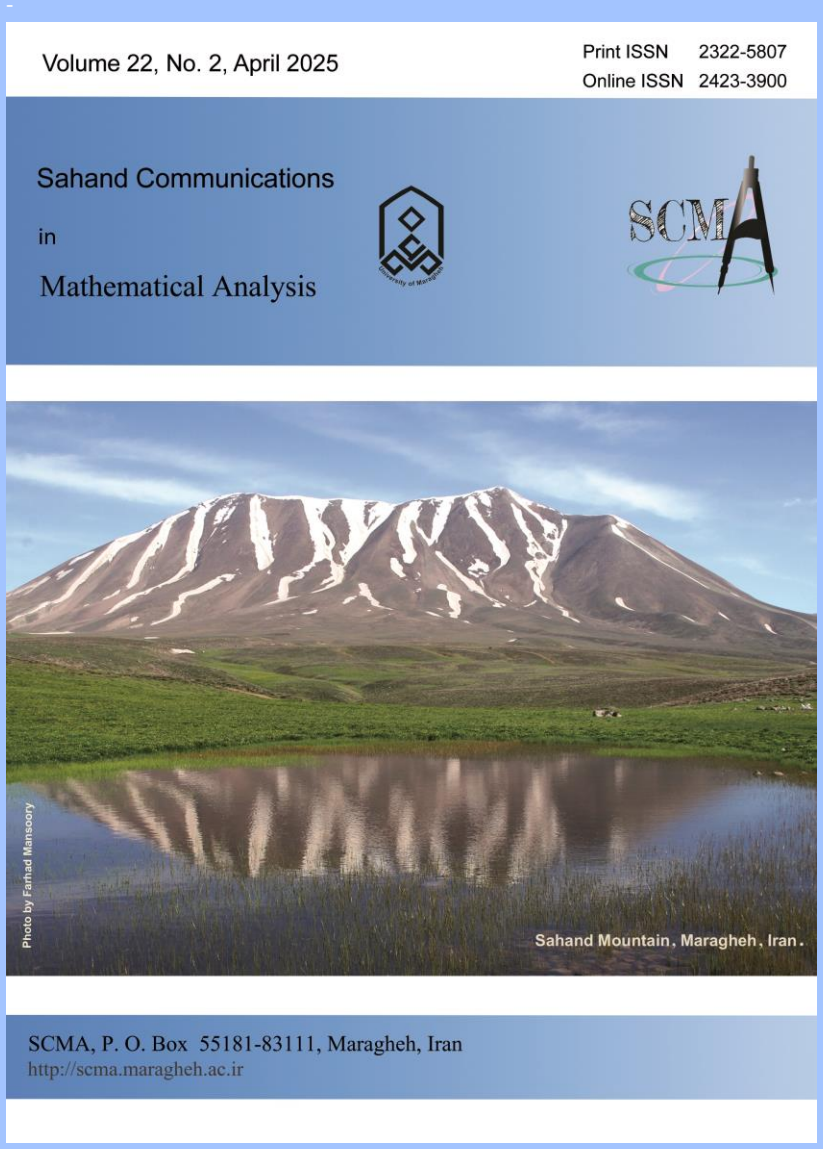
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A New Approach of Proinov-Type Contractions on Metric Spaces

Samira Rahrovi[1*], Hossein Piri[2], Christopher C. Tisdell[3] and Madi Raikhan[4]

Abstract. In this paper, we define the classes of Proinov $D-$contraction and generalized Proinov $D-$contraction which are properly larger than the classes of $(\psi,\varphi)-$contraction and generalized $(\psi,\varphi)-$contraction studied by Proinov (2020) and improved and corrected by Popescu (2021). Also, the given results are a real generalization of Banach's contraction principle, Ćirić (1971) and Boyd and Wong (1969). The given examples demonstrate that our results generalize the results announced by Popescu and Proinov effectively and robustly.

1. Introduction and Preliminaries

The classical contraction principle is a cornerstone of analysis, particularly in fixed-point theory, with extensive applications in mathematics and the applied sciences. It is fundamental to areas such as equilibrium modeling, Nash equilibria and stability analysis. Introduced by Banach in [5], the Banach Contraction Principle forms the basis of this theory. It is stated as follows:

Theorem 1.1 ([5]). *Let (Y,d) be a complete metric space and Γ be a self-mapping on Y such that: $d(\Gamma x,\Gamma y)\leq\lambda d(x,y)$,for all $x,y\in Y$, for some $\lambda\in[0,1)$, then Γ has a unique fixed point $y^*\in Y$ and the iterative sequence $\{\Gamma^n y\}$ converges to y^* for every $y\in Y$.*

The Banach Contraction Principle has been widely generalized over the years due to its foundational importance and broad applicability

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across various areas of mathematics and related disciplines. It has been extended in numerous ways across different metric spaces and with diverse types of metrics (see, for example, [1–4, 15, 16, 23]). This principle not only guarantees the existence of fixed points but also provides an iterative process for approximating them, making it indispensable in numerical methods, optimization problems and computational approaches. Its theoretical simplicity and practical relevance have inspired extensive research aimed at broadening its scope and improving its efficiency. Consequently, it finds applications in solving differential equations, optimization problems and the analysis of dynamical systems.

In 1974, Ćirić broadened Banach's principle by introducing multivalued contractions and conditions based on generalized distances. This significantly expanded its applicability to nonlinear operators and multivalued mappings. Ćirić's work paved the way for advances in set-valued analysis and further extensions, such as relaxing the contraction constant λ and modifying the distance metric, thereby enriching the theory and its applications.

Ćirić's generalization is stated as follows:

Theorem 1.2 ([9])**.** *Let (Y, d) be a complete metric space and let Γ be a self-mapping on Y such that: $d(\Gamma x, \Gamma y) \leq \lambda m(x, y)$, for all $x, y \in Y, \lambda \in [0, 1)$, where*

$$m(x,y) = \max\left\{d(x,y), d(x,\Gamma x), d(y,\Gamma y), \frac{d(x,\Gamma y) + d(\Gamma x, y)}{2}\right\}. \tag{1.1}$$

Then Γ has a unique fixed point $y^* \in Y$ and the iterative sequence $\{\Gamma^n y\}$ converges to y^* for every $y \in Y$.

Remark 1.3. The function $m(x, y)$ considers multiple aspects of the relationship between x, y and their images under Γ. By taking the maximum of these terms, $m(x, y)$ ensures that the largest deviation among these distances is accounted for, making it a robust measure for defining contraction-like behavior. This generalized distance is especially useful when Γ is non-linear or does not strictly satisfy a simple contraction condition based solely on $d(x, y)$. In summary, $m(x, y)$ captures a comprehensive and nuanced picture of the distances in the metric space, allowing Ćirić's theorem to apply to a broader class of mappings than the standard Banach contraction principle.

In 1969, Boyd and Wong [7] introduced generalized contractions by weakening the traditional contraction condition, allowing the contraction factor to vary. This innovation extended the principle to mappings satisfying more flexible conditions.

Their result is as follows:

Theorem 1.4 ([7]). *Let (Y,d) be a complete metric space and let Γ be a self-mapping on Y such that: $d(\Gamma x,\Gamma y)\leq \varrho(d(x,y))$, for all $x,y\in Y$, where $\varrho:[0,\infty)\to[0,\infty)$ satisfies:*

(i) *$\varrho(s)<s$, for all $s>0$;*
(ii) *$\limsup_{s\to\varepsilon^+}\varrho(s)<\varepsilon$, for all $\varepsilon>0$.*

Then Γ has a unique fixed point $y^*\in Y$ and the iterative sequence $\{\Gamma^n y\}$ converges to y^* for every $y\in Y$.

Remark 1.5. Since $\varrho(s)<s$ for all $s>0$, the mapping Γ satisfies the uniform continuity condition in Theorem 1.4.

Recently, Proinov introduced in [20] a broad and influential class of contractions in the context of complete metric spaces, garnering significant attention from researchers. The key strength of this new class lies in its ability to accommodate a diverse range of auxiliary functions with minimal restrictions. The relaxed conditions imposed on the function pairs considered by Proinov in the referenced work allow for a wide variety of contractivity conditions to be explored. This flexibility and generality have inspired many mathematicians to further investigate and expand upon this innovative framework (see, for example, [12, 17, 19, 21, 24, 25]).

Further developments include the introduction of condition (D_α) by Donghan et al. [10], later refined by Bejenaru and Postolache [6] as condition (D). Building on these insights, this paper introduces a new class of contractions called Proinov D-contractions and their generalizations, termed generalized Proinov D-contractions. These classes extend foundational results from Banach [5], Ćirić [9] and Boyd–Wong [7], demonstrating their robustness and effectiveness through illustrative examples.

To ensure a self-contained presentation, we now recall relevant definitions and results from the literature.

Lemma 1.6 ([11]). *Let $\{y_n\}$ be a sequence in metric space (Y,d) such that*

$$\lim_{n\to\infty} d(y_n,y_{n+1})=0.$$

If $\{y_n\}$ is not a Cauchy sequence, then there exists $\varepsilon\in(0,\infty)$ and sequences $\{p_k\}$ and $\{q_k\}$ of natural number such that $p_k>q_k>k$, for all $k\in\mathbb{N}$ and the following sequences tend to $\varepsilon+$ as $k\to+\infty$.

$$\{d(y_{p_k},y_{q_k})\},\quad \{d(y_{p_k},y_{q_k+1})\},\quad \{d(y_{p_k-1},y_{q_k})\},$$
$$\{d(y_{p_k-1},y_{q_k+1})\},\quad \{d(y_{p_k+1},y_{q_k+1})\}.$$

Definition 1.7 ([8]). A self-mapping Γ on a metric space (Y,d) is said to be an asymptotically regular at a point $y\in Y$, if

$$\lim_{n\to\infty} d\left(\Gamma^n y,\Gamma^{n+1}y\right)=0.$$

A self-mapping Γ on a metric space (Y, d) is said to be asymptotically regular if it is asymptotically regular at each point of the Y.

Lemma 1.8 ([20]). *Let $\rho : (0, \infty) \to \mathbb{R}$. Then the following conditions are equivalent:*

(i) $\inf_{s>\varepsilon} \rho(s) > -\infty$, *for any* $\varepsilon > 0$;
(ii) $\liminf_{s\to\varepsilon+} \rho(s) > -\infty$, *for any* $\varepsilon > 0$;
(iii) *if* $\lim_{s\to\infty} \rho(s_n) = -\infty$, *then* $s_n \to 0$ *as* $n \to \infty$.

Definition 1.9 ([10]). Let C be a nonempty subset of a metric space (Y, d) and $a \in \left(\frac{1}{2}, 1\right)$, the self-mapping Γ on C is said to be endowed with condition D_a if $d(\Gamma x, \Gamma y) \leq d(x, y)$ for all $x \in C$, $y \in C(\Gamma, x, \alpha)$ with $d(\Gamma x, \Gamma y) > 0$, where $\alpha \in [a, 1]$, and

$$C(\Gamma, x, \alpha) = \{(1-\alpha)p + \alpha\Gamma q : p, q \in C, d(p, \Gamma p), d(q, \Gamma q) \leq d(x, \Gamma x)\}.$$

In 2020, Bejenaru and Postolache [6] introduced a more general class of mappings by fixing $\alpha = 1$, resulting in the so-called condition (D):

Definition 1.10 ([6]). Let C be a nonempty subset of a metric space (Y, d), the self-mapping Γ on C is said to satisfy condition D, if

$$d(\Gamma x, \Gamma y) \leq d(x, y), \quad \text{for all } x \in C, y \in C(\Gamma, x) \text{ with } d(\Gamma x, \Gamma y) > 0,$$

where $C(\Gamma, x) = \{\Gamma p : p \in C, d(p, \Gamma p) \leq d(x, \Gamma x)\}$.

Definition 1.11. Let $\rho, \varrho : (0, \infty) \to \mathbb{R}$. We define the following properties for this function:

(P_0) ρ is nondecreasing;
(P_1) $\inf_{s>\varepsilon} \rho(s) > -\infty$, for any $\varepsilon > 0$;
(P_2) $\varrho(s) < \rho(s)$, for any $s > 0$;
(P_3) $\varrho(s) < \rho(s-)$, for any $s > 0$;
(P_4) $\limsup_{s\to\varepsilon+} \varrho(s) < \rho(\varepsilon+)$, for any $\varepsilon > 0$;
(P_5) $\varrho(\varepsilon) < \liminf_{s\to\varepsilon} \rho(s)$, for any $\varepsilon > 0$;
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(P_9) $\limsup_{s\to\varepsilon+} \varrho(s) < \liminf_{s\to\varepsilon+} \rho(s)$, for any $\varepsilon > 0$;
(P_{10}) $\limsup_{s\to0+} \varrho(s) < \min\{\liminf_{s\to\varepsilon} \rho(s), \rho(\varepsilon)\}$, for any $\varepsilon > 0$;
(P_{11}) $\varrho(\varepsilon) < \min\{\lim_{s\to\varepsilon} \rho(s), \rho(\varepsilon)\}$, for any $\varepsilon > 0$;
(P_{12}) if $\{\rho(s_n)\}$ and $\{\varrho(s_n)\}$ are convergent sequences with the same limit and $\{\rho(s_n)\}$ is strictly decreasing, then $s_n \to 0$ as $n \to \infty$.

Theorem 1.12 ([20]). *Let (Y,d) be a complete metric space and Γ be a self-mapping on Y such that: $\rho\left(d(\Gamma x,\Gamma y)\right)\leq\varrho\left(d(x,y)\right)$, for all $x,y\in Y$, with $d\left(\Gamma x,\Gamma y\right)>0$. If ρ and ϱ are mappings satisfy (P_0), (P_2) and (P_4), then Γ has a unique fixed point $y^*\in Y$ and the iterative sequence $\{\Gamma^n y\}$ converges to y^* for every $y\in Y$.*

Taking $\varrho(t)=\alpha(t)\rho(t)$ and $\rho(t)=t$ in Theorem 1.12, this result reduces to the well-known fixed-point theorem of Geraghty [12]. Similarly, by setting $\varrho(t)=F(\rho(t))$ in Theorem 1.12, the theorem provides an improvement over Theorem 3.6 of Moradi [17]. Furthermore, taking $\varrho(t)=\rho(t)-\tau$ in Theorem 1.12, the theorem enhances Theorem 2.1 of Wardowski [24], the results of Piri and Kumam [19] and Theorem 3.5 of Secelean [21].

Theorem 1.13 ([20]). *Let (Y,d) be a complete metric space and Γ be a self-mapping on Y such that: $\rho\left(d(\Gamma x,\Gamma y)\right)\leq\varrho\left(d(x,y)\right)$ for all $x,y\in Y$, with $d\left(\Gamma x,\Gamma y\right)>0$.*

Suppose that the functions ρ and ϱ satisfy the following conditions:

(i) *ρ and ϱ satisfy (P_0), (P_1), (P_2) and (P_{12});*
(ii) *ρ and ϱ satisfy (P_6) or (P_7);*
(iii) *Γ has a closed graph (i.e., for every sequence $\{y_n\}\subset Y$ such that $\lim_{n\to\infty}y_n=y$ and $\lim_{n\to\infty}\Gamma y_n=y^*$, it follows that $\Gamma y=y^*$) or ρ and ϱ satisfy (P_8).*

Then the mapping Γ has a unique fixed point $y^*\in Y$ and the iterative sequence $\{\Gamma^n y\}$ converges to y^* for every $y\in Y$.

Theorem 1.14 ([20]). *Let (Y,d) be a complete metric space and Γ be a self-mapping on Y such that: $\rho\left(d(\Gamma x,\Gamma y)\right)\leq\varrho\left(m(x,y)\right)$, for all $x,y\in Y$, with $d\left(\Gamma x,\Gamma y\right)>0$, where $m(x,y)$ is defined as in relation (1.1). If ρ and ϱ are mapping such that:*

(i) *ρ and ϱ satisfy (P_0), (P_2) and (P_4);*
(ii) *Γ has a closed graph or ρ and ϱ satisfy (P_3),*

then Γ has a unique fixed point $y^*\in Y$ and the iterative sequence $\{\Gamma^n y\}$ converges to y^* for every $y\in Y$.

Theorem 1.15 ([20]). *Let (Y,d) be a complete metric space and Γ be a self-mapping on Y such that: $\rho\left(d(\Gamma x,\Gamma y)\right)\leq\varrho\left(m(x,y)\right)$, for all $x,y\in Y$, with $d\left(\Gamma x,\Gamma y\right)>0$, where $m(x,y)$ is defined as in relation (1.1). Suppose that the functions ρ and ϱ satisfy the following conditions:*

(i) *ρ and ϱ satisfy (P_0), (P_1) and (P_2);*
(ii) *ρ and ϱ satisfy (P_6) or (P_7);*
(iii) *Γ has a closed graph or ρ and ϱ satisfy (P_5).*

Then the mapping Γ has a unique fixed point $y^*\in Y$ and the iterative sequence $\{\Gamma^n y\}$ converges to y^* for every $y\in Y$.

Taking $\varrho(t) = \lambda\rho(t)$ in either Theorem 1.14 or Theorem 1.15, we obtain an improvement of the main result of Skof [22]. Similarly, by taking $\varrho(t) = \lambda\rho(t)$ and $\rho(t) = \lambda t$, Theorems 1.14 and 1.15 reduce to the well-known result of Ćirić [9]. Finally, setting $\varrho(t) = \rho(t) + \tau$ in Theorem 1.14, we achieve an improvement of Theorem 2.4 of Wardowski and Dung [25].

Popescu [18] improved and corrected Theorem 1.13 and Theorem 1.15 as follows:

Theorem 1.16 ([18])**.** *Let (Y, d) be a complete metric space and Γ be a self-mapping on Y such that for all $x, y \in Y$ with $d(\Gamma x, \Gamma y) > 0$,*

$$\rho(d(\Gamma x, \Gamma y))) \leq \varrho(d(x, y)).$$

Suppose that the functions ρ and ϱ satisfy the following conditions:

(i) *ρ and ϱ satisfy (P_0), (P_1), (P_2), (P_9) and (P_{12});*
(ii) *Γ has a closed graph or ρ and ϱ satisfy (P_{10}),*

Then the mapping Γ has a unique fixed point $y^* \in Y$ and the iterative sequence $\{\Gamma^n y\}$ converges to y^* for every $y \in Y$.

Theorem 1.17 ([18])**.** *Let (Y, d) be a complete metric space and Γ be a self-mapping on Y such that: $\rho(d(\Gamma x, \Gamma y)) \leq \varrho(m(x, y))$, for all $x, y \in Y$, with $d(\Gamma x, \Gamma y) > 0$, where $m(x, y)$ is defined as in relation (1.1). Suppose that the functions ρ and ϱ satisfy the following conditions:*

(i) *ρ and ϱ satisfy (P_1), (P_2), and (P_6);*
(ii) *Γ has a closed graph or ρ and ϱ satisfy (P_{11});*

Then the mapping Γ has a unique fixed point $y^* \in Y$ and the iterative sequence $\{\Gamma^n y\}$ converges to y^* for every $y \in Y$.

Remark 1.18. Since the functions ρ and ϱ used in Theorem 1.12, Theorem 1.13 and Theorem 1.16 satisfy (P_1) and (P_2), therefore the function Γ satisfies the uniform continuity assumption of these theorems.

2. Fixed Point Theorems for New Type of (ρ, ϱ)-Contractions

We begin this section by introducing the definition of Proinov D-contraction. Following this, we present several lemmas that will serve as key tools in proving the main results of this section.

Definition 2.1. Let (Y, d) be a metric space and ρ and ϱ be real valued functions on $(0, \infty)$. A self-mapping Γ on Y is called Proinov $D-$contraction if

$$\rho(d(\Gamma x, \Gamma y)) \leq \varrho(d(x, y)), \text{ for all } x \in Y, y \in D(\Gamma, x), \tag{2.1}$$

with $d(\Gamma x, \Gamma y) > 0$, where $D(\Gamma, x) = \{\Gamma p : p \in Y, d(p, \Gamma p) \leq d(x, \Gamma x)\}$.

Lemma 2.2. *Let (Y,d) be a metric space and let Γ be a self-mapping on Y that is a Proinov D-contraction. Suppose that the functions ρ and ϱ satisfy the following conditions:*

(i) *ρ and ϱ satisfy (P_1) and (P_2);*
(ii) *ρ and ϱ satisfy either (P_0) and (P_4), or (P_{12}).*

Then Γ is an asymptotically regular mapping.

Proof. Let $y \in Y$ and set $y_n = \Gamma^n y$ and $\alpha_n = d(y_n, y_{n+1})$, for all $n \in \mathbb{N}$. We have to prove that $\lim_{n\to\infty} \alpha_n = 0$. It is trivial if $\alpha_n = 0$ for some $n \in \mathbb{N}$. So, we suppose that $\alpha_n > 0$ for all $n \in \mathbb{N}$. Since $y_{n+1} = \Gamma y_n$ and $d(y_n, \Gamma y_n) \leq d(y_n, \Gamma y_n)$ so, $y_{n+1} \in D(\Gamma, y_n)$ for all $n \in \mathbb{N}$. Applying (2.1) with $x = y_n$, $y = y_{n+1}$ and using condition (P_2) we have

$$\rho(\alpha_{n+1}) \leq \varrho(\alpha_n) \leq \rho(\alpha_n), \quad \forall n \in \mathbb{N}. \tag{2.2}$$

Now, by using a similar method as used in the proof of [20, Lemma 3.2], it can be shown that $\lim_{n\to\infty} \alpha_n = 0$ and the proof is complete. □

Lemma 2.3. *Let (Y,d) be a metric space and let Γ be a self-mapping on Y that is a Proinov D-contraction. Suppose ρ and ϱ are mappings that satisfy at least one of the following conditions:*

(i) *ρ and ϱ satisfy (P_0), (P_2) and (P_4);*
(ii) *ρ and ϱ satisfy at least one of (P_6), (P_7), or (P_9).*

If Γ is asymptotically regular at $y \in Y$, then the sequence $\{\Gamma^n y\}$ is a Cauchy sequence.

Proof. Let $y_n = \Gamma^n y$ for all $n \in \mathbb{N}$, if $d(y_n, y_{n+1}) = 0$ for some $n \in \mathbb{N}$, then $\{\Gamma^n y\}$ is convergent and hence it is a Cauchy sequence. So, we suppose that $d(y_n, y_{n+1}) > 0$ for all $n \in \mathbb{N}$ and $\{\Gamma^n y\}$ is not a Cauchy sequence. By Lemma 1.6 there exists $\varepsilon \in (0, \infty)$ and sequences $\{p_k\}$ and $\{q_k\}$ of natural numbers such that $p_k > q_k > k$, for all $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} d(y_{p_k}, y_{q_k}) = \lim_{k\to\infty} d(y_{p_k+1}, y_{q_k+1}) = \varepsilon + . \tag{2.3}$$

Since $y_{p_k} = \Gamma y_{p_k-1}$ and $d(y_{p_k-1}, \Gamma y_{p_k-1}) \leq d(y_{p_k-1}, \Gamma y_{p_k-1})$ so, $y_{p_k} \in D(\Gamma, y_{p_k-1})$ for all $k \in \mathbb{N}$. Applying (2.1) with $x = y_{p_k-1}$, $y = y_{p_k}$ and using condition (P_2) we have

$$\rho(d(y_{p_k}, y_{p_k+1})) \leq \varrho(d(y_{p_k-1}, y_{p_k})) < \rho(d(y_{p_k-1}, y_{p_k})), \quad \forall k \in \mathbb{N}.$$

Since ρ satisfies (P_0), therefore

$$d(y_{p_k}, y_{p_k+1}) \leq d(y_{p_k-1}, y_{p_k}), \quad \forall k \in \mathbb{N}.$$

It follows that

$$\begin{aligned} d(y_{p_k-1}, \Gamma y_{p_k-1}) &= d(y_{p_k-1}, y_{p_k}) \\ &\leq d(y_{p_k-2}, y_{p_k-1}) \end{aligned}$$

$$\begin{aligned} &\leq d(y_{p_k-3}, y_{p_k-2}) \\ &\vdots \\ &\leq d(y_{q_k}, y_{q_k+1}) \\ &= d(y_{q_k}, \Gamma y_{q_k}), \quad \forall k \in \mathbb{N}. \end{aligned}$$

Hence $y_{p_k} = \Gamma y_{p_k-1} \in D(\Gamma, y_{q_k})$, for all $k \in \mathbb{N}$. On the other hand from (2.3), there exists $N_1 \in \mathbb{N}$ such that for all $k > N_1$, $d(\Gamma y_{p_k}, \Gamma y_{q_k}) > 0$. Hence by applying (2.1) with $x = y_{q_k}$, $y = y_{p_k}$, we have

$$\rho\left(d(y_{q_k+1}, y_{p_k+1})\right) \leq \varrho\left(d(y_{q_k}, y_{p_k})\right), \quad \forall k > N_1. \tag{2.4}$$

By taking $\alpha_k = d(y_{q_k+1}, y_{p_k+1})$ and $\beta_k = d(y_{q_k}, y_{p_k})$, from (2.4), we have

$$\rho(\alpha_k) \leq \varrho(\beta_k), \quad \text{for all } k > N_1. \tag{2.5}$$

If the functions ρ and ϱ satisfy (P_0), (P_2) and (P_4), or (P_6), or (P_7), then by using a similar method as used in the proof of [20, Lemma 3.3], it can be shown that $\{\Gamma^n y\}$ is a Cauchy sequence.

If the functions ρ and ϱ satisfy (P_9), then by using a similar method as used in the proof of [18, Lemma 2.2], then it can be shown that $\{\Gamma^n y\}$ is a Cauchy sequence. □

Lemma 2.4. *Let (Y, d) be a metric space and let Γ be a self-mapping on Y that is a Proinov D-contraction. Assume that ρ and ϱ are mappings satisfying at least one of the following conditions:*

(i) *ρ and ϱ satisfy both (P_0) and (P_2);*
(ii) *ρ and ϱ satisfy either (P_8) or (P_{10});*
(iii) *Γ has a closed graph.*

If $\lim_{n\to\infty} \Gamma^n y = y^*$ for some $y \in Y$, then y^* is a fixed point of Γ.

Proof. If Γ has a closed graph then the proof is trivial. Therefore, we proceed with proof only in cases when ρ and ϱ satisfy (P_0) and (P_2), or (P_8), or (P_{10}).

Let $y_n = \Gamma^n y$ for all $n \in \mathbb{N}$. Assume that $d(y^*, \Gamma y^*) > 0$. We have

$$\begin{aligned} 0 &< d(y^*, \Gamma y^*) \\ &\leq d(y^*, y_{n+1}) + d(y_{n+1}, \Gamma y^*) \\ &\leq 2d(y^*, y_{n+1}) + d(y^*, \Gamma y^*). \end{aligned}$$

Since $\lim_{n\to\infty} d(y_{n+1}, y^*) = 0$, so by Squeeze theorem, we get

$$\lim_{n\to\infty} d(y_{n+1}, \Gamma y^*) = d(y^*, \Gamma y^*).$$

It follows that there exists $N_1 \in \mathbb{N}$ such that

$$d(\Gamma y_n, \Gamma y^*) > 0, \quad \forall n > N_1. \tag{2.6}$$

Since $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$ and $d(y^*, \Gamma y^*) > 0$, so that there exists $N_2 \in \mathbb{N}$ such that

$$d(y_n, \Gamma y_n) \leq d(y^*, \Gamma y^*), \quad \forall n > N_2. \tag{2.7}$$

From (2.6) and (2.7) and by applying (2.1) with $x = y^*$, $y = y_n$, we have

$$\begin{aligned}\rho(d(\Gamma y^*, y_{n+1})) &= \rho(d(\Gamma y^*, \Gamma y_n)) \\ &\leq \varrho(d(y^*, y_n)), \quad \forall n > N = \max\{N_1, N_2\}.\end{aligned}$$

If ρ and ϱ satisfy (P_0) and (P_2), or (P_8), then by using a similar method as used in the proof of [20, Lemma 3.5], it can be shown that $\Gamma y^* = y^*$.

Also if the functions ρ and ϱ satisfy (P_{10}), then by using a similar method as used in the proof of [18, Lemma 2.4], it can be shown that $\Gamma y^* = y^*$. □

2.1. **Main Results.** Building on the previously established lemmas, we now present the following theorems along with illustrative examples.

Theorem 2.5. *Let (Y, d) be a complete metric space and let Γ be a self-mapping on Y that is a Proinov D-contraction. If ρ and ϱ are mappings that satisfy (P_0), (P_2) and (P_4), then Γ has a unique fixed point $y^* \in Y$ and the iterative sequence $\{\Gamma^n y\}$ converges to y^* for every $y \in Y$.*

Proof. Let $y \in Y$ be arbitrarily fixed. Since ρ and ϱ satisfy (P_0) and (P_4), therefore they satisfy (P_1). Because they also satisfy (P_2), therefore by Lemma 2.2, Γ is asypmtotically regular at y. It follows from (P_0), (P_2), (P_4) and Lemma (2.3) that $\{\Gamma^n y\}$ is a Cauchy sequence. The completness of (Y, d) implies that there exists $y^* \in Y$ such that $\lim_{n\to\infty} \Gamma^n y = y^*$. Since ρ and ϱ satisfy (P_0) and (P_2), thus according to Lemma 2.4, y^* is a fixed point of Γ. Now, let x^* be another fixed point of Γ, therefore $d(y^*, x^*) > 0$. Since $d(y^*, \Gamma y^*) \leq d(x^*, \Gamma x^*)$, hence $y^* = \Gamma y^* \in D(\Gamma, x^*)$. So, by (2.1) we have:

$$\rho(d(x^*, y^*)) = \rho(d(\Gamma x^*, \Gamma y^*)) \leq \varrho(d(x^*, y^*) < \rho(d(x^*, y^*).$$

This is a contradiction. So, Γ has a unique fixed point. □

Theorem 2.6. *Let (Y, d) be a complete metric space and let Γ be a self-mapping on Y that is a Proinov D-contraction. Suppose that ρ and ϱ are mappings such that:*

(i) *ρ and ϱ satisfy (P_0), (P_1), (P_2) and (P_{12});*
(ii) *ρ and ϱ satisfy at least one of (P_6), (P_7), or (P_8);*
(iii) *Γ has a closed graph.*

Then Γ has a unique fixed point $y^* \in Y$ and the iterative sequence $\{\Gamma^n y\}$ converges to y^* for every $y \in Y$.

Proof. Let $y \in Y$ be arbitrarily fixed. Since ρ and ϱ satisfy (P_1), (P_2) and (P_{12}), therefore by Lemma 2.2, Γ is asypmtotically regular at y. It follows from (P_6) or (P_7) and Lemma 2.3 that $\{\Gamma^n y\}$ is a Cauchy sequence. The completness of (Y, d) implies that there exists $y^* \in Y$ such that $\lim_{n\to\infty} \Gamma^n y = y^*$. Since Γ has a closed graph or ρ and ϱ satisfy (P_8), thus according to Lemma 2.4, y^* is a fixed point of Γ. By using a similar method as used in the proof of Theorem 2.5, it can be shown that y^* is a unique fixed point of Γ. □

Theorem 2.7. *Let (Y, d) be a complete metric space and Γ be a self-mapping on Y which is a Proinov $D-$contraction. Suppose that the functions ρ and ϱ satisfy the following conditions:*

(i) *ρ and ϱ satisfy (P_0), (P_1), (P_2), (P_9), and (P_{12});*
(ii) *Γ has a closed graph or ρ and ϱ satisfy (P_{10}),*

Then Γ has a unique fixed point $y^* \in Y$ and the iterative sequence $\{\Gamma^n y\}$ converges to y^* for every $y \in Y$.

Proof. Let $y \in Y$ be arbitrarily fixed. Since the functions ρ and ϱ satisfy (P_1), (P_2) and (P_{12}), by Lemma 2.2, Γ is asymptotically regular at y. It follows from (P_9) and Lemma 2.3 that $\{\Gamma^n y\}$ is a Cauchy sequence. The completeness of (Y, d) implies that there exists $y^* \in Y$ such that $\lim_{n\to\infty} \Gamma^n y = y^*$. Since Γ has a closed graph or ρ and ϱ satisfy (P_{10}), thus according to Lemma 2.4, y^* is a fixed point of Γ. By using a similar method as used in the proof of Theorem 2.5, it can be shown that y^* is a unique fixed point of Γ. □

Corollary 2.8. *Let (Y, d) be a complete metric space and Γ be a self-mapping on Y such that $d(\Gamma x, \Gamma y) \leq \lambda d(x, y)$, for all $x \in Y, y \in D(\Gamma, x)$ with $d(\Gamma x, \Gamma y) > 0$, where, $\lambda \in (0, 1)$, then Γ has a unique fixed point $y^* \in Y$ and the iterative sequence $\{\Gamma^n y\}$ converges to y^* for every $y \in Y$.*

Proof. By substituting $\rho(s) = s$ and $\varrho(s) = \lambda s$ into Theorem 2.5, Theorem 2.6, or Theorem 2.7, a complete proof is obtained. □

Corollary 2.9. *Let (Y, d) be a complete metric space and Γ be a self-mapping on Y such that $d(\Gamma x, \Gamma y) \leq \varrho(d(x, y))$, for all $x \in Y$ and $y \in D(\Gamma, x)$, where $\varrho : [0, \infty) \to \mathbb{R}$ satisfies the following conditions:*

(i) *$\varrho(s) < s$ for any $s > 0$;*
(ii) *$\limsup_{s\to\varepsilon+} \varrho(s) < \varepsilon$ for any $\varepsilon > 0$.*

Then Γ has a unique fixed point $y^* \in Y$ and the iterative sequence $\{\Gamma^n y\}$ converges to y^* for every $y \in Y$.

Proof. By substituting $\rho(s) = s$ into Theorem 2.5, Theorem 2.6, or Theorem 2.7, a complete proof is obtained. □

Remark 2.10. Example 2.11 and Example 2.12 illustrate that Corollary 2.8 and Corollary 2.9 respectively are genuine generalizations of Theorem 1.1 and Theorem 1.4.

Example 2.11. Let $Y = [0,1]$ be endowed with the usual metric $d : Y \times Y \to [0,\infty)$, $d(x,y) = |x-y|$ and a self-mapping Γ on Y defined as follows:

$$\Gamma : Y \longrightarrow Y, \tag{2.8}$$

$$\Gamma(y) = \begin{cases} 1/2, & y \neq 1, \\ 1/4, & y = 1. \end{cases} \tag{2.9}$$

Clearly, Γ is not continuous. Therefore, according to Remark 1.18 it does not satisfy the conditions of Theorem 1.12, Theorem 1.13 and Theorem 1.16. We examine the following cases:

Case 1. Let $x = 1$ and $y \in D(\Gamma, x)$. So, there exists $p \in Y$ such that $y = \Gamma p$ and $|p - \Gamma p| \leq |x - \Gamma x| = \frac{3}{4}$.

(1-1) If $p = 1$, then $y = \frac{1}{4}$, $|\Gamma x - \Gamma y| = \frac{1}{4}$ and $|x - y| = \frac{3}{4}$.

(1-2) If $p \neq 1$, then $y = \frac{1}{2}$, $|\Gamma x - \Gamma y| = \frac{1}{4}$ and $|x - y| = \frac{1}{2}$.

Case 2. Let $x \neq 1$ and $y \in D(\Gamma, x)$. So, there exists $p \in Y$ such that $y = \Gamma p$ and $|p - \Gamma p| \leq |x - \Gamma x| = \left|x - \frac{1}{2}\right|$.

(2-1) If $p = 1$, then $|p - \Gamma p| \leq |x - \Gamma x|$ implies that $x \geq \frac{5}{4}$ or $x < -\frac{1}{4}$ and this is impossible.

(2-2) If $p \neq 1$, then $y = \frac{1}{2}$, $|\Gamma x - \Gamma y| = 0$ and $|x - y| = \left|x - \frac{1}{2}\right|$.

Let $\lambda \in \left[\frac{1}{2}, 1\right)$ and the functions ρ and ϱ are defined as follows:

$$\rho : (0,\infty) \longrightarrow \mathbb{R}, \qquad \varrho : (0,\infty) \longrightarrow \mathbb{R},$$
$$\rho(t) = t, \qquad \varrho(t) = \lambda t.$$

From **Case 1** and **Case 2**, for all $x \in Y$ and $y \in D(\Gamma, x)$, we have

$$\rho\left(d(\Gamma x, \Gamma y)\right)) \leq \varrho\left(d(x,y)\right).$$

Obviously, Γ satisfies conditions of Theorem 2.5, Theorem 2.6 and Theorem 2.7 and $\frac{1}{2}$ is a unique fixed point of Γ.

Example 2.12. Let $Y = [0,1]$ be endowed with the usual metric $d : Y \times Y \to [0,\infty)$, $d(x,y) = |x-y|$ and a self-mapping Γ on Y defined as follows:

$$\Gamma : Y \longrightarrow Y, \tag{2.10}$$

$$\Gamma(y) = \begin{cases} y/4, & y \neq 1, \\ 1/2, & y = 1. \end{cases} \tag{2.11}$$

Clearly, Γ is not continuous. Therefore, according to Remark 1.18 it does not satisfy the conditions of Theorem 1.12, Theorem 1.13 and Theorem 1.16. We examine the following cases:

Case 1. Let $x = 1$ and $y \in D(\Gamma, x)$. So, there exists $p \in Y$ such that $y = \Gamma p$ and $|p - \Gamma p| \leq |x - \Gamma x| = \frac{1}{2}$.

(1-1) If $p = 1$, then $y = \frac{1}{2}$, $|\Gamma x - \Gamma y| = \frac{3}{8}$ and $|x - y| = \frac{1}{2}$.

(1-2) If $p \neq 1$, then $y = \frac{1}{4}p \neq 1$ and $|p - \Gamma p| = \frac{3}{4}p$. Since $|p - \Gamma p| \leq \frac{1}{2}$, therefore $y \leq \frac{1}{6}$ and then $\frac{3}{4}(1 - y) \geq \frac{1}{2} - \frac{1}{4}y \geq 0$. It follows that

$$|\Gamma x - \Gamma y| = \frac{1}{2} - \frac{1}{4}y \leq \frac{3}{4}(1 - y) = \frac{3}{4}|x - y|$$

Case 2. Let $x \neq 1$ and $y \in D(\Gamma, x)$, so there exists $p \in Y$ such that $y = \Gamma p$ and $|p - \Gamma p| \leq |x - \Gamma x| = \frac{3}{4}x$.

(2-1) If $p = 1$ then $y = \frac{1}{2}p \neq 1$ and $|\Gamma x - \Gamma y| = \frac{1}{4}|x - y|$.

(2-2) If $p \neq 1$, then $y = \frac{1}{4}p \neq 1$ and $|\Gamma x - \Gamma y| = \frac{1}{4}|x - y|$.

Let $\lambda \in [\frac{3}{4}, 1)$ and the functions ρ and ϱ are defined as Example 2.11. From **Case 1** and **Case 2**, for all $x \in Y$ and $y \in D(\Gamma, x)$, we have

$$\rho\left(d(\Gamma x, \Gamma y)\right) \leq \varrho\left(d(x, y)\right).$$

Obviously, Γ satisfies in conditions of Theorem 2.5, Theorem 2.6 and Theorem 2.7 and 0 is a unique fixed point of Γ.

3. Fixed Point Theorems for New Type of Generalized (ρ, ϱ)-Contractions

We begin this section by introducing the definition of a generalized Proinov D-contraction. Following this, we present several lemmas that will serve as key tools in proving the main results of this section.

Definition 3.1. Let (Y, d) be a metric space and and $\rho, \varrho : (0, \infty) \to \mathbb{R}$ be arbitrary functions. A self-mapping Γ on Y is called a generalized Proinov $D-$contraction if

$$\rho\left(d(\Gamma x, \Gamma y)\right) \leq \varrho\left(Q(x, y)\right), \text{ for all } x \in C, y \in D(\Gamma, x), \tag{3.1}$$

with $d\left(\Gamma x, \Gamma y\right) > 0$, where

$$Q(x, y) = \max\left\{ d(x, y), d(x, \Gamma x), d(y, \Gamma y), d(\Gamma^2 x, \Gamma x), \right.$$
$$\left. \frac{d(x, \Gamma y) + d(\Gamma x, y)}{2}, \frac{d(\Gamma^2 x, \Gamma y) + d(x, \Gamma y)}{2} \right\}.$$

Remark 3.2. The generalized distances $Q(x, y)$ and $m(x, y)$ play crucial roles in extending the fixed-point theory to broader classes of mappings. While $m(x, y)$ is simpler and computationally efficient, $Q(x, y)$ offers significant advantages in specific contexts. $Q(x, y)$ offers greater flexibility and applicability in challenging scenarios, making it a valuable tool for extending fixed-point theory. However, its increased complexity comes

with higher computational costs, which may not be necessary for simpler mappings where $m(x, y)$ suffices. Below are the benefits of $Q(x, y)$ compared to $m(x, y)$:

- By incorporating terms such as $d(\Gamma^2 x, \Gamma x)$ and $\frac{d(\Gamma^2 x, \Gamma y)+d(x,\Gamma y)}{2}$, $Q(x, y)$ captures second-order interactions in the mapping Γ. This provides a more comprehensive representation of the mapping's behavior, especially for mappings with complex dynamics or those requiring higher-order continuity conditions. In contrast, $m(x, y)$ focuses solely on first-order relationships, which may be insufficient for capturing nuanced behaviors in intricate mappings.
- The additional terms in $Q(x, y)$ make it more robust for analyzing mappings that exhibit irregularities or non-linear interactions. $Q(x, y)$ accommodates cases where the mapping Γ requires a more detailed structure to ensure contraction-like properties. While $m(x, y)$ is effective for simpler mappings, it may struggle to identify contraction properties in mappings with complex dependency structures or where higher-order terms significantly affect convergence.
- The extended terms in $Q(x, y)$ enable its application to a broader range of scenarios, such as mappings that exhibit oscillatory or cyclic behavior, or where convergence depends on more than immediate deviations. In contrast, $m(x, y)$ is limited to mappings where first-order distances sufficiently represent the behavior.
- With its additional terms, $Q(x, y)$ provides stronger convergence criteria. This is particularly advantageous when dealing with iterative methods for solving equations, where stabilization of the iterative sequence relies on higher-order terms. Meanwhile, $m(x, y)$ primarily guarantees convergence for mappings with simpler contraction-like properties.
- The inclusion of second-order terms allows $Q(x, y)$ to generalize fixed-point results to mappings that may not satisfy the simpler contraction properties required by $m(x, y)$. This is especially valuable for extending fixed-point theory to new classes of problems, such as those in dynamical systems and differential equations.

Lemma 3.3. *Let (Y, d) be a complete metric space and Γ be a self-mapping on Y which is a generalized Proinov $D-$contraction. If ρ and ϱ are mappings that satisfy (P_1), (P_2) and (P_9), then Γ is an asymptotically regular mapping.*

Proof. Let $y \in Y$ and set $y_n = \Gamma^n(y)$ and $\gamma_n = d(y_n, y_{n+1})$, for all $n \in \mathbb{N}$. We have to prove that $\lim_{n\to\infty} \gamma_n = 0$. It is trivial if $\gamma_n = 0$ for some $n \in \mathbb{N}$. So, we suppose that $\gamma_n > 0$ for all $n \in \mathbb{N}$. Since $y_{n+1} = \Gamma y_n$ and $d(y_n, \Gamma y_n) \leq d(y_n, \Gamma y_n)$, therefore $y_{n+1} \in D(\Gamma, y_n)$ for all $n \in \mathbb{N}$. Applying (3.1) with $x = y_n$, $y = y_{n+1}$, we have

$$\rho\left(\gamma_{n+2}\right) \leq \varrho\left(Q(y_n, y_{n+1})\right), \quad \forall n \in \mathbb{N}. \tag{3.2}$$

From the definition of $Q(x, y)$, we have:

(3.3)

$$\begin{aligned} Q(y_n, y_{n+1}) = \max \Bigg\{ & d(y_n, y_{n+1}), d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2}), d(y_{n+2}, y_{n+1}), \\ & \frac{d(y_n, y_{n+2}) + d(y_{n+1}, y_{n+1})}{2}, \frac{d(y_{n+2}, y_{n+2}) + d(y_n, y_{n+2})}{2} \Bigg\} \\ & \leq \max\left\{d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2})\right\}. \end{aligned}$$

If $d(y_n, y_{n+1}) \leq d(y_{n+1}, y_{n+2})$ then it follows from relations (3.2) and (3.3) and condition (P_2) that:

$$\rho(\gamma_{n+1}) \leq \varrho(\gamma_{n+1}) < \rho(\gamma_{n+1}).$$

This is a contradiction. So, we assume that $d(y_n, y_{n+1}) > d(y_{n+1}, y_{n+2})$, for all $n \in \mathbb{N}$ and then $Q(y_n, y_{n+1}) = d(y_n, y_{n+1})$. Therefore, from (3.2) and (P_2), we get

$$\rho(\gamma_{n+1}) \leq \varrho(\gamma_n), \quad \forall n \in \mathbb{N}. \tag{3.4}$$

It follows that $\{\gamma_n\}$ is a decreasing sequence of real numbers, so there exists $\gamma \in [0, \infty)$ such that $\lim_{n\to\infty} \gamma_n = \gamma+$. Now, we claim that $\gamma = 0$.

If $\gamma > 0$, then (P_9) implies that:

$$\limsup_{s\to\gamma+} \varrho(s) < \liminf_{s\to\gamma+} \rho(s). \tag{3.5}$$

On the other hand from (3.4), we have

$$\begin{aligned} \liminf_{s\to\gamma+} \rho(s) &\leq \liminf_{n\to\infty} \rho(\gamma_{n+2}) \\ &\leq \limsup_{n\to\infty} \rho(\gamma_{n+2}) \\ &\leq \limsup_{n\to\infty} \varrho(\gamma_{n+2}) \\ &\leq \limsup_{s\to\gamma+} \varrho(s), \end{aligned}$$

which is contrary to (3.5). Hence $\gamma = 0$ and the proof is complete. □

Lemma 3.4. *Let (Y,d) be a complete metric space and let Γ be a self-mapping on Y that is a generalized Proinov D-contraction. Suppose the mappings ρ and ϱ satisfy at least one of the following conditions:*

(i) *ρ and ϱ satisfy (P_0) and (P_4), or they satisfy either (P_7) or (P_8);*
(ii) *ρ and ϱ satisfy (P_9).*

If Γ is asymptotically regular at some $y \in Y$, then the sequence $\{\Gamma^n y\}$ is a Cauchy sequence.

Proof. Let $y_n = \Gamma^n y$ for all $n \in \mathbb{N}$, if $d(y_n, y_{n+1}) = 0$ for some $n \in \mathbb{N}$, then $\{\Gamma^n y\}$ is convergent and hence it is a Cauchy sequence. So, we suppose that $d(y_n, y_{n+1}) > 0$ for all $n \in \mathbb{N}$ and $\{\Gamma^n y\}$ is not a Cauchy sequence. By Lemma 1.6 there exists $\varepsilon \in (0,\infty)$ and sequences $\{p_k\}$ and $\{q_k\}$ of natural numbers such that $p_k > q_k > k$, for all $k \in \mathbb{N}$ and

$$\begin{aligned} &\lim_{k\to\infty} d(y_{p_k}, y_{q_k}) = \lim_{k\to\infty} d(y_{p_k+1}, y_{q_k}) = \varepsilon+; \\ &\lim_{k\to\infty} d(y_{p_k}, y_{q_k+1}) = \lim_{k\to\infty} d(y_{p_k+1}, y_{q_k+1}) = \varepsilon + . \end{aligned} \tag{3.6}$$

Since $y_{p_k} = \Gamma y_{p_k-1}$ and $d(y_{p_k-1}, \Gamma y_{p_k-1}) \leq d(y_{p_k-1}, \Gamma y_{p_k-1})$ so, $y_{p_k} \in D(\Gamma, y_{p_k-1})$ for all $k \in \mathbb{N}$. Applying (3.1) with $x = y_{p_k-1}$, $y = y_{p_k}$, we have

$$\rho\left(d\left(y_{p_k}, y_{p_k+1}\right)\right) \leq \varrho\left(Q\left(y_{p_k-1}, y_{p_k}\right)\right), \quad \forall k \in \mathbb{N}. \tag{3.7}$$

From the definition of $Q(x,y)$, we have:

(3.8)

$$\begin{aligned} Q\left(y_{p_k-1}, y_{p_k}\right) = \max\Bigg\{ & d\left(y_{p_k-1}, y_{p_k}\right), d\left(y_{p_k-1}, y_{p_k}\right), d\left(y_{p_k}, y_{p_k+1}\right), \\ & d(y_{p_k+1}, y_{p_k}), \frac{d(y_{p_k-1}, y_{p_k+1}) + d(y_{p_k}, y_{p_k})}{2}, \\ & \frac{d(y_{p_k+1}, y_{p_k+1}) + d(y_{p_k-1}, y_{p_k+1})}{2} \Bigg\} \\ \leq \max & \left\{ d\left(y_{p_k-1}, y_{p_k}\right), d\left(y_{p_k}, y_{p_k+1}\right) \right\}. \end{aligned}$$

If $d\left(y_{p_k-1}, y_{p_k}\right) \leq d\left(y_{p_k}, y_{p_k+1}\right)$, then $Q\left(y_{p_k-1}, y_{p_k}\right) = d\left(y_{p_k}, y_{p_k+1}\right)$ and from (3.7), we get $\rho\left(d\left(y_{p_k}, y_{p_k+1}\right)\right) \leq \varrho\left(d\left(y_{p_k}, y_{p_k+1}\right)\right)$, which is a contradiction, because by (P_2), $\varrho\left(d\left(y_{p_k}, y_{p_k+1}\right)\right) < \rho\left(d\left(y_{p_k}, y_{p_k+1}\right)\right)$. Therefore, $d\left(y_{p_k-1}, y_{p_k}\right) \leq d\left(y_{p_k}, y_{p_k+1}\right)$, for all $k \in \mathbb{N}$. It follows that:

$$\begin{aligned} d\left(y_{p_k-1}, \Gamma y_{p_k-1}\right) &= d\left(y_{p_k-1}, y_{p_k}\right) \\ &\leq d(y_{p_k-2}, y_{p_k-1}) \\ &\leq d(y_{p_k-3}, y_{p_k-2}) \end{aligned}$$

$$\begin{aligned}&\vdots\\ &\leq d(y_{q_k}, y_{q_k+1})\\ &= d(y_{q_k}, \Gamma y_{q_k}), \quad k \in \mathbb{N}.\end{aligned}$$

Hence $y_{p_k} = \Gamma y_{p_k-1} \in D(\Gamma, y_{q_k})$, for all $k \in \mathbb{N}$. From (3.6), there exists $N_1 \in \mathbb{N}$ such that for all $k > N_1$, $d(y_{p_k}, y_{q_k}) > 0$. Hence by applying (3.1) with $x = y_{q_k}$, $y = y_{p_k}$, we have

$$\rho\left(d(y_{q_k+1}, y_{p_k+1})\right) \leq \varrho\left(Q(y_{q_k}, y_{p_k})\right), \quad k > N_1. \tag{3.9}$$

From the definition of $Q(x, y)$, we have:

$$\begin{aligned}&Q(y_{q_k}, y_{p_k})\\ &\quad = \max\left\{ d(y_{q_k}, y_{p_k}), d(y_{q_k}, y_{q_k+1}), d\left(y_{p_k}, y_{p_k+1}\right), d(y_{q_k+2}, y_{q_k+1}),\right.\\ &\qquad \left.\frac{d(y_{q_k+1}, y_{p_k}) + d(y_{q_k}, y_{p_k+1})}{2}, \frac{d(y_{q_k+2}, y_{p_k+1}) + d(y_{q_k}, y_{p_k+1})}{2} \right\}\\ &\quad \leq \max\left\{ d(y_{q_k}, y_{p_k}), d(y_{q_k}, y_{q_k+1}), d\left(y_{p_k}, y_{p_k+1}\right),\right.\\ &\qquad d(y_{q_k+2}, y_{q_k+1}), \frac{d(y_{q_k+1}, y_{p_k}) + d(y_{q_k}, y_{p_k+1})}{2},\\ &\qquad \left.\frac{d(y_{q_k+2}, y_{q_k+1}) + d(y_{q_k+1}, y_{p_k+1}) + d(y_{q_k}, y_{p_k+1})}{2} \right\}.\end{aligned}$$

So, from (3.6) and the Squeeze Theorem, we have

$$\lim_{k\to\infty} Q(y_{q_k}, y_{p_k}) = \varepsilon + . \tag{3.10}$$

If ρ and ϱ satisfy (P_0) and (P_4) then by taking the limit superior in (3.9) as $k \to \infty$ and using (3.6), we get

$$\rho(\varepsilon+) = \lim_{k\to\infty} \rho\left(d(y_{q_k+1}, y_{p_k+1})\right) \leq \limsup_{k\to\infty} \varrho\left(Q(y_{q_k}, y_{p_k})\right). \tag{3.11}$$

On the othe hand, from (3.10), we have

$$\limsup_{k\to\infty} \varrho(Q(y_{q_k}, y_{p_k})) \leq \limsup_{s\to\varepsilon+} \varrho(s). \tag{3.12}$$

It follows from (3.11), (3.12) and (P_4) that $\rho(\varepsilon+) < \rho(\varepsilon+)$ and this is a contradiction. Hence $\{\Gamma^n y\}$ is a Cauchy sequence.

If ρ and ϱ satisfy (P_7) or (P_8), then by using a similar method as used in the proof of [20, Lemma 5.2], it can be shown that $\{\Gamma^n y\}$ is a Cauchy sequence.

If the functions ρ and ϱ satisfy (P_9), then by using a similar method as used in the proof of [18, Lemma 3.3], it can be shown that $\{\Gamma^n y\}$ is a Cauchy sequence. □

Remark 3.5. If we replace $Q(x,y)$ with $m(x,y)$ in Lemma 3.4, then the proof becomes simpler and there is no need for mapping ρ and ϱ to satisfy (P_2), while Proinove has proven the [20, Lemma 5.2] for $m(x,y)$ by utilizing condition (P_2) for mappings ρ and ϱ.

Lemma 3.6. *Let (Y,d) be a complete metric space and let Γ be a self-mapping on Y that is a generalized Proinov D-contraction. Suppose ρ and ϱ are mappings that satisfy at least one of the following conditions:*

(i) *Γ has a closed graph;*
(ii) *ρ and ϱ satisfy (P_5), or (P_0) and (P_2), or (P_{10}).*

If $\lim_{n\to\infty} \Gamma^n y = y^*$ for some $y \in Y$, then y^* is a fixed point of Γ.

Proof. If Γ has a closed graph, then the proof is trivial. Therefore, we proceed to prove only in cases when ρ and ϱ satisfy (P_0) and (P_2), or (P_5), or (P_{10}).

Let $y_n = \Gamma^n y$ for all $n \in \mathbb{N}$. Assume that $d(y^*, \Gamma y^*) > 0$. We have

$$\begin{aligned} 0 < d(y^*, \Gamma y^*) &\leq d(y^*, y_{n+1}) + d(y_{n+1}, \Gamma y^*) \\ &\leq 2d(y^*, y_{n+1}) + d(y^*, \Gamma y^*). \end{aligned}$$

Since $\lim_{n\to\infty} d(y_{n+1}, y^*) = 0$, so by applying the Squeeze theorem, we get

$$\lim_{n\to\infty} d(y_{n+1}, \Gamma y^*) = d(y^*, \Gamma y^*).$$

Hence, there exists $N_1 \in \mathbb{N}$ such that

$$d(\Gamma y_n, \Gamma y^*) > 0, \quad \forall n > N_1. \tag{3.13}$$

Since $d(y^*, \Gamma y^*) > 0$, so that there exists $N_2 \in \mathbb{N}$ such that

$$d(y_{n-1}, \Gamma y_{n-1}) \leq d(y^*, \Gamma y^*), \quad \forall n > N_2. \tag{3.14}$$

From (3.13) and (3.14) and by applying (3.1) with $x = y^*$, $y = y_n$, we have

$$\begin{aligned} \rho(d(\Gamma y^*, y_{n+1})) &= \rho(d(\Gamma y^*, \Gamma y_n)) \\ &\leq \varrho(Q(y^*, y_n)), \quad \forall n > \max\{N_1, N_2\}. \end{aligned} \tag{3.15}$$

From definition of $Q(x,y)$, we have:

(3.16)

$$Q(y^*, y_n) = \max\left\{ d(y^*, y_n), d(y^*, \Gamma y^*), d(y_n, y_{n+1}), d\left(\Gamma^2 y^*, \Gamma y^*\right), \right.$$

$$\left.\frac{d(y^*,y_{n+1})+d(\Gamma y^*,y_n)}{2},\frac{d\left(\Gamma^2y^*,y_{n+1}\right)+d(y^*,y_{n+1})}{2}\right\}$$

$$\leq \max\left\{d\left(y^*,y_n\right),d\left(y^*,\Gamma y^*\right),d(y_n,y_{n+1}),d\left(\Gamma^2y^*,\Gamma y^*\right),\right.$$

$$\frac{d(y^*,y_{n+1})+d(\Gamma y^*,y_n)}{2},$$

$$\left.\frac{d(\Gamma^2y^*,\Gamma y^*)+d(\Gamma y^*,y_{n+1})+d(y^*,y_{n+1})}{2}\right\}.$$

Now, we claim that $d\left(\Gamma y^*,\Gamma^2y^*\right)<d\left(y^*,\Gamma y^*\right)$.

If $d\left(\Gamma y^*,\Gamma^2y^*\right)=0$, it is trivial. We suppose that $d\left(\Gamma y^*,\Gamma^2y^*\right)>0$. Hence by applying (3.1) with $x=y^*$, $y=\Gamma y^*$, we have

$$\rho\left(d\left(\Gamma y^*,\Gamma^2y^*\right)\right)\leq\varrho\left(Q(y^*,\Gamma y^*)\right). \tag{3.17}$$

From definition of $Q(x,y)$, we have:

$$Q\left(y^*,\Gamma y^*\right)$$

$$=\max\left\{d\left(y^*,\Gamma y^*\right),d\left(y^*,\Gamma y^*\right),d\left(\Gamma y^*,\Gamma^2y^*\right),d\left(\Gamma^2y^*,\Gamma y^*\right),\right.$$

$$\left.\frac{d\left(y^*,\Gamma^2y^*\right)+d\left(\Gamma y^*,\Gamma y^*\right)}{2},\frac{d\left(\Gamma^2y^*,\Gamma^2y^*\right)+d(y^*,\Gamma^2y^*)}{2}\right\}$$

$$\leq\max\left\{d\left(y^*,\Gamma y^*\right),\frac{d\left(y^*,\Gamma y^*\right)+d\left(\Gamma y^*,\Gamma^2y^*\right)+0}{2},\right.$$

$$\left.\frac{0+d\left(y^*,\Gamma y^*\right)+d\left(\Gamma y^*,\Gamma^2y^*\right)}{2}\right\}$$

$$=\max\left\{d\left(y^*,\Gamma y^*\right),d\left(\Gamma y^*,\Gamma^2y^*\right)\right\}.$$

If $d\left(y^*,\Gamma y^*\right)\leq d\left(\Gamma y^*,\Gamma^2y^*\right)$, then $Q\left(y^*,\Gamma y^*\right)=d\left(\Gamma y^*,\Gamma^2y^*\right)$. So, (3.17) implies that

$$\rho\left(d\left(\Gamma y^*,\Gamma^2y^*\right)\right)\leq\varrho\left(d\left(\Gamma y^*,\Gamma^2y^*\right)\right).$$

Since ρ and ϱ satisfy (P_2) and $d\left(\Gamma y^*,\Gamma^2y^*\right)>0$, therefore

$$\varrho\left(d\left(\Gamma y^*,\Gamma^2y^*\right)\right)<\rho\left(d\left(\Gamma y^*,\Gamma^2y^*\right)\right),$$

and this is a contradiction. Therefore, $d\left(\Gamma y^*,\Gamma^2y^*\right)<d\left(y^*,\Gamma y^*\right)$. So, from (3.16), we obtain:

$$Q\left(y^*,y_n\right)\leq\max\left\{d\left(y^*,y_n\right),d\left(y^*,\Gamma y^*\right),d(y_n,y_{n+1}),\right.$$

$$\left.\begin{array}{l} \dfrac{d(y^*, y_{n+1}) + d(\Gamma y^*, y_n)}{2}, \\ \dfrac{d\left(\Gamma^2 y^*, \Gamma y^*\right) + d(\Gamma y^*, y_{n+1}) + d(y^*, y_{n+1})}{2} \end{array}\right\}.$$

Since $d\left(\Gamma y^*, \Gamma^2 y^*\right) < d\left(y^*, \Gamma y^*\right)$, $\lim\limits_{n\to\infty} d\left(y^*, y_n\right) = \lim\limits_{n\to\infty} d(y_n, y_{n+1}) = 0$ and $\lim\limits_{n\to\infty} d(\Gamma y^*, y_n) = d(\Gamma y^*, y^*)$, so that there exists $N_3 \in \mathbb{N}$ such that

$$Q\left(y^*, y_n\right) = d\left(y^*, \Gamma y^*\right), \quad \forall n > N_3.$$

It follows from (3.15) that:

$$\rho\left(d(\Gamma y^*, y_{n+1})\right) \leq \varrho\left(d\left(y^*, \Gamma y^*\right)\right), \quad \forall n > \max\{N_1, N_2, N_3\}.$$

If the functions ρ and ϱ satisfy (P_0) and (P_2), then by using a similar method as used in the proof of [18, Lemma 3.6], it can be shown that $y^* = \Gamma y^*$.

If the functions ρ and ϱ satisfy in (P_5), then by using a similar method as used in the proof of [20, Lemma 5.3], it can be shown that $y^* = \Gamma y^*$.

If the functions ρ and ϱ satisfy in (P_{10}), then by using a similar method as used in the proof of [18, Lemma 3.6], it can be shown that $y^* = \Gamma y^*$. □

Remark 3.7. If in Lemma 3.6, $Q(x, y)$ is replaced with $m(x, y)$ then the proof will be simpler and there is no need for the functions ρ and ϱ to satisfy (P_2).

3.1. **Main results.** Building on the established lemmas above, we now present the following theorems accompanied by an illustrative example

Theorem 3.8. *Let (Y, d) be a complete metric space and Γ be a self-mapping on Y which is a generalized Proinov $D-$contraction. Suppose that the functions ρ and ϱ satisfy the following conditions:*

(i) *ρ and ϱ satisfy (P_0), (P_2) and (P_4);*
(ii) *Γ has a closed graph or ρ and ϱ satisfy (P_3),*

Then Γ has a unique fixed point $y^* \in Y$ and the iterative sequence $\{\Gamma^n y\}$ converges to y^* for every $y \in Y$.

Proof. Let $y \in Y$ arbitrarily fixed. Since the functions ρ and ϱ satisfy in (P_0), (P_2) and (P_4), so by Lemma (3.3), Γ is asymptotically regular at y. Since the functions ρ and ϱ satisfy in (P_0) and (P_4), so by Lemma 3.4 that $\{\Gamma^n y\}$ is a Cauchy sequence and the completeness of (Y, d) implies that there exists $y^* \in Y$ such that $\lim\limits_{n\to\infty} \Gamma^n y = y^*$.

If Γ has a closed graph, then by Lemma 3.6, y^* is a fixed point of Γ.

If ρ and ϱ satisfy (P_3), because they also satisfy (P_0), therefore they satisfy (P_5). Thus, according to Lemma 3.6, y^* is a fixed point of Γ.

Now, let x^* be another fixed point of Γ, therefore $d(y^*, x^*) > 0$. Since $d(y^*, \Gamma y^*) \leq d(x^*, \Gamma x^*)$, hence $y^* = \Gamma y^* \in D(\Gamma, x^*)$. So, by (3.1) we have:

$$\rho\left(d\left(x^*, y^*\right)\right) = \rho\left(d(\Gamma x^*, \Gamma y^*)\right) \leq \varrho\left(Q(x^*, y^*\right). \tag{3.18}$$

From the definition of $Q(x, y)$, we have:

$$\begin{aligned} Q\left(x^*, y^*\right) = \max \Bigg\{ & d(x^*, y^*), d(x^*, \Gamma x^*), d\left(y^*, \Gamma y^*\right), \\ & d(\Gamma^2 x^*, \Gamma x^*), \frac{d(x^*, \Gamma y^*) + d(\Gamma x^*, y^*)}{2} \Bigg\}, \\ = d(x^*, y^*). \end{aligned}$$

Since the functions ρ and ϱ satisfy in (P_2), therefore:

$$\varrho\left(Q(x^*, y^*\right) = \varrho\left(d(x^*, y^*\right) < \rho\left(d(x^*, y^*\right).$$

This is a contradiction with (3.18). So, Γ has a unique fixed point. □

Theorem 3.9. *Let (Y, d) be a complete metric space and let Γ be a self-mapping on Y that is a generalized Proinov D-contraction. Suppose ρ and ϱ are mappings that satisfy the following condition:*

(i) *ρ and ϱ satisfy (P_1), (P_2) and either (P_6) or (P_7);*
(ii) *Γ has a closed graph or ρ and ϱ satisfy (P_5).*

Then Γ has a unique fixed point $y^* \in Y$ and the iterative sequence $\{\Gamma^n y\}$ converges to y^* for every $y \in Y$.

Proof. Let $y \in Y$ be arbitrarily fixed. Since ρ and ϱ satisfy (P_6) or (P_7), therefore they satisfy (P_9). Also, ρ and ϱ satisfy (P_1) and (P_2), so by Lemma 3.3, Γ is asymptotically regular at y. Since ρ and ϱ satisfy (P_6) or (P_7), so by Lemma 3.4, $\{\Gamma^n y\}$ is a Cauchy sequence. The completeness of (Y, d) implies that there exists $y^* \in Y$ such that $\lim_{n\to\infty} \Gamma^n y = y^*$. Since Γ has a closed graph or ρ and ϱ satisfy in (P_5), so by Lemma 3.6, y^* is a fixed point of Γ. By using a similar method as used in the proof of Theorem 3.8, it can be shown that y^* is a unique fixed point of Γ. □

Theorem 3.10. *Let (Y, d) be a complete metric space and Γ be a self-mapping on Y which is a generalized Proinov $D-$contraction. Suppose that the functions ρ and ϱ satisfy the following conditions:*

(i) *ρ and ϱ satisfy in (P_1), (P_2) and (P_6);*
(ii) *Γ has a closed graph or ρ and ϱ satisfy in (P_{11}).*

Then Γ has a unique fixed point $y^* \in Y$ and the iterative sequence $\{\Gamma^n y\}$ converges to y^* for every $y \in Y$.

Proof. Let $y \in Y$ be arbitrarily fixed. Since ρ and ϱ satisfy (P_6), therefore they satisfy (P_9). Also, ρ and ϱ satisfy (P_1) and (P_2), so by Lemma 3.3, Γ is asypmtotically regular at y. Since the functions ρ and ϱ satisfy in(P_9), so by Lemma 3.4, $\{\Gamma^n y\}$ is a Cauchy sequence. The completeness of (Y, d) implise that there exists $y^* \in Y$ such that $\lim_{n\to\infty} \Gamma^n y = y^*$. If Γ has a closed graph, then by Lemma (3.6), y^* is a fixed point of Γ and if ρ and ϱ satisfy (P_{11}), therefore they satisfy (P_5) and again by Lemma 3.6, y^* is a fixed point of Γ. By using similar method as used in the proof of Theorem 3.8, it can be shown that y^* is a unique fixed point of Γ. □

By substituting $\rho = t$ and $\varrho = \lambda t$ into Theorem 3.8, Theorem 3.9, or Theorem 3.10, the following corollary is obtained:

Corollary 3.11. *Let (Y, d) be a complete metric space and Γ be a self-mapping on Y such that $d(\Gamma x, \Gamma y) \leq \lambda Q(x, y)$, for all $x, y \in Y$ where $\lambda \in [0, 1)$. Then Γ has a unique fixed point $y^* \in Y$ and the iterative sequence $\{\Gamma^n y\}$ converges to y^* for every $y \in Y$.*

Remark 3.12. Since $m(x, y) \leq Q(x, y)$, so Corollary 3.11 is a genuine generalization of Theorem 1.2 of Ćirić.

Remark 3.13. Example 3.14 shows that, Theorem 3.8, Theorem 3.9 and Theorem 3.10 are improvement and generalization of Theorem 1.14, Theorem 1.15 and Theorem 1.17, respectively.

Example 3.14. Let $Y = \{0, 1, 2, 3, 4\}$ and the function d be defined as follows:

$$d : Y \times Y \longrightarrow [0, \infty),$$

$$d(x, y) = \begin{cases} 0, & x = y, \\ 2, & (x, y) \in \{(1, 3), (3, 1), (2, 0), (0, 2), (2, 3), (3, 2)\}, \\ 1, & \text{otherwise.} \end{cases}$$

Obviously, (Y, d) is a complete metric space. Let Γ be a self mapping on Y defined as follows:

$$\Gamma : Y \longrightarrow Y,$$

$$\Gamma(y) = \begin{cases} y + 2, & y \in \{0, 1, 2\}, \\ 4, & y \in \{3, 4\}. \end{cases}$$

We examine the following cases:

Case 1. Let $x = 0$ and $y \in D(\Gamma, x)$. So, there exists $p \in Y$ such that $y = \Gamma p$ and $d(p, \Gamma p) \leq d(x, \Gamma x) = d(0, 2) = 2$. It follows that $y \in \{2, 3, 4\}$.

(1-1) If $y = 2$, then $d(\Gamma x, \Gamma y) = d(2,4) = 1$ and

$$Q(x,y) = \left\{ d(0,2), d(0,2), d(2,4), d(4,2), \frac{d(0,4)+d(2,2)}{2}, \frac{d(4,4)+d(0,4)}{2} \right\} = 2.$$

(1-2) If $y = 3$, then $d(\Gamma x, \Gamma y) = d(2,4) = 1$ and

$$Q(x,y) = \left\{ d(0,3), d(0,2), d(3,4), d(4,2), \frac{d(0,4)+d(2,3)}{2}, \frac{d(4,4)+d(0,4)}{2} \right\} = 2.$$

(1-3) If $y = 4$, then $d(\Gamma x, \Gamma y) = d(2,4) = 1$ and

$$Q(x,y) = \left\{ d(0,4), d(0,2), d(4,4), d(4,2), \frac{d(0,4)+d(2,4)}{2}, \frac{d(4,4)+d(0,4)}{2} \right\} = 2.$$

Case 2. Let $x = 1$ and $y \in D(\Gamma, x)$. So, there exists $p \in Y$ such that $y = \Gamma p$ and $d(p, \Gamma p) \leq d(x, \Gamma x) = d(1,3) = 2$. It follows that $y \in \{2,3,4\}$.

(2-1) If $y = 2$, then $d(\Gamma x, \Gamma y) = d(3,4) = 1$ and

$$Q(x,y) = \left\{ d(1,2), d(1,3), d(2,4), d(4,3), \frac{d(1,4)+d(3,2)}{2}, \frac{d(4,4)+d(1,4)}{2} \right\} = 2.$$

(2-2) If $y = 3$, then $d(\Gamma x, \Gamma y) = d(3,4) = 1$ and

$$Q(x,y) = \left\{ d(1,3), d(1,3), d(3,4), d(4,3), \frac{d(1,4)+d(3,3)}{2}, \frac{d(4,4)+d(1,4)}{2} \right\} = 2.$$

(2-3) If $y = 4$, then $d(\Gamma x, \Gamma y) = d(3,4) = 1$ and

$$Q(x,y) = \left\{ d(1,4), d(1,3), d(4,4), d(4,3), \frac{d(1,4)+d(3,4)}{2}, \frac{d(4,4)+d(1,4)}{2} \right\} = 2.$$

Case 3. Let $x \in \{2,3,4\}$ and $y \in D(\Gamma, x)$. So, there exists $p \in Y$ such that $y = \Gamma p$ and $d(p, \Gamma p) \leq d(x, \Gamma x) = 1$. It follows that $y = 4$ and therefore,

$$d(\Gamma x, \Gamma y) = d(4,4) = 0.$$

Let $\lambda \in \left[\frac{1}{2}, 1\right)$ and the functions ρ and ϱ are defined as follows:

$$\rho : (0,\infty) \longrightarrow \mathbb{R}, \qquad \varrho : (0,\infty) \longrightarrow \mathbb{R},$$
$$\rho(t) = t, \qquad \varrho(t) = \lambda t.$$

From **Case 1**, **Case 2** and **Case 3**, for all $x \in Y$ and $y \in D(\Gamma, x)$, we have

$$\rho\left(d(\Gamma x, \Gamma y)\right) \leq \varrho\left(Q(x,y)\right).$$

The above calculation shows that the function Γ satisfies conditions of Theorem 3.8, Theorem 3.9 and Theorem 3.10 and 4 is a unique fixed point of Γ.

Since the functions ρ and ϱ used in [20, Theorem 5.4], [20, Theorem 5.5] and [18, Theorem 3.7] satisfy (P_2), therefore in this theorems for all $x, y \in Y$ with $d(\Gamma x, \Gamma y) > 0$, we have:

$$\rho\left(d(\Gamma x, \Gamma y)\right) \leq \varrho\left(Q(x, y)\right) < \rho(Q(x, y).$$

For $x = 0$ and $y = 1$, we have:

$$\rho(2) = \rho(d(\Gamma 0, \Gamma 1)) < \rho(Q(0, 1) = \rho(2).$$

This is a contradiction, so [20, Theorem 5.4], [20, Theorem 5.5] and [18, Theorem 3.7] cannot be used to find a fixed point of Γ.

4. Conclusions

In this paper, we have defined the classes of Proinov D-contraction and generalized Proinov D-contraction, which significantly broaden the scope of (ψ, φ)-contractions and generalized (ψ, φ)-contractions studied by Proinov (2020) and subsequently improved and corrected by Popescu (2021). These newly introduced classes serve as a true generalization of foundational fixed-point results, including Banach's contraction principle, Ćirić (1971) and Boyd and Wong (1969). The illustrative examples provided demonstrate that our results effectively and robustly generalize those announced by Proinov and Popescu, offering enhanced versatility and applicability. From our point of view, some future research directions are as follows:

- The Proinov D-contraction framework could be expanded to more generalized spaces such as modular metric spaces, fuzzy metric spaces, or partial metric spaces. This would enable the development of fixed-point results in more diverse and abstract settings.
- Future studies could investigate the utility of Proinov D-contractions in proving the existence and uniqueness of solutions to differential and integral equations, particularly under weak or generalized contraction conditions.
- Incorporating stochastic elements into Proinov D-contractions could lead to novel results in random fixed-point theory, which may find applications in areas like stochastic programming, probabilistic algorithms and operations research.
- The new classes of contractions could be explored in the context of non-linear dynamical systems, where they may provide new tools for stability analysis, control systems and modeling complex real-world phenomena.

- Investigating the iterative processes derived from these generalized contractions could yield new insights into numerical methods, optimization algorithms and their convergence behavior under weak or generalized conditions.
- The results could be extended to multi-valued mappings or hybrid fixed-point scenarios, potentially impacting fields like game theory, optimization and economics.
- The results can be applied to fixed-point iterations in optimization, particularly in non-convex or non-linear optimization problems.
- The generalized contractions can improve the robustness and applicability of iterative methods for solving complex equations in applied mathematics and engineering.

In conclusion, the introduction of Proinov D-contractions and generalized Proinov D-contractions lays a strong foundation for advancing fixed-point theory. These new classes not only generalize existing results but also open up exciting avenues for future research and practical applications across mathematics and interdisciplinary fields.

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