned}
&\left|\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta}+\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta}}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph\left(\varsigma^{\eth}\right)\right.\\
&\quad+\frac{\Gamma(\beta)\mathrm{M}(\beta)}{\eth^{1-\beta}\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[{}^{ABK_{\eth}}_{\wp_1^+}I_{\varsigma}^{\beta}\aleph(\wp_2^{\eth})+{}^{ABK_{\eth}}_{\wp_2^-}I_{\varsigma}^{\beta}\aleph(\wp_1^{\eth})\right]\\
&\quad\left.-\frac{(1-\beta)\,\Gamma(\beta)}{\eth^{1-\beta}\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\aleph(\wp_1^{\eth})+\aleph(\wp_2^{\eth})\right]\right|\\
&\leq\left[\frac{1}{\eth\left(\beta+1\right)}\right]^{1-\frac{1}{q}}\frac{M}{\eth(\beta+1)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+1}+\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+1}\right].
\end{aligned}$$

3. Ostrowski Type Integral Inequalities For Twice Differentiable Convex Function

This section proves an Atangana-Baleanu Katugampola fractional identity for twice differentiable functions. After that, we present our main results by employing this and some fundamental integral inequalities.

Lemma 3.1. *Let* $\beta > 0$ *and* $\eth > 0$. *Let* $\aleph : I \subset \mathbb{R} \to \mathbb{R}$ *is twice differentiable on* I^0,*(Interior of* I*) such that* $\aleph \in X_c^p(\wp_1^{\eth}, \wp_2^{\eth})$, *for all* $\wp_1^{\eth}, \wp_2^{\eth} \in I$ *and* $\wp_1^{\eth} < \wp_2^{\eth}$. *If* $\aleph'' \in L_1[\wp_1^{\eth}, \wp_2^{\eth}]$, *then for all* $\varsigma^{\eth} \in [\wp_1^{\eth}, \wp_2^{\eth}]$ *and* $\beta \in (0,1]$, *then the following identity holds:*

(3.1)

$$\begin{aligned}
&\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+1}-\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+1}}{\eth\left(\beta+1\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph'\left(\varsigma^{\eth}\right)\\
&\quad-\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta}-\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta}}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph\left(\varsigma^{\eth}\right)-\frac{\Gamma(\beta)\mathrm{M}(\beta)}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[{}^{ABK_{\eth}}_{\wp_1^+}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})\right.\\
&\quad\left.+{}^{ABK_{\eth}}_{\wp_2^-}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})\right]-\frac{2\,(1-\beta)\,\Gamma(\beta)}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\aleph(\varsigma^{\eth})\\
&=\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+2}}{\eth^{\beta-1}\left(\beta+1\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\int_0^1\left(\ell^{\eth}\right)^{\beta+1}\ell^{\eth-1}\aleph''\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_1^{\eth}\right)d\ell
\end{aligned}$$

$$+\frac{\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+2}}{\eth^{\beta-1}\left(\beta+1\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\int_0^1\left(\ell^{\eth}\right)^{\beta+1}\ell^{\eth-1}\aleph''\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_2^{\eth}\right)d\ell.$$

Proof. For easier computations, let us write

(3.2)

$$\begin{aligned}I&=\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+2}}{\eth^{\beta-1}\left(\beta+1\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\int_0^1\left(\ell^{\eth}\right)^{\beta+1}\ell^{\eth-1}\aleph''\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_1^{\eth}\right)d\ell\\&\quad+\frac{\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+2}}{\eth^{\beta-1}\left(\beta+1\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\int_0^1\left(\ell^{\eth}\right)^{\beta+1}\ell^{\eth-1}\aleph''\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_2^{\eth}\right)d\ell\\&=\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+2}}{\eth^{\beta-1}\left(\beta+1\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}I_1+\frac{\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+2}}{\eth^{\beta-1}\left(\beta+1\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}I_2.\end{aligned}$$

Employing integration by parts, gives

(3.3)

$$\begin{aligned}I_1&=\int_0^1\left(\ell^{\eth}\right)^{\beta+1}\ell^{\eth-1}\aleph''\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_1^{\eth}\right)d\ell\\&=\left(\ell^{\eth}\right)^{\beta+1}\frac{\aleph'\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_1^{\eth}\right)}{\eth\left(\varsigma^{\eth}-\wp_1^{\eth}\right)}\Bigg|_0^1\\&\quad-(\beta+1)\int_0^1\left(\ell^{\eth}\right)^{\beta}\ell^{\eth-1}\frac{\aleph'\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_1^{\eth}\right)}{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)}d\ell\\&=\frac{\aleph'\left(\varsigma^{\eth}\right)}{\eth\left(\varsigma^{\eth}-\wp_1^{\eth}\right)}-\frac{\beta+1}{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)}\int_0^1\left(\ell^{\eth}\right)^{\beta}\ell^{\eth-1}\aleph'\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_1^{\eth}\right)d\ell\\&=\frac{\aleph'\left(\varsigma^{\eth}\right)}{\eth\left(\varsigma^{\eth}-\wp_1^{\eth}\right)}-\frac{\beta+1}{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)}\\&\quad\times\left[\frac{\aleph\left(\varsigma^{\eth}\right)}{\eth\left(\varsigma^{\eth}-\wp_1^{\eth}\right)}-\beta\int_0^1\left(\ell^{\eth}\right)^{\beta-1}\ell^{\eth-1}\frac{\aleph\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_1^{\eth}\right)}{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)}d\ell\right].\end{aligned}$$

By changing the variable technique with $\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_1^{\eth}=\hbar^{\varrho}$, the right side of (3.3) becomes

$$\begin{aligned}I_1&=\frac{\aleph'\left(\varsigma^{\eth}\right)}{\eth\left(\varsigma^{\eth}-\wp_1^{\eth}\right)}-\frac{(\beta+1)}{\eth\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^2}\aleph\left(\varsigma^{\eth}\right)\\&\quad+\frac{\beta\left(\beta+1\right)}{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^2}\int_{a^{\eth}}^{\varsigma^{\eth}}\left(\frac{\hbar^{\varrho}-\wp_1^{\eth}}{\varsigma^{\eth}-\wp_1^{\eth}}\right)^{\beta-1}\frac{\hbar^{\eth-1}}{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)}\aleph\left(\hbar^{\varrho}\right)d\hbar\\&=\frac{\aleph'\left(\varsigma^{\eth}\right)}{\eth\left(\varsigma^{\eth}-\wp_1^{\eth}\right)}-\frac{(\beta+1)}{\eth\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^2}\aleph\left(\varsigma^{\eth}\right)\end{aligned}\tag{3.4}$$

$$-\frac{\beta\left(\beta+1\right)}{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+2}}\int_{a^{\eth}}^{\varsigma^{\eth}}\frac{\hbar^{\eth-1}}{\left(\wp_1^{\eth}-\hbar^{\eth}\right)^{1-\beta}}\aleph\left(\hbar^{\varrho}\right)d\hbar.$$

By adding and subtracting $\frac{1-\beta}{\mathrm{M}(\beta)}\aleph\left(\varsigma^{\eth}\right)$ on the right side of (3.4), we get

$$\begin{aligned}
I_1 &= \frac{\aleph'\left(\varsigma^{\eth}\right)}{\eth\left(\varsigma^{\eth}-\wp_1^{\eth}\right)}-\frac{(\beta+1)}{\eth\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^2}\aleph\left(\varsigma^{\eth}\right)\\
&\quad-\frac{\mathrm{M}(\beta)\Gamma(\beta+2)}{\varrho\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+2}}\left[{}^{ABK_{\eth}}_{\wp_1^+}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})+\frac{1-\beta}{\mathrm{M}(\beta)}\aleph\left(\varsigma^{\eth}\right)\right]\\
&= \frac{\aleph'\left(\varsigma^{\eth}\right)}{\eth\left(\varsigma^{\eth}-\wp_1^{\eth}\right)}-\frac{(\beta+1)}{\eth\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^2}\aleph\left(\varsigma^{\eth}\right)-\frac{\mathrm{M}(\beta)\Gamma(\beta+2)}{\eth\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+2}}\left[{}^{ABK_{\eth}}_{\wp_1^+}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})\right]\\
&\quad-\frac{{}^{1-\beta}}{\eth\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+2}}\Gamma(\beta+2)\aleph\left(\varsigma^{\eth}\right).
\end{aligned}$$

By using similar steps, we have

$$\begin{aligned}
I_2 &= -\frac{\aleph'\left(\varsigma^{\eth}\right)}{\eth\left(\wp_2^{\eth}-\varsigma^{\eth}\right)}+\frac{(\beta+1)}{\eth\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^2}\aleph\left(\varsigma^{\eth}\right)\\
&\quad-\frac{\mathrm{M}(\beta)\Gamma(\beta+2)}{\eth\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+2}}\left[{}^{ABK_{\eth}}_{\wp_2^-}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})\right]-\frac{{}^{1-\beta}}{\eth\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+2}}\Gamma(\beta+2)\aleph\left(\varsigma^{\eth}\right).
\end{aligned}$$

By putting the values of I_1 and I_2 in (3.2), we get (3.1), which is our final result. □

Theorem 3.2. *Let $\beta>0$ and $\eth>0$. Let $\aleph: I\subset\mathbb{R}\to\mathbb{R}$ is twice differentiable on I^0, such that $\aleph\in X_c^p(\wp_1^{\eth},\wp_2^{\eth})$, for all $\wp_1^{\eth},\wp_2^{\eth}\in I$ and $\wp_1^{\eth}<\wp_2^{\eth}$. If $\aleph^{''}\in L_1[\wp_1^{\eth},\wp_2^{\eth}]$, and $\left|\aleph''\right|$ is a convex function, then for all $\varsigma^{\eth}\in[\wp_1^{\eth},\wp_2^{\eth}]$ and $\beta\in(0,1]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+1}-\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+1}}{\eth\left(\beta+1\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph'\left(\varsigma^{\eth}\right)\right.\\
&\quad-\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta}-\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta}}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph\left(\varsigma^{\eth}\right)\\
&\quad-\frac{\Gamma(\beta)\mathrm{M}(\beta)}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[{}^{ABK_{\eth}}_{\wp_1^+}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})+{}^{ABK_{\eth}}_{\wp_2^-}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})\right]\\
&\quad\left.-\frac{2\left(1-\beta\right)\Gamma(\beta)}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\aleph(\varsigma^{\eth})\right|
\end{aligned}$$

$$\leq \frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+2}}{\eth^{\beta}\left(\beta+1\right)\left(\beta+3\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\left|\aleph^{''}\left(\varsigma^{\eth}\right)\right|+\frac{\left|\aleph^{''}\left(\wp_1^{\eth}\right)\right|}{(\beta+2)}\right]$$
$$+\frac{\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+2}}{\eth^{\beta}\left(\beta+1\right)\left(\beta+3\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\left|\aleph^{''}\left(\varsigma^{\eth}\right)\right|+\frac{\left|\aleph^{''}\left(\wp_2^{\eth}\right)\right|}{(\beta+2)}\right].$$

Proof. By using Lemma 3.1 and convexity of $\left|\aleph^{''}\right|$, we get

$$\begin{aligned}
&\left|\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+1}-\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+1}}{\eth\left(\beta+1\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph'\left(\varsigma^{\eth}\right)\right.\\
&\quad-\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta}-\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta}}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph\left(\varsigma^{\eth}\right)\\
&\quad-\frac{\Gamma(\beta)\mathrm{M}(\beta)}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[{}^{ABK_{\eth}}_{\wp_1^{+}}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})+{}^{ABK_{\eth}}_{\wp_2^{-}}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})\right]\\
&\quad\left.-\frac{2\left(1-\beta\right)\Gamma(\beta)}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\aleph(\varsigma^{\eth})\right|\\
&\leq\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+2}}{\eth^{\beta-1}\left(\beta+1\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\\
&\quad\times\int_0^1\left(\ell^{\eth}\right)^{\beta+1}\ell^{\eth-1}\left|\aleph^{''}\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_1^{\eth}\right)\right|d\ell\\
&\quad+\frac{\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+2}}{\eth^{\beta-1}\left(\beta+1\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\\
&\quad\times\int_0^1\left(\ell^{\eth}\right)^{\beta+1}\ell^{\eth-1}\left|\aleph^{''}\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_2^{\eth}\right)\right|d\ell\\
&\leq\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+2}}{\eth^{\beta-1}\left(\beta+1\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\\
&\quad\times\int_0^1\left(\ell^{\eth}\right)^{\beta+1}\ell^{\eth-1}\left(\ell^{\eth}\left|\aleph^{''}(\varsigma^{\eth})\right|+(1-\ell^{\eth})\left|\aleph^{''}(\wp_1^{\eth})\right|\right)d\ell\\
&\quad+\frac{\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+2}}{\eth^{\beta-1}\left(\beta+1\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\\
&\quad\times\int_0^1\left(\ell^{\eth}\right)^{\beta+1}\ell^{\eth-1}\left(\ell^{\eth}\left|\aleph^{''}(\varsigma^{\eth})\right|+(1-\ell^{\eth})\left|\aleph^{''}(\wp_2^{\eth})\right|\right)d\ell
\end{aligned}$$

$$= \frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+2}}{\eth^{\beta}(\beta+1)(\beta+3)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\left|\aleph^{''}\left(\varsigma^{\eth}\right)\right|+\frac{\left|\aleph^{''}\left(\wp_1^{\eth}\right)\right|}{(\beta+2)}\right]$$
$$+\frac{\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+2}}{\eth^{\beta}(\beta+1)(\beta+3)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\left|\aleph^{''}\left(\varsigma^{\eth}\right)\right|+\frac{\left|\aleph^{''}\left(\wp_2^{\eth}\right)\right|}{(\beta+2)}\right].$$

So the proof is completed. □

Corollary 3.3. *If we take* $\left|\aleph^{''}\right| \leq M$ *in Theorem 3.2, we will obtain the following inequality:*

$$\left|\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+1}-\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+1}}{\eth(\beta+1)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph'\left(\varsigma^{\eth}\right)\right.$$
$$-\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta}-\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta}}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph\left(\varsigma^{\eth}\right)$$
$$-\frac{\Gamma(\beta)\mathrm{M}(\beta)}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[{}^{ABK_{\eth}}_{\wp_1^{+}}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})+{}^{ABK_{\eth}}_{\wp_2^{-}}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})\right]$$
$$\left.-\frac{2(1-\beta)\Gamma(\beta)}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\aleph(\varsigma^{\eth})\right|$$
$$\leq \frac{M}{\eth^{\beta}(\beta+1)(\beta+2)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+2}+\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+2}\right].$$

Theorem 3.4. *Let* $\beta>0$ *and* $\eth>0$. *Let* $\aleph: I\subset\mathbb{R}\to\mathbb{R}$ *is twice differentiable on* I^0, *such that* $\aleph\in X_c^p(\wp_1^{\eth},\wp_2^{\eth})$, *for all* $\wp_1^{\eth},\wp_2^{\eth}\in I$ *and* $\wp_1^{\eth}<\wp_2^{\eth}$. *If* $\aleph^{''}\in L_1[\wp_1^{\eth},\wp_2^{\eth}]$, *and* $\left|\aleph^{''}\right|^q$ *is a convex function, then for all* $\varsigma^{\eth}\in[\wp_1^{\eth},\wp_2^{\eth}]$ *and* $\beta\in(0,1]$, *then the following inequality holds:*

(3.5)

$$\left|\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+1}-\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+1}}{\eth(\beta+1)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph'\left(\varsigma^{\eth}\right)-\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta}-\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta}}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph\left(\varsigma^{\eth}\right)\right.$$
$$\left.-\frac{\Gamma(\beta)\mathrm{M}(\beta)}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[{}^{ABK_{\eth}}_{\wp_1^{+}}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})+{}^{ABK_{\eth}}_{\wp_2^{-}}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})\right]-\frac{2(1-\beta)\Gamma(\beta)}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\aleph(\varsigma^{\eth})\right|$$
$$\leq\left[\frac{1}{\eth\left((\beta+1)p+1\right)}\right]^{\frac{1}{p}}\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+2}}{\eth^{\beta-1}(\beta+1)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\frac{\left|\aleph^{''}\left(\wp_1^{\eth}\right)\right|^q+\left|\aleph^{''}\left(\varsigma^{\eth}\right)\right|^q}{2\eth}\right]^{\frac{1}{q}}\right.$$

$$+\frac{\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+2}}{\eth^{\beta-1}\left(\beta+1\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\frac{\left|\aleph''\left(\varsigma^{\eth}\right)\right|^q+\left|\aleph''\left(\wp_2^{\eth}\right)\right|^q}{2\eth}\right]^{\frac{1}{q}}\Bigg],$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and* $q>1$.

Proof. By using Lemma 3.1 and Hölder's inequality, we get

(3.6)

$$\begin{aligned}
&\left|\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+1}-\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+1}}{\eth\left(\beta+1\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph'\left(\varsigma^{\eth}\right)-\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta}-\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta}}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph\left(\varsigma^{\eth}\right)\right.\\
&\quad\left.-\frac{\Gamma(\beta)\mathrm{M}(\beta)}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[{}^{ABK_{\eth}}_{\wp_1^+}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})+{}^{ABK_{\eth}}_{\wp_2^-}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})\right]-\frac{2\left(1-\beta\right)\Gamma(\beta)}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\aleph(\varsigma^{\eth})\right|\\
&\leq\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+2}}{\eth^{\beta-1}\left(\beta+1\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\int_0^1\left(\ell^{\eth}\right)^{\beta+1}\ell^{\eth-1}\left|\aleph''\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_1^{\eth}\right)\right|d\ell\\
&\quad+\frac{\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+2}}{\eth^{\beta-1}\left(\beta+1\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\int_0^1\left(\ell^{\eth}\right)^{\beta+1}\ell^{\eth-1}\left|\aleph''\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_2^{\eth}\right)\right|d\ell\\
&\leq\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+2}}{\eth^{\beta-1}\left(\beta+1\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\int_0^1\ell^{p\eth\beta}\ell^{\eth-1}d\ell\right]^{\frac{1}{p}}\\
&\quad\times\left[\int_0^1\ell^{\eth-1}\left|\aleph''\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_1^{\eth}\right)\right|^q d\ell\right]^{\frac{1}{q}}\\
&\quad+\frac{\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+2}}{\eth^{\beta-1}\left(\beta+1\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\int_0^1\ell^{p\eth\beta}\ell^{\eth-1}d\ell\right]^{\frac{1}{p}}\\
&\quad\times\left[\int_0^1\ell^{\eth-1}\left|\aleph''\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_2^{\eth}\right)\right|^q d\ell\right]^{\frac{1}{q}}.
\end{aligned}$$

By the convexity of $\left|\aleph''\right|^q$, we get

$$\begin{aligned}
&\int_0^1\ell^{\eth-1}\left|\aleph''\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_1^{\eth}\right)\right|^q d\ell\\
&\quad\leq\int_0^1\ell^{\eth-1}\left(\ell^{\eth}\left|\aleph''\left(\varsigma^{\eth}\right)\right|^q+(1-\ell^{\eth})\left|\aleph''\left(\wp_1^{\eth}\right)\right|^q\right)d\ell\\
&\quad=\frac{\left|\aleph''\left(\wp_1^{\eth}\right)\right|^q+\left|\aleph''\left(\varsigma^{\eth}\right)\right|^q}{2\eth},
\end{aligned}\tag{3.7}$$

and

$$\int_0^1\ell^{\eth-1}\left|\aleph''\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_2^{\eth}\right)\right|^q d\ell\tag{3.8}$$

$$\leq \int_0^1 \ell^{\eth-1}\left(\ell^{\eth}\left|\aleph''\left(\varsigma^{\eth}\right)\right|^q + (1-\ell^{\eth})\left|\aleph''\left(\wp_2^{\eth}\right)\right|^q\right) d\ell$$
$$= \frac{\left|\aleph''\left(\varsigma^{\eth}\right)\right|^q + \left|\aleph''\left(\wp_2^{\eth}\right)\right|^q}{2\eth}.$$

Combining (3.7) and (3.8) with (3.6), we get (3.5). So the proof is completed. □

Corollary 3.5. *If we take* $\left|\aleph''\right| \leq M$ *in Theorem 3.4, we will obtain the following inequality:*

$$\left|\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+1}-\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+1}}{\eth(\beta+1)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph'\left(\varsigma^{\eth}\right)-\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta}-\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta}}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph\left(\varsigma^{\eth}\right)$$
$$-\frac{\Gamma(\beta)\mathrm{M}(\beta)}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[{}^{ABK_{\eth}}_{\wp_1^+}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})+{}^{ABK_{\eth}}_{\wp_2^-}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})\right]-\frac{2(1-\beta)\Gamma(\beta)}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\aleph(\varsigma^{\eth})\Bigg|$$
$$\leq \left[\frac{1}{\eth\left((\beta+1)p+1\right)}\right]^{\frac{1}{p}}$$
$$\times\frac{M}{\eth^{\beta}(\beta+1)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+2}+\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+2}\right].$$

Theorem 3.6. *Let* $\beta > 0$ *and* $\eth > 0$. *Let* $\aleph : I \subset \mathbb{R} \to \mathbb{R}$ *is twice differentiable on* I^0, *such that* $\aleph \in X_c^p(\wp_1^{\eth}, \wp_2^{\eth})$, *for all* $\wp_1^{\eth}, \wp_2^{\eth} \in I$ *and* $\wp_1^{\eth} < \wp_2^{\eth}$. *If* $\aleph'' \in L_1[\wp_1^{\eth}, \wp_2^{\eth}]$, *and* $\left|\aleph''\right|^q$ *is a convex function, then for all* $\varsigma^{\eth} \in [\wp_1^{\eth}, \wp_2^{\eth}]$ *and* $\beta \in (0,1]$, *then the following inequality holds:*

(3.9)

$$\left|\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+1}-\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+1}}{\eth(\beta+1)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph'\left(\varsigma^{\eth}\right)-\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta}-\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta}}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph\left(\varsigma^{\eth}\right)$$
$$-\frac{\Gamma(\beta)\mathrm{M}(\beta)}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[{}^{ABK_{\eth}}_{\wp_1^+}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})+{}^{ABK_{\eth}}_{\wp_2^-}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})\right]-\frac{2(1-\beta)\Gamma(\beta)}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\aleph(\varsigma^{\eth})\Bigg|$$
$$\leq \frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+2}}{\eth^{\beta}(\beta+1)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\frac{1}{p\left(p(\beta+1)+1\right)}+\frac{\left|\aleph''\left(\wp_1^{\eth}\right)\right|^q+\left|\aleph''\left(\varsigma^{\eth}\right)\right|^q}{2q}\right]$$
$$+\frac{\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+2}}{\eth^{\beta-1}(\beta+1)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\frac{1}{p\left(p(\beta+1)+1\right)}+\frac{\left|\aleph''\left(\varsigma^{\eth}\right)\right|^q+\left|\aleph''\left(\wp_2^{\eth}\right)\right|^q}{2q}\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and* $q > 1$.

Proof. By using Lemma 3.1 and Young's inequality, we get

(3.10)

$$\begin{aligned}
&\left|\left[\frac{\left(\varsigma^{\mathfrak{d}}-\wp_1^{\mathfrak{d}}\right)^{\beta+1}-\left(\wp_2^{\mathfrak{d}}-\varsigma^{\mathfrak{d}}\right)^{\beta+1}}{\mathfrak{d}\left(\beta+1\right)\left(\wp_2^{\mathfrak{d}}-\wp_1^{\mathfrak{d}}\right)}\right]\aleph'\left(\varsigma^{\mathfrak{d}}\right)-\left[\frac{\left(\varsigma^{\mathfrak{d}}-\wp_1^{\mathfrak{d}}\right)^{\beta}-\left(\wp_2^{\mathfrak{d}}-\varsigma^{\mathfrak{d}}\right)^{\beta}}{\mathfrak{d}\left(\wp_2^{\mathfrak{d}}-\wp_1^{\mathfrak{d}}\right)}\right]\aleph\left(\varsigma^{\mathfrak{d}}\right)\\
&\quad-\frac{\Gamma(\beta)\mathrm{M}(\beta)}{\mathfrak{d}\left(\wp_2^{\mathfrak{d}}-\wp_1^{\mathfrak{d}}\right)}\left[{}^{ABK_{\mathfrak{d}}}_{\wp_1^{+}}I_{\varsigma}^{\beta}\aleph(\varsigma^{\mathfrak{d}})+{}^{ABK_{\mathfrak{d}}}_{\wp_2^{-}}I_{\varsigma}^{\beta}\aleph(\varsigma^{\mathfrak{d}})\right]-\frac{2\left(1-\beta\right)\Gamma(\beta)}{\mathfrak{d}\left(\wp_2^{\mathfrak{d}}-\wp_1^{\mathfrak{d}}\right)}\aleph(\varsigma^{\mathfrak{d}})\Bigg|\\
&\leq\frac{\left(\varsigma^{\mathfrak{d}}-\wp_1^{\mathfrak{d}}\right)^{\beta+2}}{\mathfrak{d}^{\beta-1}\left(\beta+1\right)\left(\wp_2^{\mathfrak{d}}-\wp_1^{\mathfrak{d}}\right)}\int_0^1\left(\ell^{\mathfrak{d}}\right)^{\beta+1}\ell^{\mathfrak{d}-1}\left|\aleph''\left(\ell^{\mathfrak{d}}\varsigma^{\mathfrak{d}}+(1-\ell^{\mathfrak{d}})\wp_1^{\mathfrak{d}}\right)\right|d\ell\\
&\quad+\frac{\left(\wp_2^{\mathfrak{d}}-\varsigma^{\mathfrak{d}}\right)^{\beta+2}}{\mathfrak{d}^{\beta-1}\left(\beta+1\right)\left(\wp_2^{\mathfrak{d}}-\wp_1^{\mathfrak{d}}\right)}\int_0^1\left(\ell^{\mathfrak{d}}\right)^{\beta+1}\ell^{\mathfrak{d}-1}\left|\aleph''\left(\ell^{\mathfrak{d}}\varsigma^{\mathfrak{d}}+(1-\ell^{\mathfrak{d}})\wp_2^{\mathfrak{d}}\right)\right|d\ell\\
&\leq\frac{\left(\varsigma^{\mathfrak{d}}-\wp_1^{\mathfrak{d}}\right)^{\beta+2}}{\mathfrak{d}^{\beta-1}\left(\beta+1\right)\left(\wp_2^{\mathfrak{d}}-\wp_1^{\mathfrak{d}}\right)}\left[\frac{1}{p}\int_0^1\left(\ell^{\mathfrak{d}}\right)^{p(\beta+1)}\ell^{\mathfrak{d}-1}d\ell\right.\\
&\quad\left.+\frac{1}{q}\int_0^1\ell^{\mathfrak{d}-1}\left|\aleph''\left(\ell^{\mathfrak{d}}\varsigma^{\mathfrak{d}}+(1-\ell^{\mathfrak{d}})\wp_1^{\mathfrak{d}}\right)\right|^q d\ell\right]\\
&\quad+\frac{\left(\wp_2^{\mathfrak{d}}-\varsigma^{\mathfrak{d}}\right)^{\beta+2}}{\mathfrak{d}^{\beta-1}\left(\beta+1\right)\left(\wp_2^{\mathfrak{d}}-\wp_1^{\mathfrak{d}}\right)}\left[\frac{1}{p}\int_0^1\left(\ell^{\mathfrak{d}}\right)^{p(\beta+1)}\ell^{\mathfrak{d}-1}d\ell\right.\\
&\quad\left.+\frac{1}{q}\int_0^1\ell^{\mathfrak{d}-1}\left|\aleph''\left(\ell^{\mathfrak{d}}\varsigma^{\mathfrak{d}}+(1-\ell^{\mathfrak{d}})\wp_2^{\mathfrak{d}}\right)\right|^q d\ell\right].
\end{aligned}$$

By the convexity of $\left|\aleph''\right|^q$, we get

$$\begin{aligned}
&\int_0^1\ell^{\mathfrak{d}-1}\left|\aleph''\left(\ell^{\mathfrak{d}}\varsigma^{\mathfrak{d}}+(1-\ell^{\mathfrak{d}})\wp_1^{\mathfrak{d}}\right)\right|^q d\ell\\
&\quad\leq\int_0^1\ell^{\mathfrak{d}-1}\left(\ell^{\mathfrak{d}}\left|\aleph''\left(\varsigma^{\mathfrak{d}}\right)\right|^q+(1-\ell^{\mathfrak{d}})\left|\aleph''\left(\wp_1^{\mathfrak{d}}\right)\right|^q\right)d\ell\\
&\quad=\frac{\left|\aleph''\left(\wp_1^{\mathfrak{d}}\right)\right|^q+\left|\aleph''\left(\varsigma^{\mathfrak{d}}\right)\right|^q}{2\mathfrak{d}},
\end{aligned}\tag{3.11}$$

and

$$\begin{aligned}
&\int_0^1\ell^{\mathfrak{d}-1}\left|\aleph''\left(\ell^{\mathfrak{d}}\varsigma^{\mathfrak{d}}+(1-\ell^{\mathfrak{d}})\wp_2^{\mathfrak{d}}\right)\right|^q d\ell\\
&\quad\leq\int_0^1\ell^{\mathfrak{d}-1}\left(\ell^{\mathfrak{d}}\left|\aleph''\left(\varsigma^{\mathfrak{d}}\right)\right|^q+(1-\ell^{\mathfrak{d}})\left|\aleph''\left(\wp_2^{\mathfrak{d}}\right)\right|^q\right)d\ell\\
&\quad=\frac{\left|\aleph''\left(\varsigma^{\mathfrak{d}}\right)\right|^q+\left|\aleph''\left(\wp_2^{\mathfrak{d}}\right)\right|^q}{2\mathfrak{d}}.
\end{aligned}\tag{3.12}$$

Combining (3.11) and (3.12) with (3.10), we get (3.9). So the proof is completed. □

Corollary 3.7. *If we choose* $\left|\aleph''\right|^q \leq M$ *in Theorem 3.6, we will have the following inequality:*

$$\begin{aligned}
&\left|\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+1}-\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+1}}{\eth\left(\beta+1\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph'\left(\varsigma^{\eth}\right)-\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta}-\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta}}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph\left(\varsigma^{\eth}\right)\\
&\quad-\frac{\Gamma(\beta)\mathrm{M}(\beta)}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[{}_{\wp_1^+}^{ABK_{\eth}}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})+{}_{\wp_2^-}^{ABK_{\eth}}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})\right]-\frac{2\left(1-\beta\right)\Gamma(\beta)}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\aleph(\varsigma^{\eth})\Bigg|\\
&\leq\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+2}}{\eth^{\beta}\left(\beta+1\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\frac{1}{p\left(p\left(\beta+1\right)+1\right)}+\frac{M}{q}\right]\\
&\quad+\frac{\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+2}}{\eth^{\beta}\left(\beta+1\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\frac{1}{p\left(p\left(\beta+1\right)+1\right)}+\frac{M}{q}\right].
\end{aligned}$$

Corollary 3.8. *If we take* $\eth=1$ *in Corollary 3.7, we will obtain the following inequality:*

$$\begin{aligned}
&\left|\left[\frac{\left(\varsigma-\wp_1\right)^{\beta+1}-\left(\wp_2-\varsigma\right)^{\beta+1}}{\left(\beta+1\right)\left(\wp_2-\wp_1\right)}\right]\aleph'\left(\varsigma\right)-\left[\frac{\left(\varsigma-\wp_1\right)^{\beta}-\left(\wp_2-\varsigma\right)^{\beta}}{\wp_2-\wp_1}\right]\aleph\left(\varsigma\right)\right.\\
&\quad\left.-\frac{\Gamma(\beta)\mathrm{M}(\beta)}{\wp_2-\wp_1}\left[{}_{\wp_1^+}^{ABK}I_{\varsigma}^{\beta}\aleph(\varsigma)+{}_{\wp_2^-}^{ABK}I_{\varsigma}^{\beta}\aleph(\varsigma)\right]-\frac{2\left(1-\beta\right)\Gamma(\beta)}{\wp_1-\wp_1}\aleph(\varsigma)\right|\\
&\leq\frac{\left(\varsigma-\wp_1\right)^{\beta+2}}{\left(\beta+1\right)\left(\wp_2-\wp_1\right)}\left[\frac{1}{p\left(p\left(\beta+1\right)+1\right)}+\frac{M}{q}\right]\\
&\quad+\frac{\left(\wp_2-\varsigma\right)^{\beta+2}}{\left(\beta+1\right)\left(\wp_2-\wp_1\right)}\left[\frac{1}{p\left(p\left(\beta+1\right)+1\right)}+\frac{M}{q}\right].
\end{aligned}$$

Theorem 3.9. *Let* $\beta>0$ *and* $\eth>0$. *Let* $\aleph: I\subset\mathbb{R}\to\mathbb{R}$ *is twice differentiable on* I^0, *such that* $\aleph\in X_c^p(\wp_1^{\eth},\wp_2^{\eth})$, *for all* $\wp_1^{\eth},\wp_2^{\eth}\in I$ *and* $\wp_1^{\eth}<\wp_2^{\eth}$. *If* $\aleph''\in L_1[\wp_1^{\eth},\wp_2^{\eth}]$, *and* $\left|\aleph''\right|^q$ *is a convex function, then for all* $\varsigma^{\eth}\in[\wp_1^{\eth},\wp_2^{\eth}]$ *and* $\beta\in(0,1]$, *then the following inequality holds:*

(3.13)

$$\begin{aligned}
&\left|\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+1}-\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+1}}{\eth\left(\beta+1\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph'\left(\varsigma^{\eth}\right)-\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta}-\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta}}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph\left(\varsigma^{\eth}\right)\\
&\quad-\frac{\Gamma(\beta)\mathrm{M}(\beta)}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[{}_{\wp_1^+}^{ABK_{\eth}}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})+{}_{\wp_2^-}^{ABK_{\eth}}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})\right]-\frac{2\left(1-\beta\right)\Gamma(\beta)}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\aleph(\varsigma^{\eth})\Bigg|\\
&\leq\left[\frac{1}{\eth\left(\beta+2\right)}\right]^{1-\frac{1}{q}}\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+2}}{\eth^{\beta-1}(\beta+1)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right.
\end{aligned}$$

$$\times \left[\frac{\left|\aleph''\left(\varsigma^{\eth}\right)\right|^q}{\eth\left(\beta+3\right)}+\frac{\left|\aleph''\left(\wp_1^{\eth}\right)\right|^q}{\eth\left(\beta+2\right)\left(\beta+3\right)}\right]^{\frac{1}{q}}$$

$$+\frac{\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+2}}{\eth^{\beta-1}(\beta+1)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\frac{\left|\aleph''\left(\varsigma^{\eth}\right)\right|^q}{\eth\left(\beta+3\right)}+\frac{\left|\aleph''\left(\wp_2^{\eth}\right)\right|^q}{\eth\left(\beta+2\right)\left(\beta+3\right)}\right]^{\frac{1}{q}}\Bigg],$$

where $q \geq 1$.

Proof. By using Lemma 3.1 and power-mean integral inequality, we get

(3.14)

$$\begin{aligned}
&\left|\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+1}-\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+1}}{\eth\left(\beta+1\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph'\left(\varsigma^{\eth}\right)-\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta}-\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta}}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph\left(\varsigma^{\eth}\right)\right.\\
&\quad\left.-\frac{\Gamma(\beta)\mathrm{M}(\beta)}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[{}^{ABK_{\eth}}_{\wp_1^+}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})+{}^{ABK_{\eth}}_{\wp_2^-}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})\right]-\frac{2\left(1-\beta\right)\Gamma(\beta)}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\aleph(\varsigma^{\eth})\right|\\
&\leq\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+2}}{\eth^{\beta-1}\left(\beta+1\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\int_0^1\left(\ell^{\eth}\right)^{\beta+1}\ell^{\eth-1}\left|\aleph''\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_1^{\eth}\right)\right|d\ell\\
&\quad+\frac{\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+2}}{\eth^{\beta-1}\left(\beta+1\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\int_0^1\left(\ell^{\eth}\right)^{\beta+1}\ell^{\eth-1}\left|\aleph''\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_2^{\eth}\right)\right|d\ell\\
&\leq\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+2}}{\eth^{\beta-1}\left(\beta+1\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\int_0^1\left(\ell^{\eth}\right)^{(\beta+1)}\ell^{\eth-1}d\ell\right]^{1-\frac{1}{q}}\\
&\quad\times\left[\int_0^1\left(\ell^{\eth}\right)^{(\beta+1)}\ell^{\eth-1}\left|\aleph''\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_1^{\eth}\right)\right|^q d\ell\right]^{\frac{1}{q}}\\
&\quad+\frac{\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+2}}{\eth^{\beta-1}\left(\beta+1\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\int_0^1\left(\ell^{\eth}\right)^{(\beta+1)}\ell^{\eth-1}d\ell\right]^{1-\frac{1}{q}}\\
&\quad\times\left[\int_0^1\left(\ell^{\eth}\right)^{(\beta+1)}\ell^{\eth-1}\left|\aleph''\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_2^{\eth}\right)\right|^q d\ell\right]^{\frac{1}{q}}.
\end{aligned}$$

By the convexity of $\left|\aleph''\right|^q$, we get

$$\begin{aligned}
(3.15)\quad&\int_0^1\left(\ell^{\eth}\right)^{(\beta+1)}\ell^{\eth-1}\left|\aleph''\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_1^{\eth}\right)\right|^q d\ell\\
&\leq\int_0^1\left(\ell^{\eth}\right)^{(\beta+1)}\ell^{\eth-1}\left(\ell^{\eth}\left|\aleph''\left(\varsigma^{\eth}\right)\right|+(1-\ell^{\eth})\left|\aleph''\left(\wp_1^{\eth}\right)\right|\right)^q d\ell\\
&=\frac{\left|\aleph''\left(\varsigma^{\eth}\right)\right|^q}{\eth(\beta+3)}+\frac{\left|\aleph''\left(\wp_1^{\eth}\right)\right|^q}{\eth(\beta+2)(\beta+3)},
\end{aligned}$$

and

$$\begin{aligned}(3.16)\quad &\int_0^1 \left(\ell^{\eth}\right)^{(\beta+1)} \ell^{\eth-1} \left|\aleph''\left(\ell^{\eth}\varsigma^{\eth}+(1-\ell^{\eth})\wp_2^{\eth}\right)\right|^q d\ell \\ &\leq \int_0^1 \left(\ell^{\eth}\right)^{(\beta+1)} \ell^{\eth-1}\left(\ell^{\eth}\left|\aleph''\left(\varsigma^{\eth}\right)\right|+(1-\ell^{\eth})\left|\aleph''\left(\wp_2^{\eth}\right)\right|\right)^q d\ell \\ &= \frac{\left|\aleph''\left(\varsigma^{\eth}\right)\right|^q}{\eth(\beta+3)}+\frac{\left|\aleph''\left(\wp_2^{\eth}\right)\right|^q}{\eth(\beta+2)(\beta+3)}.\end{aligned}$$

Combining (3.15) and (3.16) with (3.14), we get (3.13). So the proof is completed. □

Corollary 3.10. *If we choose* $\left|\aleph''\right| \leq M$ *in Theorem 3.9, we will obtain the following inequality:*

$$\begin{aligned}&\left|\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+1}-\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+1}}{\eth\left(\beta+1\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph'\left(\varsigma^{\eth}\right)-\left[\frac{\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta}-\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta}}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\right]\aleph\left(\varsigma^{\eth}\right)\\ &\quad-\frac{\Gamma(\beta)\mathrm{M}(\beta)}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[{}^{ABK_{\eth}}_{\wp_1^+}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})+{}^{ABK_{\eth}}_{\wp_2^-}I_{\varsigma}^{\beta}\aleph(\varsigma^{\eth})\right]-\frac{2\left(1-\beta\right)\Gamma(\beta)}{\eth\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\aleph(\varsigma^{\eth})\Bigg|\\ &\leq \frac{M}{\eth^{\beta}(\beta+1)\left(\beta+2\right)\left(\wp_2^{\eth}-\wp_1^{\eth}\right)}\left[\left(\varsigma^{\eth}-\wp_1^{\eth}\right)^{\beta+2}+\left(\wp_2^{\eth}-\varsigma^{\eth}\right)^{\beta+2}\right].\end{aligned}$$

4. Applications To Special Means

Let us consider the following special means for positive real numbers $\wp_1, \wp_2$ such that $\wp_1 \neq \wp_2$.

The arithmetic mean:

$$A(\wp_1,\wp_2)=\frac{\wp_1+\wp_2}{2}.$$

The logarithmic mean:

$$L(\wp_1,\wp_2)=\frac{\wp_2-\wp_1}{\ln\wp_2-\ln\wp_1}.$$

The generalized logarithmic mean:

$$L_c(\wp_1,\wp_2)=\left[\frac{\wp_2^{c+1}-\wp_1^{c+1}}{\left(\wp_2-\wp_1\right)(c+1)}\right]^{\frac{1}{c}},\quad c\in\mathbb{Z}\backslash\{-1,0\}.$$

Proposition 4.1. *Let* $m\in\mathbb{Z}\backslash\{-1,0\}$ *and* $\wp_1,\wp_2\in\mathbb{R}$, *such that* $0<\wp_1<\wp_2$. *Then for* $\varsigma\in[\wp_1,\wp_2]$ *and* $\frac{1}{p}+\frac{1}{q}=1$ *with* $q>1$, *the following inequality holds:*

$$\left|\varsigma^m-L_m^m\left(\wp_1,\wp_2\right)\right|\leq\frac{\left(\varsigma-\wp_1\right)^2}{\left(\wp_2-\wp_1\right)}\left[\frac{1}{p\left(p+1\right)}+\frac{m}{q}A\left(\wp_1^{(m-1)q},\varsigma^{(m-1)q}\right)\right]$$

$$+\frac{(\wp_2-\varsigma)^2}{(\wp_2-\wp_1)}\left[\frac{1}{p\,(p+1)}+\frac{m}{q}A\left(\varsigma^{(m-1)q},\wp_2^{(m-1)q}\right)\right].$$

Proof. Taking $\beta=\eth=1$ in Theorem 2.6 and $\aleph(\hbar)=\hbar^m$, we obtain the desired result. □

Proposition 4.2. *Let $m\in\mathbb{Z}\backslash\{-1,0\}$ and $\wp_1,\wp_2\in\mathbb{R}$, such that $0<\wp_1<\wp_2$. Then for $\varsigma\in[\wp_1,\wp_2]$ and $\frac{1}{p}+\frac{1}{q}=1$ with $q>1$, the following inequality holds:*

$$\begin{aligned}&\left|L_m^m(\wp_1,\wp_2)-\varsigma^m+m\,(\varsigma-A(\wp_1,\wp_2))\,\varsigma^{m-1}\right|\\&\quad\le\frac{(\varsigma-\wp_1)^3}{2\,(\wp_2-\wp_1)}\left[\frac{1}{p\,(2p+1)}+\frac{m(m-1)}{q}A\left(\wp_1^{(m-2)q},\varsigma^{(m-2)q}\right)\right]\\&\qquad+\frac{(\wp_2-\varsigma)^3}{2\,(\wp_2-\wp_1)}\left[\frac{1}{p\,(2p+1)}+\frac{m\,(m-1)}{q}A\left(\varsigma^{(m-2)q},\wp_2^{(m-2)q}\right)\right].\end{aligned}$$

Proof. Choosing $\beta=\eth=1$ in Theorem 3.6 and $\aleph(\hbar)=\hbar^m$, we get the desired result. □

Proposition 4.3. *Let $\wp_1,\wp_2\in\mathbb{R}$, such that $0<\wp_1<\wp_2$. Then for $\varsigma\in[\wp_1,\wp_2]$ and $\frac{1}{p}+\frac{1}{q}=1$ with $q>1$, the following inequality holds:*

$$\begin{aligned}\left|\varsigma^{-1}-L^{-1}(\wp_1,\wp_2)\right|&\le\frac{(\varsigma-\wp_1)^2}{(\wp_2-\wp_1)}\left[\frac{1}{p\,(p+1)}+\frac{1}{q}A\left(\wp_1^{-2q},\varsigma^{-2q}\right)\right]\\&\quad+\frac{(\wp_2-\varsigma)^2}{(\wp_2-\wp_1)}\left[\frac{1}{p\,(p+1)}+\frac{1}{q}A\left(\varsigma^{-2q},\wp_2^{-2q}\right)\right].\end{aligned}$$

Proof. Taking $\beta=\eth=1$ in Theorem 2.6 and $\aleph(\hbar)=\frac{1}{\hbar}$, we obtain the desired result. □

Proposition 4.4. *Let $\wp_1,\wp_2\in\mathbb{R}$, such that $0<\wp_1<\wp_2$. Then for $\varsigma\in[\wp_1,\wp_2]$ and $\frac{1}{p}+\frac{1}{q}=1$ with $q>1$, the following inequality holds:*

$$\begin{aligned}&\left|L^{-1}(\wp_1,\wp_2)-\varsigma^{-1}-(\varsigma-A(\wp_1,\wp_2))\,\varsigma^{-2}\right|\\&\quad\le\frac{(\varsigma-\wp_1)^3}{2\,(\wp_2-\wp_1)}\left[\frac{1}{p\,(2p+1)}+\frac{2}{q}A\left(\wp_1^{-3q},\varsigma^{-3q}\right)\right]\\&\qquad+\frac{(\wp_2-\varsigma)^3}{2\,(\wp_2-\wp_1)}\left[\frac{1}{p\,(2p+1)}+\frac{2}{q}A\left(\varsigma^{-3q},\wp_2^{-3q}\right)\right].\end{aligned}$$

Proof. Choosing $\beta=\eth=1$ in Theorem 3.6 and $\aleph(\hbar)=\frac{1}{\hbar}$, we get the desired result. □

5. Conclusions

The study focuses on exploring new Ostrowski-type inequalities for ABK-fractional integral operators. Several integral identities were derived, which are associated with first-and second-order derivatives. Employing ABK-fractional integral operators made significant advancements in the study of Ostrowski-type integral inequalities. In this paper, we extend the investigation of Ostrowski-type integral inequalities by incorporating convexity through ABK-fractional integral operators. We establish several new inequalities for various types of convex functions. Additionally, we present applications to special means for positive real numbers.

6. Future scope

We believe that our results can help interested researcher working in this field to continue examining different classes of convexity using fractional and quantum calculus. Ostrowski-type inequalities can also be derived using interval-valued functions and fuzzy number-valued functions. Also, from an application point of view, it will be exciting to see the utility of these types of inequalities in information theory, entropy and probability density functions.

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