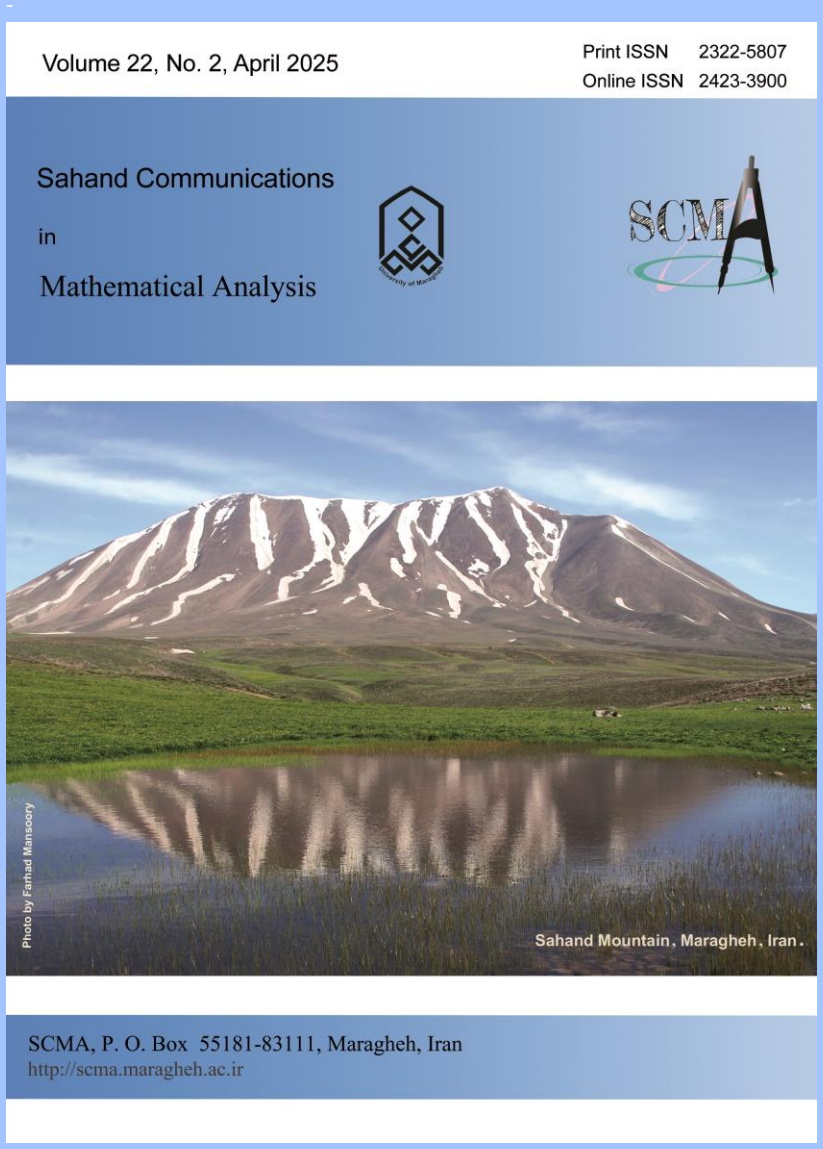
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Quantum Calculus-Based Error Estimations in Hermite-Hadamard Type Inequalities

Muhammad Samraiz[1], Kanwal Saeed[2], Gauhar Rahman[3*] and Saima Naheed[4]

Abstract. The main aim of this article is to present error approximations for Hermite-Hadamard inequalities, which are connected with ${}_{\omega_1}\mathfrak{q}$-fractional calculus operators and are based on (ρ, m)-convex functions. The investigation revolves around our primary results, which are established using a specific identity. Moreover, we demonstrate that the results from earlier literature can be derived as special cases from our more comprehensive findings.

1. Introduction

Convexity is a fundamental mathematical concept that has wide-ranging applications in various fields. For more deeply information on this topic, interested readers can explore references [25] and [22]. The study of optimization theory and inequalities associated with convex functions has generated a large number of persuasive outcomes. As a result, many scholars have been motivated to investigate these functions more deeply, expanding their knowledge through creative concepts and effective methodologies.[21, 29]. Ghulam et al. developed Hermite inequalities for m-convex functions. These inequalities were developed through the utilization of Riemann-Liouville fractional integrals [13]. In a related context, the work of Khan et al. [17] has significantly advanced the field of inequalities by extending the scope and introducing a comprehensive class of novel and generalized Hermite-Hadamard type inequalities. Their contributions have been particularly influential in the study of both s-convex and s-concave functions, representing an

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important expansion in the mathematical tools available for analyzing these types of functions. In their research, Wang et al. [42] made new advancements in the field of mathematical analysis by enhancing Hardy-type and Jensen's inequalities for convex functions. They achieved this by introducing Riemann-Liouville delta fractional integrals, which provided a unique and innovative perspective on the behavior and properties of convex functions. In a related context, Bohner et al. [7] made a noteworthy advancement in the area of mathematical inequalities. Their contribution focused on the introduction of Hermite-Hadamard type inequalities designed to deal with the framework of conformable fractional calculus. In a related framework, P'ales et al. in [24] achieved a significant milestone by extending decomposition theorem that represents an important development, offering fresh insights and applications for higher-order convexity principles in mathematical analysis.
In this section, we provide the preliminaries relevant to our main results.

Definition 1.1. Let function $\digamma : [\omega_1, \omega_2] \to \Re$ is considered as convex if

$$\digamma\left(\varphi\psi_1 + (1-\varphi)\psi_2\right) \leq \varphi\digamma(\psi_1) + (1-\varphi)\digamma(\psi_2)$$

holds for every $\psi_1, \psi_2 \in [\omega_1, \omega_2]$ and $\varphi \in [0, 1]$.

If a function $\digamma$ is convex on an interval, its inverse function $\digamma^{-1}$ is concave on that same interval. Toader et al. in [40] establish the class of m-convex mapping through the next definition.

Definition 1.2. A function $\digamma : [\omega_1, \omega_2] \to \Re$ is termed as m-convex if for all ψ_1, ψ_2 in the domain of $\digamma$ and for all m, φ in the interval $[0, 1]$, the following inequality holds:

$$\digamma\left(\varphi\psi_1 + \mathrm{m}(1-\varphi)\psi_2\right) \leq \varphi\digamma(\psi_1) + \mathrm{m}(1-\varphi)\digamma(\psi_2).$$

m-convexity is the same as classical convexity for $\mathrm{m} = 1$.

Many researchers have explored integral inequalities related to m-convex functions due to their wide-ranging applications in various fields, including mathematics and optimization. In their work, Tariq et al. [39] developed a novel category of convex mappings termed as exponential type m-convex mappings. In [41], Wang et al. introduced Hadamard inequalities for m-convex functions via integrals with exponential kernels.

Mihesan's et al. [20] introduced the concept of (ρ, m)-convexity.

Definition 1.3. A function $\digamma : [\omega_1, \omega_2] \to \Re$ is termed as an (ρ, m)-convex if for all ψ_1, ψ_2 in the domain of $\digamma$ and for all φ in the interval $[0, 1]$, (ρ, m) in the interval $[0, 1]^2$, the following inequality

$$\digamma\left(\varphi\psi_1 + \mathrm{m}(1-\varphi)\psi_2\right) \leq \varphi^{\rho}\digamma(\psi_1) + \mathrm{m}(1-\varphi^{\rho})\digamma(\psi_2)$$

holds. It is noted that when $\rho = 1$, (ρ, m)-convexity simplifies to m-convexity, and when $\rho = \mathrm{m} = 1$, it reduces to the traditional concept of classical convexity.

Safdar et al. [30] examined conclusions related to generalized (ρ, m)-convex functions, specifically those that reveal absolute differentiability. Erhan et al. [35] introduced several Hadamard-type integral inequalities applicable to (ρ, m)-convex functions.

Theory of convex functions is often used to establish a broad range of inequalities [31, 32]. The Hermite-Hadamard inequality is a well-known classical inequality with several geometric consequences [19]. It is classically stated by the following.

Theorem 1.4. *If $\digamma : [\omega_1, \omega_2] \to \Re$ is a convex mapping and $\psi_1, \psi_2 \in [\omega_1, \omega_2]$ with $\psi_1 < \psi_2$, then the following inequality*

$$\digamma\left(\frac{\psi_1 + \psi_2}{2}\right) \leq \frac{1}{\psi_2 - \psi_1} \int_{\psi_1}^{\psi_2} \digamma(\xi) d\xi \leq \frac{\digamma(\psi_1) + \digamma(\psi_2)}{2}$$

holds. If $\digamma$ is concave the inequalities are reversed.

Many researchers have explored the Hermite-Hadamard inequality using fractional integrals and derivatives, leading to numerous refinements and extensions in the literature [27, 28, 33, 36]. Additionally, Euler first provided a wide variety of $\mathfrak{q}$-analysis fields. These efforts are directed towards establishing a strong mathematical framework, ultimately contributing to the progression of $\mathfrak{q}$-calculus in the field of quantum computing. The importance of this field lies in its crucial role as a link connecting the domains of mathematical physics [11, 16]. Quantum calculus plays a crucial role in quantum information theory, a field that encompasses various aspects, including computer science, information theory, philosophy and cryptography [5, 8]. Newton utilized the $\mathfrak{q}$-parameter in his explorations of infinite series. Moreover, in [12, 14] Jackson is recognized as the pioneer who systematically introduced quantum calculus, also referred to as limitless calculus. Bermudo et al. [6] have investigated various adaptations of the Hermite-Hadamard inequality using $\mathfrak{q}$-calculus, specifically for general convex functions. Furthermore, Chu et al. [10] introduced the concepts of ${}^{\omega_2}D_{\mathfrak{p},\mathfrak{q}}$ derivative and ${}^{\omega_2}(\mathfrak{p}, \mathfrak{q})$-integral. Quantum and post-quantum integrals are frequently applied in the analysis of various integral inequalities. Jhanthanam et al. [15] have made significant contributions by deriving several novel findings related to the left-hand side of the $\mathfrak{q}$-Hermite-Hadamard inequality for differentiable convex functions exhibiting a critical point. Budak et al. [9] introduced an innovative quantum integral and derived two new quantum inequalities of the Hermite-Hadamard type. They further expanded

their research to establish several quantum Hermite-Hadamard inequalities. In [37], Sitthiwirattham et al. conducted research on $\mathfrak{q}$-Hermite-Hadamard type inequalities, utilizing newly introduced integrals, with a particular emphasis on convex stochastic processes and coordinated stochastic processes. Prabseang et al. [26] investigated quantum analogues of Hermite-Hadamard inequalities for double integrals, and they also proposed improved versions for $\mathfrak{q}$-differentiable convex functions. Moreover, Ali et al. [3] have formulated new Hadamard inequalities by applying the recently defined $R(\mathfrak{p},\mathfrak{q})$-integral for convex functions.
Let's revisit the subject of $\mathfrak{q}$-calculus and explore the standard definitions, inequalities, and notation as discussed in [16].

$$[r]_{\mathfrak{q}}=\frac{1-\mathfrak{q}^{r}}{1-\mathfrak{q}}=1+\mathfrak{q}+\mathfrak{q}^{2}+\cdots+\mathfrak{q}^{r-1}$$

for $\mathfrak{q}\in(0,1)$. In [14], Jackson presented the $\mathfrak{q}$-Jackson integral of a function $\digamma$, ranging from 0 to ω_2, using the following formulation:

$$\int_{0}^{\omega_2}\digamma(\xi)d_{\mathfrak{q}}\xi=(1-\mathfrak{q})\omega_2\sum_{r=0}^{\infty}\mathfrak{q}^{r}\digamma(\omega_2\mathfrak{q}^{r}), \tag{1.1}$$

where $\mathfrak{q}\in(0,1)$, provided that the sum converges absolutely. Jackson [14] introduced the $\mathfrak{q}$-Jackson integral for a function defined on the interval $[\omega_1,\omega_2]$ in the subsequent manner:

$$\int_{\omega_1}^{\omega_2}\digamma(\xi)d_{\mathfrak{q}}\xi=\int_{0}^{\omega_2}\digamma(\xi)d_{\mathfrak{q}}\xi-\int_{0}^{\omega_1}\digamma(\xi)d_{\mathfrak{q}}\xi.$$

Definition 1.5 ([38])**.** Consider the mapping $\digamma:[\omega_1,\omega_2]\to\Re$ to be continuous. Then, for $\digamma$ at $\varphi\in[\omega_1,\omega_2]$, the ${}_{\omega_1}\mathfrak{q}$-derivative is found to be

$${}_{\omega_1}D_{\mathfrak{q}}\digamma(\xi)=\frac{\digamma(\xi)-\digamma(\mathfrak{q}\xi+(1-\mathfrak{q})\omega_1)}{(1-\mathfrak{q})(\xi-\omega_1)},\quad \xi\neq\omega_1. \tag{1.2}$$

Since $\digamma:[\omega_1,\omega_2]\to\Re$ is a continuous function, we can provide

$${}_{\omega_1}D_{\mathfrak{q}}\digamma(\omega_1)=\lim_{\xi\to\omega_1}{}_{\omega_1}D_{\mathfrak{q}}\digamma(\xi).$$

If ${}_{\omega_1}D_{\mathfrak{q}}\digamma(\xi)$ exists for any $\xi\in[\omega_1,\omega_2]$, then the mapping $\digamma$ is known to be ${}_{\omega_1}\mathfrak{q}$-differentiable on $[\omega_1,\omega_2]$. If we assume that $\omega_1=0$ in equation (1.2), then we get ${}_{0}D_{\mathfrak{q}}\digamma(\xi)=D_{\mathfrak{q}}\digamma(\xi)$, where $D_{\mathfrak{q}}\digamma(\xi)$ is the $\mathfrak{q}$-derivative of $\digamma$ at $0\leq\xi\leq\omega_2$ provided by

$$D_{\mathfrak{q}}\digamma(\xi)=\frac{\digamma(\xi)-\digamma(\mathfrak{q}\xi)}{(1-\mathfrak{q})\xi},\quad \xi\neq 0.$$

see [16] for further information.

Definition 1.6. Consider the function $\digamma : [\omega_1, \omega_2] \to \Re$ to be continuous. Then, for $\digamma$ at $\varphi \in [\omega_1, \omega_2]$, the second ${}_{\omega_1}\mathfrak{q}$-derivative is given as

$$\begin{aligned}{}_{\omega_1}D_{\mathfrak{q}}^2\digamma(\xi) &= {}_{\omega_1}D_{\mathfrak{q}}\left({}_{\omega_1}D_{\mathfrak{q}}\digamma(\xi)\right)\\ &= \frac{\digamma(\mathfrak{q}^2\varphi\omega_2 + (1-\varphi\mathfrak{q}^2)\omega_1) - (1+\mathfrak{q})\digamma(\mathfrak{q}\varphi\omega_2 + (1-\varphi\mathfrak{q})\omega_1)}{(1-\mathfrak{q})^2\mathfrak{q}(\omega_2-\omega_1)^2\varphi^2}\\ &\quad + \frac{\mathfrak{q}\digamma(\varphi\omega_2 + (1-\varphi)\omega_1)}{(1-\mathfrak{q})^2\mathfrak{q}(\omega_2-\omega_1)^2\varphi^2}.\end{aligned}$$

Definition 1.7 ([6])**.** Consider the mapping $\digamma : [\omega_1, \omega_2] \to \Re$ to be continuous. Then, for $\digamma$ at $\varphi \in [\omega_1, \omega_2]$, the ${}^{\omega_2}\mathfrak{q}$-derivative is defined as

$${}^{\omega_2}D_{\mathfrak{q}}\digamma(\xi) = \frac{\digamma(\mathfrak{q}\xi + (1-\mathfrak{q})\omega_2) - \digamma(\xi)}{(1-\mathfrak{q})(\omega_2-\xi)}, \quad \text{for } \xi \neq \omega_2.$$

Definition 1.8 ([38])**.** Consider the mapping $\digamma : [\omega_1, \omega_2] \to \Re$ to be continuous. Then the ${}_{\omega_1}\mathfrak{q}$-definite integral on $[\omega_1, \omega_2]$ is given as

$$\begin{aligned}\int_{\omega_1}^{\omega_2}\digamma(\xi){}_{\omega_1}d_{\mathfrak{q}}\xi &= (1-\mathfrak{q})(\omega_2-\omega_1)\sum_{r=0}^{\infty}\mathfrak{q}^r\digamma(\mathfrak{q}^r\omega_2 + (1-\mathfrak{q}^r)\omega_1)\\ &= (\omega_2-\omega_1)\int_0^1 \digamma\left((1-\varphi)\omega_1 + \varphi\omega_2\right)d_{\mathfrak{q}}\varphi.\end{aligned}$$

In their research, Alp et al. [4] presented the ${}_{\omega_1}\mathfrak{q}$-Hermite-Hadamard inequality for convex functions in the realm of $\mathfrak{q}$-calculus.

Theorem 1.9. *For a convex mapping $\digamma : [\omega_1, \omega_2] \to \Re$ that is differentiable on $[\omega_1, \omega_2]$, then following holds*

$$\digamma\left(\frac{\mathfrak{q}\omega_1 + \omega_2}{1+\mathfrak{q}}\right) \leq \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\digamma(\xi){}_{\omega_1}d_{\mathfrak{q}}\xi \leq \frac{\mathfrak{q}\digamma(\omega_1) + \digamma(\omega_2)}{1+\mathfrak{q}}, \tag{1.3}$$

where $0 < \mathfrak{q} < 1$.

Authors established specific estimates for the previous inequality in [4, 23]. The next formulation was given by Bermudo et al. [6], who also established the corresponding Hermite-Hadamard-type inequalities.

Definition 1.10. Consider the mapping $\digamma : [\omega_1, \omega_2] \to \Re$ to be continuous. Then the ${}^{\omega_2}\mathfrak{q}$-definite integral on $[\omega_1, \omega_2]$ is given as

$$\begin{aligned}\int_{\omega_1}^{\omega_2}\digamma(\xi){}^{\omega_2}d_{\mathfrak{q}}\xi &= (1-\mathfrak{q})(\omega_2-\omega_1)\sum_{r=0}^{\infty}\mathfrak{q}^r\digamma(\mathfrak{q}^r\omega_1 + (1-\mathfrak{q}^r)\omega_2)\\ &= (\omega_2-\omega_1)\int_0^1 \digamma\left(\varphi\omega_1 + (1-\varphi)\omega_2\right)d_{\mathfrak{q}}\varphi.\end{aligned}$$

Theorem 1.11 ([6]). *For a convex mapping $F : [\omega_1, \omega_2] \to \Re$ that is differentiable on $[\omega_1, \omega_2]$, the following $\mathfrak{q}$-Hermite-Hadamard inequality holds:*

$$F\left(\frac{\omega_1 + \mathfrak{q}\omega_2}{1+\mathfrak{q}}\right) \leq \frac{1}{\omega_2 - \omega_1}\int_{\omega_1}^{\omega_2} F(\xi)^{\omega_2} d_{\mathfrak{q}}\xi \leq \frac{F(\omega_1) + \mathfrak{q}F(\omega_2)}{1+\mathfrak{q}}, \tag{1.4}$$

where $\mathfrak{q} \in (0,1)$.

The following inequalities are derived from Theorem 1.9 and 1.11.

Corollary 1.12 ([6]). *For convex function $F : [\omega_1, \omega_2] \to \Re$ and $0 < \mathfrak{q} < 1$, we have*

$$\begin{aligned} F\left(\frac{\mathfrak{q}\omega_1 + \omega_2}{1+\mathfrak{q}}\right) + F\left(\frac{\omega_1 + \mathfrak{q}\omega_2}{1+\mathfrak{q}}\right) & \\ \leq \frac{1}{\omega_2 - \omega_1}\left[\int_{\omega_1}^{\omega_2} F(\xi)_{\omega_1} d_{\mathfrak{q}}\xi + \int_{\omega_1}^{\omega_2} F(\xi)^{\omega_2} d_{\mathfrak{q}}\xi\right] & \\ \leq F(\omega_1) + F(\omega_2) & \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} F\left(\frac{\omega_1 + \omega_2}{2}\right) &\leq \frac{1}{2(\omega_2 - \omega_1)}\left[\int_{\omega_1}^{\omega_2} F(\xi)_{\omega_1} d_{\mathfrak{q}}\xi + \int_{\omega_1}^{\omega_2} F(\xi)^{\omega_2} d_{\mathfrak{q}}\xi\right] \\ &\leq \frac{F(\omega_1) + F(\omega_2)}{2}. \end{aligned} \tag{1.6}$$

In the following we recall the Hölder's inequality which is given in [1, p. 604].

Theorem 1.13. *Let $\xi > 0, 0 < \mathfrak{q} < 1$ and $\sigma > 1$. If $\frac{1}{\kappa} + \frac{1}{\sigma} = 1$, then*

$$\int_0^{\xi} |F(\xi)\Theta(\xi)|\, d_{\mathfrak{q}}\xi \leq \left(\int_0^{\xi} |F(\xi)|^{\kappa}\, d_{\mathfrak{q}}\xi\right)^{\frac{1}{\kappa}} \left(\int_0^{\xi} |\Theta(\xi)|^{\sigma}\, d_{\mathfrak{q}}\xi\right)^{\frac{1}{\sigma}}.$$

2. Main Results

In this section, our main objective is to examine original Hermite-Hadamard inequalities specifically designed for functions demonstrating absolute convexity in their second ${}_{\omega_1}\mathfrak{q}$-derivatives. We shall apply the following lemma to determine our main results [18, Lemma 4.1].

Lemma 2.1. *A mapping $F : [\omega_1, \omega_2] \to \Re$ that is twice ${}_{\omega_1}\mathfrak{q}$-differentiable on (ω_1, ω_2), where $[\omega_1, \omega_2] \subset \Re$ and ${}_{\omega_1}D_{\mathfrak{q}}^2 F$ is both integrable and continuous on $[\omega_1, \omega_2]$, then we have*

$$\frac{\mathfrak{q}F(\omega_1) + F(\omega_2)}{1+\mathfrak{q}} - \frac{1}{(\omega_2 - \omega_1)}\int_{\omega_1}^{\omega_2} F(\xi)_{\omega_1} d_{\mathfrak{q}}\xi$$

$$= \frac{\mathfrak{q}^2(\omega_2-\omega_1)^2}{1+\mathfrak{q}} \int_0^1 \varphi(1-\mathfrak{q}\varphi)_{\omega_1} D_{\mathfrak{q}}^2 F\left((1-\varphi)\omega_1+\varphi\omega_2\right) d_{\mathfrak{q}}\varphi,$$

where $0 < \mathfrak{q} < 1$.

The objective of this article is to establish some estimates for the inequality's right-hand side (1.4).

Theorem 2.2. *Suppose a mapping* $F : [\omega_1, \omega_2] \to \Re$ *that is twice* ${}_{\omega_1}\mathfrak{q}$*-differentiable on* (ω_1, ω_2)*, where* $[\omega_1, \omega_2] \subset \Re$ *and* ${}_{\omega_1} D_{\mathfrak{q}}^2 F$ *is both integrable and continuous on* $[\omega_1, \omega_2]$*, provided that* $|{}_{\omega_1} D_{\mathfrak{q}}^2 F|$ *is* (ρ, m)*-convex on* $[\omega_1, \omega_2]$*, then the following inequality holds.*

(2.1)

$$\left|\frac{\mathfrak{q}F(\omega_1)+F(\omega_2)}{1+\mathfrak{q}} - \frac{1}{(\omega_2-\omega_1)}\int_{\omega_1}^{\omega_2} F(\xi)_{\omega_1} d_{\mathfrak{q}}\xi\right|$$
$$\leq \frac{\mathfrak{q}^2(\omega_2-\omega_1)^2(1-\mathfrak{q})^2}{(1+\mathfrak{q})(1-\mathfrak{q}^{\rho+2})(1-\mathfrak{q}^{\rho+3})}\left[a_1\left|{}_{\omega_1} D_{\mathfrak{q}}^2 F\left(\omega_1\right)\right| + \mathrm{m}\left|{}_{\omega_1} D_{\mathfrak{q}}^2 F\left(\frac{\omega_2}{\mathrm{m}}\right)\right|\right],$$

where

$$a_1 = \frac{(1-\mathfrak{q}^{\rho+2})(1-\mathfrak{q}^{\rho+3})}{(1-\mathfrak{q}^2)(1-\mathfrak{q}^3)} - 1. \tag{2.2}$$

Proof. Upon applying the modulus operation in Lemma 2.1 and subsequently utilizing the (ρ, m)-convexity property of $|\omega 1 D_{\mathfrak{q}}^2 F|$, we arrive at the following result:

$$\begin{aligned}
&\left|\frac{\mathfrak{q}F(\omega_1)+F(\omega_2)}{1+\mathfrak{q}} - \frac{1}{(\omega_2-\omega_1)}\int_{\omega_1}^{\omega_2} F(\xi)_{\omega_1} d_{\mathfrak{q}}\xi\right| \\
&\leq \frac{\mathfrak{q}^2(\omega_2-\omega_1)^2}{1+\mathfrak{q}}\int_0^1 \varphi(1-\mathfrak{q}\varphi)\left|{}_{\omega_1} D_{\mathfrak{q}}^2 F\left((1-\varphi)\omega_1+\varphi\omega_2\right)\right| d_{\mathfrak{q}}\varphi \\
&= \frac{\mathfrak{q}^2(\omega_2-\omega_1)^2}{1+\mathfrak{q}}\int_0^1 \varphi(1-\mathfrak{q}\varphi)\left|{}_{\omega_1} D_{\mathfrak{q}}^2 F\left((1-\varphi)\omega_1+\mathrm{m}\varphi\frac{\omega_2}{\mathrm{m}}\right)\right| d_{\mathfrak{q}}\varphi \\
&\leq \frac{\mathfrak{q}^2(\omega_2-\omega_1)^2}{1+\mathfrak{q}}\int_0^1 \varphi\left(1-\mathfrak{q}\varphi\right)\Big[\left(1-\varphi^{\rho}\right)\left|{}_{\omega_1} D_{\mathfrak{q}}^2 F\left(\omega_1\right)\right| \\
&\quad + \mathrm{m}\varphi^{\rho}\left|{}_{\omega_1} D_{\mathfrak{q}}^2 F\left(\frac{\omega_2}{\mathrm{m}}\right)\right|\Big] d_{\mathfrak{q}}\varphi \\
&= \frac{\mathfrak{q}^2(\omega_2-\omega_1)^2}{1+\mathfrak{q}}\Bigg[\left|{}_{\omega_1} D_{\mathfrak{q}}^2 F\left(\omega_1\right)\right|\int_0^1 (\varphi-\varphi^{\rho+1})(1-\mathfrak{q}\varphi) d_{\mathfrak{q}}\varphi \\
&\quad + \mathrm{m}\left|{}_{\omega_1} D_{\mathfrak{q}}^2 F\left(\frac{\omega_2}{\mathrm{m}}\right)\right|\int_0^1 \varphi^{\rho+1}\left(1-\mathfrak{q}\varphi\right) d_{\mathfrak{q}}\varphi\Bigg].
\end{aligned}$$

After simplification, we obtain the required result. □

Corollary 2.3. *By setting $\rho = 1$ inequality (2.1) becomes*

$$\begin{aligned}&\frac{\mathfrak{q}\digamma(\omega_1)+\digamma(\omega_2)}{1+\mathfrak{q}}-\frac{1}{(\omega_2-\omega_1)}\int_{\tau_1}^{\omega_2}\digamma(\xi)_{\omega_1}d_{\mathfrak{q}}\xi\\&\leq\frac{\mathfrak{q}^2(\omega_2-\omega_1)^2}{(1+\mathfrak{q})(\mathfrak{q}^2+\mathfrak{q}+1)(\mathfrak{q}^3+\mathfrak{q}^2+\mathfrak{q}+1)}\\&\quad\times\left[\mathfrak{q}^2\left|{}_{\omega_1}D_{\mathfrak{q}}^2\digamma\left(\omega_1\right)\right|+\mathrm{m}\left|{}_{\omega_1}D_{\mathfrak{q}}^2\digamma\left(\frac{\omega_2}{\mathrm{m}}\right)\right|\right].\end{aligned}$$

Corollary 2.4. *By choosing* $\mathrm{m} = \rho = 1$ *inequality* (2.1) *becomes*

$$\begin{aligned}&\frac{\mathfrak{q}\digamma(\omega_1)+\digamma(\omega_2)}{1+\mathfrak{q}}-\frac{1}{(\omega_2-\omega_1)}\int_{\omega_1}^{\omega_2}\digamma(\xi)_{\omega_1}d_{\mathfrak{q}}\xi\\&\leq\frac{\mathfrak{q}^2(\omega_2-\omega_1)^2}{(1+\mathfrak{q})(\mathfrak{q}^2+\mathfrak{q}+1)(\mathfrak{q}^3+\mathfrak{q}^2+\mathfrak{q}+1)}\\&\quad\times\left[\mathfrak{q}^2\left|{}_{\omega_1}D_{\mathfrak{q}}^2\digamma\left(\omega_1\right)\right|+\left|{}_{\omega_1}D_{\mathfrak{q}}^2\digamma\left(\omega_2\right)\right|\right].\end{aligned}$$

Remark 2.5. Under the same assumptions of Theorem 2.2 with the limit as $\mathfrak{q}\to 1^-$ and $\mathrm{m} = \rho = 1$, we get the next trapezoidal inequality

$$\begin{aligned}&\left|\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\digamma(\xi)d\xi-\frac{\digamma(\omega_1)+\digamma(\omega_2)}{2}\right|\\&\leq\frac{(\omega_2-\omega_1)^2}{12}\left[\frac{|\digamma''(\omega_1)|+|\digamma''(\omega_2)|}{2}\right],\end{aligned}$$

which was published in [34, Proposition 2].

Theorem 2.6. *Suppose a function $\digamma:[\omega_1,\omega_2]\to\Re$ that is twice ${}_{\omega_1}\mathfrak{q}$-differentiable on (ω_1,ω_2) where $[\omega_1,\omega_2]\subset\Re$ and ${}_{\omega_1}D_{\mathfrak{q}}^2\digamma$ is both integrable and continuous on $[\omega_1,\omega_2]$. If $|{}_{\omega_1}D_{\mathfrak{q}}^2\digamma|^{\sigma}$, $\sigma>1$, is (ρ,m)-convex on $[\omega_1,\omega_2]$, then the following inequality holds*

$$\begin{aligned}&\left|\frac{\mathfrak{q}\digamma(\omega_1)+\digamma(\omega_2)}{1+\mathfrak{q}}-\frac{1}{(\omega_2-\omega_1)}\int_{\omega_1}^{\omega_2}\digamma(\xi)_{\omega_1}d_{\mathfrak{q}}\xi\right|\\&\leq\frac{\mathfrak{q}^2(\omega_2-\omega_1)^2}{(1+\mathfrak{q})^{2-\frac{1}{\sigma}}(1+\mathfrak{q}+\mathfrak{q}^2)}\left[\frac{(1-\mathfrak{q}^3)(1-\mathfrak{q})}{(1-\mathfrak{q}^{\rho+2})(1-\mathfrak{q}^{\rho+3})}\right]^{\frac{1}{\sigma}}\\&\quad\times\left[a_1\left|{}_{\omega_1}D_{\mathfrak{q}}^2\digamma\left(\omega_1\right)\right|^{\sigma}+\mathrm{m}\left|{}_{\omega_1}D_{\mathfrak{q}}^2\digamma\left(\frac{\omega_2}{\mathrm{m}}\right)\right|^{\sigma}\right]^{\frac{1}{\sigma}},\end{aligned}\tag{2.3}$$

where a_1 is defined in (2.2) *and $0<\mathfrak{q}<1$.*

Proof. After applying the modulus function then utilizing the power mean inequality from Lemma 2.1, we obtain the following result:

$$\left|\frac{\mathfrak{q}\digamma(\omega_1)+\digamma(\omega_2)}{1+\mathfrak{q}}-\frac{1}{(\omega_2-\omega_1)}\int_{\omega_1}^{\omega_2}\digamma(\xi)_{\omega_1}d_{\mathfrak{q}}\xi\right|$$

$$\leq \frac{\mathfrak{q}^2(\omega_2-\omega_1)^2}{1+\mathfrak{q}}\int_0^1 \varphi(1-\mathfrak{q}\varphi)\left|{}_{\omega_1}D_{\mathfrak{q}}^2F\left((1-\varphi)\omega_1+\varphi\omega_2\right)\right| d_{\mathfrak{q}}\varphi$$

$$\leq \frac{\mathfrak{q}^2(\omega_2-\omega_1)^2}{1+\mathfrak{q}}\left(\int_0^1 \varphi(1-\mathfrak{q}\varphi)d_{\mathfrak{q}}\varphi\right)^{1-\frac{1}{\sigma}}$$

$$\times\left(\int_0^1 \varphi(1-\mathfrak{q}\varphi)\left|{}_{\omega_1}D_{\mathfrak{q}}^2F\left((1-\varphi)\omega_1+\varphi\omega_2\right)\right|^{\sigma} d_{\mathfrak{q}}\varphi\right)^{\frac{1}{\sigma}}$$

$$= \frac{\mathfrak{q}^2(\omega_2-\omega_1)^2}{1+\mathfrak{q}}\left(\int_0^1 \varphi(1-\mathfrak{q}\varphi)d_{\mathfrak{q}}\varphi\right)^{1-\frac{1}{\sigma}}$$

$$\times\left(\int_0^1 \varphi(1-\mathfrak{q}\varphi)\left|{}_{\omega_1}D_{\mathfrak{q}}^2F\left((1-\varphi)\omega_1+\mathrm{m}\varphi\frac{\omega_2}{\mathrm{m}}\right)\right|^{\sigma} d_{\mathfrak{q}}\varphi\right)^{\frac{1}{\sigma}}.$$

By the (ρ,m)-convexity of $|{}_{\omega_1}D_{\mathfrak{q}}^2F|^{\sigma}$, we have

$$\left|\frac{\mathfrak{q}F(\omega_1)+F(\omega_2)}{1+\mathfrak{q}}-\frac{1}{(\omega_2-\omega_1)}\int_{\omega_1}^{\omega_2}F(\xi)_{\omega_1}d_{\mathfrak{q}}\xi\right|$$

$$\leq \frac{\mathfrak{q}^2(\omega_2-\omega_1)^2}{1+\mathfrak{q}}\left(\int_0^1 \varphi(1-\mathfrak{q}\varphi)d_{\mathfrak{q}}\varphi\right)^{1-\frac{1}{\sigma}}$$

$$\times\left(\int_0^1 \varphi(1-\mathfrak{q}\varphi)\left[(1-\varphi^{\rho})\left|{}_{\omega_1}D_{\mathfrak{q}}^2F\left(\omega_1\right)\right|^{\sigma}+\mathrm{m}\varphi^{\rho}\left|{}_{\omega_1}D_{\mathfrak{q}}^2F\left(\frac{\omega_2}{\mathrm{m}}\right)\right|^{\sigma}\right] d_{\mathfrak{q}}\varphi\right)^{\frac{1}{\sigma}}$$

$$= \frac{\mathfrak{q}^2(\omega_2-\omega_1)^2}{1+\mathfrak{q}}\left(\int_0^1 \varphi(1-\mathfrak{q}\varphi)d_{\mathfrak{q}}\varphi\right)^{1-\frac{1}{\sigma}}\left(\left|{}_{\omega_1}D_{\mathfrak{q}}^2F\left(\omega_1\right)\right|^{\sigma}\right.$$

$$\left.\times\int_0^1(\varphi-\varphi^{\rho+1})(1-\mathfrak{q}\varphi)d_{\mathfrak{q}}\varphi+\mathrm{m}\left|{}_{\omega_1}D_{\mathfrak{q}}^2F\left(\frac{\omega_2}{\mathrm{m}}\right)\right|^{\sigma}\int_0^1\varphi^{\rho+1}(1-\mathfrak{q}\varphi)d_{\mathfrak{q}}\varphi\right)^{\frac{1}{\sigma}}$$

$$= \frac{\mathfrak{q}^2(\omega_2-\omega_1)^2}{(1+\mathfrak{q})}\left(\frac{1}{(1+\mathfrak{q})(1+\mathfrak{q}+\mathfrak{q}^2)}\right)^{1-\frac{1}{\sigma}}$$

$$\times\left(\frac{(1-\mathfrak{q})^2}{(1-\mathfrak{q}^{\rho+2})(1-\mathfrak{q}^{\rho+3})}\right)^{\frac{1}{\sigma}}\left[a_1\left|{}_{\omega_1}D_{\mathfrak{q}}^2F\left(\omega_1\right)\right|^{\sigma}+\mathrm{m}\left|{}_{\omega_1}D_{\mathfrak{q}}^2F\left(\frac{\omega_2}{\mathrm{m}}\right)\right|^{\sigma}\right]^{\frac{1}{\sigma}}.$$

After simplification, the required inequality is obtained. □

Corollary 2.7. *By putting* $\rho=1$ *inequality* (2.3) *becomes*

$$\left|\frac{\mathfrak{q}F(\omega_1)+F(\omega_2)}{1+\mathfrak{q}}-\frac{1}{(\omega_2-\tau_1)}\int_{\omega_1}^{\omega_2}F(\xi)_{\omega_1}d_{\mathfrak{q}}\xi\right|$$

$$\leq \frac{\mathfrak{q}^2(\omega_2-\omega_1)^2}{(1+\mathfrak{q})^{2-\frac{1}{\sigma}}(1+\mathfrak{q}+\mathfrak{q}^2)}\left(\frac{1}{(1+\mathfrak{q}+\mathfrak{q}^2+\mathfrak{q}^3)}\right)^{\frac{1}{\sigma}}$$

$$\times\left[\mathfrak{q}^2\left|{}_{\omega_1}D_{\mathfrak{q}}^2F\left(\omega_1\right)\right|^{\sigma}+\mathrm{m}\left|{}_{\omega_1}D_{\mathfrak{q}}^2F\left(\frac{\omega_2}{\mathrm{m}}\right)\right|^{\sigma}\right]^{\frac{1}{\sigma}}.$$

Corollary 2.8. *By choosing* $\mathrm{m} = \rho = 1$ *inequality* (2.3) *becomes*

$$\begin{aligned}&\left|\frac{\mathfrak{q}\digamma(\omega_1)+\digamma(\omega_2)}{1+\mathfrak{q}}-\frac{1}{(\omega_2-\omega_1)}\int_{\omega_1}^{\omega_2}\digamma(\xi)_{\omega_1}d_{\mathfrak{q}}\xi\right|\\&\leq\frac{\mathfrak{q}^2(\omega_2-\omega_1)^2}{(1+\mathfrak{q})^{2-\frac{1}{\sigma}}(1+\mathfrak{q}+\mathfrak{q}^2)}\left(\frac{1}{(1+\mathfrak{q}+\mathfrak{q}^2+\mathfrak{q}^3)}\right)^{\frac{1}{\sigma}}\\&\quad\times\left[\mathfrak{q}^2\left|{}_{\omega_1}D_{\mathfrak{q}}^2\digamma\left(\omega_1\right)\right|^{\sigma}+\left|{}_{\omega_1}D_{\mathfrak{q}}^2\digamma\left(\omega_2\right)\right|^{\sigma}\right]^{\frac{1}{\sigma}}.\end{aligned}$$

Remark 2.9. By setting $\mathrm{m} = \rho = 1$ and taking the limit as $\mathfrak{q} \to 1^-$ in Theorem 2.6, we obtain an integral inequality that was previously discussed in [2, Remark 3].

Theorem 2.10. *Suppose a function* $\digamma : [\omega_1, \omega_2] \to \Re$ *that is twice* ${}_{\omega_1}\mathfrak{q}$*-differentiable on* (ω_1, ω_2)*, where* $[\omega_1, \omega_2] \subset \Re$ *and* ${}_{\omega_1}D_{\mathfrak{q}}^2\digamma$ *is both continuous and integrable on* $[\omega_1, \omega_2]$*. If* $|{}_{\omega_1}D_{\mathfrak{q}}^2\digamma|^{\sigma}$ *is* (ρ, m)*-convex on* $[\omega_1, \omega_2]$ *for some* $\sigma > 1$ *and* $\frac{1}{\kappa} + \frac{1}{\sigma} = 1$*, then*

$$\begin{aligned}(2.4)\quad&\left|\frac{\mathfrak{q}\digamma(\omega_1)+\digamma(\omega_2)}{1+\mathfrak{q}}-\frac{1}{(\omega_2-\omega_1)}\int_{\omega_1}^{\omega_2}\digamma(\xi)_{\omega_1}d_{\mathfrak{q}}\xi\right|\\&\leq\frac{\mathfrak{q}^2(\omega_2-\omega_1)^2}{(1+\mathfrak{q})}(a_2)^{\frac{1}{\kappa}}\\&\quad\times\left(\frac{\mathfrak{q}(1-\mathfrak{q}^{\rho})\left|{}_{\omega_1}D_{\mathfrak{q}}^2\digamma\left(\omega_1\right)\right|^{\sigma}+\mathrm{m}(1-\mathfrak{q})\left|{}_{\omega_1}D_{\mathfrak{q}}^2\digamma\left(\frac{\omega_2}{\mathrm{m}}\right)\right|^{\sigma}}{1-\mathfrak{q}^{\rho+1}}\right)^{\frac{1}{\sigma}},\end{aligned}$$

where

$$(2.5)\qquad a_2=(1-\mathfrak{q})\sum_{r=0}^{\infty}(\mathfrak{q}^r)^{\kappa+1}(1-\mathfrak{q}^{r+1})^{\kappa},\quad \textit{for } \mathfrak{q}\in(0,1).$$

Proof. After applying the modulus function then utilizing the Hölder's inequality from Lemma 2.1, we have the following result:

$$\begin{aligned}&\left|\frac{\mathfrak{q}\digamma(\omega_1)+\digamma(\omega_2)}{1+\mathfrak{q}}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\digamma(\xi)_{\omega_1}d_{\mathfrak{q}}\xi\right|\\&\leq\frac{\mathfrak{q}^2(\omega_2-\omega_1)^2}{1+\mathfrak{q}}\int_0^1\varphi\left(1-\mathfrak{q}\varphi\right)\left|{}_{\omega_1}D_{\mathfrak{q}}^2\digamma\left(\left(1-\varphi\right)\omega_1+\varphi\omega_2\right)\right|d_{\mathfrak{q}}\varphi\\&\leq\frac{\mathfrak{q}^2(\omega_2-\omega_1)^2}{1+\mathfrak{q}}\left(\int_0^1\left(\varphi\left(1-\mathfrak{q}\varphi\right)\right)^{\kappa}d_{\mathfrak{q}}\varphi\right)^{\frac{1}{\kappa}}\\&\quad\times\left(\left|{}_{\omega_1}D_{\mathfrak{q}}^2\digamma\left(\left(1-\varphi\right)\omega_1+\varphi\omega_2\right)\right|^{\sigma}d_{\mathfrak{q}}\varphi\right)^{\frac{1}{\sigma}}\end{aligned}$$

$$= \frac{\mathfrak{q}^2(\omega_2-\omega_1)^2}{1+\mathfrak{q}} \left(\int_0^1 \left(\varphi\left(1-\mathfrak{q}\varphi\right)\right)^{\kappa} d_{\mathfrak{q}}\varphi \right)^{\frac{1}{\kappa}}$$
$$\times \left(\left| {}_{\omega_1}D_{\mathfrak{q}}^2 F\left(\left(1-\varphi\right)\omega_1 + \mathrm{m}\varphi\frac{\omega_2}{\mathrm{m}} \right) \right|^{\sigma} d_{\mathfrak{q}}\varphi \right)^{\frac{1}{\sigma}}.$$

Since $|{}_{\omega_1}D_{\mathfrak{q}}^2 F|^{\sigma}$ is (ρ,m)-convex, therefore we have

$$\left| \frac{\mathfrak{q}F(\omega_1)+F(\omega_2)}{1+\mathfrak{q}} - \frac{1}{(\omega_2-\omega_1)} \int_{\omega_1}^{\omega_2} F(\xi)_{\omega_1} d_{\mathfrak{q}}\xi \right|$$
$$\leq \frac{\mathfrak{q}^2(\omega_2-\omega_1)^2}{1+\mathfrak{q}} \left(\int_0^1 \left(\varphi(1-\mathfrak{q}\varphi)\right)^{\kappa} d_{\mathfrak{q}}\varphi \right)^{\frac{1}{\kappa}}$$
$$\times \left(\left|{}_{\omega_1}D_{\mathfrak{q}}^2 F\left(\omega_1\right)\right|^{\sigma} \int_0^1 (1-\varphi^{\rho}) d_{\mathfrak{q}}\varphi + \mathrm{m} \left|{}_{\omega_1}D_{\mathfrak{q}}^2 F\left(\frac{\omega_2}{\mathrm{m}}\right)\right|^{\sigma} \int_0^1 \varphi^{\rho} d_{\mathfrak{q}}\varphi \right)^{\frac{1}{\sigma}}.$$

After simplification we get the required inequality. □

Corollary 2.11. *By setting $\rho = 1$ inequality (2.4) becomes*

$$\left| \frac{\mathfrak{q}F(\omega_1)+F(\omega_2)}{1+\mathfrak{q}} - \frac{1}{(\omega_2-\omega_1)} \int_{\omega_1}^{\omega_2} F(\varphi)_{\omega_1} d_{\mathfrak{q}}\xi \right|$$
$$\leq \frac{\mathfrak{q}^2(\omega_2-\omega_1)^2}{(1+\mathfrak{q})} \left(a_2\right)^{\frac{1}{\kappa}} \left(\frac{\mathfrak{q}\left|{}_{\omega_1}D_{\mathfrak{q}}^2 F\left(\omega_1\right)\right|^{\sigma} + \mathrm{m}\left|{}_{\omega_1}D_{\mathfrak{q}}^2 F\left(\frac{\omega_2}{\mathrm{m}}\right)\right|^{\sigma}}{1+\mathfrak{q}} \right)^{\frac{1}{\sigma}},$$

where a_2 is define in (2.5).

Corollary 2.12. *By setting $\mathrm{m} = \rho = 1$ inequality (2.4) becomes*

$$\left| \frac{\mathfrak{q}F(\omega_1)+F(\omega_2)}{1+\mathfrak{q}} - \frac{1}{(\omega_2-\omega_1)} \int_{\omega_1}^{\omega_2} F(\xi)_{\omega_1} d_{\mathfrak{q}}\xi \right|$$
$$\leq \frac{\mathfrak{q}^2(\omega_2-\omega_1)^2}{(1+\mathfrak{q})} \left(a_2\right)^{\frac{1}{\kappa}} \left(\frac{\mathfrak{q}\left|{}_{\omega_1}D_{\mathfrak{q}}^2 F\left(\omega_1\right)\right|^{\sigma} + \left|{}_{\omega_1}D_{\mathfrak{q}}^2 F\left(\omega_2\right)\right|^{\sigma}}{1+\mathfrak{q}} \right)^{\frac{1}{\sigma}},$$

where a_2 is define in (2.5).

Remark 2.13. By setting $\mathrm{m} = \rho = 1$ and taking the limit as $\mathfrak{q} \to 1^-$ in Theorem 2.10, we obtain the integral inequality which is presented in [2, Remark 4].

Another Hermite-Hadamard-type inequality for powers is obtained in terms of the second $\mathfrak{q}$-derivatives.

Theorem 2.14. *If the assumptions of Theorem 2.10 are correct, then the following inequality is valid*

$$\left| \frac{\mathfrak{q}F(\omega_1)+F(\omega_2)}{1+\mathfrak{q}} - \frac{1}{\omega_2-\omega_1} \int_{\omega_1}^{\omega_2} F(\xi)_{\omega_1} d_{\mathfrak{q}}\xi \right| \tag{2.6}$$

$$\leq \frac{\mathfrak{q}^2(\omega_2-\omega_1)^2}{(1+\mathfrak{q})}\left(\frac{1}{[\kappa+1]_{\mathfrak{q}}}\right)^{\frac{1}{\kappa}} \times \left(a_3\left|{}_{\omega_1}D_{\mathfrak{q}}^2\digamma\left(\omega_1\right)\right|^{\sigma}+\mathrm{m}a_4\left|{}_{\omega_1}D_{\mathfrak{q}}^2\digamma\left(\frac{\omega_2}{\mathrm{m}}\right)\right|^{\sigma}\right)^{\frac{1}{\sigma}},$$

where

$$a_3=(1-\mathfrak{q})\sum_{r=0}^{\infty}\mathfrak{q}^r(1-\mathfrak{q}^{r\rho})(1-\mathfrak{q}^{r+1})^{\sigma}$$

and

$$a_4=(1-\mathfrak{q})\sum_{r=0}^{\infty}\mathfrak{q}^{r(\rho+1)}(1-\mathfrak{q}^{r+1})^{\sigma}.$$

Proof. By initially employing the modulus function and subsequently invoking Hölder's inequality in Lemma 2.1, we attain the following result:

$$\begin{aligned}
&\left|\frac{\mathfrak{q}\digamma\left(\omega_1\right)+\digamma\left(\omega_2\right)}{1+\mathfrak{q}}-\frac{1}{(\omega_2-\omega_1)}\int_{\omega_1}^{\omega_2}\digamma\left(\xi\right)_{\omega_1}d_{\mathfrak{q}}\xi\right| \\
&\quad\leq\frac{\mathfrak{q}^2(\omega_2-\omega_1)^2}{1+\mathfrak{q}}\int_0^1\varphi\left(1-\mathfrak{q}\varphi\right)\left|{}_{\omega_1}D_{\mathfrak{q}}^2\digamma\left(\left(1-\varphi\right)\omega_1+\varphi\omega_2\right)\right|d_{\mathfrak{q}}\varphi \\
&\quad\leq\frac{\mathfrak{q}^2(\omega_2-\omega_1)^2}{1+\mathfrak{q}}\left(\int_0^1\varphi^{\kappa}d_{\mathfrak{q}}\varphi\right)^{\frac{1}{\kappa}} \times\left(\left(1-\mathfrak{q}\varphi\right)^{\sigma}\left|{}_{\omega_1}D_{\mathfrak{q}}^2\digamma\left(\left(1-\varphi\right)\omega_1+\varphi\omega_2\right)\right|^{\sigma}d_{\mathfrak{q}}\varphi\right)^{\frac{1}{\sigma}} \\
&\quad=\frac{\mathfrak{q}^2(\omega_2-\omega_1)^2}{1+\mathfrak{q}}\left(\int_0^1\varphi^{\kappa}d_{\mathfrak{q}}\varphi\right)^{\frac{1}{\kappa}} \times\left(\left(1-\mathfrak{q}\varphi\right)^{\sigma}\left|{}_{\omega_1}D_{\mathfrak{q}}^2\digamma\left(\left(1-\varphi\right)\omega_1+\mathrm{m}\varphi\frac{\omega_2}{\mathrm{m}}\right)\right|^{\sigma}d_{\mathfrak{q}}\varphi\right)^{\frac{1}{\sigma}}.
\end{aligned}$$

Since $|{}_{\omega_1}D_{\mathfrak{q}}^2\digamma|^{\sigma}$ is (ρ,m)-convex, we have

$$\begin{aligned}
&\left|\frac{\mathfrak{q}\digamma\left(\omega_1\right)+\digamma\left(\omega_2\right)}{1+\mathfrak{q}}-\frac{1}{(\omega_2-\omega_1)}\int_{\omega_1}^{\omega_2}\digamma\left(\xi\right)_{\omega_1}d_{\mathfrak{q}}\xi\right| \\
&\quad\leq\frac{\mathfrak{q}^2(\omega_2-\omega_1)^2}{1+\mathfrak{q}}\left(\int_0^1\varphi^{\kappa}d_{\mathfrak{q}}\varphi\right)^{\frac{1}{\kappa}} \times\left(\left|{}_{\omega_1}D_{\mathfrak{q}}^2\digamma\left(\omega_1\right)\right|^{\sigma}\int_0^1\left(1-\mathfrak{q}\varphi\right)^{\sigma}\left(1-\varphi^{\rho}\right)d_{\mathfrak{q}}\varphi\right. \\
&\quad\left.+\ m\left|{}_{\omega_1}D_{\mathfrak{q}}^2\digamma\left(\frac{\omega_2}{m}\right)\right|^{\sigma}\int_0^1\left(1-\mathfrak{q}\varphi\right)^{\sigma}\varphi^{\rho}d_{\mathfrak{q}}\varphi\right)^{\frac{1}{\sigma}}
\end{aligned}$$

$$= \frac{\mathfrak{q}^2(\omega_2-\omega_1)^2}{(1+\mathfrak{q})}\left(\frac{1}{[\kappa+1]_{\mathfrak{q}}}\right)^{\frac{1}{\kappa}} \times \left(a_3\left|{}_{\omega_1}D_{\mathfrak{q}}^2 F\left(\omega_1\right)\right|^{\sigma} + \mathrm{m}a_4\left|{}_{\omega_1}D_{\mathfrak{q}}^2 F\left(\frac{\omega_2}{\mathrm{m}}\right)\right|^{\sigma}\right)^{\frac{1}{\sigma}},$$

where

$$a_3 = \int_0^1 (1-\mathfrak{q}\varphi)^{\sigma}(1-\varphi^{\rho})d_{\mathfrak{q}}\varphi$$

and

$$a_4 = \int_0^1 (1-\mathfrak{q}\varphi)^{\sigma}\varphi^{\rho}d_{\mathfrak{q}}\varphi.$$

Hence the proof is done. □

Corollary 2.15. *By setting* $\rho = 1$ *in inequality* (2.6) *becomes*

$$\left|\frac{\mathfrak{q}F(\omega_1)+F(\omega_2)}{1+\mathfrak{q}} - \frac{1}{(\omega_2-\omega_1)}\int_{\omega_1}^{\omega_2} F(\xi)_{\omega_1}d_{\mathfrak{q}}\xi\right| \leq \frac{\mathfrak{q}^2(\omega_2-\omega_1)^2}{(1+\mathfrak{q})}\left(\frac{1}{[\kappa+1]_{\mathfrak{q}}}\right)^{\frac{1}{\kappa}} \times \left(a_3\left|{}_{\omega_1}D_{\mathfrak{q}}^2 F\left(\omega_1\right)\right|^{\sigma} + \mathrm{m}a_4\left|{}_{\omega_1}D_{\mathfrak{q}}^2 F\left(\frac{\omega_2}{\mathrm{m}}\right)\right|^{\sigma}\right)^{\frac{1}{\sigma}},$$

where

$$a_3 = (1-\mathfrak{q})\sum_{r=0}^{\infty}\mathfrak{q}^r(1-\mathfrak{q}^r)(1-\mathfrak{q}^{r+1})^{\sigma}$$

and

$$a_4 = (1-\mathfrak{q})\sum_{r=0}^{\infty}\mathfrak{q}^{2r}(1-\mathfrak{q}^{r+1})^{\sigma}.$$

Corollary 2.16. *By setting* $\mathrm{m} = \rho = 1$ *inequality* (2.6) *becomes*

$$\left|\frac{\mathfrak{q}F(\omega_1)+F(\omega_2)}{1+\mathfrak{q}} - \frac{1}{(\omega_2-\omega_1)}\int_{\omega_1}^{\omega_2} F(\xi)_{\omega_1}d_{\mathfrak{q}}\xi\right| \leq \frac{\mathfrak{q}^2(\omega_2-\omega_1)^2}{(1+\mathfrak{q})}\left(\frac{1}{[\kappa+1]_{\mathfrak{q}}}\right)^{\frac{1}{\kappa}} \times \left(a_3\left|{}_{\omega_1}D_{\mathfrak{q}}^2 F\left(\omega_1\right)\right|^{\sigma} + a_4\left|{}_{\omega_1}D_{\mathfrak{q}}^2 F\left(\omega_2\right)\right|^{\sigma}\right)^{\frac{1}{\sigma}},$$

where

$$a_3 = (1-\mathfrak{q})\sum_{r=0}^{\infty}\mathfrak{q}^r(1-\mathfrak{q}^r)\left(1-\mathfrak{q}^{r+1}\right)^{\sigma}$$

and

$$a_4 = (1-\mathfrak{q})\sum_{r=0}^{\infty}\mathfrak{q}^{2r}(1-\mathfrak{q}^{r+1})^{\sigma}.$$

Remark 2.17. By setting $\mathrm{m} = \rho = 1$ and taking the limit as $\mathfrak{q} \to 1^-$ in Theorem 2.14, we obtain

$$\begin{aligned}\left|\frac{F(\omega_1)+F(\omega_2)}{2} - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2} F(\xi)d\xi\right| &\leq \frac{(\omega_2-\omega_1)^2}{2}\left(\frac{1}{\kappa+1}\right)^{\frac{1}{\kappa}}\left(\frac{1}{(\sigma+1)(\sigma+2)}\right)^{\frac{1}{\sigma}} \\ &\times \left((\sigma+1)\left|F''(\omega_1)\right|^{\sigma} + \left|F''(\omega_2)\right|^{\sigma}\right)^{\frac{1}{\sigma}}.\end{aligned}$$

3. Conclusions

The current study examines the quantum inequalities of Hermite-Hadamard type. Hermite-Hadamard estimates, or quadrature rules, play a critical role in numerical integration. The main objective of this article is to explore the connection between Hermite-Hadamard inequalities and integral calculus without limits, often referred to as quantum calculus, through error approximations. We present our findings and relate them to earlier research in the form of remarks, offering readers insights into potential avenues for extending similar estimations to different forms of convexities in the future. Our discoveries provide valuable insights and hold significant implications for future research in this field.

Declarations

Availability of Data and Material. No data were used to support this study.

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References

1. G.A. Anastassiou, *Intelligent Mathematics: Computational Analysis*, Springer, New York, NY, USA, 2011.
2. M.A. Ali, H. Budak, M. Abbas and Y.M. Chu, *Quantum Hermite-Hadamard-type inequalities for functions with convex absolute values of second ${}^{\omega_2}\mathfrak{q}$-derivatives,* Adv. Differ. Equ., 2021 (2021), pp. 1-12.

3. M.A. Ali, H. Budak, G. Murtaza and Y.M. Chu, *Post-quantum Hermite-Hadamard inequalities for interval-valued convex functions,* J. Inequal. Appl., 2021 (2021), 84.
4. N. Alp, M.Z. Sarikaya, M. Kunt and İ.İşcan, $\mathfrak{q}$-*Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions,* J. King Saud Univ. Sci., 30 (2018), pp. 193-203.
5. F. Benatti, M. Fannes, R. Floreanini and D. Petritis, *Quantum Information, Computation and Cryptography: An Introductory Survey of Theory, Technology and Experiments*, Springer Science and Business Media: Berlin, Germany, 2010.
6. S. Bermudo, P. Kórus and J.E. N. Valdés, *On* $\mathfrak{q}$-*Hermite-Hadamard inequalities for general convex functions,* Acta Math. Hung., 162 (2020), pp. 364-374.
7. M. Bohner, A. Kashuri, P.O. Mohammad and J.E.N. Valdés, *Hermite-Hadamard-type inequalities for conformable integrals,* Hacet. J. Math. Stat., 51 (3) (2022), pp. 775-786.
8. A. Bokulich and G. Jaeger, *Philosophy of Quantum Information Theory and Entaglement*, Cambridge University Press: Cambridge, UK, 2010.
9. H. Budak, S. Khan, M.A. Ali and Y.M. Chu, *Refinements of quantum Hermite-Hadamard type inequalities,* Open. Math., 19 (2021), pp. 724-734.
10. Y.M. Chu, M. U. Awan, S. Talib, M.A. Noor and K.I. Noor, *New post quantum analogues of Ostrowski-type inequalities using new definitions of left-right* $(\mathfrak{p},\mathfrak{q})$*-derivatives and definite integrals,* Adv. Differ. Equ., 634 (2020), pp. 1-15.
11. T. Ernst, *A Comprehensive Treatment of* q*-Calculus*, Springer: Besel, Switzerland, 2012.
12. T. Ernst, *The History of* $\mathfrak{q}$*-Calculus and a New Methods*, Department of Mathematics, Uppsala University, 2000.
13. F. Ghulam, S.B. Akbar, L. Rathour, L.N. Mishra and V.N. Mishra, *Riemann-Liouville Fractional versions of Hadamard inequality for strongly* m*-convex functions,* Int. J. Anal. Appl., 20 (5) (2022), pp. 1-12.
14. F.H. Jackson, On $\mathfrak{q}$-definite integrals, Q.J. Pure Appl. Math., 41 (1910), pp. 193-203.
15. S. Jhanthanam, J. Tariboon, S.K. Ntouyas and K. Nonlaopon, *On* $\mathfrak{q}$*-Hermite-Hadamard inequalities for differentiable convex functions,* Mathematics, 7 (2019), pp. 1-9.
16. V. Kac and P. Cheung, *Quantum Calculus*, Springer: Berlin, Germany, 2001.

17. M.A. Khan, Y.M. Chu, T.U. Khan and J. Khan, *Some new inequalities of Hermite-Hadamard type for s-convex functions with applications,* Open Math., 15 (2017), pp. 1414-1430.
18. W. Liu and H. Zhuang, *Some quantum estimates of Hermite-Hadamard inequalities for convex functions,* J. Appl. Anal. Comput., 7 (2) (2017), pp. 501-522.
19. S.L. Lyu, *On the Hermite-Hadamard inequality for convex functions of two variables,* Numer. Algebra, Control. Optim., 4 (1) (2014), pp. 1-8.
20. V.G. Mihesan, *A Generalization of the Convexity: Seminar on Functional Equations*, Approximation and Convexity, Cluj-Napoca, Romania, 1993.
21. P.O. Mohammad, T. Abdeljawad and A. Kashuri, *Fractional Hermite-Hadamard-Fejer inequalities for a convex function with respect to an increasing function involving a positive weighted symmetric function,* Symmetry., 12 (9) (2020), pp. 1-17.
22. C.P. Niculescu and L.E. Persson, *Convex functions and their applications: A contemporary approach*, 1 Ed., New York: CMC Books in Mathematics, 2004.
23. M.A. Noor, K.I. Noor and M.U. Awan, *Some quantum estimates for Hermite-Hadamard inequalities,* Appl. Math. Comput., 251 (2015), pp. 675-679.
24. Z. Páles and M.K. Shihab, *Decomposition of higher-order Wright convex functions revisited,* Results Math., 77 (2022), pp. 1-11.
25. J. Pečarić, F. Proschan and Y.L. Tong, *Convex functions, partial orderings and statistical application*, 1 Ed., New York: Academic Press, 1992.
26. J. Prabseang, K. Nonlaopon and J. Tariboon, *Quantum Hermite-Hadamard inequalities for double integral and* $\mathfrak{q}$*-differentiable convex functions,* J. Math. Inequalities., 13 (3) (2019), pp. 675-686.
27. S. Rashid, T. Abdeljawad, F. Jarad and M.A. Noor, *Some estimates for generalized Riemann-Liouville fractional integrals of exponentially convex functions and their applications,* Mathematics, 7 (2019), pp. 1-18.
28. S. Rashid, A.O. Akdemir, F. Jarad, M.A. Noor and K.I. Noor, *Simpson type integral inequalities for* κ*-fractional integrals and their applications,* AIMS Math., 4 (4) (2019), pp. 1087-1100.
29. S. Rashid, I. Işcan, D. Baleanu and Y.M. Chu, *Generation of new fractional inequalities via n-polynomials s-type convexity with applications,* Adv. Differ. Equ., 264 (2020), pp. 1-20.
30. F. Safdar, M.A. Noor and K.I. Noor, *Some new inequalities related to* (α, m)*-convex functions,* J. Prime Res. Math., 15 (2019), pp.

49-62.
31. M. Samraiz, F. Nawaz, S. Iqbal, T. Abdeljawad, G. Rahman and K.S. Nisar, *Certain mean-type fractional integral inequalities via different convexities with applications,* J. Inequal. Appl., 208 (2020), pp. 1-19.
32. M.Z. Sarikaya, E. Set, H. Yaldiz and N. Başak, *Hermite-Hadamard inequalities for fractional integrals and related fractional inequalities,* Math. Comp. Model., 57 (2013), pp. 2403-2407.
33. M.Z. Sarikaya and H. Yildirim, *On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals,* Miskolc Math. Notes., 17 (2) (2017), pp. 1049-1059.
34. M.Z. Sarikaya and N. Aktan, *On the generalization of some integral inequalities and their applications,* Math. Comput. Model., 54 (2011), pp. 2175-2182.
35. E. Set, M. Sardari, M.E. Ozdemir and J. Rooin, *On generalizations of the Hadamard inequality for* (α, m)*-convex functions,* Kyungpook Math. J., 52 (2012), pp. 307-317.
36. E. Set, A.O. Akdemir, A. Gözpinar and F. Jarad, *Ostrowski type inequalities via new fractional conformable integrals,* AIMS. Math., 4 (6) (2019), pp. 1684-1697.
37. T. Sitthiwirattham, M.A. Ali, H. Budak and S. Chasreechai, *Quantum Hermite-Hadamard type integral inequalities for convex stochastic processes,* AIMS Math., 6 (11) (2021), pp. 11989-12010.
38. J. Tariboon and S.K. Ntouyas, *Quantum calculus on finite intervals and applications to impulsive difference equations,* Adv. Differ. Equ., 282 (2013), pp. 1-19.
39. M. Tariq, H. Ahmad, C. Cesarano, H.A. Zinadah, A.E. Abouelregal and S. Askar, *Novel analysis of Hermite-Hadamard type integral inequalities via generalized exponential type* m*-convex functions,* Mathematics, 10 (31) (2022), pp. 1-21.
40. G. Toader, *Some generalizations of the convexity,* In: Proc. Colloq. Approx. Optim. Cluj-Napoca, Romania, Univ. Cluj-Napoca, (1985), pp. 329-338.
41. H. Wang and Z. Wu, *Certain* m*-convexity inequalities related to fractional integrals with exponential kernels,* Open Access Libr., 8 (5) (2021), pp. 1-10.
42. F. Wang, U. Hanif, A. Nosheen, K.A. Khan, H. Ahmad and K. Nonlaopon, *Some Hardy-type inequalities for convex functions via delta fractional integrals,* Fractals., 30 (1) (2022), pp. 1-15.

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