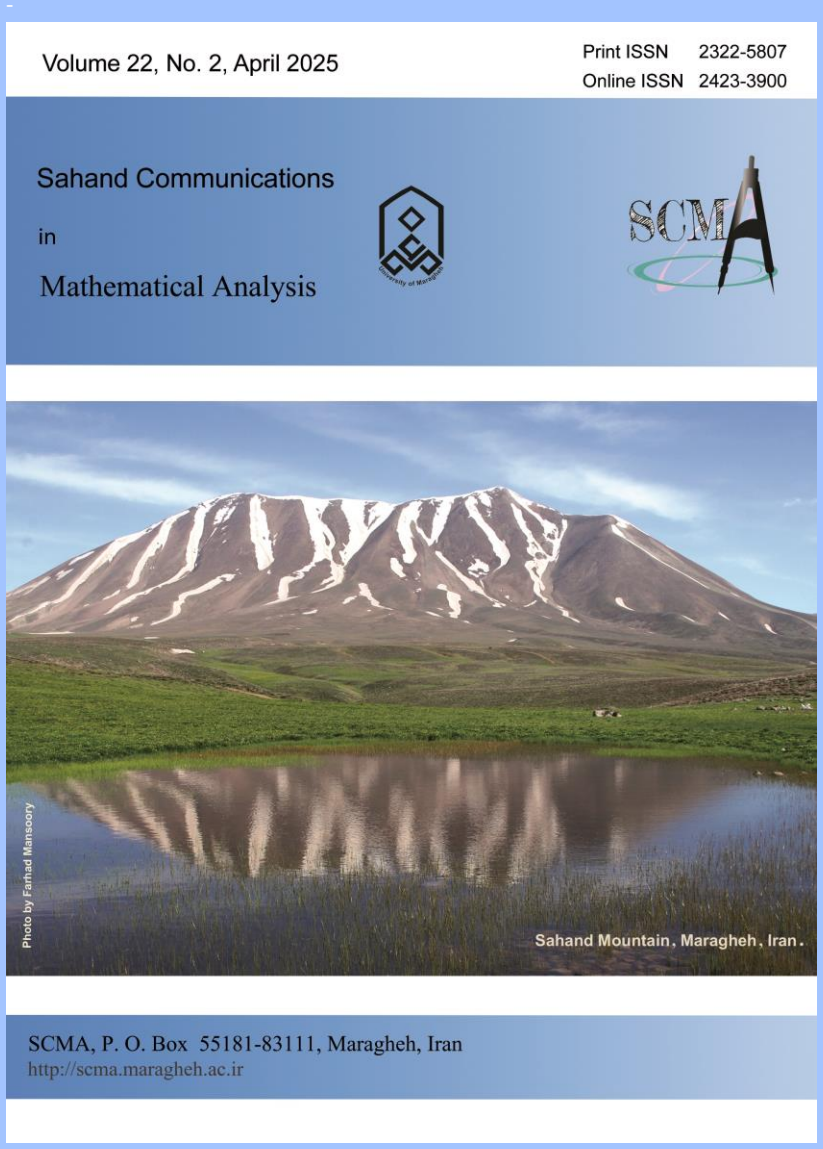
Contractive Fixed Points in a Rectangular Metric Space and Applications

G. Shanmuga Sundari and T. Phaneendra

Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 22
Number: 2
Pages: 47-72

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2024.2031468.1757



Sahand Communications in Mathematical Analysis (SCMA) Vol. 22 No. 2 (2025), 47-72
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2024.2031468.1757

Contractive Fixed Points in a Rectangular Metric Space and Applications

G. Shanmuga Sundari[1] and T. Phaneendra[2*]

Abstract. Branciari (2000) introduced the notion of a rectangular metric space (*rms*) as a generalization of a metric space and proved the well-known Banach's contraction mapping theorem in an *rms*, which was further generalized by Sarma et al. (2009) through a Ciric contraction. A fixed point p of a self-map f is a contractive fixed point (Edelstein, 1962), provided all the Picard's iterates $x, fx, f^2x, \ldots$ converge to p. In the first part of the paper, contractive fixed points of Banach and Ciric contractions are established in a rectangular metric space. Usually, it is shown that an appropriate Picard's iterative sequence with an arbitrary seed converges to a point, which is a unique fixed point. Rather than relying on the standard iterative procedure, in the next part of the paper, unique fixed points are obtained for Banach, Hardy-Roger and Ciric's contractions in a rectangular metric space through the rectangle inequality and the greatest lower bound property of real numbers. In the last part of the paper, two elegant problems of Volterra integral equations are presented with the necessary MATLAB interpretation.

1. Introduction

Let f be a self-map on a nonempty set X. The image $f(x)$ of $x \in X$ under f is denoted by fx. A point $u \in X$ is a fixed point of f, provided $fu = u$. Fixed points of a self-map f are given by the equation $fx - x = 0$. Whereas, a fixed point of a real-valued function f is identified graphically as the x-coordinate of the point P of intersection of the curve $y = f(x)$ with the straight line $y = x$.

2020 *Mathematics Subject Classification.* 54H25,47H10.
Key words and phrases. Rectangular metric space, Contraction type mappings, Infimum Property, Fixed point, Contractive fixed point, Volterra integral equation.
Received: 2024-06-07, Accepted: 2024-10-02.
* Corresponding author.

Example 1.1. The constant function $fx = 3/7$ has a unique fixed point, namely $3/7$. For $gx = x^3$ for all $x \in \mathbb{R}$, 0, 1 and -1 are three fixed points. However, the translation tx has no fixed points.

If (X, d) is a metric space, an obvious choice for $f : X \to X$ is

$$d(fx, fy) \leq \alpha d(x, y), \quad \text{for all } x, y \in X, \tag{1.1}$$

where $\alpha \in (0, 1)$, called a Banach contraction on X.

Remark 1.2. A contraction is necessarily continuous.

However, not all continuous mappings are contractions. For instance, $fx = x$ forall $x \in \mathbb{R}$ is continuous but not a contraction.

Theorem 1.3 (Banach contraction theorem)**.** *Let (X, d) be a complete metric space. Then a Banach contraction given by* (1.1) *has a unique fixed point.*

Remark 1.4. If u and v are distinct fixed points of f, (1.1) would imply a contradiction that $d(u, v) = d(fu, fv) \leq \alpha d(u, v) < d(u, v)$. Therefore, the fixed point of f is unique, if it exists.

The proof of Theorem 1.3 begins with the definition of an f-orbit. If $x_0 \in X$ is arbitrary, the *f-orbit* $\mathcal{O}_f(x_0)$ at x_0 is the sequence $\left\{x_0, fx_0, f^2x_0, ^3x_0, \ldots\right\}$ of iterates, where $f^n = f \circ f \circ \cdots \circ f$, $n \geq 1$. Repeatedly using (1.1), it was shown that

$$d\left(f^{n+1}x_0, f^n x_0\right) \leq \alpha^n d\left(fx_0, x_0\right), \quad \text{for all } n \geq 1$$

and

$$d(f^m x_0, f^n x_0) \leq \frac{\alpha^n d\left(fx_0, x_0\right)}{1 - \alpha}, \quad \text{for all } m \geq 1, n \geq 1.$$

Then for any $\varepsilon > 0$, choosing a natural number n_0 such that $\alpha^{n_0} < \varepsilon(1 - \alpha)/d\left(fx_0, x_0\right)$, it follows that $d(f^m x_0, f^n x_0) < \varepsilon$ for all $m \geq n_0$, $n \geq n_0$. Thus $\langle f^n x_0\rangle_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{O}_f(x_0) \subset X$. Since X is complete, $f^n x_0 \to u$ for some $u \in X$.

In view of Remark 1.2 and Remark 1.4, u will be a unique fixed point for f. The iterative procedure suggests that all the bounded orbits converge to u. This leads to the introduction of a contractive-fixed point of a self-map in a metric space.

Definition 1.5 (Edelstein, [5])**.** A fixed point u of f is a *contractive fixed point*, if every $\mathcal{O}_f(x_0)$ converges to u.

Example 1.6. Consider the complete metric space $(\mathbb{R}, d_{\mathrm{u}})$ and $f : \mathbb{R} \to \mathbb{R}$, given by $fx = 2x/3$. Then f is a contraction on $\mathbb{R}$ with constant $\alpha = 2/3$. By Theorem 1.3, there exists exactly one $u \in X$ with $fu = u$. Note that $\mathcal{O}_f(x_0) = \left\{x_0, \frac{2}{3}x_0, \left(\frac{2}{3}\right)^2 x_0, \ldots, \left(\frac{2}{3}\right)^n x_0, \ldots\right\} \to 0$ as $n \to \infty$

for each $x_0 \in X$, and 0 is the only fixed point, showing that 0 is a contractive fixed point for f.

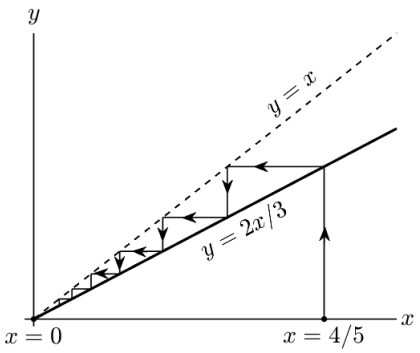


FIGURE 1. $\mathcal{O}_f(x_0)$ converging to the fixed point $x = 0$

Banach's fixed point theorem has several applications. It has been a source of existence and uniqueness theorems for Fredholm and Volterra integral equations, and plays a vital role in solving systems of linear algebraic equations and its sufficient conditions for convergence and error bounds [10]. Also the proofs of Inverse Function Theorem and Implicit Function Theorem involve Banach's theorem ([26], p. 221-223).

Bakhtin [1] introduced a b-metric by generalizing the triangle inequality of the metric d as

$$d_b(x, y) \leq b(d_b(x, z) + d_b(z, y)), \quad \text{for all } x, y, z \in X, \tag{1.2}$$

where $b \geq 1$. A rectangular metric space is another generalization, proposed by Branciari [2], wherein the rectangle inequality

$$d_r(x, y) \leq d_r(x, u) + d_r(u, v) + d_r(v, y), \tag{1.3}$$

is used in place of the triangle inequality. Note that u or v is distinct from x and y in (1.3). If $u = v$, then (1.3) reduces to the triangle inequality.

A b-metric space and a rectangular metric space are abbreviated as *bms* and *rms* respectively, through out the paper.

The notions of *bms* and *rms* are independent of each other, as asserted in the following examples:

Example 1.7 (*bms* - not *rms*)**.** Let $X = \{1, 2, 3, 4, 5\}$ with $d_b : X \times X \to [0, \infty)$ given by

$$d_b(x, y) = 0, \quad \text{for } x = y,$$
$$d_b(1, 5) = d_b(5, 1) = 25,$$

$$d_b(1,3) = d_b(3,1) = d_b(2,4) = d_b(4,2) = 10,$$
$$d_b(1,4) = d_b(4,1) = d_b(2,3) = d_b(3,2) = 15,$$
$$d_b(2,5) = d_b(5,2) = d_b(3,5) = d_b(5,3) = 20,$$
$$d_b(1,2) = d_b(2,1) = d_b(3,4) = d_b(4,3) = d_b(4,5) = d_b(5,4) = 5.$$

Then (X, d_b) is a b-metric space with $b = 2$. However, since $d_b(1,5) = 25 > 20 = d_b(1,2) + d_b(2,4) + d_b(4,5)$, X is not an *rms*.

Example 1.8 (*rms* - not *bms*)**.** Let $X = \{1,2,3,4\}$, defining $d_r : X \times X \to [0,\infty)$ by

$$d_r(x,y) = 0, \quad \text{for } x = y,$$
$$d_r(1,2) = d_r(2,1) = d_r(3,4) = d_r(4,3) = 20,$$
$$d_r(1,3) = d_r(3,1) = d_r(2,4) = d_r(4,2) = 110,$$
$$d_r(1,4) = d_r(4,1) = d_r(2,3) = d_r(3,2) = 280.$$

Then (X, d_r) is an *rms* but not a *bms* with $b = 2$, since

$$d_r(1,4) = 280 > 260 = 2(d_r(1,2) + d_r(2,4)).$$

Note that the domains of metric, b-metric and rectangular metric are the Cartesian products $X \times X$. However, Gahler [6] and Mustafa and Sims [12] introduced 2-metric space and G-metric space respectively, whereas Sedhgi et al. [23] introduced an S-metric space as generalizations of a metric space to three dimensions, with the domain $X \times X \times X$. A 2-metric represents the area [13], whereas a G-metric represents the perimeter of a triangle in the Euclidean 3-space [14].

The second author did an extensive research with others on contractive fixed points in the above mentioned generalized metric spaces. For instance, one can refer to [8], [9], [13], [14], [15], [16], [17], [18], [19], [20] and [21] for contractive-fixed points of Banach and other various generalized contractions in 2-metric and G-metric spaces. Contractive fixed points in a b-metric space can be found in [25].

Usually, fixed point theorems are proved by iterative procedure. However, Joseph and Kwack [7] initiated the idea of using the greatest lower bound property (or the infimum property) of real numbers to obtain Cauchy sequences. In proving some fixed point theorems, including Banach's one in a metric space, they employed the triangle inequality repeatedly along with the infimum property, and no Picard's iterations are involved. Fixed points were obtained for several contraction conditions in G, 2 and S-metric spaces in the references cited earlier.

In search of applications of the infimum property, the authors of [8], [9], [13], [14], [15], [16], [17], [18], [19], [20] and [21] refered to [7] for

obtaining contractive fixed points. The simplicity in this technique definitely paves the way for contractive fixed points in various generalized metric structures. This is the main motivation for us to establish unique fixed points in the next section, for Banach, Hardy-Roger and Ciric's contractions in a rectangular metric space through the repeated use of the rectangle inequality.

Useful topological properties of a rectangular metric space (*rms*) are presented and contractive-fixed points of Banach and Ciric contractions on an *rms* are obtained in Section 2 of the paper. Fixed point theorems are proved in Section 3 for Banach, Hardy-Roger and Ciric contractions on an *rms* with repeated use of the rectangle inequality without involving any iterations. The infimum property of real numbers is only employed to establish Cauchy sequences in the proofs. Finally, the existence and uniqueness of solutions of Volterra integral equations are solved as related applications with necessary MATLAB interpretation.

2. Contractive Fixed Points in a Rectangular Metric Space

We recall from the previous section that a rectangular metric was introduced by Branciari [2] as follows:

Definition 2.1. Let X be a non-empty set. A mapping $d_r : X \times X \to [0, \infty)$ is a *rectangular metric* on X, provided the following conditions hold good:

(rm1) $d_r(x, y) = 0$ implies that $x = y$
(rm2) $x = y$ implies that $d_r(x, y) = 0$
(rm3) $d_r(x, y) = d_r(y, x)$ for all $x, y \in X$
(rm4) $d_r(x, y) \leq d_r(x, u) + d_r(u, v) + d_r(v, y)$ for all $x, y, u, v \in X$, where $u \neq v$, and each being distinct from x and y.

The condition (rm4) is called the *rectangle inequality* of d_r, and the pair (X, d_r) denotes a *rectangular metric space* (*rms*).

If (X, d) is a metric space, from the triangle inequality, we get

$$d(x, y) \leq d(x, u) + d(u, y) < d(x, u) + d(u, v) + d(v, y),$$

where $u \neq v$. Thus X is an *rms*.

Example 2.2. Write $X = \{1, 3, 5, 7\}$. Define

$$d_r(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } (x \in \{1, 3\}, y = 5) \text{ or } (x = 5, y \in \{1, 3\}) \\ 2, & \text{if } (x \in \{1, 3, 5\}, y = 7) \text{ or } (x = 7, y \in \{1, 3, 5\}) \\ 3, & \text{if } (x = 1, y = 3) \text{ or } (x = 3, y = 1). \end{cases} \tag{2.1}$$

Then (X, d_r) is an *rms*. Note that $d_r(1,5) + d_r(3,5) = 1 + 1 < 3 = d_r(1,3)$, that is the triangle inequality fails to hold good. Thus (X, d_r) is not a metric space.

The topological notions related to an *rms* can be extracted in a standard way similar to those of a metric space.

The following is an interesting example from [22], related to an *rms* (X, d_r):

Example 2.3. Let $A = \{0, 2\}$, $B = \{1/m : m \in \mathbb{N}\}$, $X = A \cup B$ and

$$d_r(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y \text{ and } \{x,y\} \subset A \text{ or } \{x,y\} \subset B, \\ y, & \text{if } x \in A \text{ and } y \in B, \\ x, & \text{if } x \in B \text{ and } y \in A. \end{cases} \tag{2.2}$$

(a) Then $\mathcal{B}_{2/3}(1/3) = \{0, 2, 1/3\}$ is the open ball with centre at $1/3$ and radius $2/3$, which is not open, since there exists no $\delta > 0$ such that $\mathcal{B}_\delta(0) \subset \mathcal{B}_{2/3}(1/3)$.
(b) Since there exists no $\delta > 0$ such that $\mathcal{B}_\delta(0) \cap \mathcal{B}_\delta(2) = \phi$, the respective topology is not Hausdorff in an *rms*.
(c) The sequence $\langle 1/n \rangle_{n=1}^{\infty}$ converges to both 0 and 2, and it is not a Cauchy sequence.
(d) Since $\lim_{n\to\infty} 1/n = 0$ and $\lim_{n\to\infty} d_r(1/n, 1/2) \neq d_r(0, 1/2)$, d_r is not jointly continuous at $(0, 1/2)$.

Example 2.3 reveals that a convergent sequence need not be Cauchy and may have more than one limit in an *rms*. However, the uniqueness of the limit of a convergent sequence is ensured, whenever it is a Cauchy sequence, as shown in the following:

Lemma 2.4. *Let (X, d_r) be a rectangular metric space, and $\langle u_n \rangle_{n=1}^{\infty} \subset X$ be both Cauchy and convergent in X. Then its limit is unique.*

Proof. Let $u_n \to u$ as $n \to \infty$ in X. Then for every $\varepsilon > 0$, there exists a natural number n_1 such that

$$d_r(u_n, u) < \frac{\varepsilon}{3}, \quad \text{for all } n \geq n_1. \tag{2.3}$$

Suppose that $\langle u_n \rangle_{n=1}^{\infty}$ also converges to some $q \in X$ so that

$$d_r(u_n, q) < \frac{\varepsilon}{3}, \text{for all } n \geq n_2 \text{ for some natural number } n_2. \tag{2.4}$$

Since $\langle u_n \rangle_{n=1}^{\infty}$ is Cauchy, finally we can find a natural number n_3 with

$$d_r(u_n, u_m) < \frac{\varepsilon}{3}, \quad \text{for all } n \geq n_2, m \geq n_3. \tag{2.5}$$

Choose $n_0 = \min\{n_1, n_2, n_3\}$. Then from the rectangle inequality (rm4), and (2.3)-(2.5), it follows that

$$d_r(u,q) \leq d_r(u,u_n) + d_r(u_n,u_m) + d_r(u_m,q) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

for all $n > n_0$. This proves that $u = q$. □

Lemma 2.4 is validated by the following example:

Example 2.5. Consider the *rms* given in Example 2.2. Let $\langle u_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in X. Given ε, $\exists$ a positive integer n_0 with $d_r(u_n, u_m) < \varepsilon$ whenever $n \geq n_0$ and $m \geq n_0$. In view of (2.1), $d_r(u_n, u_m) < \varepsilon$ only if $d_r(u_n, u_m) = 0$ for all $n, m \geq n_0$. Thus $d_r(u_n, u_m) < \varepsilon$ only if $u_{n_0} = u_{n_0+1} = u_{n_0+2} = \cdots = l$ for some $l \in X$. In other words, $u_n \to l$. Therefore, from Lemma 2.4, the limit l is unique.

A brief comparison of the properties of a metric space and an *rms* mentioned above is summarized below:

(a) Every open ball is an open set in a metric space, whereas an open ball is not necessarily be open set in an *rms* (Ex. 2.3 (a)).
(b) Rectangular metric topology is not Hausdorff unlike the metric topology (Ex. 2.3 (b)).
(c) The limit of a convergent sequence is unique in a metric space. However, a convergent sequence in an *rms* may have more than one limit (Ex. 2.3 (c)).
(d) Convergent sequences are Cauchy in a metric space. But a convergent sequence need not be Cauchy in an *rms* (Ex. 2.3 (c)).
(e) A metric $d(x, y)$ is known to be jointly continuous, whereas a rectangular metric $d_r(x, y)$ is not jointly continuous (Ex. 2.3 (d)).

We obtain contractive fixed points for some contraction types in an *rms*:

Theorem 2.6. *Let (X, d_r) be an rms and f, a Banach's contraction satisfying*

$$d_r(fx, fy) \leq \alpha\, d_r(x, y), \quad \textit{for all } x, y \in X, \tag{2.6}$$

where $\alpha \in (0, 1)$. If $fu = u$ for exactly one $u \in X$, then u will be contractive fixed point for f.

Proof. Note that $u = fu = f^2u = \cdots = f^nu = \cdots$. Let $\xi_0 \in X$ be arbitrary. Define $\xi_n = f^n\xi_0$ for $n \geq 1$. Writing $x = f^{n-1}\xi_0 = \xi_{n-1}$ and $y = f^{n-1}p$ in (2.6), we see that

$$d_r(\xi_n, u) \leq d_r\left(ff^{n-1}\xi_0, ff^{n-1}u\right) \leq \alpha\, d_r\left(f^{n-1}\xi_0, f^{n-1}p\right) = \alpha\, d_r(\xi_{n-1}, u),$$

and hence by induction,

$$d_r(\xi_n, u) \leq \alpha^n d_r(\xi_0, u), \quad \text{for } n \geq 1. \tag{2.7}$$

Now (rm4) with $x = \xi_0$, $y = u$, $y = f\xi_0$, $z = fu$ and (2.6) with $x = \xi_0$, $y = u$ imply that

$$d_r(\xi_0, u) \leq d_r(\xi_0, f\xi_0) + d_r(f\xi_0, fu) \leq d_r(\xi_0, f\xi_0) + \alpha\, d_r(\xi_0, u)$$

or $d_r(\xi_0, u) \leq \frac{1}{1-\alpha} d_r(\xi_0, f\xi_0)$ for each $\xi_0 \in X$. Inserting this in (2.7), we get

$$d_r(\xi_n, u) \leq \left\{\frac{\alpha^n}{1-\alpha}\right\} d_r(\xi_0, f\xi_0), \quad \text{for each } \xi_0 \in X,\ n = 1, 2, \ldots. \tag{2.8}$$

As $n \to \infty$, (2.8) implies that $d_r(\xi_n, u) = d_r(f^n\xi_0, u) \to 0$ for each $\xi_0 \in X$, proving the contractive fixed point. □

Example 2.7. Define $f : X \to X$ by

$$fx = \begin{cases} 5, & \text{if } x \in \{1, 3, 5\} \\ 1, & \text{if } x = 7, \end{cases}$$

where X as given in Example 2.2. Then

$$d_r(fx, fy) = \begin{cases} 0, & \text{if } (x \in \{1,3,5\}, y \in \{1,3,5\}) \\ & \text{or } (x = 7, y = 7) \\ 1, & \text{if } (x \in \{1,3,5\}, y = 7) \\ & \text{or } (x = 7, y \in \{1,3,5\}). \end{cases} \tag{2.9}$$

It follows from (2.1) and (2.9) that $d_r(fx, fy) \leq \frac{1}{2} d_r(x, y)$ for all $x, y \in X$. That is f is a Banach's contraction with $\alpha = 1/2$. Also,

$$\mathcal{O}_f(x) = \begin{cases} \langle 1, f1, f^2 1, f^3 1, \ldots \rangle, & \text{if } x = 1 \\ \langle 3, f3, f^2 3, f^3 3, \ldots \rangle, & \text{if } x = 3 \\ \langle 5, f1, f^2 5, f^3 5, \ldots \rangle, & \text{if } x = 5 \\ \langle 7, f7, f^2 7, f^3 7, \ldots \rangle, & \text{if } x = 7 \end{cases}$$

$$= \begin{cases} \langle 1, 5, 5, 5, \ldots \rangle, & \text{if } x = 1 \\ \langle 5, 5, 5, 5, \ldots \rangle, & \text{if } x = 3 \\ \langle 5, 5, 5, 5, \ldots \rangle, & \text{if } x = 5 \\ \langle 7, 1, 5, 5, \ldots \rangle, & \text{if } x = 7. \end{cases}$$

This yields that $\mathcal{O}_f(x) \to 5$ for each $x \in X$. Thus, 5 is a contractive-fixed point, since it is the only fixed point for f.

The generalized setting of Theorem 2.6 is:

Theorem 2.8. *Let (X, d_r) be an rms and u be a unique fixed point of the Hardy-Roger contraction $f : X \to X$ which satisfies*

$$d_r(fx, fy) \le a_1 d_r(x, fx) + a_2 d_r(y, fy) + a_3 d_r(x, fy) + a_4 d_r(y, fx) + a_5 d_r(x, y), \quad \text{for all } x, y \in X, \tag{2.10}$$

where a_j, $1 \le j \le 5$ are nonnegative, not all being zero such that $0 < \sum_{j=1}^{5} a_j < 1$. Then u will be a contractive-fixed point.

Proof. Let $u_0 \in X$. Write $x = f^{n-1}u_0$ and $y = u = f^{n-1}u$ in (2.10),

$$\begin{aligned} d_r\left(f^n u_0, u\right) &= d_r\left(ff^{n-1}u_0, ff^{n-1}u\right) \\ &\le a_1 d_r\left(f^{n-1}u_0, ff^{n-1}u_0\right) + a_2 d_r\left(f^{n-1}u, ff^{n-1}u\right) \\ &\quad + a_3 d_r\left(f^{n-1}u_0, ff^{n-1}u\right) + a_4 d_r\left(f^{n-1}u, ff^{n-1}u_0\right) \\ &\quad + a_5 d_r\left(f^{n-1}u_0, f^{n-1}u\right) \\ &\le a_1 d_r\left(f^{n-1}u_0, f^n u_0\right) + a_2.0 \\ &\quad + a_3 d_r\left(f^{n-1}u_0, u\right) + a_4 d_r\left(u, f^n u_0\right) + a_5 d_r\left(f^{n-1}u_0, u\right) \\ &\le a_1\left[d_r\left(f^{n-1}u_0, u\right) + d_r\left(u, f^n u_0\right)\right] \\ &\quad + (a_3 + a_5) d_r\left(f^{n-1}u_0, u\right) + a_4 d_r\left(u, f^n u_0\right) \end{aligned}$$

or

$$(1 - a_1 - a_4) d_r\left(f^n u_0, u\right) \le (a_1 + a_3 + a_5) d_r\left(f^{n-1}u_0, u\right). \tag{2.11}$$

Similarly, we have

$$(1 - a_2 - a_3) d_r\left(u, f^n u_0\right) \le (a_2 + a_4 + a_5) d_r\left(u, f^{n-1}u_0\right). \tag{2.12}$$

Adding (2.11) and (2.12), and then simplifying,

$$d_r\left(u, f^n u_0\right) \le \eta\, d_r\left(u, f^{n-1}u_0\right), \quad \text{for } n \ge 1.$$

Thus by induction,

$$d_r\left(u, f^n u_0\right) \le \eta^n d_r(u, u_0), \quad \text{for } n \ge 1, \tag{2.13}$$

where $\eta = \frac{a_1 + a_2 + a_3 + a_4 + 2a_5}{2 - a_1 - a_2 - a_3 - a_4} < 1$, in view of the choice of a_j's. Finally, by (rm4) and $x = u$ and $y = u_0$ in (2.10), we obtain

$$\begin{aligned} d_r(u, u_0) &\le d_r(u, fu) + d_r(fu, fu_0) + d_r(fu_0, u_0) \\ &\le 0 + d_r(fu_0, u_0) + a_1 d_r(u, fu) + a_2 d_r\left(u_0, fu_0\right) \\ &\quad + a_3 d_r(u, fu_0) + a_4 d_r(u_0, fu) + a_5 d_r(u, u_0) \\ &\le (1 + a_2) d_r(fu_0, u_0) + a_1.0 + (a_4 + a_5) d_r(u, u_0) \\ &\quad + a_3[d_r(u, u_0) + d_r\left(u_0, fu_0\right)] \end{aligned}$$

or

$$d_r(u, u_0) \leq \nu d_r(fu_0, u_0), \tag{2.14}$$

where $\nu = \frac{1+a_2+a_3}{1-a_3-a_4-a_5}$. Inserting (2.14) in (2.13), and proceeding with the limit as $n \to \infty$, it follows that $d_r\left(f^n u_0, u\right) \to 0$ for every $u_0 \in X$, proving that u is a contractive-fixed point. □

With $a_1 = \cdots = a_4 = 0$, the condition (2.10) becomes the Banach's contraction (2.6), and Theorem 2.6 is a special case of Theorem 2.8. Now with $a_3 = a_4 = 0$ and in (2.10), we have Reich-type contraction

$$d_r(fx, fy) \leq a_1 d_r(x, fx) + a_2 d_r(y, fy) + a_5 d_r(x, y) \tag{2.15}$$

for all $x, y \in X$, where $a_1 + a_2 + a_5 < 1$ and we have

Corollary 2.9. *Let u be a unique fixed point for f given in* (2.15)*. Then it is a contractive-fixed point.*

If $a_1 = a_2 = \kappa$ and $a_3 = a_4 = a_5 = 0$ in (2.10), we have Kannan-type

$$d_r(fx, fy) \leq \kappa \left(d_r(x, fx) + d_r(y, fy)\right), \quad \text{for all } x, y \in X, \tag{2.16}$$

where $0 \leq \kappa < 1/2$ and

Corollary 2.10. *Let u be the only fixed point for f given in* (2.16)*. Then it is a contractive-fixed point.*

Whereas, (2.10) with $a_1 = a_2 = a_5 = 0$ and $a_3 = a_4 = \kappa$ gives a Chatterjea-type

$$d_r(fx, fy) \leq \kappa \left(d_r(x, fy) + d_r(y, fx)\right), \quad \text{for all } x, y \in X, \tag{2.17}$$

where $0 \leq \kappa < 1/2$ and

Corollary 2.11. *Let u be a unique fixed point for f given in* (2.17)*. Then it is a contractive-fixed point.*

Finally, if $a_1 = a_2 = \kappa_1$, $a_3 = a_4 = \kappa_2$ and $a_5 = \kappa_3$ in Theorem 2.8 we have

Corollary 2.12. *Let (X, d_r) be an rms and u be the only fixed point of $f : X \to X$, with*

$$\begin{aligned} d_r(fx, fy) \leq \kappa_1 (d_r(x, fx) + d_r(y, fy)) + \kappa_2 (d_r(x, fy) \\ + d_r(y, fx)) + \kappa_3 d_r(x, y), \quad \textit{for all } x, y \in X, \end{aligned} \tag{2.18}$$

where $0 \leq 2\kappa_1 + 2\kappa_2 + \kappa_3 < 1$. Then u is a contractive-fixed point.

Since the maximum of a finite set of numbers cannot be less than each of its elements, the right hand side of (2.10) is less than or equal to qM, where $q = \sum_{i=1}^{5} a_i$ so that

$$d_r(fx, fy) \leq qM, \quad \text{for all } x, y \in X, \tag{2.19}$$

and

$$M = \max\left\{d_r(x,y), d_r(x,fx), d_r(y,fy), d_r(x,fy), d_r(y,fx)\right\}. \tag{2.20}$$

In other words, (2.19) is weaker than (2.10). Naturally, an additional condition is required in Theorem 2.8 when (2.10) is replaced by (2.19).

Theorem 2.13. *Let u be a unique fixed point of a Ciric's quasi-contraction f on an rms (X, d_r) with the choice* (2.19)*. Suppose that $\mathcal{O}_f(u_0)$ is bounded for each $u_0 \in X$. Then $\mathcal{O}_f(u_0) \to u$ for each $u_0 \in X$. That is, u is a contractive-fixed point.*

Proof. Suppose that $\mathcal{O}_f(u_0)$ is unbounded. Choose a positive integer n with

$$d_r\left(f^n u_0, u_0\right) > \kappa \max\left\{d_r\left(f^i u_0, f^j u_0\right) : 0 \leq i \leq n-1, 0 \leq j \leq n-1\right\}, \tag{2.21}$$

where $\kappa = \max\left\{\frac{2q}{1-2q}, 2q\right\}$.

Now

$$\begin{aligned} d_r\left(f^n u_0, u_0\right) &\leq d_r\left(f f^{n-1} u_0, f u_0\right) + d_r(f u_0, u_0) \\ &\leq q \max\left\{d_r\left(f^{n-1} u_0, u_0\right), d_r\left(f^{n-1} u_0, f f^{n-1} u_0\right),\right. \\ &\quad d_r\left(u_0, f u_0\right), d_r\left(f^{n-1} u_0, f u_0\right), d_r(f f^{n-1} u_0, u_0)\Big\} \\ &\quad + d_r(f u_0, u_0) \\ &= 2qR, \end{aligned} \tag{2.22}$$

where

$$\begin{aligned} R = \max\left\{d_r\left(f^{n-1} u_0, u_0\right), d_r\left(f^{n-1} u_0, f^n u_0\right), d_r\left(u_0, f u_0\right),\right. \\ \left. d_r\left(f^{n-1} u_0, f u_0\right), d_r\left(f^n u_0, u_0\right)\right\}. \end{aligned}$$

If $R = d_r\left(f^{n-1} u_0, u_0\right), d_r\left(u_0, f u_0\right), d_r\left(f^{n-1} u_0, f u_0\right)$ or $d_r\left(f^n u_0, u_0\right)$, then (2.21) and (2.22) imply that

$$d_r\left(f^n u_0, u_0\right) \leq 2qR < \kappa R < d_r\left(f^n u_0, u_0\right), \tag{2.23}$$

which is a contradiction. Whereas, if $R = d_r\left(f^{n-1} u_0, f^n u_0\right)$, by the rectangle inequality, (2.21) and (2.22)

$$\begin{aligned} d_r\left(f^n u_0, u_0\right) &\leq 2q d_r\left(f^{n-1} u_0, f^n u_0\right) \\ &\leq 2q\left[d_r\left(f^{n-1} u_0, u_0\right) + d_r(u_0, f^n u_0)\right] \end{aligned}$$

$$\begin{aligned}
&\leq \frac{2q}{1-2q} d_r\left(f^{n-1}u_0, u_0\right) \\
&\leq \kappa d_r\left(f^{n-1}u_0, u_0\right) \\
&< d_r\left(f^n u_0, u_0\right),
\end{aligned}$$

which is also a contradiction. Thus for each $u_0 \in X$, $\mathcal{O}_f(u_0)$ must be bounded so that its diameter $\delta[\mathcal{O}_f(u_0)] = \sup_{i\geq 0,\, j\geq 0} d_r(f^i u_0, f^j u_0)$ is finite, say η. Then for each n,

$$\begin{aligned}
\delta\left[\mathcal{O}_f(u_0, n)\right] &= \max\{d_r\left(f^i u_0, f^j u_0\right) : 0 \leq i \leq n, 0 \leq j \leq n\} \\
&= \max\{d_r(u_0, f^i u_0), d_r\left(f^j u_0, f^{j+1} u_0\right) : \\
&\quad 1 \leq i \leq n,\, 1 \leq j \leq n-1\} \\
&\leq \eta.
\end{aligned} \tag{2.24}$$

Since f is a quasi-contraction,

$$\begin{aligned}
d_r\left(f^j u_0, f^{j+1} u_0\right) &\leq q \max\left\{d_r\left(f^{j-1}u_0, f^j u_0\right), d_r\left(f^{j-1}u_0, f^j u_0\right),\right. \\
&\quad d_r\left(f^j u_0, f^{j+1} u_0\right), d_r\left(f^{j-1}u_0, f^{j+1} u_0\right), \\
&\quad \left. d_r\left(f^j u_0, f^j u_0\right)\right\} \\
&= q \max\left\{d_r\left(f^{j-1}u_0, f^j u_0\right),\right. \\
&\quad \left. d_r\left(f^{j-1}u_0, f^{j+1} u_0\right) : 1 \leq j \leq n-1\right\} \\
&< \max\left\{d_r\left(u_0, f^j u_0\right) : 1 \leq j \leq n\right\}.
\end{aligned} \tag{2.25}$$

From (2.24) and (2.25), it follows that

$$\delta\left[\mathcal{O}_f(u_0, n)\right] = \max\left\{d_r\left(u_0, f^j u_0\right) : 1 \leq j \leq n\right\},$$

for each n, $\exists$ a positive integer $\kappa \leq n$ with

$$\delta\left[\mathcal{O}_f(u_0, n)\right] = d_r\left(u_0, f^\kappa u_0\right). \tag{2.26}$$

But, by the rectangle inequality of d_r, (2.19) and (2.26), we have

$$\begin{aligned}
\delta\left[\mathcal{O}_f(u_0, n)\right] &= d_r\left(u_0, f^\kappa u_0\right) \\
&\leq d_r\left(u_0, f u_0\right) + d_r\left(f u_0, f^2 u_0\right) + d_r\left(f^2 u_0, f^\kappa u_0\right) \\
&\leq d_r\left(u_0, f u_0\right) + q \max\left\{d_r\left(u_0, f u_0\right), d_r\left(u_0, f^2 u_0\right)\right\} \\
&\quad + q\delta\left[\mathcal{O}_f\left(f u_0, \kappa - 1\right)\right] \\
&\leq (1+q) \max\left\{d_r\left(u_0, f u_0\right), d_r\left(u_0, f^2 u_0\right)\right\} \\
&\quad + q d_r\left(f u_0, f^\mu f u_0\right) \qquad \text{for some } \mu \leq \kappa - 1 \\
&\leq (1+q) \max\left\{d_r\left(u_0, f u_0\right), d_r\left(u_0, f^2 u_0\right)\right\} \\
&\quad + q^2 \delta\left[\mathcal{O}_f\left(u_0, \mu + 1\right)\right]
\end{aligned}$$

$$\leq (1+q)\max\left\{d_r\left(u_0, fu_0\right), d_r\left(u_0, f^2u_0\right)\right\} + q^2\delta\left[\mathcal{O}_f(u_0, n)\right],$$

which on simplification, implies that

(2.27)

$$\delta\left[\mathcal{O}_f(u_0, n)\right] \leq \frac{1}{1-q}\max\left\{d_r\left(u_0, fu_0\right), d_r\left(u_0, f^2u_0\right)\right\}, \quad \text{for each } n.$$

Also, by induction and (2.27), one can easily show that

(2.28)

$$\begin{aligned} d_r\left(f^nu_0, f^{n+1}u_0\right) &\leq q^n\,\delta[\mathcal{O}_f(u_0, n+1)] \\ &\leq \frac{q^n}{1-q}\max\{d_r\left(u_0, fu_0\right), d_r\left(u_0, f^2u_0\right)\}, n \geq 1. \end{aligned}$$

Finally, by the rectangle inequality of d_r, (2.19) and (2.28), we have

$$\begin{aligned} d_r\left(f^nu_0, u\right) &= d_r\left(f^nu_0, fu\right) \\ &\leq d_r\left(f^nu_0, f^{n+1}u_0\right) + d_r(f^{n+1}u_0, fu) \\ &\leq \frac{q^n}{1-q}\max\left\{d_r\left(u_0, fu_0\right), d_r\left(u_0, f^2u_0\right)\right\} \\ &\quad + q.\max\left\{d_r\left(f^nu_0, u\right), d_r\left(f^nu_0, f^{n+1}u_0\right), d_r\left(f^{n+1}u_0, u\right)\right\} \\ &\leq \frac{q^n}{1-q}\max\left\{d_r\left(u_0, fu_0\right), d_r\left(u_0, f^2u_0\right)\right\} + q.\delta\left[\mathcal{O}_f(u_0, n+1)\right] \\ &\leq \left(\frac{q^n}{1-q} + \frac{q^{n+1}}{1-q}\right)\max\left\{d_r\left(u_0, fu_0\right), d_r\left(u_0, f^2u_0\right)\right\}, \quad n \geq 1. \end{aligned}$$

Since $q^n \to 0$ as $n \to \infty$, this in the limiting case yields $d_r\left(f^nu_0, u\right) \to 0$ for each u_0. Thus the fixed point u is a contractive fixed point for f. □

The following example validates the result obtained in Theorem 2.13:

Example 2.14. Let $X = \{1, 2, 3, 4\}$ and d_r be a rectangular metric defined by

$$\begin{cases} d_r(1,2) = d_r(2,1) = d_r(3,4) = d_r(4,3) = 20, \\ d_r(1,3) = d_r(3,1) = d_r(2,4) = d_r(4,2) = 40, \\ d_r(1,4) = d_r(4,1) = d_r(2,3) = d_r(3,2) = 120. \end{cases} \tag{2.29}$$

Consider $f : X \to X$ defined by

$$fx = \begin{cases} 3, & \text{if } x \in \{3,4\} \\ 4, & \text{if } x \in \{1,2\}. \end{cases} \tag{2.30}$$

Then

$$d_r(fx, fy) = \begin{cases} 0, & \text{if } (x \in \{1,2\}, y \in \{1,2\}) \\ & \quad \text{or } (x \in \{3,4\}, y \in \{3,4\}) \\ 20, & \text{if } (x \in \{1,2\}, y \in \{3,4\}) \\ & \quad \text{or } (x \in \{3,4\}, y \in \{1,2\}). \end{cases} \tag{2.31}$$

It follows from (2.29) and (2.31) that

$$d_r(fx, fy) \leq q \max \{d_r(x,y), d_r(x,fx), d_r(y,fy), d_r(x,fy), d_r(y,fx)\}$$

for all $x, y \in X$. That is f is a Ciric's quasi-contraction with $q = 1/6$, and 3 is a unique fixed point of f. Also,

$$\mathcal{O}_f(x) = \begin{cases} \langle 1, f1, f^2 1, f^3 1, \ldots \rangle, & \text{if } x = 1 \\ \langle 2, f2, f^2 2, f^3 2, \ldots \rangle, & \text{if } x = 2 \\ \langle 3, f3, f^2 3, f^3 3, \ldots \rangle, & \text{if } x = 3 \\ \langle 4, f4, f^2 4, f^3 4, \ldots \rangle, & \text{if } x = 4 \end{cases}$$

$$= \begin{cases} \langle 1, 4, 3, 3, \ldots \rangle, & \text{if } x = 1 \\ \langle 2, 4, 3, 3, \ldots \rangle, & \text{if } x = 2 \\ \langle 3, 3, 3, 3, \ldots \rangle, & \text{if } x = 3 \\ \langle 4, 3, 3, 3, \ldots \rangle, & \text{if } x = 4. \end{cases}$$

This yields that $\mathcal{O}_f(x) \to 3$ for each $x \in X$ and therefore, 3 is a contractive fixed point.

3. Fixed Points Without Iterations

Banach's contraction mapping theorem in an *rms*, due to Branciari [2] is given below:

Theorem 3.1. *Let (X, d_r) be a complete rectangular metric space. If $f : X \to X$ is a Banach's contraction given in* (2.6)*, then there exists exactly one $u \in X$ with $fu = u$.*

In the proof of Theorem 3.1, Branciari assumed that the *rms* X is Hausdorff without mentioning it in the statement. Indeed an *rms* is not Hausdorff in general. For instance, in the *rms* of Example 2.3, we see that $\mathcal{B}_\delta(0) = \mathcal{B}_\delta(2) = \{0\} \cup \{1/n : n \geq 1\}$ for any $\delta > 0$. Thus the neighbourhoods $\mathcal{B}_\delta(0)$ and $\mathcal{B}_\delta(2)$ corresponding to distinct points are not disjoint. Naturally, subsequent proofs need an additional condition. Assuming that the *rms* X is Hausdorff, Rao et al. [22] obtained a unique fixed point in Theorem 3.1. Their proof involves the notion of periodic point (A point $u \in X$ is called a periodic point of $f : X \to X$, if $f^s(u) = u$ for some positive integer s. A periodic point u becomes a fixed point of f, if $s = 1$).

Joseph and Kwack [7] used greatest lower bound property (or the infimum property) of real numbers to obtain a fixed point of a Banach's contraction in a metric space. Inspired by these works, we obtain a fixed point for Hardy-Roger and Ciric contractions on an *rms* without the Hausdorff-ness and the notion of periodic point.

The infimum property states that a non-empty set of real numbers, which is bounded below, has an infimum in $\mathbb{R}$. As a consequence, it follows that the infimum of a non-empty set of nonnegative real numbers which is bounded below is nonnegative. That is, if A is a non-empty set of nonnegative real numbers, which is bounded below, then $\inf A = c \geq 0$.

Lemma 3.2. *Let (X, d_r) be an rms and $f : X \to X$ satisfy*

$$d_r(fx, f^2x) \leq \eta d_r(x, fx), \quad \text{for all } x \in X, \tag{3.1}$$

where $0 < \eta < 1$. If $A_f = \{d_r(x, fx) : x \in X\}$, then $\inf A_f = 0$.

Proof. Suppose that the infimum c of A_f is positive. In view of the choice $\eta < 1$, c/η being more than the infimum c cannot be a lower bound of A_f, and hence exceeds some number of A_f. That is there exists some $\omega \in X$ such that

$$d_r(\omega, f\omega) < c/\eta, \quad \text{that is } \eta\, d_r(\omega, f\omega) < c. \tag{3.2}$$

But then from (3.1) with $x = \omega$,

$$d_r\left(f\omega, f^2\omega\right) \leq \eta d_r(\omega, f\omega). \tag{3.3}$$

Now from (3.2) and (3.3), it follows that

$$d_r\left(f\omega, f^2\omega\right) < c. \tag{3.4}$$

Since $d_r\left(f\omega, f^2\omega\right) \in A_f$, (3.4) shows that c is not a lower bound of A_f, contradicting the infimality of c and therefore, c must be zero. □

Lemma 3.3. *Let ζ be the infimum of a nonempty set S of nonnegative real numbers. Then there exists a sequence $\langle \zeta_n \rangle_{n=1}^{\infty} \subset S$ such that $\zeta_n \to \zeta$ as $n \to \infty$.*

Proof. Since ζ is the greatest lower bound of S, any number which is even slightly greater than ζ cannot be a lower bound for S. Given $\varepsilon > 0$, thus $\zeta + \varepsilon$ is greater than ζ and hence is not lower bound of S. Therefore, $\zeta \leq s < \zeta + \varepsilon$ for some $s \in S$. For each $\varepsilon = 1/n$, choose $\zeta_1, \zeta_2, \ldots, \zeta_n, \ldots$ in S such that $\zeta \leq \zeta_n < \zeta + 1/n$ for $n \geq 1$. Then $\zeta_n \to \zeta$ as $n \to \infty$. □

Theorem 3.4. *Let $f : X \to X$ be a Hardy-Roger's contraction on a complete rms (X, d_r) satisfying* (2.10)*. Then there exists exactly one $\mu \in X$ with $f\mu = \mu$.*

Proof. Let $A = \{d_r(x, fx) : x \in X\}$. Interchanging the roles of x and y in (2.10), we see that

$$d_r(fy, fx) \leq a_1 d_r(y, fy) + a_2 d_r(x, fx) + a_3 d_r(y, fx) + a_4 d_r(x, fy) + a_5 d_r(y, x). \tag{3.5}$$

Adding (2.10) and (3.5), using (rm3) and then dividing by 2, we obtain

$$d_r(fx, fy) \leq \frac{a_1 + a_2}{2} (d_r(x, fx) + d_r(y, fy)) + \frac{a_3 + a_4}{2} (d_r(x, fy) + d_r(y, fx)) + a_5 d_r(x, y).$$

Writing $b_1 = (a_1 + a_2)/2$, $b_2 = (a_3 + a_4)/2$ and $b_3 = a_5$ in the above

$$d_r(fx, fy) \leq b_1(d_r(x, fx) + d_r(y, fy)) + b_2(d_r(x, fy) + d_r(y, fx)) + b_3 d_r(x, y), \tag{3.6}$$

where $2b_1 + 2b_2 + b_3 < 1$. With $x = u$ and $y = fu$, (3.6) in view of (rm4) gives

$$d_r\left(fu, f^2u\right) \leq (b_1 + b_2 + b_3)\, d_r(u, fu) + (b_1 + b_2) d_r\left(fu, f^2u\right)$$

so that

$$d_r\left(fu, f^2u\right) \leq \frac{b_1 + b_2 + b_3}{1 - b_1 - b_2} d_r(u, fu) \quad \text{or} \quad d_r\left(fu, f^2u\right) \leq \eta \cdot d_r(u, fu), \tag{3.7}$$

where $\eta = (b_1+b_2+b_3)/(1-b_1-b_2)$. Note that $\eta < 1$, since $b_1+b_2+b_3 < 1$. Therefore by Lemma 3.2, $\inf A = 0$, and by Lemma 3.3, there exists a sequence $\langle \mu_n \rangle_{n=1}^{\infty}$ in X such that

$$d_r(\mu_n, f\mu_n) \to 0, \quad \text{as } n \to \infty. \tag{3.8}$$

Now we prove that $\langle \mu_n \rangle_{n=1}^{\infty}$ is a Cauchy sequence. For all positive integers n and m, from (rm4), (3.6) with $x = \mu_n$ and $y = \mu_m$ and (m4), we have

$$\begin{aligned}
&d_r(\mu_n, \mu_m) \\
&\quad \leq d_r(\mu_n, f\mu_n) + d_r(f\mu_n, f\mu_m) + d_r(f\mu_m, \mu_m) \\
&\quad \leq d_r(\mu_n, f\mu_n) + d_r(f\mu_m, \mu_m) + b_1 (d_r(\mu_n, f\mu_n) + d_r(\mu_m, f\mu_m)) \\
&\qquad + b_2 (d_r(\mu_n, f\mu_m) + d_r(\mu_m, f\mu_n)) + b_3 d_r(\mu_n, \mu_m) \\
&\quad \leq (1 + b_1) (d_r(\mu_n, f\mu_n) + d_r(f\mu_m, \mu_m)) \\
&\qquad + b_2 (d_r(\mu_n, \mu_m) + d_r(\mu_m, f\mu_m) + d_r(\mu_m, \mu_n) + d_r(\mu_n, f\mu_n)) \\
&\qquad + b_3 d_r(\mu_n, \mu_m),
\end{aligned}$$

which on rearranging the terms and then simplifying, gives

$$d_r(\mu_n, \mu_m) \leq \left\{ \frac{1 + b_1 + b_2}{1 - 2b_2 - b_3} \right\} (d_r(\mu_n, f\mu_n) + d_r(\mu_m, f\mu_m)) .$$

This and (3.8) imply that $d_r(\mu_n, \mu_m) \to 0$ as $n, m \to \infty$. Thus $\langle \mu_n \rangle_{n=1}^{\infty}$ is a Cauchy-sequence in the complete *rms* X and hence converges to a unique $\mu \in X$, in view of Lemma 2.4.

Now, by (rm4), (2.10) with $x = \mu_n$ and $y = \mu$,

$$\begin{aligned} d_r(\mu, f\mu) &\leq d_r(\mu, \mu_n) + d_r(\mu_n, f\mu_n) + d_r(f\mu_n, f\mu) \\ &\leq d_r(\mu, \mu_n) + d_r(\mu_n, f\mu_n) + b_1 (d_r(\mu_n, f\mu_n) + d_r(\mu, f\mu)) \\ &\quad + b_2 (d_r(\mu_n, f\mu) + d_r(\mu, f\mu_n)) + b_3 d_r(\mu_n, \mu) \\ &\leq (1 + b_3) d_r(\mu, \mu_n) + (1 + b_1) d_r(\mu_n, f\mu_n) + b_1 d_r(\mu, f\mu) \\ &\quad + b_2 (d_r(\mu_n, \mu) + d_r(\mu, f\mu) + d_r(\mu, \mu_n) + d_r(\mu_n, f\mu_n)) \\ &\leq (1 + 2b_2 + b_3) d_r(\mu, \mu_n) + (1 + b_1 + b_2) d_r(\mu_n, f\mu_n) \\ &\quad + (b_1 + b_2) d_r(\mu, f\mu). \end{aligned}$$

As $n \to \infty$, this in view of (3.8) and convergence of $\langle \mu_n \rangle_{n=1}^{\infty}$ gives

$$d_r(\mu, f\mu) \leq (1 + 2b_2 + b_3) \cdot 0 + (1 + b_1 + b_2) \cdot 0 + (b_1 + b_2) d_r(\mu, f\mu)$$

or $0 \leq (1 - b_2 - b_3) d_r(\mu, f\mu) \leq 0$ so that $d_r(\mu, f\mu) = 0$ or $f\mu = \mu$. It is not difficult to establish that μ is unique. □

Since Banach's condition (2.6) is stronger than the condition (2.10), Banach's contraction mapping theorem in an *rms* (given below) follows as a special case of Theorem 3.4:

Corollary 3.5. *Let (X, d_r) be a complete rms and $f : X \to X$ be a Banach's contraction satisfying* (2.6)*. Then $\exists$ exactly one $\mu \in X$ with $f\mu = \mu$.*

Now $a_3 = a_4 = 0$ in (2.10) gives Reich-type

Corollary 3.6. *Let (X, d_r) be a complete rms and $f : X \to X$ satisfy* (2.15)*. Then $\exists$ exactly one $\mu \in X$ with $f\mu = \mu$.*

Writing $a_1 = a_2 = \kappa$ and $a_3 = a_4 = a_5 = 0$ in (2.10), we have Kannan-type

Corollary 3.7. *Let (X, d_r) be a complete rms and $f : X \to X$ satisfy* (2.16)*. Then $\exists$ exactly one $\mu \in X$ with $f\mu = \mu$.*

Whereas, (2.10) with $a_1 = a_2 = a_5 = 0$ and $a_3 = a_4 = \kappa$ gives a Chatterjea-type

Corollary 3.8. *Let (X, d_r) be a complete rms and $f : X \to X$ satisfy* (2.17)*. Then $\exists$ exactly one $\mu \in X$ with $f\mu = \mu$.*

Finally, writing $a_1 = a_2 = \kappa_1$, $a_3 = a_4 = \kappa_2$ and $a_5 = \kappa_3$ in Theorem 3.4, we have

Corollary 3.9. *Let (X, d_r) be a complete rms and $f : X \to X$ satisfy* (2.18)*. Then $\exists$ exactly one $\mu \in X$ with $f\mu = \mu$.*

Lahiri and Das [11] proved the following result:

Theorem 3.10. *Let (X, d_r) be a complete rms. Then every quasi-contraction satisfying the condition* (2.19) *will have a unique fixed point.*

Authors of [11] initiated the proof of Theorem 3.10 with the sequence $\langle x_n \rangle_{n=1}^{\infty}$ of f-iterates at an initial seed $x_0 \in X$, where $x_n = f^n x_0$ for $n \geq 1$. They used a routine iterative procedure to show that it is a Cauchy sequence, converging to some $u \in X$. The proof technique in this regard is similar to that of Ciric in metric space [4].

Also, they derived that

(3.9)

$$d_r\left(f^n x_0, f^m x_0\right) \leq \frac{q^n}{1-q} \max\left\{d_r(x_0, fx_0), d_r\left(x_0, f^2 x_0\right)\right\} \text{ for } m > n$$

and

$$\lim_{n\to\infty} d_r(f^n x_0, u) = 0. \tag{3.10}$$

Then proceeding with the limit as $m \to \infty$ in (3.9) and using (3.10), they claimed that

$$d_r(f^n x_0, u) \leq \frac{q^n}{1-q} \max\left\{d_r(x_0, fx_0), d_r\left(x_0, f^2 x_0\right)\right\}. \tag{3.11}$$

It is important to observe that (3.11) is valid only if the rectangular metric d_r is continuous at u, because d_r is continuous at u if and only if $d_r(f^n x_0, v) \to d_r(u, v)$ for each $v \in X$, whenever (3.10) holds good. However, d_r is not continuous in general (see [22]). Thus the authors are intrinsic in using the continuity of d_r without mentioning it. Also, Lemma 2.4 is instrumental for the uniqueness of the limit u, which would have been used in the proof.

The following is our non-iterative proof, which does not use the continuity of d_r.

Proof. Let c be the infimum of $\mathcal{D}_f = A \cup B$, where $A = \left\{d_r(u, fu) : u \in X\right\}$ and $B = \left\{d_r(u, f^2 u) : u \in X\right\}$. Then c is nonnegative and is less than or equal to $\inf A$ and $\inf B$. If $c > 0$, then c/q exceeds c and hence cannot be a lower bound for $\mathcal{D}_f$, since $q < 1$. In other words, $c/q > l$ or $ql < c$ for some $l \in \mathcal{D}_f$. That is there exist points $u_1, u_2 \in X$ such that

$$d_r(u_1, fu_1) < q d_r(u_1, fu_1) < c \tag{3.12}$$

or

$$d_r\left(u_2, f^2u_2\right) < qd_r\left(fu_2, f^2u_2\right) < c. \tag{3.13}$$

Since $d_r(u_1, fu_1)$ and $d_r\left(u_2, f^2u_2\right)$ being elements of $\mathcal{D}_f$, are greater than or equal to c, (3.12) and (3.13) give a contradiction that $c < c$, where $c > 0$. Therefore, c must be zero. Therefore, by Lemma 3.3, there exists a sequence $\langle\zeta_n\rangle_{n=1}^{\infty} \subset \mathcal{D}_f$ such that $\zeta_n \to \zeta$ as $n \to \infty$ so that the subsequence $\langle\omega_n\rangle_{n=1}^{\infty}$ of $\langle\zeta_n\rangle_{n=1}^{\infty} \subset \mathcal{D}_f$ is also convergent with the limit zero. Choose points $u_1, u_2, u_3, \ldots, u_n, \ldots$ in X such that

$$\omega_n = d_r(u_n, fu_n) \to 0. \tag{3.14}$$

For any $n \geq 1$ and $m \geq 1$, by (rm4) and (2.19) with $x = u_n$ and $y = u_m$, we have

$$\begin{aligned} d_r(u_n, u_m) &\leq d_r(u_n, fu_n) + d_r(fu_n, fu_m) + d_r(fu_m, u_m) \\ &\leq d_r(u_n, fu_n) + qQ + d_r(fu_m, u_m), \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} Q = \max\{&d_r(u_n, u_m), d_r(u_n, fu_n), d_r(u_m, fu_m), \\ &d_r(u_n, fu_m), d_r(u_m, fu_n)\}. \end{aligned} \tag{3.16}$$

Case (a): If $Q = d_r(u_n, u_m)$, then (3.15) and (3.14) imply that

$$d_r(u_n, u_m) \leq \frac{1}{1-q}(d_r(u_n, fu_n) + d_r(u_m, fu_m)).$$

Case (b): If $Q = d_r(u_n, fu_n)$, (3.15) and (3.14) give

$$d_r(u_n, u_m) \leq (1+q)d_r(u_n, fu_n) + d_r(u_m, fu_m).$$

Case (c): If $Q = d_r(u_m, fu_m)$, interchanging the roles of the indices m and n in Case (b),

$$d_r(u_n, u_m) \leq d_r(u_n, fu_n) + (1+q)d_r(u_m, fu_m).$$

Case (d): If $Q = d_r(u_n, fu_m)$, (3.15) followed by (3.14) gives

$$d_r(u_n, u_m) \leq \frac{1}{1-q} \cdot d_r(u_n, fu_n) + \frac{1+q}{1-q} \cdot d_r(u_m, fu_m).$$

Case (e): If $Q = d_r(u_m, fu_n)$, interchanging the roles of the indices m and n in Case (d),

$$d_r(u_n, u_m) \leq \frac{1+q}{1-q} \cdot d_r(u_n, fu_n) + \frac{1}{1-q} \cdot d_r(u_m, fu_m).$$

Thus in all the five cases, we see that $d_r(u_n, u_m) \to 0$ as $n, m \to \infty$, showing that $\langle u_n\rangle_{n=1}^{\infty} \subset X$ is Cauchy. Since X is complete,

$$u_n \to u \in X. \tag{3.17}$$

We claim that u is a fixed point. In fact, from (rm4) and (2.19),

$$\begin{aligned} d_r(u, fu) &\leq d_r(u, u_n) + d_r(u_n, fu_n) + d_r(fu_n, fu) \\ &\leq d_r(u, u_n) + d_r(u_n, fu_n) + q \max \big\{ d_r(u_n, fu_n), d_r(u, fu), \\ &\quad d_r(u_n, fu), d_r(u, fu_n), d(u, u_n) \big\} \\ &\leq d_r(u, u_n) + d_r(u_n, fu_n) + q \max \big\{ d_r(u_n, fu_n), d_r(u, fu), \\ &\quad (d_r(u_n, u) + d_r(u, fu)), (d_r(u, u_n) + d_r(u_n, fu_n)), d_r(u, u_n) \big\} \\ &\leq d_r(u, u_n) + d_r(u_n, fu_n) + q \max \big\{ (d_r(u_n, u) + d_r(u, fu)), \\ &\quad (d_r(u, u_n) + d_r(u_n, fu_n)) \big\}. \end{aligned}$$

As $n \to \infty$, this implies that $d_r(u, fu) \leq 0 + 0 + q \max\{0 + d_r(u, fu), 0 + 0\} = q d_r(u, fu)$ so that $d_r(u, fu) = 0$ or $fu = u$. That is u is a fixed point.

Suppose that $fu = u$ and $fv = v$. Then by (2.19) with $x = u$ and $y = v$,

$$\begin{aligned} d_r(u, v) &= d_r(fu, fv) \\ &\leq q \max\{d_r(u, v), d_r(u, fu), d_r(v, fv), d_r(u, fv), d_r(v, fu)\} \\ &\leq q \max\{d_r(u, v), 0, 0, d_r(u, v), d_r(v, u)\} = q d_r(u, v), \end{aligned}$$

which implies that $d_r(u, v) = 0$, that is $u = v$. □

Remark 3.11. Our proof is two-fold significant. It needs no f-iterations but only employs the greatest lower bound property to establish the sequence $\langle d_r(u_n, fu_n) \rangle_{n=1}^{\infty}$ of nonnegative numbers converging to zero. Also, the rectangle inequality is repeatedly used to prove that $\langle u_n \rangle_{n=1}^{\infty}$ is a Cauchy sequence, which is not a Picard's sequence.

Replacing (2.19) with the stronger condition:

$$\begin{aligned} d_r(fx, fy) \leq q \max \{ & d_r(x, y), d_r(x, fx), d_r(y, fy), \\ & (d_r(x, fy) + d_r(y, fx))/2 \}, \quad \text{for all } x, y \in X, \end{aligned} \tag{3.18}$$

where $0 < q < 1$, we have

Corollary 3.12. *Let (X, d_r) be a complete rms and $f : X \to X$, a generalized contraction satisfying* (3.18). *Then $\exists$ exactly one $u \in X$ with $fu = u$.*

Since every metric space is an *rms*, we have

Corollary 3.13 (Theorem 4, Ciric [3])**.** *Let (X, d) be a complete metric space and $f : X \to X$, satisfies*

$$\begin{aligned} d(fx, fy) \leq q \max \{ & d(x, y), d(x, fx), d(y, fy), \\ & (d(x, fy) + d(y, fx))/2 \}, \quad \textit{for all } x, y \in X, \end{aligned} \tag{3.19}$$

where $q \in (0, 1)$. Then there exists exactly one $u \in X$ with $fu = u$.

4. Applications

Consider the Volterra integral equation

$$x(t) = u(t) + \lambda \int_a^t \eta(t,s)x(s)ds, \quad \text{for all } t \in [a,b], \tag{4.1}$$

where $u : [a,b] \to \mathbb{R}$ and $\eta : [a,b] \times [a,b] \to \mathbb{R}$ are continuous and λ is any real number. Writing the right hand side of (4.1) as $\varphi(x(t))$, it can be shown that $\varphi : \mathcal{C}[a,b] \to \mathcal{C}[a,b]$ is a contraction, for an appropriate choice of λ, where the function space $\mathcal{C}[a,b]$ of all continuous functions on $[a,b]$ is complete. Therefore, by Banach's theorem, φ has a unique fixed point $x(t)$ and all the iterations, given by

$$x_{n+1}(t) = u(t) + \lambda \int_a^t \eta(t,s)x_n(s)ds, \quad \text{for all } t \in [a,b], \tag{4.2}$$

converge to $x(t)$. The approximations to the analytical solution can be demonstrated with the use of MATLAB as follows:

Application 1. To solve the Volterra integral equation

$$x(t) = t - \int_0^t (t-s)x(s)ds, \quad \text{for } t \in [0,1]. \tag{4.3}$$

Solusion. Denoting the right hand side of (4.3) by $f(x(t))$, we observe that f is a Banach's contraction on $\mathcal{C}[0,1]$ and hence by Theorem 3.1, f has exactly one fixed point $x(t)$, and hence is a contractive fixed point, which is, indeed a unique solution of (4.3), and all the iterative sequences $\langle x_n \rangle_{n=1}^{\infty}$, given by

$$\begin{aligned} x_n(t) &= f(x_{n-1}(t)) \\ &= t - \int_{s=0}^t (t-s)x_{n-1}(s)ds, \quad t \in [0,1], n \geq 1, \end{aligned} \tag{4.4}$$

converge to $x(t)$, where $x_0(t)$ is a zeroth approximation.
For instance, with $x_0(t) = t$, we have the following iterative scheme:

$$x_1(t) = t - \int_0^t (t-s)sds = t - \frac{t^3}{3!},$$

$$x_2(t) = t - \int_0^t (t-s)\left(s - \frac{s^3}{6}\right)ds = t - \frac{t^3}{3!} + \frac{t^5}{5!},$$

$$x_3(t) = t - \int_0^t (t-s)\left(s - \frac{s^3}{6} + \frac{s^5}{120}\right)ds = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!},$$

$$x_4(t) = t - \int_0^t (t-s)\left(s - \frac{s^3}{6} + \frac{s^5}{120}\right)ds = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!},$$

$$x_5(t) = t - \int_0^t (t-s)\left(s - \frac{s^3}{6} + \frac{s^5}{120}\right) ds = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} - \frac{t^{11}}{11!}$$

and so on. We realize that x_i's are the partial sums of the Taylor series $\sum_{n=0}^{\infty}(-1)^n \frac{t^{2n+1}}{(2n+1)!}$, which converge to $\sin t$ for all real t and hence for $t \in [0,1]$. In other words, $x_n(t) \to \sin t$ for all $t \in [0,1]$. Note that $\varphi(t) = \sin t$ is the analytic solution of (4.3).

```
t = .1:.2:1;
sum = 0;
for i = 1:5
sum = round(vpa(sum + ((−1)^(i−1))∗t.^(2∗i−1)/factorial(2∗i−1),4),5);
disp(['x',num2str(i), ' = '])
disp(sum)
end
phi = round(vpa(sin(t),5),5)
```

FIGURE 2. MATLAB Code

Description. The first line gives the row vector t of the test values of the variable t from 0.1 to 1.0 in steps 0.2. Note that the last test value is 0.9 and not 1.0. Line 2 initializes the computations at 0. Then the *for* loop evaluates the row vectors of the first five approximations (upto 5 decimal places) in succession, which are displayed in MATLAB output as $x1, x2, x3, x4$ and $x5$ respectively . Finally, the analytical solution $phi = \sin(t)$ is evaluated to at least 5 significant digits, which is rounded off to 5 decimal places in line 8.

Results and Discussion.

TABLE 1. Comparison Table

i	$x_1(i)$	$x_2(i)$	$x_3(i)$	$x_4(i)$	$x_5(i)$	$\varphi(i)$
0.1	0.1	0.09983	0.09983	0.09983	0.09983	0.09983
0.3	0.3	0.29550	0.29552	0.29552	0.29552	0.29552
0.5	0.5	0.47917	0.47943	0.47943	0.47943	0.47943
0.7	0.7	0.64283	0.64423	0.64421	0.64421	0.64422
0.9	0.9	0.77850	0.78342	0.78333	0.78333	0.78333

The columns $x_4(i)$ and $x_5(i)$ in Table 1 coincide. Therefore, $x_4(t)$ is regarded as a better approximation for the exact solution $\varphi(t) = \sin t$.

The last three columns reveal that the error in computing φ by the numerical approximations is relatively small.

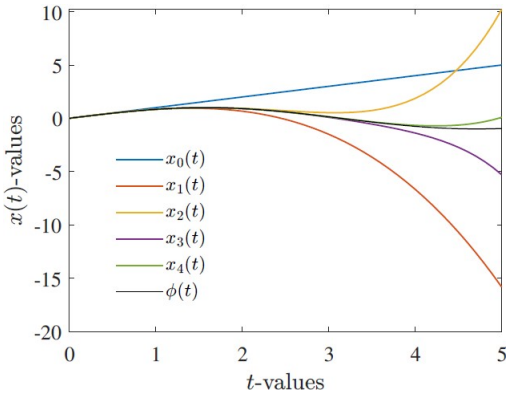


FIGURE 3. Graphical Comparison

The authors would like to acknowledge the work of Sihombing and Lia [24], which motivated in formulating the above problem.

Integral equations can be solved through other generalized forms of a Banach's contraction also. The following is such a nice application of Theorem 3.4 (in a metric space) to the existence and uniqueness of the solution of an integral equation:

Application 2. Consider the complete metric space (X, d), where X is the set $\mathcal{C}[a, b]$, of all real valued continuous functions on $[a, b]$ with metric $d : X \times X \to [0, \infty)$, defined by

$$d(x, y) = \max\{|x(t) - y(t)| : t \in [a, b]\}, \quad \text{for all } x, y \in X \tag{4.5}$$

Let $\eta : [a, b] \times [a, b] \to \mathbb{R}$ be such that

$$|\eta(t, s)| \leq \frac{1}{(t-a)d(x, y)} \left(\frac{9}{80}(d(x, fx) + d(y, fy)) + \frac{9}{80}(d(x, fy) + d(y, fx)) + \frac{6}{20} d(x, y) \right), \quad x \neq y, t > a. \tag{4.6}$$

If $u : [a, b] \to \mathbb{R}$ and η are continuous, then the integral equation

$$f(x(t)) = u(t) + \int_a^t \eta(t, s)x(s)ds, \quad \text{for all } t \in [a, b], \tag{4.7}$$

has a unique solution $x(t)$ on $[a, b]$.

Solusion. For all $x, y \in X$, from (4.5), (4.6) and (4.7), we have

$$\begin{aligned}
d(fx, fy) &= \max\{|f(x(t)) - f(y(t))| : t \in [a, b]\} \\
&= \max\left\{\left|\int_a^t \eta(t, s)(x(s) - y(s))ds\right| : t \in [a, b]\right\} \\
&\leq \frac{1}{(t-a)d(x, y)}\left(\frac{9}{80}(d(x, fx) + d(y, fy))\right. \\
&\quad \left. + \frac{9}{80}(d(x, fy) + d(y, fx)) + \frac{6}{20}d(x, y)\right) \\
&\quad \cdot \max\left\{\int_a^t |x(s) - y(s)|\, ds : t \in [a, b]\right\} \\
&\leq \frac{1}{(t-a)d(x, y)}\left(\frac{9}{80}(d(x, fx) + d(y, fy))\right. \\
&\quad \left. + \frac{9}{80}(d(x, fy) + d(y, fx)) + \frac{6}{20}d(x, y)\right)\left(d(x, y)\int_a^t ds\right) \\
&= \frac{9}{80}(d(x, fx) + d(y, fy)) + \frac{9}{80}(d(x, fy) + d(y, fx)) \\
&\quad + \frac{6}{20}d(x, y)
\end{aligned}$$

where $b_1 = b_2 = 9/80$ and $b_3 = 6/20$. Since every metric space is an *rms*, f has unique fixed point $x(t)$, by Theorem 3.4, which is the unique solution of the integral equation (4.7).

5. Conclusions

The introductory section has been about a brief history of Banach's contraction theorem and its applications, followed by contractive fixed points in a metric space and in some of its generalizations: b, G, S and 2-metric spaces, as applications of the infimum property. Some topological properties of an *rms* were presented, and contractive fixed points of Banach, Hardy-Roger and Ciric contractions on an *rms* were presented in Section 2 of this paper. Fixed point were obtained in Section 3 for these contractions on an *rms* with the repeated use of the rectangle inequality without involving any iterations. The infimum property of real numbers is only employed to establish Cauchy sequences in the proofs. Section 4 contains two applications: Application 4.1 is about solving a Volterra integral equation using Banach's theorem with necessary MATLAB interpretation. Whereas Application 4.2 is about the existence and uniqueness of the solution of a Volterra integral equation through a generalized form of a Banach's contraction. The authors of this paper feel that Section 4 is definitely a source of inspiration for

employing the MATLAB tools in obtaining approximate solutions of integral equations through generalized contractions in generalized settings of a metric space.

Acknowledgment. The authors express their sincere thanks to the reviewers for their valuable suggestions in improving the paper.

References

1. I.A. Bakhtin, *The contraction mapping principle in quasi-metric spaces*, Funct. Anal. Unianowsk Gos. Ped. Inst., 30 (1989), pp. 26-37.
2. A. Branciari, *A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces*, Publ. Math. Debrecen, 57 (2000), pp. 31-37.
3. Lj. B. Ciric, *Fixed point theorems for mappings with a generalized contractive iterate at a point*, Publ. Inst. Math. (Beograd) (N.S.), 13(27) (1972), pp. 11-16.
4. Lj. B. Ciric, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc., 45 (2) (1974), pp. 271-273.
5. M. Edelstein, *On fixed and periodic points under contractive mappings*, J. Lond. Math. Soc., 37 (1962), pp. 74-79.
6. S. Gahler, *2-metrische Raume und ihre topologische Struktur*, Math. Nachr., 26 (1963), pp. 115-148.
7. J.E. Joseph and M.H. Kwack, *Alternative approaches to proofs of contraction mapping fixed point theorems*, Missouri J. Math. Sci., 11 (1999), pp. 167-175.
8. K. Kumara Swamy and T. Phaneendra, *Another proof of Mohantas fixed point theorem in G-metric space*, Math. Sci. Inter. Res. Journal, 4(2) (2015), pp. 284-287.
9. K. Kumara Swamy and T. Phaneendra, *Fixed point of 2-contraction through an alternative technique*, Proc. Jangjeon Math. Soc., 18 (1) (2015), pp. 49-54.
10. Kreyszig, *Introductory Functional Analysis with Applications*, Wiley Classics Library Edition Published (1989).
11. B.K. Lahiri and P. Das, *Fixed point of a Ljubomir Ciric's quasi-contraction mapping in a generalized metric space*, Publ. Math. Debrecen, 61 (3-4) (2002), pp. 584-594.
12. Z. Mustafa and B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal., 7 (2) (2006), pp. 289-297.
13. T. Phaneendra and K. Kumara Swamy, *A Unique Common Fixed Point of a pair of Self-maps on a 2-metric space*, Math. Aeterna, 3(4) (2013), pp. 271-277.

14. T. Phaneendra and K. Kumara Swamy, *Unique fixed point in G-metric space through greatest lower bound properties*, Novi Sad J. Math., 43 (2) (2013), pp. 107-115.
15. T. Phaneendra, *Banach and Kannan contraction on S-metric space*, Ital. J. Pure Appl. Math., 39 (2018), pp. 243-247.
16. T. Phaneendra and K. Kumara Swamy, *Some elegant proofs in 2-metric space and G-metric space*, Ital. J. Pure and Appl. Math., 36 (2016), pp. 801-818.
17. T. Phaneendra and K. Kumara Swamy, *Fixed points of Chatterjea and Ciric contractions in S-metric space*, Int. J. Pure Appl. Math., 115 (2) (2017), pp. 361-367.
18. T. Phaneendra and S. Saravanan, *On Some Misconceptions and Chatterjee-type G-contraction*, Int. J. Pure Appl. Math., 109 (4), 2016, pp. 789-797.
19. T. Phaneendra and S. Saravanan, *On G-contractive fixed points*, Jnanabha, 46 (2016), pp. 105-112.
20. T. Phaneendra and S. Saravanan, *Fixed point under general contractions in G-metric space*, J. Anal., 25 (2) (2017), pp. 15-234.
21. S. Saravanan and T. Phaneendra, *Fixed Point as a G-contractive fixed point*, Int. J. Appl. Engrg. Res., 11 (1), 2016, pp. 316-319.
22. I.R. Sarma, J.M. Rao and S.S. Rao, *Contractions over generalized metric spaces*, J. Nonlinear Sci. Appl., 2 (3) (2009), pp. 180-182.
23. Sh. Sedghi, N. Shobe and A. Aliouche, *A generalization of fixed point theorems in S-metric spaces*, Mat. Vesnik, 64 (3) (2012), pp. 258-266.
24. S.C. Sihombing and L. Lia, *Fixed point theorem on Volterra integral equation*, J. Physics: Conference Series., 1375 (2019), IOP Publ., doi:10.1088/1742-6596/1375/1/012064.
25. P. Swapna and T. Phaneendra, *Generalized fixed point theorems in a b-metric space*, Nonlinear Funct. Anal. and Appl., 25 (2) (2020), pp. 215-230.
26. W. Rudin, *Principles of Mathematical Analysis*, Mc Graw Hill International Editions, Mathematics Series.

---

[1] Department of Mathematics, Vellore Institute of Technology, Vellore-632 014, Tamil Nadu, India.
Email address: poonkuzhali1896@gmail.com

[2] Department of Mathematics, Vellore Institute of Technology, Vellore-632 014, Tamil Nadu, India.
Email address: drtp.indra@gmail.com