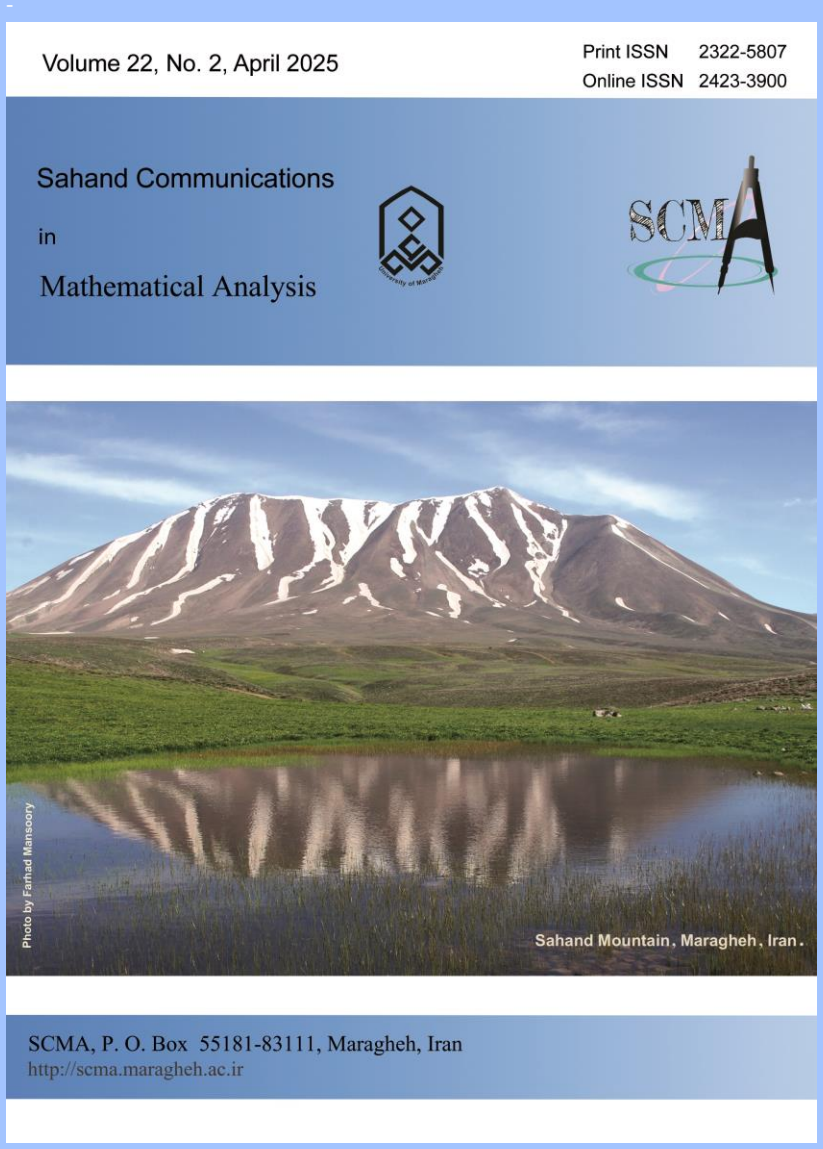
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Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 22
Number: 2
Pages: 247-260

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2025.2042611.1915



Sahand Communications in Mathematical Analysis (SCMA) Vol. 22 No. 2 (2025), 247-260
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2025.2042611.1915

Jordan and Local Multipliers on Certain Banach Algebras are Multipliers

Abbas Zivari-Kazempour[1*] and Ahmad Minapoor[2]

Abstract. We prove that every continuous Jordan multiplier T from a C^*-algebra A into a Banach A-bimodule X is a multiplier. We also characterize continuous linear maps on C^*-algebras and standard operator algebras determined by preserving some identities involving zero products. Additionally, we show that each continuous local multiplier from a Banach algebra A with property $(\mathbb{B})$ into a Banach A-bimodule X is a multiplier.

1. Introduction

Let A be an associative algebra and X be an A-bimodule. A linear map $T : A \to X$ is called a left multiplier (right multiplier) if for every $a, b \in A$,

$$T(ab) = T(a)b, \qquad (T(ab) = aT(b)),$$

and T is called a multiplier if it is both a left and right multiplier. Also, T is called a left Jordan multiplier (right Jordan multiplier) if for all $a \in A$,

$$T\left(a^2\right) = T(a)a, \qquad \left(T\left(a^2\right) = aT(a)\right),$$

and T is a Jordan multiplier if T is a left and a right Jordan multiplier.

The concept of a multiplier first appears in harmonic analysis in connection with the theory of summability for Fourier series. Subsequently, the notion has been employed in other areas of harmonic analysis, such as the investigation of homomorphisms of group algebras, in the general theory of Banach algebras, and so on; see [13].

2020 *Mathematics Subject Classification.* 47B47, 47B49.
Key words and phrases. Multiplier, Jordan multiplier, Local multiplier, Group algebra, Standard operator algebra.
Received: 2024-10-08, Accepted: 2025-02-15.
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It should be pointed out that the term multiplier was introduced by Helgason in [9]. Even nowadays some authors use the term centralizer instead of multiplier. This terminology was introduced by Wendel in [14], who prove that if T is a left multiplier acting on group algebra $L^1(G)$, then there exists a unique regular measure $\mu \in M(G)$ of bounded variation such that $T(f) = f \star \mu$ for all $f \in L^1(G)$.

The general theory of (centralizers) multipliers on Banach algebras has been developed by Johnson [10]. One may refer to the monograph [13] for the background and history of these notions.

Clearly, every multiplier is a Jordan multiplier, but the converse fails, in general, as it was proved by Ghahramani in [8, Example 2.6], (see also Example 2.1 below). Now we arrived at the problem of whether a (continuous) linear map $T : A \to X$ which satisfies the weaker condition $T\left(a^2\right) = aT(a)$ for all $a \in A$ is automatically a right multiplier.

A linear map T from A into X is called a left local multiplier (right local multiplier) if for every $a \in A$ there is a left (right) multiplier $T_a : A \to X$ such that $T(a) = T_a(a)$, and T is called a local multiplier if it is a left and a right local multiplier. It is clear that every left (right) local multiplier satisfies

$$T(a)b = T_a(a)b = T_a(ab), \qquad \left(aT(b) = aT_b(b) = T_b(ab)\right),$$

for all $a, b \in A$. A remarkable result due to Johnson [12, Corollary 5.4], states that a bounded local multiplier T from a C^*-algebra A into a Banach A-bimodule is a multiplier.

We say that a map T is a right multiplier at zero point if

$$a, b \in A, \quad ab = 0 \quad \Rightarrow \quad aT(b) = 0. \tag{1.1}$$

Characterization of T at zero point is also a matter of concern. In [6], Brešar proves that if A is a prime ring and $T : A \to A$ is an additive map satisfying (1.1), then T is a right multiplier. The same result was obtained in [17] for every continuous linear map $T : A \to X$ where A is a unital C^*-algebra and X is a unital Banach A-bimodule. See also ([16, 18]).

The more general condition of (1.1), which is considered, is as follows:

$$a, b \in A, \quad ab = 0 \quad \Rightarrow \quad aT_1(b) + T_2(a)b = 0, \tag{1.2}$$

where $T_1, T_2 : A \to X$ are linear maps. The condition (1.2) has also been studied by some authors and the mappings T_1 and T_2 have been characterized on different algebras. For example, the authors in [3] consider linear maps $T_1, T_2 : A \to X$ satisfying (1.2) and prove that if the unital algebra A is generated by idempotents, then T_1 and T_2 are of the form $T_1(a) = \Delta(a) + aT_1(e_A)$ and $T_2(a) = \Delta(a) + T_2(e_A)a$ where $\Delta : A \to X$ is a derivation.

Also, characterizations of the maps T_1 and T_2 are given if A is assumed to be standard operator algebras [5].

In this paper, we study the Jordan multiplier from C^*-algebras and group algebras $L^1(G)$, where G is a locally compact group, into a Banach A-bimodule X. We also consider the weaker condition

$$a, b \in A, \quad ab = ba = 0 \quad \Rightarrow \quad aT_1(b) + T_2(a)b = 0, \tag{1.3}$$

and characterize linear maps $T_1 : A \to X$ and $T_2 : A \to X$ whether A is a C^*-algebra or standard operator algebra. Finally, we prove that each continuous local multiplier T from a Banach algebra A with property $(\mathbb{B})$ into a Banach A-bimodule X is exactly a multiplier.

2. Characterization of Jordan Multipliers

The next example, obtained by the first author in [17], shows that Jordan multipliers are not always multipliers.

Example 2.1. Let

$$A = \left\{ \begin{bmatrix} z & w \\ 0 & 0 \end{bmatrix} : \ z, w \in \mathbb{C} \right\}.$$

We make $X = \mathbb{C}$ an A-bimodule by defining

$$a\lambda = 0, \qquad \lambda a = \lambda z, \quad \lambda \in \mathbb{C}, \ a \in A.$$

Define $T : A \to X$ by

$$T\left(\begin{bmatrix} z & w \\ 0 & 0 \end{bmatrix} \right) = w.$$

Then $T\left(a^2\right) = T(a)a$ for all $a \in A$, but neither T is a left multiplier nor a right multiplier. Note that A is not unital, but it has a left identity.

It is known that on the second dual space, A^{**}, of a Banach algebra A there are at least two products extending the multiplication on A, called the first and second Arens products which make A^{**} into a Banach algebra [7]. If these products coincide on A^{**}, then A is said to be Arens regular. Note that every C^*-algebra is Arens regular [7].

For each Banach A-bimodule X, the second dual, X^{**}, turns into a Banach A^{**}-bimodule, when A^{**} is equipped with the first Arens product. The module actions satisfy:

$$\Phi \cdot u = \lim_i \lim_j a_i \cdot x_j, \qquad u \cdot \Phi = \lim_j \lim_i x_j \cdot a_i, \quad \Phi \in A^{**}, \ u \in X^{**},$$

where $\{a_i\}_{i \in I}$ and $\{x_i\}_{j \in I}$ are nets in A and X that converge in the w^*-topology, to Φ and u, respectively. One may refer to the monograph of Dales [7] for a full account of Arens product and w^*-continuity of the above products.

We bring the following characterization of Jordan multipliers on C^*-algebras.

Theorem 2.2. *Every continuous Jordan multiplier T from a C^*-algebra A into a Banach A-bimodule X is a multiplier.*

Proof. Without loss of generality, we may assume that A is a von Neumann algebra and hence it is unital. Let $T : A \to X$ be a right Jordan multiplier. Then

$$T\left(a^2\right) = aT(a), \quad a \in A. \tag{2.1}$$

Changing the role of a by $a + b$ in (2.1), we get

$$T(ab + ba) = aT(b) + bT(a), \quad a, b \in A. \tag{2.2}$$

Replacing a by $ab + ba$ in (2.2), we obtain

$$T\left(ab^2 + b^2a + 2bab\right) = abT(b) + baT(b) + bT(ab + ba). \tag{2.3}$$

From (2.1), (2.2) and (2.3), we arrive at

$$T(bab) = baT(b), \quad a, b \in A.$$

Letting $b = e_A$, we get $T(a) = aT(e_A)$ for all $a \in A$. Hence T is a right multiplier.

For C^*-algebras we extending $T : A \to X$ to the second adjoint $T^{**} : A^{**} \to X^{**}$ and based on the Arens regularity of A, the w^*-w^*-continuity of T^{**} and the separate weak continuity of the module operations on X^{**}, we get

$$T^{**}(bab) = baT^{**}(b), \quad a, b \in A^{**}.$$

Setting $\xi = T^{**}(e_{A^{**}}) \in X^{**}$. Then it follows from the above equality with $b = e_{A^{**}}$ that $T^{**}(a) = a\xi$, for all $a \in A^{**}$. In particular, we have

$$T(a) = a\xi, \quad a \in A. \tag{2.4}$$

Note that $a\xi \in X$ for all $a \in A$. Of course, it suffices to prove it for each positive element $a \in A$. Suppose that $a \in A$ be a positive element and let $b \in A$ with $a = b^2$. According to (2.4),

$$a\xi = b^2\xi = T\left(b^2\right) \in X.$$

Consequently, it follows from (2.4) that T is a right multiplier. Similarly, we can prove that T is a left multiplier. □

Theorem 2.3 ([2, Theorem 2.2])**.** *Let A be a C^*-algebra and X be a Banach space and let $\phi : A \times A \to X$ be a continuous bilinear map such that*

$$a, b \in A, \qquad ab = ba = 0 \quad \Rightarrow \quad \phi(a, b) = 0.$$

Then, for all $a, b, x, y \in A$,

$$\phi(xa, yb) + \phi(bx, ay) = \phi(x, ayb) + \phi(bxa, y).$$

A linear map $D : A \to X$ is called a derivation (Jordan derivation) if

$$D(ab) = D(a)b + aD(b), \qquad \left(D\left(a^2\right) = D(a)a + aD(a)\right), \quad a, b \in A.$$

It should be noted that every derivation is a Jordan derivation, but the converse is not true in general [11]. It is proved by Johnson in [11] that every continuous Jordan derivation from C^*-algebra A into any Banach A-bimodule X is a derivation.

Theorem 2.4. *Let T_1 and T_2 be continuous linear maps from C^*-algebra A into an essential Banach A-bimodule X such that*

$$a, b \in A, \qquad ab = ba = 0, \qquad \Rightarrow \quad aT_1(b) + T_2(a)b = 0.$$

Then there exist $\xi, \eta \in X^{**}$ such that for all $a \in A$,

$$T_1(a) - T_2(a) = a\xi - \eta a. \tag{2.5}$$

Moreover, there exist derivations $\Delta_1, \Delta_2 : A \to X$ such that $T_1(a) = \Delta_1(a) + a\xi$, $T_2(a) = \Delta_2(a) + \eta a$ and $\Delta_1 = \Delta_2$.

Proof. First, we consider the case A is a von-Neumann algebra. Define $\phi : A \times A \to X$ by $\phi(a, b) = aT_1(b) + T_2(a)b$. Then $\phi(a, b) = 0$ whether $ab = ba = 0$. Applying Theorem 2.3 for $x = y = e_A$, we get

$$\phi(a, b) + \phi(b, a) = \phi(ab, e_A) + \phi(e_A, ba), \quad a, b \in A.$$

Replacing b by a in the above equality, we obtain

$$2\left(aT_1(a) + T_2(a)a\right) = T_1\left(a^2\right) + T_2\left(a^2\right) + a^2T_1(e_A) + T_2(e_A)a^2, \tag{2.6}$$

for all $a \in A$. Letting $a = p$ in (2.6), where p is an idempotent in A, we arrive at

$$2\left(pT_1(p) + T_2(p)p\right) = T_1(p) + T_2(p) + pT_1(e_A) + T_2(e_A)p. \tag{2.7}$$

As $p(e_A - p) = (e_A - p)p = 0$ for every idempotent $p \in A$, our assumption implies that

$$pT_1(e_A) + T_2(p) = pT_1(p) + T_2(p)p. \tag{2.8}$$

From (2.7) and (2.8), we have

$$T_1(p) - pT_1(e_A) = T_2(p) - T_2(e_A)p.$$

Since each self-adjoint element $x \in A$ is the limit of a sequence of linear combinations of projections in A, so

$$T_1(x) - xT_1(e_A) = T_2(x) - T_2(e_A)x,$$

for each self-adjoint element x in A. Now each arbitrary element $a \in A$ can be written as $a = x + iy$ where x, y are self-adjoint elements of A. Hence

$$T_1(a) - aT_1(e_A) = T_2(a) - T_2(e_A)a, \quad a \in A. \tag{2.9}$$

By taking $\xi = T_1(e_A)$ and $\eta = T_2(e_A)$ the equality (2.5) follows.

Define $\Delta_1 : A \to X$ by $\Delta_1(a) = T_1(a) - a\xi$. Then it follows from (2.6) and (2.9) that Δ_1 is a Jordan derivation and so it is a derivation by [11, Theorem 6.3]. In the same manner, Δ_2 is a derivation. For every projection p, we have

$$p\Delta_1(p) = p\,(T_1(p) - p\xi) = p\,(T_2(p) - \eta p) = p\Delta_2(p).$$

Similarly, $(e_A - p)\Delta_1(p) = (e_A - p)\Delta_2(p)$. Thus, $\Delta_1(p) = \Delta_2(p)$ for each projection $p \in A$ and hence by linearity and continuity, $\Delta_1(a) = \Delta_2(a)$ for every $a \in A$.

The result for a general C^*-algebra follows by considering T_1^{**}, T_2^{**} and using what we have proved in Theorem 2.2. □

Corollary 2.5. *Let T be a continuous linear map from a commutative C^*-algebra A into an essential Banach A-bimodule X such that*

$$a, b \in A, \qquad ab = ba = 0, \quad \Rightarrow \quad aT(b) + T(a)b = 0.$$

Then T is a multiplier.

Proof. By Theorem 2.4, $a\xi = \xi a$ and $T(a) = \Delta(a) + a\xi$ for all $a \in A$, where $\Delta : A \to X$ is a derivation. On account of [7, Theorem 2.8.63], Δ is zero. Therefore, T is a multiplier. □

Let $B(X)$ be the operator algebra of all bounded linear operators on Banach space X. A standard operator algebra is any subalgebra of $B(X)$ which contains the identity, and the ideal $F(X)$ of all finite rank operators. Note that $F(X)$ is dense in $B(X)$ with respect to the strong operator topology(denoted by "so", for short).

Recall that a non-zero ideal I of A is called essential if it has a non-zero intersection with every non-zero ideal of A. For a semisimple Banach algebra A this is equivalent to the condition $aI = 0$, for all $a \in A$, implies $a = 0$. In particular, if A is a standard operator algebra on a Banach space X, then A is primitive and $F(X)$ is essential.

Let $\mathfrak{J}(A)$ denote the subalgebra of A generated by all idempotents in A.

Theorem 2.6 ([4, Lemma 2.2])**.** *If ϕ is a bilinear map from $A \times A$ into a vector space X such that*

$$a, b \in A, \qquad ab = ba = 0 \quad \Rightarrow \quad \phi(a, b) = 0,$$

then

$$\phi(a, x) + \phi(x, a) = \phi(ax, e_A) + \phi(e_A, xa),$$

for all $a \in A$ and $x \in \mathfrak{J}(A)$.

Theorem 2.7. *Let A be a standard operator algebra on Banach space X with $dimX \geq 2$, and let T_1 and T_2 be so-continuous linear maps from A into $B(X)$ satisfying*

$$uv = vu = 0, \quad \Rightarrow \quad uT_1(v) + T_2(u)v = 0.$$

Then T_1 is a right multiplier and T_2 is a left multiplier.

Proof. Define $\phi : A \times A \to B(X)$ by $\phi(u, v) = uT_1(v) + T_2(u)v$. Thus, $\phi(u, v) = 0$ for all $u, v \in A$ with $uv = vu = 0$. Applying Theorem 2.6, we get

$$\phi(u, p) + \phi(p, u) = \phi(up, e_A) + \phi(e_A, pu), \quad u \in A,\ p \in \mathfrak{J}(A).$$

Hence

$$\begin{aligned} uT_1(p) + T_2(u)p + pT_1(u) + T_2(p)u \\ = upT_1(e_A) + T_2(up) + T_1(pu) + T_2(e_A)pu, \end{aligned} \tag{2.10}$$

for all $u \in A$ and $p \in \mathfrak{J}(A)$. Suppose that p is an idempotent of rank one. Noticing that $p(e_A - p) = (e_A - p)p = 0$, so

$$pT_1(e_A) + T_2(p) = pT_1(p) + T_2(p)p. \tag{2.11}$$

Replacing u by p in (2.10) we get

$$2\left(pT_1(p) + T_2(p)p\right) = pT_1(e_A) + T_2(p) + T_1(p) + T_2(e_A)p. \tag{2.12}$$

By (2.11) and (2.12), we arrive at

$$pT_1(p) + T_2(p)p = T_1(p) + T_2(e_A)p. \tag{2.13}$$

Multiplying p from the left of (2.13), we obtain

$$pT_2(p)p = pT_2(e_A)p. \tag{2.14}$$

Multiplying p from the right of (2.11), we have

$$pT_1(e_A)p = pT_1(p)p, \tag{2.15}$$

Since $F(X)$ is essential ideal in A, (2.14) and (2.15) yields that

$$T_2(p) = T_2(e_A), \qquad T_1(p) = T_1(e_A), \tag{2.16}$$

for all rank one idempotent p. Now it follows from (2.11), (2.13) and (2.16) that

$$T_1(p) = pT_1(e_A), \qquad T_2(p) = T_2(e_A)p. \tag{2.17}$$

Since each element $u \in F(X)$ is a linear combinations of rank-one idempotents, thus (2.17) imply that

$$T_1(u) = uT_1(e_A), \qquad T_2(u) = T_2(e_A)u, \quad u \in F(X). \tag{2.18}$$

Let $u \in A \subseteq B(X)$. Then there is a net u_i in $F(X)$ such that $u_i \to u$. From (2.18) together the separate continuity of product in $B(X)$, we get

$$T_1(u) = \lim_i T_1(u_i) = \lim_i u_i T_1(e_A) = uT(e_A), \quad u \in A.$$

Therefore, $T_1(u) = uT(e_A)$ for all $u \in A$. Similarly, $T_2(u) = T_2(e_A)u$. Consequently, T_1 is a right multiplier and T_2 is a left multiplier. □

As a consequence of Theorem 2.7, we have the following result which is similar to the [5, Theorem 2.1]. Of course, our condition is weaker than of mentioned result.

Note that in [5] the authors do not assume any continuity on the map, but our approach is quite different from that in [5].

Corollary 2.8. *Let A be a standard operator algebra on Banach space X with $dimX \geq 2$ and let T_1 and T_2 be so-continuous linear maps from A into $B(X)$ satisfying*

$$uv = vu = 0, \quad \Rightarrow \quad uT_1(v) + T_2(u)v = 0.$$

Then there exist $\delta, \tau, \sigma \in B(X)$ such

$$T_1(u) = u\tau - \delta u, \qquad T_2(u) = u\delta - \sigma u,$$

for all $u \in A$.

Proof. It is enough to set $\tau = T_1(e_A)$, $\delta = 0$ and $\sigma = -T_2(e_A)$. □

We say that a Banach algebra A is generated by idempotents, whenever $A = \overline{\mathfrak{J}(A)}$. Von-Neumann algebra, the group algebra $L^1(G)$ for a compact group G, and topologically simple Banach algebras containing a non-trivial idempotent are examples of such Banach algebras. Other classes of Banach algebras with this property can be found in [1].

Note that Theorem 2.4 remains valid for every unital Banach algebra A which is generated by idempotents instead of C^*-algebra. Indeed, since A is generated by idempotents, it follows from Theorem 2.6 that

$$\phi(a, b) + \phi(b, a) = \phi(ab, e_A) + \phi(e_A, ba), \quad a, b \in A,$$

where $\phi(a, b) = aT_1(b) + T_2(a)b$. Now similar to the proof of Theorem 2.4, we can prove that $T_1(a) - T_2(a) = a\xi - \eta a$ for some $\xi, \eta \in X$ and for all $a \in A$. In particular, if $T_2(a) = -T_1(a)$, then we get the next result.

Corollary 2.9. *Let A be a unital Banach algebra which is generated by idempotents, and let T be a continuous linear map from A into X such that*

$$a, b \in A, \qquad ab = ba = 0, \quad \Rightarrow \quad aT(b) = T(a)b.$$

If $a\xi = \xi a$ for all $a \in A$, then T is a multiplier.

Our next result characterizes Jordan multipliers on group algebra.

Theorem 2.10. *Let $A = L^1(G)$ for a locally compact amenable group G, X be a Banach A-bimodule, and let $T : A \to X$ be a continuous right Jordan multiplier. Then T is a right multiplier.*

Proof. We make X an A-bimodule by putting $xa = 0$ for all $a \in A$ and $x \in X$. Then $T : A \to X$ is a right Jordan multiplier if and only if it is a Jordan derivation. Since G is amenable, by [11, Theorem 6.2] every Jordan derivation from A into X is a derivation. Consequently, T is a derivation, and as a result, it is a right multiplier. □

Because of Theorem 2.10, the next question can be raised. Is Theorem 2.10 valid for any locally compact group? Next, we give an affirmative answer to this question for the special case $X = L^1(G)$ and $X = L^1(G)^*$.

For locally compact group G, we write $M(G)$ for the Banach algebra of all complex, Borel regular measures on G with respect to the convolution product $\star$. Of course, $L^1(G)$ can be seen as the two-sided closed ideal of $M(G)$ consisting of all measures in $M(G)$ which are continuous with respect to the Haar Measure.

Proposition 2.11. *Let T be a continuous linear map on $L^1(G)$. Then the following are equivalent.*

(i) *T is a Jordan multiplier.*
(ii) *T is a multiplier.*
(iii) *There is a unique measure $\mu \in M(G)$ such that*

$$T(f) = f \star \mu = \mu \star f, \quad f \in L^1(G).$$

Proof. The statement (i) ⇒ (ii) follows from [15, Proposition 1.4], and the implication (ii) ⇒ (iii) is Wendel's theorem [14, Theorem 1]. (iii) ⇒ (i) is clear. □

Recall that a bounded approximate identity for A is a bounded net $\{e_\alpha\}_{\alpha\in I}$ in A such that $e_\alpha a \to a$ and $ae_\alpha \to a$ for every $a \in A$. It seems appropriate to mention that the group algebra $L^1(G)$ for a locally compact group G and every C^*-algebra has a bounded approximate identity bounded by one, see [7].

Theorem 2.12. *Let A be a Banach algebra with a bounded approximate identity, and let $T : A \to A^*$ be a continuous right Jordan multiplier. Suppose that B is a Banach algebra which contains A as a closed two-sided ideal. Then there exists a continuous right Jordan multiplier Δ from B into A^* that extends T.*

Proof. First note that A^* is a Banach A-bimodule with the usual structures. Let (e_α) be a bounded approximate identity for A of bound C. Let $x \in A$ and $b \in B$. For every α, we have

$$|T(e_\alpha be_\alpha)(x)| \le \|T\|\|e_\alpha be_\alpha\|\|x\| \le C^2\|T\|\|b\|\|x\|.$$

Thus, the net $T\left(e_\alpha b e_\alpha\right)(x)$ is bounded and so it has a unique accumulation point. To extend T, let $\Delta : B \to A^*$ defined by

$$\Delta(b)(x) = \lim_\alpha T\left(e_\alpha b e_\alpha\right)(x),$$

for all $b \in B$ and $x \in A$. Then $\Delta(b)$ is a continuous linear functional on A and that Δ is a continuous linear map with $\|\Delta\| \leq C^2\|T\|$.

For each $b \in A$, $e_\alpha b e_\alpha \to b$ and hence the continuity of T imply that

$$\Delta(b)(x) = \lim_\alpha T\left(e_\alpha b e_\alpha\right)(x) = T(b)(x).$$

Therefore, Δ extends T. Since T is a right Jordan multiplier, so

$$aT(b) = T(ab + ba) - bT(a), \quad a, b \in A. \tag{2.19}$$

From (2.19) together the continuity of T we get

$$\begin{aligned}
\Delta\left(b^2\right)(xy) &= \lim_\alpha T\left(e_\alpha b^2 e_\alpha\right)(xy) \\
&= \lim_\alpha \left(y \cdot T\left(e_\alpha b^2 e_\alpha\right)\right)(x) \\
&= \lim_\alpha \left(T\left(e_\alpha b^2 e_\alpha y + y e_\alpha b^2 e_\alpha\right) - \left(e_\alpha b^2 e_\alpha\right) \cdot T(y)\right)(x) \\
&= \lim_\alpha T\left(e_\alpha b^2 e_\alpha y + y e_\alpha b^2 e_\alpha\right)(x) - T(y)\left(x e_\alpha b^2 e_\alpha\right) \\
&= T\left(b^2 y + y b^2\right)(x) - T(y)\left(x b^2\right).
\end{aligned}$$

Hence,

$$\Delta\left(b^2\right)(xy) = T\left(b^2 y + y b^2\right)(x) - T(y)\left(x b^2\right).$$

In the same manner, we have

$$\begin{aligned}
T\left(\left(e_\alpha b e_\alpha\right)^2\right)(xy) &= T\left(\left(e_\alpha b e_\alpha^2 b e_\alpha\right)\right)(xy) \\
&= \left(y \cdot T\left(e_\alpha b e_\alpha^2 b e_\alpha\right)\right)(x) \\
&= \left(T\left(e_\alpha b e_\alpha^2 b e_\alpha y + y e_\alpha b e_\alpha^2 b e_\alpha\right) - \left(e_\alpha b e_\alpha^2 b e_\alpha\right) \cdot T(y)\right)(x) \\
&= T\left(e_\alpha b e_\alpha^2 b e_\alpha y + y e_\alpha b e_\alpha^2 b e_\alpha\right)(x) - T(y)\left(x e_\alpha b e_\alpha^2 b e_\alpha\right) \\
&\to T\left(b^2 y + y b^2\right)(x) - T(y)\left(x b^2\right) \\
&= \Delta\left(b^2\right)(xy).
\end{aligned}$$

Thus,

$$\lim_\alpha T\left(\left(e_\alpha b e_\alpha\right)^2\right)(xy) = \Delta\left(b^2\right)(xy). \tag{2.20}$$

On the other hand, since T is a right Jordan multiplier we arrive at

$$\begin{aligned}
T\left(\left(e_\alpha b e_\alpha\right)^2\right)(xy) &= \left(\left(e_\alpha b e_\alpha\right) \cdot T\left(e_\alpha b e_\alpha\right)\right)(xy) \\
&= T\left(e_\alpha b e_\alpha\right)(x y e_\alpha b e_\alpha).
\end{aligned}$$

Take $u_\alpha = xye_\alpha be_\alpha$ and $u = xyb$. Then $u_\alpha \to u$ and hence

$$|T(e_\alpha be_\alpha)(u_\alpha) - T(e_\alpha be_\alpha)(u)| \leq C^2\|T\|\|b\|\|u_\alpha - u\| \to 0.$$

Since $\lim_\alpha T(e_\alpha be_\alpha)(u) = \Delta(b)(y)$, from (2.20) we get

(2.21)

$$\lim_\alpha T\left((e_\alpha be_\alpha)^2\right)(xy) = \lim_\alpha T(e_\alpha be_\alpha)(u_\alpha) = \Delta(b)(u) = \Delta(b)(xyb).$$

By (2.20) and (2.21),

$$\Delta\left(b^2\right)(xy) - b\Delta(b)(xy) = 0,$$

for all $x, y \in A$. Since A has a bounded approximate identity, each $a \in A$ can be written as $a = xy$ for some $x, y \in A$. This implies that $\Delta\left(b^2\right)(a) - b\Delta(b)(a) = 0$ for all $a \in A$ and consequently, $\Delta\left(b^2\right) = b\Delta(b)$. This finishes the proof. □

Now suppose that $T : L^1(G) \to L^1(G)^*$ is a Jordan multiplier. Then by Theorem 2.12 we can extend T to a Jordan multiplier Δ from $M(G)$ into $L^1(G)^*$. Hence we get

$$\Delta(bab) = ba\Delta(b), \quad a, b \in M(G).$$

Since $M(G)$ is unital with unit δ_e the point mass measure at e, the identity of G, so by letting $b = \delta_e$ in the above equality we obtain $\Delta(a) = a\Delta(\delta_e)$ for each $a \in M(G)$. Thus, Δ is a multiplier. Since Δ is an extension of T, we get the following result.

Corollary 2.13. *Every continuous Jordan multiplier from $L^1(G)$ into $L^1(G)^*$ is a multiplier.*

3. Local Multipliers

In this section, we characterize continuous local multipliers which are necessary multipliers.

The following fundamental example which demonstrated by Johnson in [12], shows that local multipliers are not multipliers, in general.

Example 3.1. Let $A = C^1[0,1]$ be the algebra of complex valued functions on $[0,1]$ with continuous first derivative. Then by pointwise product and the norm

$$\|f\| = \|f\|_\infty + \|f'\|_\infty,$$

A makes into a Banach algebra. Define $T : A \to A^*$ via $T(f) = f'(0)\delta$ where $\delta : A \to \mathbb{C}$ given by $\delta(g) = g(0)$. To show that T is a local multiplier it is enough to prove that for each $f \in A$, there is an element $T_f \in A^*$ such that

$$T_f(fg) = f'(0)g(0) = (T(f))\,g, \quad g \in A. \tag{3.1}$$

Define

$$T_f(g) = \left(\frac{f'(0)}{f(0)}\right)\delta(g), \quad g \in A,$$

whenever $f(0) \neq 0$. If $f(0) = 0$, then we can define $T_f(g) = g(0)$ and therefore in both case the equality (3.1) is satisfied. Thus, T is a local multiplier, but it is not a multiplier because

$$T(fg) = (fg)'(0)\delta \neq f(0)g'(0)\delta = fT(g).$$

We say that the Banach algebra A has property $(\mathbb{B})$ if for every continuous bilinear map $\phi : A \times A \to X$, where X is an arbitrary Banach space, the condition

$$a, b \in A, \qquad ab = 0 \quad \Rightarrow \quad \phi(a, b) = 0,$$

implies that $\phi(ab, c) = \phi(a, bc)$, for all $a, b, c \in A$.

This concept was introduced in [1], and it turns out that many important examples of Banach algebras, including C^*-algebras and group algebras $L^1(G)$ where G is any locally compact group, have property $(\mathbb{B})$, see [1, Example 1.3].

Theorem 3.2. *Let A be a Banach algebra with property $(\mathbb{B})$ and having a bounded approximate identity. Then every continuous local right multiplier T from A into essential Banach left A-module X is a right multiplier.*

Proof. Let $\phi : A \times A \to X$ be a continuous bilinear map defined by $\phi(a, b) = aT(b)$. Then

$$\phi(a, b) = aT(b) = aT_b(b) = T_b(ab) = 0,$$

for all $a, b \in A$ with $ab = 0$. Hence property $(\mathbb{B})$ gives

$$aT(bc) = \phi(a, bc) = \phi(ab, c) = abT(c), \quad a, b, c \in A.$$

Taking $a = e_\alpha$ in the above equality, we conclude that $T(bc) = bT(c)$ for all $b, c \in A$. So, T is a right multiplier. □

From Theorem 3.2, we get the next results.

Corollary 3.3 ([12, Corollary 5.4])**.** *Suppose that A is a C^*-algebra and X is a Banach A-bimodule. If $T : A \to X$ is a continuous local multiplier, then T is a multiplier.*

Corollary 3.4. *Every continuous local multiplier T from $L^1(G)$ into an essential Banach $L^1(G)$-bimodule X is a multiplier.*

Since every Banach algebra which is generated by idempotents has property $(\mathbb{B})$, thus we obtain the following result.

Corollary 3.5. *Let A be a Banach algebra with a bounded approximate identity. If A is generated by idempotents, then every continuous local multiplier T from A into essential Banach A-bimodule X is a multiplier.*

Acknowledgment. The authors gratefully acknowledges the helpful comments of the anonymous referees.

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