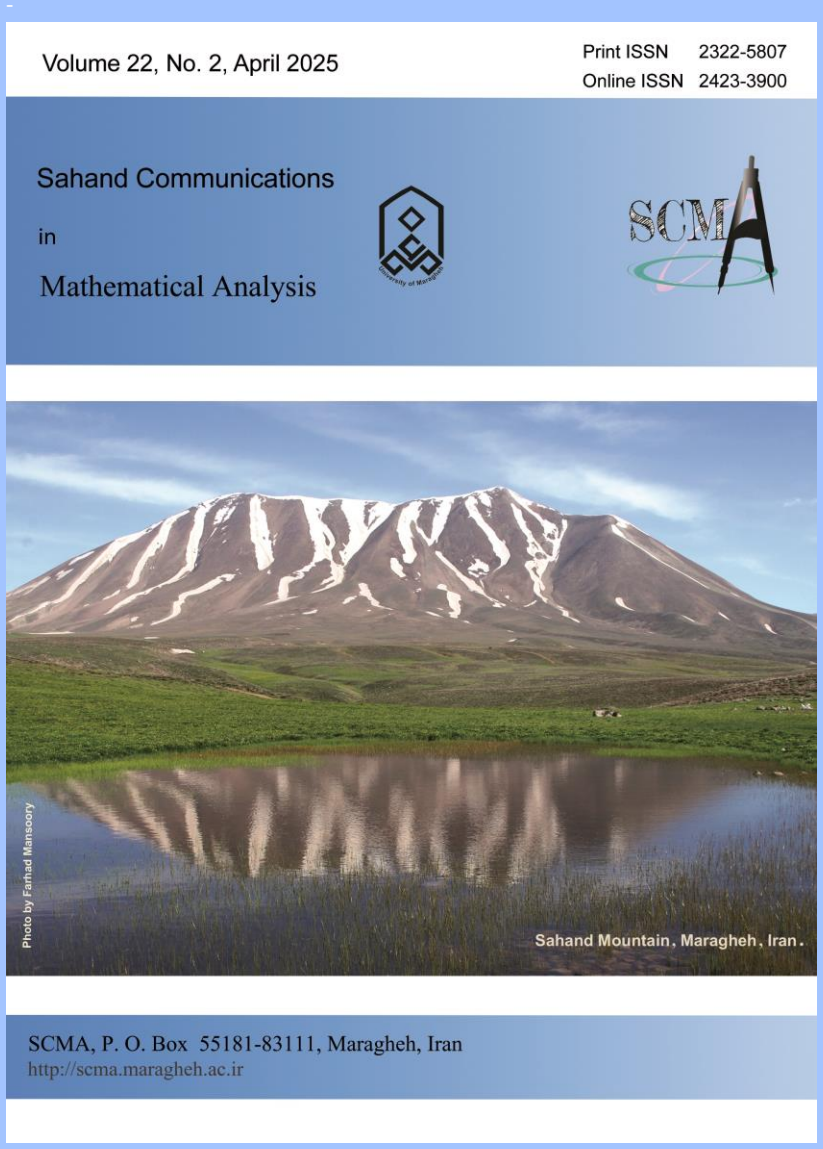
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An Explicit Structure for Duals of Frames in Krein Spaces

Elnaz Osgooei[1*] and Asghar Rahimi[2]

Abstract. In this study, motivating the explanation of Esmeral, Ferrer, and Wagner, similar findings regarding frames in Hilbert spaces were attempted to be extended to Krein spaces. The objective of the paper appears to be to explore the properties and characteristics of frames, particularly in the context of Krein spaces and various categories of operators. The paper discusses the implications of removing elements from a frame and the conditions under which a sequence remains a frame or becomes incomplete. In particular, it is shown that the value of $\left[f_j, S^{-1}f_j\right]$ is important in recognizing which sequences $\{f_k\}_{k=1}^{\infty}$ qualify as frames or incomplete sets. Subsequently, the importance of the synthesis operator in characterizing frames and establishing the connection between frame bounds and the properties of the synthesis operator was investigated. Additionally, for certain operators U, conditions have been established under which the sequence $\{Uf_k\}_{k=1}^{\infty}$ will be a frame sequence, provided that $\{f_k\}_{k=1}^{\infty}$ is already a frame. The concept of a Riesz basis in Krein spaces was introduced, followed by determining the conditions equivalent for a frame to qualify as minimal, a Riesz basis, or an exact frame. Lastly, a clear structure for the duals of frames in Krein spaces has been established. It was discovered that the dual frames of the sequence $\{f_k\}_{k=1}^{\infty}$ correspond to the families $\{JV\delta_k\}_{k=1}^{\infty}$ where $V : (\pounds^2(\mathbb{N}, [.,.]_{\tilde{J}}) \to (K, [.,.]_J)$ serves as a bounded left inverse of T_0^* and J is a fundamental symmetry in K.

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1. Introduction

In 1946, Gabor [19] introduced a foundational method to signal decomposition in terms of elementary signals. Subsequently, in 1952, Duffin and Schaeffer [14] expanded on Gabor's approach by formalizing the concept of frames within Hilbert spaces. These concepts remained relatively overlooked outside the realms of non-harmonic Fourier series and signal processing until Daubechies, Grossmann, and Meyer revived the idea of frames in their influential paper [12]. Frames can be viewed as a generalization of orthonormal bases in Hilbert spaces. Due to their redundant nature, frame expansions offer greater utility and benefits than basis expansions in many practical applications. Specifically, frames are extensively applied in areas such as sampling theory [1, 15], wavelet theory [13], wireless communication [23, 37], signal processing and image processing [9].

The duality principle is a finding that enables a more straightforward verification of the frame condition for what is known as Gabor systems. Principles of duality in Gabor theory, including the Ron-Shen duality principle [36] and the Wexler-Raz biorthogonality relations [39], are crucial for the analyzing of Gabor systems. In their work, Casazza et al. proposed a comprehensive method in [8, 9] to derive duality principles within the framework of abstract frame theory. Osgooei and Arefijamaal compared the duality of discrete and fusion frames [32]. Osgooei et al. [34] proposed a clear structure for duals of g-frames.

Han and Larson [22] introduced the concept of Schauder frames for Banach spaces. Meanwhile, Casazza et al. [7] provided several definitions of frames for Banach spaces, including the definition of Schauder frames. In 2008, Casazza et al. [6] examined the coefficient quantization of Schauder frames in Banach spaces. Beanland et al. [4] demonstrated that the upper and lower estimate theorems for finite-dimensional decompositions of Banach spaces can be extended and adapted to Schauder frames. They also provided a comprehensive characterization of duality for Schauder frames. Marin and Sama [29] investigated cones related to Schauder bases in Banach spaces. Jahan et al. [26] connected the concept of retro Banach frames with the geometric idea of a support cone in Banach spaces.

Additionally, Jahan et al. [25] linked Banach frames to another geometric concept known as a cone, associating it with Banach frames. In [24], Jahan and Kumar connected Schauder frames to another geometric concept within Banach spaces. Specifically, they investigated cones related to different types of Schauder frames and provided an example of a cone, C_F, which is minihedral but not normal.

The strong connection between Krein spaces and various areas of mathematics, including mathematical physics, quantum physics, and the analysis of measured spaces with signed measures, makes it a logical decision to extend frame theory from Hilbert spaces and Banach spaces to Krein spaces. Also, frames existing in Hilbert spaces and Banach spaces are considered an attractive tool in signal processing and wavelet analysis [17, 18, 21, 31, 33–35], which makes it significant to extend the findings in Hilbert spaces to other types of spaces that are rich in application, including Krein spaces.

Krein spaces hold significant importance in fields such as physics involving high energy, cosmology at the quantum level, and Krein space-based filtering. Various studies have recently focused on introducing frame theory to Krein spaces. For instance, Giribet et al. [20] extended the concept of J-orthonormal bases to define frames on Krein spaces called J-frames. Afterward, Esmeral et al. [16] established frames existing for Krein spaces by substituting the definite inner product that is positive in defining frames for Hilbert spaces with an indefinite inner product and proved that these two concepts are equivalent [27, 28]. This research uses Esmeral's definition to approach the new results of frames in Krein spaces. The following sections of this paper are organized in the manner outlined below:

Section 2 offers an overview of some definitions and important results regarding Krein spaces and operators on Krein spaces. In the third section, Esmeral's definition is utilized to derive results that resemble those fulfilled in Hilbert spaces but specifically apply to Krein spaces. For example, researchers reported that by considering some special conditions on operator U, $\{Uf_k\}_{k=1}^{\infty}$ constitutes a frame sequence or a frame in the case that $\{f_k\}_{k=1}^{\infty}$ is identified as a frame sequence or a frame. Furthermore, it is revealed that the value of $a_j = \left[f_j, S^{-1}f_j\right]$ is important for indicating that the sequence $\{f_k\}_{k=1}^{\infty}$ may be classified as a frame or incomplete. The frames in Krein spaces are characterized by the synthesis operator, and a clear structure for frames is provided in the fourth section. In addition, Section 5, emphasizes introducing the concept of Riesz basis in Krein spaces, followed by identifying equivalent conditions under which a frame to be minimal, Riesz basis, or an exact frame. Finally, an explicit structure for the duals of frames existing in Krein spaces is obtained in the sixth section.

Within this paper, K and $\widetilde{K}$ are Krein spaces that possess fundamental symmetries J and $\widetilde{J}$, respectively. The densely defined operator subset from K to $\tilde{K}$ will be represented by $OP\left(K, \tilde{K}\right)$. Moreover, the subsets of closed operators, bounded operators, and bounded operators

on K will be represented by $C\left(K,\tilde{K}\right)$, $B\left(K,\tilde{K}\right)$, and $B(K)$, respectively. The collection of operators that are defined everywhere (though not necessarily continuous) will be denoted by $L\left(K,\tilde{K}\right)$ as well as operators that are everywhere defined and continuous will be represented by $\beta\left(K,\tilde{K}\right)$. It should be noted that $D(T)$ will be denoted regarding the domain of T, $N(T)$ denotes the null space of T, while $R(T)$ represents its range.

2. Krein Space

An abstract vector space denoted as $(K,[.,.])$ is identified as a Krein space when it confirms the following conditions [2, 16, 20]:

(i) K is a linear space established over the field F, where F may be either $\mathbb{R}$ or $\mathbb{C}$.
(ii) An indefinite inner product $[.,.] \in F$ exists on K in such a manner that
  (1) $[y,x]=\overline{[x,y]}, x,y \in K$.
  (2) $[ax+by,z]=a[x,z]+b[y,z], x,y,z \in K, a,b \in F$.
(iii) The vector space K possesses a fundamental decomposition given by $K = K^+ \left[\dot{+}\right] K^-$ in a way that $(K^+,[.,.])$ and $(K^-,-[.,.])$ constitute Hilbert spaces defined by the norms $\|x\| = [x,x]^{\frac{1}{2}} (x \in K^+)$, $\|x\| = -[x,x]^{\frac{1}{2}} (x \in K^-)$ and K features a decomposition into a direct sum of two J-orthonormal subspaces K^+ and K^-.

Each fundamental decomposition of K creates two mutually complementary projections P^+ and P^- $(P^+ + P^- = I)$, such that P^+ mapping K onto K^+ and P^- mapping K onto K^-. P^+ and P^- are known as fundamental projections.

The linear operator $J : K \to K$ mapping from K to K is defined as $J = P^+ - P^-$ is recognized as the fundamental symmetry inherent in the Krein space K. This symmetry satisfies $J^2 = I$ and $J^* = J$. Also, the J-metric as described by $[x,y]_J = [x,Jy] = \langle x,y \rangle$.

The indefinite inner product $[.,.]$ creates a topological structure on K using the J-norm

$$\|x\|_J = \sqrt{[x,x]_J} = \sqrt{[x,Jx]} = \sqrt{\langle x,x \rangle}, \quad x \in K. \tag{2.1}$$

Topological concepts such as convergence and bounded linear operators pertain to the topology generated by the norm $\|.\|_J$. The Krein space K linked to the J-norm is a Hilbert space and is referred to as the associated Hilbert space belonging to the Krein space K and denoted by $(K,[.,.]_J)$.

Let us consider that $(K, [.,.])$ is a Krein space. The identity operator

$$I_J : (K, [.,.]) \to (K, [.,.]_J), \quad I_J x = x, \tag{2.2}$$

possesses the adjoint $J : (K, [.,.]_J) \to (K, [.,.])$ [16].

Let L, L_1, L_2 be subspaces within the Krein space K. The subspace $L^{[\perp]} = \{x \in K,\ [x, y] = 0,\quad \forall\ y \in L\}$ is referred to as J-orthogonal companion of L, $L^\circ = L \cap L^{[\perp]}$ is the isotropic part of L. If $L^\circ = \{0\}$, L is said to be non-degenerate, otherwise it is degenerate. The subspace L is termed regular provided that it is closed and serves as a Krein space with the restriction of the indefinite inner product. In the case that $[x, y] = 0$ for each $x \in L_1$, $y \in L_2$, then the two subspaces are called J-orthogonal and are denoted as $L_1[\perp]L_2$. If $L = L_1 + L_2$ where $L_1[\perp]L_2$, then L can be written as $L = L_1[+]L_2$. If in addition, this is a direct sum then it can be represented as $L = L_1 \left[\dot{+}\right] L_2$ [3].

Example 2.1 ([20])**.** The space $\left(\pounds^2(\mathbb{N}), [.,.]\right)$ constitutes a Krein space characterized by fundamental symmetry $\widetilde{J}e_k = \sigma_k e_k$ for each $k \in \mathbb{N}$, where $\sigma_k = sgn([e_k, e_k])$ and $\mathbb{N}^+ = \{k \in \mathbb{N};\ \ [e_k, e_k] \geq 0\}$ and $\mathbb{N}^- = \{k \in \mathbb{N};\ \ [e_k, e_k] < 0\}$. Also $\pounds^2(\mathbb{N}) = \pounds^2(\mathbb{N}^+) \left[\dot{+}\right] \pounds^2(\mathbb{N}^-)$.

Given a Krein space $(K, [.,.])$ with fundamental symmetry J, a J-orthonormalized system is a family $\epsilon = \{e_k\}_{k=1}^{\infty}$ where $[e_i, e_j] = \pm\delta_{i,j}$, $i, j \in \mathbb{N}$. A J-orthonormal basis is a J-orthonormalized system that is also a Schauder basis for K. If $\epsilon = \{e_k\}_{k=1}^{\infty}$ represents a J-orthonormal basis for K, then the vectors in K can be represented in the following way:

$$f = \sum_{k=1}^{\infty} \sigma_k [f, e_k] e_k, \quad f \in K,$$

where $\sigma_k = [e_k, e_k] = \pm 1$.

The sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ in the Krein space K are J-biorthogonal to each other if $[f_k, g_j] = \delta_{k,j}$.

In the following remark, we denote J-orthogonality using the symbol $[\perp]$, and the J-orthogonal complement is indicated by the same symbol with an index raised. In contrast, the conventional (Hilbert) orthogonality, based on the scalar product $\langle x, y\rangle = [Jx, y]$, is represented by the symbol $\perp$, and the usual orthogonal complement is shown by the same symbol with an index raised.

Remark 2.2 ([2, Remark 7.1, 7.2])**.** For any subset $L \subset K$ it's J-orthogonal complement $L^{[\perp]}$ and it's orthogonal complement $L^{\perp}$, are subspaces, linked by the formulae:

$$(i) \qquad L^{[\perp]} = JL^{\perp},$$
$$(ii) \qquad L^{\perp} = JL^{[\perp]}.$$

For any set $L \subset K$

$$(iii) \qquad L^{[\perp]} = (\overline{L})^{[\perp]} = (LinL)^{[\perp]},$$

where the notation $\overline{L}$ typically denotes the closure of a set L in a topological space, including in a Hilbert space (i.e., with the norm defined in (2.1)) and $LinL$ is the set of all finite linear combinations of elements from M. This set is interpreted as the smallest subspace of K containing all vectors in L, and in particular, for every linear subspace $L \subset K$,

$$L^{[\perp][\perp]} = \overline{L}.$$

For any subspace L contained in K, the space K can be expressed as an orthogonal sum

$$K = \overline{(L + L^{[\perp]})} \left[\dot{+}\right] JL^0. \tag{2.3}$$

While it is always the case that $L \cap L^{\perp} = 0$ and $L^{\perp} \left[\dot{+}\right] \overline{L} = K$ relative to the Hilbert metric, similar equalities concerning an indefinite metric are generally not true. This is primarily because L might represent a degenerate subspace and then $L \cap L^{[\perp]} = L^0 \neq 0$. In this case the algebraic sum $L + L^{[\perp]}$ is no longer a direct sum. The following lemma shows that this is true when L is a non-degenerate subspace.

Lemma 2.3 ([2, Lemma 7.7])**.** *In order for $\overline{L + L^{[\perp]}} = K$, it is necessary and sufficient for the subspace L to be non-degenerate.*

2.1. **Operators and Adjoints on the Krein Space K.** Let $T \in B\left(K, \tilde{K}\right)$ be a bounded operator. Therefore, there exists a linear operator $T^{*^{\tilde{J}}}$ for which

$$[Tx, y]_{\tilde{J}} = \left[x, T^{*^{\tilde{J}}} y\right]_J, \quad x \in K, y \in \widetilde{K}.$$

Also, there exists a linear operator $T^{\sharp}$ in such a way that for each $x \in K$, $y \in \widetilde{K}$,

$$[Tx, y] = \left[x, T^{\sharp} y\right].$$

This implies that

$$[Tx, y] = \left[\widetilde{J}Tx, y\right]_{\tilde{J}} = \left[Tx, \widetilde{J}y\right]_{\tilde{J}} = \left[x, T^{*^{\tilde{J}}} \widetilde{J}y\right]_J = \left[x, JT^{*^{\tilde{J}}} \widetilde{J}y\right].$$

Consequently, $T^{\sharp} = JT^{*^{\tilde{J}}} \widetilde{J}$. The operator $T^{\sharp}$ is known as the adjoint operator associated with T in the Krein space K. The operator T is considered self-adjoint if $T = T^{\sharp}$ and is called unitary within the Krein space K, provided that $TT^{\sharp} = T^{\sharp}T = I$. The operator $T : K \to K$ is densely defined if $\overline{D(T)} = K$ [2, 20]. Also the operator T is designated as a positive operator if $[Tf, f] \geq 0$ for each $f \in K$.

Theorem 2.4 ([38]). *Any bounded operator is closable. A bounded operator T is closed if and only if its domain $D(T)$ is closed. For a bounded operator T, this implies that $D(\overline{T}) = \overline{D(T)}$, the closure $\overline{T}$ acts as the bounded extension of the operator T onto the set $\overline{D(T)}$.*

Proposition 2.5 ([2]). *If for an operator $T : K \to K$ that is densely defined, an inverse operator that is densely defined exists, then $\left(T^{-1}\right)^{\sharp} = \left(T^{\sharp}\right)^{-1}$.*

Proposition 2.6 ([2]). *In order for the operator $T : K \to K$, $\left(T^{\sharp}\right)^{\sharp} = T^{\sharp\sharp}$ should exist, it is necessary and sufficient that T is densely defined and admits closure; in addition, $\overline{T}^{\sharp} = T^{\sharp}$, $T^{\sharp\sharp} = \overline{T}$.*

Proposition 2.7 ([2]). *Assume that $T_1, T_2 : K \to K$ and $\overline{D(T_2)} = \overline{D(T_2T_1)}$. Then $(T_2T_1)^{\sharp} \supset T_1^{\sharp}T_2^{\sharp}$, and if either T_2 is bounded or if the operator T_1^{-1}, defined everywhere and bounded, then $(T_2T_1)^{\sharp} = T_1^{\sharp}T_2^{\sharp}$.*

Proposition 2.8 ([5]). *For the operator $T \in B(K)$, the following statements are applied:*

(i) $0^{\sharp} = 0$.
(ii) $I^{\sharp} = I$.
(iii) $T^{\sharp}$ *is closed.*
(iv) $T \in B(K)$ *implies* $T^{\sharp} \in B(K)$.

Proposition 2.9. *Assume that $T \in B(K)$ is both a densely defined and closed operator then the following requirements are satisfied:*

(i) $N\left(T^{\sharp}\right) = R(T)^{[\perp]}$.
(ii) $N(T) = R\left(T^{\sharp}\right)^{[\perp]}$.
(iii) $\|T\| = \left\|T^{\sharp}\right\|$.

Proof. (i) If $x \in N\left(T^{\sharp}\right)$, then $T^{\sharp}x = 0$. For each $y \in R(T)$, there is an element $k \in K$ where $Tk = y$. By Theorem 3.10 and based on Propositions 2.6 and 2.8

$$[x, y] = [x, Tk] = \left[T^{\sharp}x, k\right] = 0.$$

Therefore $x \in R(T)^{[\perp]}$.

On the other hand, let's consider that $x \in R(T)^{[\perp]}$. Subsequently, for every $k \in K$

$$0 = [x, Tk] = \left[T^{\sharp}x, k\right].$$

So for each $k \in K$

$$\left[T^{\sharp}x, Jk\right] = \left\langle T^{\sharp}x, k\right\rangle = 0.$$

This indicates that $T^\sharp x = 0$ and so $x \in N\left(T^\sharp\right)$.

(ii) The reasoning is similar to (i).

(iii)

$$\begin{aligned}\left\|T^\sharp\right\| &= \sup_{x\neq 0} \frac{\left\|T^\sharp x\right\|_J}{\|x\|_J} \\ &= \sup_{x\neq 0} \frac{\left\|J(T^*)^J Jx\right\|_J}{\|x\|_J} \\ &= \sup_{x\neq 0} \frac{\left\|(T^*)^J Jx\right\|_J}{\|Jx\|_J} \\ &= \left\|(T^*)^J\right\| \\ &= \|T\|. \end{aligned}$$

□

Proposition 2.10 ([2, Remark 1.12])**.** *Assuming T is an operator that is closed and densely defined, then, $R\left(T^\sharp\right) = N(T)^{[\perp]}$ if and only if $R(T) = \overline{R(T)}$.*

Theorem 2.11 ([5, Theorem 2.9])**.** *Suppose that T is an operator that is densely defined which is closed and whose domain is dense in K. Then if $R(T)$ is closed, it implies that $R\left(T^\sharp\right)$ is closed.*

2.2. Generalized $*$-inverse Applicable to Unbounded Operators.

Theorem 2.12 ([30, Theorem 2.1])**.** *Let T be an operator that is densely defined, mapping from the Krein space K to the Krein space $\tilde{K}$. Subsequently, the following two statements are interchangeable:*

1. *There exists $V \in OP\left(\tilde{K},\right)$ in such a way that:*
   - (i) *$R(T) \subset D(V)$; $R(V) \subset D(T)$;*
   - (ii) *for each $h \in D(T)$, $TVTh = Th$;*
   - (iii) *for each $k \in D(V)$, $VTVk = Vk$;*
   - (iv) *TV and VT are symmetric.*
2. *$\overline{N(T)}$ and $\overline{R(V)}$ are non-degenerate, and*

$$D(T) \subset N(T)\left[\dot{+}\right] N(T)^{[\perp]}$$

If V confirms $(i), (ii), (iii)$ and (iv), then V is known as a $*$-generalized inverse of T and is represented by $T(*-inv)V$.

Theorem 2.13 ([30, Theorem 2.3])**.** *Consider $T \in OP\left(K, \tilde{K}\right)$ such that $\overline{N(T)}$ and $\overline{R(T)}$ are non-degenerate and $D(T) \subset N(T)\left[\dot{+}\right] N(T)^{[\perp]}$. Describe*

$$T^\dagger : D\left(T^\dagger\right) = R(T)\left[\dot{+}\right] R(T)^{[\perp]} \to \tilde{K}$$

$$k + k' \to \left(T \Big|_{N(T)^{[\perp]} \cap D(T)}\right)^{-1}(k).$$

Subsequently, $T(* - inv)V \Rightarrow V$ *is a restriction of* $T^\dagger$.

The operator $T^\dagger$ is referred to as the maximal inverse of T, or Moore-Penrose inversed of T.

Definition 2.14 ([30])**.** An operator $T \in L\left(K, \tilde{K}\right)$ is called regular if both $N(T)$ and $R(T)$ are regular. The collection of such regular operators is referred to as $R\left(K, \tilde{K}\right)$.

Theorem 2.15 ([30, Theorem 2.12])**.** *Suppose* $T \in L\left(K, \tilde{K}\right)$.

$$\exists V \in L\left(K, \tilde{K}\right), T(* - inv)V \quad \Leftrightarrow \quad T \text{ is regular.}$$

If it can be found, $T^\dagger$ *is continuous* $\Leftrightarrow$ T *is continuous.*

Corollary 2.16 ([30, Corollary 2.13])**.** *Consider* T *as a continuous and regular operator. Then*

(i) *the operator* $TT^\dagger : \tilde{K} \to \tilde{K}$ *is a self-adjoint projection onto the range* $R(T)$*. Its kernel is defined by* $N\left(T^\dagger\right) = R(T)^{[\perp]}$.
(ii) $T^\dagger T : K \to K$ *is a self-adjoint projection on to the range* $R\left(T^\dagger\right) = N(T)^{[\perp]}$*, having kernel* $N(T)$.

Definition 2.17 ([30])**.** An operator $T \in OP\left(K, \tilde{K}\right)$ is called weakly regularred, if $N(T)$ and $\overline{R(T)}$ are both regular. The collection of such weakly regular operators is denoted by $WR\left(K, \tilde{K}\right)$.

Theorem 2.18 ([30, Theorem 2.15])**.** *Suppose that* $T \in C\left(K, \tilde{K}\right)$. *Then* $T \in WR\left(K, \tilde{K}\right) \Leftrightarrow T^* \in WR\left(K, \tilde{K}\right)$ *and in this case* $\left(T^\sharp\right)^\dagger = \left(T^\dagger\right)^\sharp$.

3. Frames and Operators on Krein Spaces

Consider H as a Hilbert space equipped with the inner product $\langle ., . \rangle$. A countable sequence $\{f_k\}_{k=1}^\infty$ belongs to the Hilbert space H is considered a frame if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f_k, f \rangle|^2 \leq B\|f\|^2, \quad f \in H. \tag{3.1}$$

If just the right hand inequality in (3.1) satisfied, then $\{f_k\}_{k=1}^\infty$ is called a Bessel sequence.

Given a Bessel sequence $\{f_k\}_{k=1}^{\infty}$ for H, the definition for the synthesis operator is given by

$$T_0 : \ell^2(\mathbb{N}) \to H, \qquad T_0\left(\{\alpha_k\}_{k=1}^{\infty}\right) = \sum_{k=1}^{\infty} \alpha_k f_k, \tag{3.2}$$

which constitutes an operator that is both well-defined and bounded. The adjoint operator is obtained directly as

$$T_0^* : H \to \ell^2(\mathbb{N}), \qquad T_0^*(f) = \{\langle f, f_k\rangle\}_{k=1}^{\infty}. \tag{3.3}$$

The frame operator S is defined by $S = T_0 T_0^*$. So by (3.2) and (3.3)

$$Sf = \sum_{k=1}^{\infty} \langle f, f_k\rangle f_k, \quad f \in H. \tag{3.4}$$

Also inequality (3.1) implies that $A \leq S \leq B$, which shows that S is an operator that is positive, bounded, invertible and $f = SS^{-1}f = S^{-1}Sf$. So the frame decomposition theorem is arrived such that

$$\begin{aligned} f &= \sum_{k=1}^{\infty} \left\langle f, S^{-1}f_k\right\rangle f_k \\ &= \sum_{k=1}^{\infty} \langle f, f_k\rangle S^{-1}f_k, \quad f \in H. \end{aligned} \tag{3.5}$$

Also, $B^{-1} \leq S^{-1} \leq A^{-1}$, which implies that $\left\{S^{-1}f_k\right\}_{k=1}^{\infty}$ is a frame for H [10, 11].

Krein spaces have beneficial applications in known areas such as high energy physics, quantum, and Krein space filtering. Accordingly, it is essential to obtain the outcomes of frames in Hilbert spaces that are applicable in Krein spaces. Researchers have studied frames in Krein spaces in various ways. For example, Giribet et al. [20] presented frames within Krein spaces referred to as J-frames. Relevant to their definition, not every spanning set of a finite-dimensional Krein space K is necessarily a J-frame for K. Similarly, Esmeral et al.[16] introduced an independent concept of frames existing in Krein spaces by substituting the definite inner product with the indefinite inner product and proposed a useful characterization to switch between frames existing in Krein spaces to frames in associated Hilbert spaces. To study the frame theory within Krein spaces, the definition provided by Esmeral et al. [16] is used in this study as follows:

Definition 3.1 ([16, Definition 3.1])**.** Let K be a Krein space. A countable sequence $\{f_k\}_{k=1}^{\infty} \subseteq K$ is called a frame for K, if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|_J^2 \leq \sum_{k=1}^{\infty} |[f, f_k]|^2 \leq B\|f\|_J^2. \quad f \in K. \tag{3.6}$$

If solely the upper condition in (3.4) is fulfilled, then $\{f_k\}_{k=1}^{\infty}$ is called a Bessel sequence for K. As in the Hilbert space, A and B are referred to as frame bounds. The greatest constant A and the smallest constant B satisfying (3.4) are called optimal lower frame bound and optimal upper frame bound, respectively. A frame is considered tight if one can choose $A = B$.

Example 3.2. Let $\left(\mathbb{C}^2, [.,.]\right)$ be a Krein space with the indefinite inner product

$$[(x_1, x_2), (y_1, y_2)] = x_1\overline{y_1} - x_2\overline{y_2}, \quad (x_1, x_2),\, (y_1, y_2) \in \mathbb{C}^2,$$

and the fundamental symmetry J as

$$J(x_1, x_2) = (x_1, -x_2), \quad (x_1, x_2) \in \mathbb{C}^2.$$

Consider $f_1 = \left(1, \sqrt{2}\right)$, $f_2 = \left(\sqrt{2}, 1\right)$, then $\{f_1, f_2\}$ is a frame for $\mathbb{C}^2$ with upper and lower bounds, $B = 3 + 2\sqrt{2}$ and $A = 3 - 2\sqrt{2}$, respectively.

Remark 3.3. For a frame $\{f_k\}_{k=1}^{\infty}$ of a Krein space K to also qualify as a frame for either K^- or K^+, it is crucial that the vectors are entirely confined to the respective subspace (either all within K^- or K^+). If the frame $\{f_k\}_{k=1}^{\infty}$ in the Krein space K, consists solely of vectors from the negative subspace K^-, it can be demonstrated that the inequalities defining a frame will still apply when projecting onto K^-. In other words, for $\{f_k\}_{k=1}^{\infty}$ to be a valid frame for K^-, the vectors must remain completely within K^- without encroaching on K^+. But, if $\{f_k\}_{k=1}^{\infty}$ included vectors from both K^- and K^+, the frame conditions could break down, as the inner product wouldn't necessarily maintain the necessary stability for representing every vector in K^- or K^+.

The following example shows that how the sequence $\{f_k\}_{k=1}^{\infty}$ will form a frame for K^- or K^+ under the indefinite inner product

Example 3.4. Suppose that $\left(\pounds^2(\mathbb{N}), [.,.]\right)$ represents a Krein space characterized by the fundamental symmetry $\tilde{J}$ where $[.,.]_{\tilde{J}}$ corresponds to the standard inner product $\langle .,. \rangle$ on $\ell^2(\mathbb{N})$. Let $\tilde{J}e_{2k} = -e_{2k}$ and $\tilde{J}e_{2k+1} = e_{2k+1}$, for each $k \in \mathbb{N}$, where $\{e_k\}_{k=1}^{\infty}$ is the canonical orthonormal basis for the associated Hilbert space $\left(\pounds^2(\mathbb{N}), [.,.]_{\tilde{J}}\right)$. Let $f = (\alpha_1, \alpha_2, \alpha_3, \ldots)$, $g = (\beta_1, \beta_2, \beta_3, \ldots) \in \pounds^2(\mathbb{N})$. Then

$$[f, g] = \alpha_1\beta_1 - \alpha_2\beta_2 + \alpha_3\beta_3 - \cdots$$

So for $f \in \mathcal{L}^2(\mathbb{N})$ and $\{f_k\}_{k=1}^{\infty} = \{e_1, e_1, e_2, e_2, \ldots\}$,

$$\sum_{k=1}^{\infty} |[f, f_k]|^2 = 2\alpha_1^2 + 2\alpha_2^2 + 2\alpha_3^2 + \cdots$$

and

$$\|f\|_J^2 = [f, Jf] = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \cdots ,$$

which shows that

$$\sum_{k=1}^{\infty} |[f, f_k]|^2 = 2[f, Jf] = 2\|f\|_J^2 .$$

Therefore, $\{f_k\}_{k=1}^{\infty} = \{e_1, e_1, e_2, e_2, \ldots\}$ forms a frame in the Krein space $\mathcal{L}^2(\mathbb{N})$ with frame bounds $A = B = 2$.

Now, for

$$f^- = (0, \alpha_2, 0, \alpha_4, \ldots) \in \mathcal{L}^2(\mathbb{N}^-) \quad \text{and} \quad \{f_k^-\}_{k=1}^{\infty} = \{e_2, e_2, e_4, e_4, \ldots\}$$

have

$$\sum_{k=1}^{\infty} \left|\langle f^-, f_k^- \rangle\right|^2 = 2\alpha_2^2 + 2\alpha_4^2 + \cdots$$

and

$$\left\|f^-\right\|^2 = -[f^-, f^-] = \alpha_2^2 + \alpha_4^2 + \cdots .$$

Therefore, $\{f_k^-\}_{k=1}^{\infty} = \{e_2, e_2, e_4, e_4, \ldots\}$, forms a frame for the Hilbert space $\mathcal{L}^2(\mathbb{N}^-)$ with frame bounds $A = B = 2$. Similarly, it can be shown that $\{f_k^+\}_{k=1}^{\infty} = \{e_1, e_1, e_3, e_3, \ldots\}$, forms a frame for $\mathcal{L}^2(\mathbb{N}^+)$, with frame bounds $A = B = 2$.

Also, for $f \in \mathcal{L}^2(\mathbb{N})$ and $\{f_k\}_{k=1}^{\infty} = \{e_1, e_1, e_2, e_3, \ldots\}$

$$\sum_{k=1}^{\infty} |[f, f_k]|^2 = \alpha_1^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \cdots$$

and

$$\|f\|^2 = [f, Jf] = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \cdots ,$$

Considering $A = 1$, $B = 2$,

$$\|f\|_J^2 \leq \sum_{k=1}^{\infty} |[f, f_k]|^2 \leq 2\|f\|_J^2 .$$

Therefore $\{f_k\}_{k=1}^{\infty} = \{e_1, e_1, e_2, e_3, \ldots\}$ forms a frame in the Krein space $\mathcal{L}^2(\mathbb{N})$ with frame bounds set as $A = 1$ and $B = 2$.

Now, for $f^- = (0, \alpha_2, 0, \alpha_4, \ldots) \in \mathcal{L}^2(\mathbb{N}^-)$ and $\{f_k^-\}_{k=1}^{\infty} = \{e_2, e_4, \ldots\}$,

$$\sum_{k=1}^{\infty} \left|\langle f^-, f_k^- \rangle\right|^2 = \alpha_2^2 + \alpha_4^2 + \cdots$$

and

$$\left\|f^-\right\|^2 = -[f^-, f^-] = \alpha_2^2 + \alpha_4^2 + \cdots .$$

Therefore, $\{f_k^-\}_{k=1}^{\infty} = \{e_2, e_4, \ldots\}$, forms a frame for the Hilbert space $\mathcal{L}^2(\mathbb{N}^-)$ with frame bounds $A = B = 1$. Similarly, it can be shown that $\{f_k^+\}_{k=1}^{\infty} = \{e_1, e_1, e_3, e_5, \ldots\}$, forms a frame for the Hilbert space $\mathcal{L}^2(\mathbb{N}^+)$ with frame bounds $A = 1$ and $B = 2$.

The following is a useful theorem that shows that the concept of a frame for a Krein space is essentially the same object as a frame for the associated Hilbert space.

Theorem 3.5 ([16, Theorem 3.3])**.** *Let K be a Krein space and $\{f_k\}_{k=1}^{\infty}$ be a sequence in K. Then the following statements are equivalent:*

(i) *The sequence $\{f_k\}_{k=1}^{\infty}$ is a frame for the Krein space K with frame bounds A and B such that $A \leq B$.*
(ii) *The sequence $\{Jf_k\}_{k=1}^{\infty}$ is a frame for the Krein space K with frame bounds A and B such that $A \leq B$.*
(iii) *The sequence $\{f_k\}_{k=1}^{\infty}$ is a frame for the Hilbert space $(K, [.,.]_J)$ with frame bounds A and B such that $A \leq B$.*
(iv) *The sequence $\{Jf_k\}_{k=1}^{\infty}$ is a frame for the Hilbert space $(K, [.,.]_J)$ with frame bounds A and B such that $A \leq B$.*

Lemma 3.6. *Let K represent a Krein space with fundamental symmetry J such that $Je_k = \sigma_k e_k$ for each $k \in \mathbb{N}$. The subsequent conditions are interchangeable:*

(i) *$\{e_k\}_{k=1}^{\infty}$ is a J-orthonormal basis for the Krein space $(K, [.,.])$.*
(ii) *$\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for the associated Hilbert space $(K, [.,.]_J)$.*

Proof. $(i) \rightarrow (ii)$ Let $\{e_k\}_{k=1}^{\infty}$ is a J-orthonormal basis concerning the Krein space K. Then $[e_k, e_j] = \pm\delta_{k,j}$. So

$$\langle e_k, e_j\rangle = [e_k, Je_j] = [e_k, \sigma_j e_j] = \sigma_j[e_k, e_j] = \sigma_j(\pm\delta_{k,j}) = \delta_{k,j}.$$

$(ii) \rightarrow (i)$ Suppose that $\langle e_k, e_j\rangle = \delta_{k,j}$. Then

$$[e_k, e_j] = \langle e_k, Je_j\rangle = \langle e_k, \sigma_j e_j\rangle = \pm\delta_{k,j}. \qquad \square$$

The following example shows that the above lemma is not always correct when J is defined as in Example 3.7. It means that the validity of the Lemma 3.6 relies on the definition of J. If the definition of J is modified to $J(x_1, x_2) = (x_1, -x_2)$, the following example demonstrates that the lemma is incorrect.

Example 3.7. Suppose that $\left(\mathbb{C}^2, [.,.]\right)$ be a Krein space as defined in Example 3.2. Consider $f_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $f_2 = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$. Then $\{f_1, f_2\}$

is an orthonormal basis for $\mathbb{C}^2$, while is not a J-orthonormal basis for $\mathbb{C}^2$.

Lemma 3.8. *Suppose that the sequence $\{f_k\}_{k=1}^{\infty}$ serves as a frame for the dense subset of the Krein space K, then $\{f_k\}_{k=1}^{\infty}$ constitutes a frame for the Krein space K.*

Proof. The proof is akin to the Hilbert space case. □

It can be deduced that, according to the definition, if $\{f_k\}_{k=1}^{\infty}$ constitutes a frame for the Krein space K, then

$$\overline{span}\{f_k\}_{k=1}^{\infty} = K. \tag{3.7}$$

The associated Hilbert space K can be expressed as

$$K = \overline{span}\{f_k\}_{k=1}^{\infty} \oplus (\overline{span}\{f_k\}_{k=1}^{\infty})^{\perp}.$$

Thus each $f \in K$ can be expressed as $f = f_1 + f_2$, where $f_1 \in \overline{span}\{f_k\}_{k=1}^{\infty}$ and $f_2 \in (\overline{span}\{f_k\}_{k=1}^{\infty})^{\perp}$. Therefore

$$0 = \langle f_2, f_k\rangle = [f_2, Jf_k] = [Jf_2, f_k], \quad k \in \mathbb{N}.$$

Now (3.4) implies that $f_2 = 0$ and so $f = f_1 \in \overline{span}\{f_k\}_{k=1}^{\infty}$.

A sequence $\{f_k\}_{k=1}^{\infty}$ that satisfies the condition in (3.7) is referred to as complete or total.

Definition 3.9. The sequence $\{f_k\}_{k=1}^{\infty}$ within the Krein space K is referred to as a frame sequence if it serves as a frame for $\overline{span}\{f_k\}_{k=1}^{\infty}$.

In this section similar results satisfied into the Hilbert spaces are investigated for the Krein cases as Lemma 3.11, 3.13 and Proposition 3.15.

Definition 3.10 ([16])**.** Consider $(K, [.,.])$ as a Krein space equipped with fundamental symmetry J and $\big(\pounds^2(\mathbb{N}), [.,.]\big)$ as another Krein space with fundamental symmetry $\widetilde{J}$ where $[.,.]_{\widetilde{J}}$ coincides with the standard inner product $\langle .,.\rangle$ on $\ell^2(\mathbb{N})$. Given a Bessel sequence $\{f_k\}_{k=1}^{\infty}$ for K, the linear map

$$T : \pounds^2(\mathbb{N}) \to K, \qquad T\{\alpha_k\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} \alpha_k f_k, \tag{3.8}$$

is called a synthesis operator.

It is important to observe that T is well-defined and bounded because, $T = I_J^{-1} T_0 I_{\tilde{J}}$, where the operators $I_{\tilde{J}}$, $(I_J)^{-1}$ and T_0 are as

$$\big(\pounds^2(\mathbb{N}), [.,.]\big) \to^{I_{\tilde{J}}} \big(\ell^2(\mathbb{N}), [.,.]_{\tilde{J}}\big) \to^{T_0} (K, [.,.]_J) \to^{I_J^{-1}} (K, [.,.]). \tag{3.9}$$

are well-defined and bounded. It follows from the Hilbert space frame theories applied to T_0 and Theorem 3.5, that a sequence $\{f_k\}_{k=1}^{\infty}$ serves

as a frame for K, if and only if the operator T is well-defined, bounded and surjective.

The adjoint of T can be expressed as

$$T^\sharp f = \widetilde{J}\{[f, f_k]\}_{k=1}^{\infty}, \quad f \in K. \tag{3.10}$$

The operator $S = T\widetilde{J}T^\sharp$ is called a frame operator. Since $\widetilde{J}^2 = I$, (3.8) and (3.10) results that

$$Sf = \sum_{k=1}^{\infty} [f, f_k] f_k, \quad f \in K. \tag{3.11}$$

Now by (3.11) for each $f \in K$,

$$[Sf, f] = \sum_{k=1}^{\infty} |[f, f_k]|^2. \tag{3.12}$$

Lemma 3.11. *Regard $\{f_k\}_{k=1}^{\infty}$ as a frame for K that has frame bounds A and B and denote S as the frame operator. The following statements are true:*

(i) *S is bounded, self-adjoint, positive, and invertible.*
(ii) *The sequence $\left\{S^{-1} f_k\right\}_{k=1}^{\infty}$ constitutes a frame along with its frame operator S^{-1}.*

Proof. The operator $S = T\tilde{J}T^\sharp$ is bounded because it results from the combination of bounded operators. By Theorem 3.10 and Propositions 2.6, 2.7, S is self-adjoint. Since

$$S^\sharp = \left(T\tilde{J}T^\sharp\right)^\sharp = \left(T^\sharp\right)^\sharp \left(\tilde{J}\right)^\sharp T^\sharp = \overline{T}\tilde{J}T^\sharp = T\tilde{J}T^\sharp.$$

Also since $\{f_k\}_{k=1}^{\infty}$ constitutes a frame for K and

$$[Sf, f] = \sum_{k=1}^{\infty} |[f, f_k]|^2, \quad f \in K,$$

S is a positive operator.

Because $\{f_k\}_{k=1}^{\infty}$ constitutes a frame for the Krein space K, it serves as a frame for the associated Hilbert space $(K, [.,.]_J)$ by Theorem 3.5. Considering $S = (I_J)^{-1} S_0 J$, where S_0 represents the frame operator associated with $\{f_k\}_{k=1}^{\infty}$ in the associated Hilbert space $(K, [.,.]_J)$, the operator S is invertible.

(ii) Since S is self-adjoint, by Theorem 3.10 and Propositions 2.5 and 2.6

$$A\left(\|S\|^{-1}\right)^2 \|f\|_J^2 \leq \sum_{k=1}^{\infty} \left|\left[f, S^{-1} f_k\right]\right|^2$$

$$= \sum_{k=1}^{\infty} \left| \left[S^{-1} f, f_k \right] \right|^2$$
$$\leq B \left\| S^{-1} \right\|^2 \|f\|_J^2 \qquad \square$$

The frame $\left\{ S^{-1} f_k \right\}_{k=1}^{\infty}$ is referred to as the canonical dual frame of $\{f_k\}_{k=1}^{\infty}$ within the Krein space K.

Theorem 3.12 ([16] Frame Decomposition Theorem)**.** *Consider $\{f_k\}_{k=1}^{\infty}$ serves as a frame for the Krein space K, and allow S to be the frame operator. Then*

$$f = \sum_{k=1}^{\infty} [f, f_k] S^{-1} f_k, \quad f \in K, \tag{3.13}$$

$$f = \sum_{k=1}^{\infty} \left[f, S^{-1} f_k \right] f_k, \quad f \in K. \tag{3.14}$$

Lemma 3.11 states that if $\{f_k\}_{k=1}^{\infty}$ constitutes a frame, then $\left\{ S^{-1} f_k \right\}_{k=1}^{\infty}$ additionally qualifies as a frame. This represents a specific instance of a general statement that $\{U f_k\}_{k=1}^{\infty}$ is a frame or a frame sequence for some category of operators U as the following:

Proposition 3.13. *Consider $\{f_k\}_{k=1}^{\infty}$ serving as a frame for the Krein space K that has bounds A and B and let $U \in B(K)$ be a regular operator in a way that $\overline{N(U)}$ and $\overline{R(U)}$ are non-degenerate and $D(U) \subseteq N(U) \left[\dot{+}\right] N(U)^{[\perp]}$. Consequently the sequence $\{U f_k\}_{k=1}^{\infty}$ constitutes a frame sequence with frame bounds $A\|U^+\|^2$ and $B\|U\|^2$.*

Proof. In the event that f belongs to K, then

$$\sum_{k=1}^{\infty} |[f, U f_k]|^2 \leq B \left\| U^{\sharp} f \right\|_J^2 \leq B \left\| U^{\sharp} \right\|^2 \|f\|_J^2 = B \|U\|^2 \|f\|_J^2,$$

this demonstrates that the sequence $\{U f_k\}_{k=1}^{\infty}$ is a Bessel sequence for K.

Concerning the lower frame condition, assume that g is an element of the span of the sequence $\{U f_k\}_{k=1}^{\infty}$, then g can be written as $g = Uf$ for $f \in span\{f_k\}_{k=1}^{\infty}$. By Corollary 2.16, the operator UU^+ is self-adjoint projection on $R(U)$ with kernel $N(U^{\dagger}) = R(U)^{[\perp]}$. This implies that

$$g = Uf = \left(UU^+\right)^{\sharp} Uf = \left(U^+\right)^{\sharp} U^{\sharp} Uf.$$

So

$$\|g\|_J^2 \leq \left\| \left(U^+\right)^{\sharp} \right\|^2 \left\| U^{\sharp} U f \right\|_J^2$$

$$\leq \frac{\|U^+\|^2}{A} \sum_{k=1}^{\infty} \left| \left[U^\sharp U f, f_k \right] \right|^2$$
$$= \frac{\|U^+\|^2}{A} \sum_{k=1}^{\infty} |[Uf, Uf_k]|^2 .$$

Therefore the lower frame condition fullfilled for each $g \in span\{Uf_k\}_{k=1}^{\infty}$. By Lemma 3.8 the lower condition holds for $\overline{span}\{Uf_k\}_{k=1}^{\infty}$ and the proof is complete. □

Corollary 3.14. *Consider $\{f_k\}_{k=1}^{\infty}$ serving as a frame for the Krein space K that has bounds A and B and consider $U \in B(K)$ be a regular and surjective operator in such a way that $\overline{N(U)}$ and $\overline{R(U)}$ are non-degenerate and $D(U) \subseteq N(U) \oplus N(U)^{[\perp]}$. Then $\{Uf_k\}_{k=1}^{\infty}$ serves as a frame for the Krein space K having frame bounds $A\|U^+\|^2$, $B\|U\|^2$.*

Lemma 3.15. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame sequence for the Krein space K with bounds A, B and consider $U \in B(K)$ be a unitary operator. Then $\{Uf_k\}_{k=1}^{\infty}$ constitutes a frame sequence with frame bounds $A\|U\|^{-2}$, $B\|U\|^2$.*

Proof. Let $g \in span\{Uf_k\}_{k=1}^{\infty}$ then $g = Uf$ where $f \in span\{f_k\}_{k=1}^{\infty}$.

$$\sum_{k=1}^{\infty} |[g, Uf_k]|^2 = \sum_{k=1}^{\infty} \left| \left[U^\sharp g, f_k \right] \right|^2$$
$$= \sum_{k=1}^{\infty} \left| \left[U^\sharp U f, f_k \right] \right|^2$$
$$= \sum_{k=1}^{\infty} |[f, f_k]|^2$$
$$\leq B \left\| U^\sharp g \right\|_J^2$$
$$\leq B \|U\|^2 \|g\|_J^2 .$$

Alternatively, for $g \in span\{Uf_k\}_{k=1}^{\infty}$

$$\|g\|_J^2 = \|Uf\|_J^2$$
$$\leq \|U\|^2 \|f\|_J^2$$
$$\leq \frac{1}{A} \|U\|^2 \sum_{k=1}^{\infty} |[f, f_k]|^2$$
$$= \frac{1}{A} \|U\|^2 \sum_{k=1}^{\infty} \left| \left[U^\sharp g, f_k \right] \right|^2$$

$$= \frac{1}{A} \|U\|^2 \sum_{k=1}^{\infty} |[g, Uf_k]|^2 .$$

Now by Lemma 3.8 $\{Uf_k\}_{k=1}^{\infty}$ is a frame sequence. A similar proof demonstrates that $\{Uf_k\}_{k=1}^{\infty}$ is a frame if $\{f_k\}_{k=1}^{\infty}$ is a frame for K. □

Lemma 3.16. *Consider the sequence $\{f_k\}_{k=1}^{\infty}$ as a frame for the Krein space K and let f represent an element of K. Then, there is a unique sequence $\{a_k\}_{k=1}^{\infty}$ in such a way that $f = \sum_{k=1}^{\infty} a_k f_k$ and $a_k = [g, f_k]$ where $Sg = f$. If there is another sequence $\{b_k\}_{k=1}^{\infty}$ so that $f = \sum_{k=1}^{\infty} b_k f_k$ then*

$$\sum_{k=1}^{\infty} |b_k|^2 = \sum_{k=1}^{\infty} |a_k|^2 + \sum_{k=1}^{\infty} |a_k - b_k|^2 . \tag{3.15}$$

Proof. The first part is immediate, since S is invertible and $f = S(S^{-1}f)$. Let's assume that

$$\sum_{k=1}^{\infty} a_k f_k = \sum_{k=1}^{\infty} b_k f_k.$$

By choosing the indefinite inner product of both sides with g, it is obtained that

$$\sum_{k=1}^{\infty} |a_k|^2 = \sum_{k=1}^{\infty} \overline{a_k} b_k,$$

and (3.15) is obtained quickly. □

In the context of Krein spaces, our statement regarding frames and the impact of removing an element f_j is indeed accurate and highlights some important nuances compared to frames in standard Hilbert spaces. Let us explore this concept more deeply, focusing on the specific aspects of Krein spaces and the interpretation of the $\left[f_j, S^{-1}f_j\right]$. When an element f_j is removed from the frame in a Krein space, the consequences can similarly be assessed based on the modified frame properties and the inner product concerning, to the frame operator. This assessment emphasizes the interplay between redundancy and the stability of frames within the indefinite inner product framework of Krein spaces.

Lemma 3.17. *Removing a vector from a frame leads to either another frame or an incomplete set, depending on the value of $\left[f_j, S^{-1}f_j\right]$.*

Proof. Let the sequence $\{f_k\}_{k=1}^{\infty}$ be a frame for the space K with bounds A, B and suppose that f_m is removed. By Lemma 3.16, $f_m = \sum_{n=1}^{\infty} a_n f_n$,

where $a_n = [g_m, f_n]$ and $Sg_m = f_m$. Two cases considered: $a_m = 1$ and $a_m \neq 1$.

If $a_m = 1$, then $\sum\limits_{n\neq m} a_n f_n = 0$ and so $\sum\limits_{n=1}^{\infty} \tilde{a_n} f_n = 0$, where $\tilde{a_n} = a_n$ when $n \neq m$ and $\tilde{a_m} = 0$. Applying Lemma 3.16, with every b_n equal to zero, it is obtained that

$$\sum_{n=1}^{\infty} |\tilde{a_n}|^2 = 0.$$

So $a_n = 0$, $n \neq m$. Therefore, g_m is J-orthogonal to f_n for $n \neq m$. Since $[g_m, f_m] = 1$, it implies that $g_m \neq 0$ and so $\{f_n\}_{n\neq m}$ is incomplete.

Now suppose that $a_m \neq 1$. Then

$$f_m = \frac{1}{1 - a_m} \sum_{n\neq m} a_n f_n.$$

Therefore

$$|[f, f_m]|^2 \leq \frac{1}{|1 - a_m|^2} \sum_{n\neq m} |a_n|^2 \sum_{n\neq m} |[f, f_n]|^2 .$$

So

$$\begin{aligned} \sum_{n=1}^{\infty} |[f, f_n]|^2 &\leq \sum_{n\neq m} |[f, f_n]|^2 + \frac{1}{|1 - a_m|^2} \sum_{n\neq m} |a_n|^2 \sum_{n\neq m} |[f, f_n]|^2 \\ &= \left(1 + \frac{1}{|1 - a_m|^2} \sum_{n\neq m} |a_n|^2\right) \sum_{n\neq m} |[f, f_n]|^2 . \end{aligned}$$

Considering $M = 1 + \frac{1}{|1-a_m|^2} \sum\limits_{n\neq m} |a_n|^2$,

$$\frac{A}{M} \|f\|_J^2 \leq \sum_{n\neq m} |[f, f_n]|^2 \leq B \|f\|_J^2 .$$

The proof has been concluded. □

4. Characterization of Frames in Krein Spaces

Checking that the sequence $\{f_k\}_{k=1}^{\infty}$ confirms in (3.4) is not easy, particularly since the lower frame condition is the most difficult one to verify. The synthesis operator helps us to characterize frames through the subsequent theorem:

Theorem 4.1. *The sequence $\{f_k\}_{k=1}^{\infty}$ constitutes a frame for the Krein space K if and only if the operator $T : \pounds^2(\mathbb{N}) \to K$ is a well-defined operator that is both bounded and surjective.*

Proof. Assuming that $\{f_k\}_{k=1}^{\infty}$ serves as a frame for the Krein space K, then it constitutes a frame for the associated Hilbert space $(K, [.,.]_J)$ by Theorem 3.5. Using [10, Theorem 5.5.1], $T_0 : \big(\pounds^2(\mathbb{N}), [.,.]_{\tilde{J}}] \to (K, [.,.]_J)\big)$ is characterized as a bounded operator that is both well-defined and surjective and so is $T = (I_J)^{-1} T_0 I_{\tilde{J}}$. Conversely, suppose that T is a well-defined, bounded and surjective operator, so is $T_0 = I_J T (I_{\tilde{J}})^{-1}$. Again in the base of [10, Theorem 5.5.1], $\{f_k\}_{k=1}^{\infty}$ serves as a frame for the associated Hilbert space $(K, [.,.]_J)$ and so is for the Krein space K. □

In the following theorem the frames on Krein space K are characterized and a clear structure of frames is given:

Theorem 4.2. *Consider $\{e_k\}_{k=1}^{\infty}$ as a random J-orthonormal basis for the Krein space K such that $Je_k = \sigma_k e_k$. The frames for K are explicitly in the form of $\{Ue_k\}_{k=1}^{\infty}$ in which U represents a bounded operator that is surjective on K.*

Proof. Assume that $\{f_k\}_{k=1}^{\infty}$ serves as a frame for K, so constitutes a frame for the associated Hilbert space $(K, [.,.]_J)$ by Theorem 3.5. In the base of [11, Theorem 5.4.4], $\{f_k\}_{k=1}^{\infty} = \{Ue_k\}_{k=1}^{\infty}$ in which U is defined as a bounded operator that is surjective on K and $\{e_k\}_{k=1}^{\infty}$ constitutes an orthonormal basis for the associted Hilbert space $(K, [.,.]_J)$. Since Lemma 3.6 results that $\{e_k\}_{k=1}^{\infty}$ forms a J-orthonormal basis for K, the proof is obtained. Similarly, the opposite direction is satisfied. □

Via [11, Theorem 5.4.5], in Hilbert spaces, the presence of upper and lower frame bounds is equal to that of the upper and lower bounds of the synthesis operators of frames, while in the Krein spaces, the researchers could only prove one part. This implies that the upper and lower bounds reassured the existence of the upper and lower bounds associated with the synthesis operator while not confirming the opposite part since the generalized *-inverse of the synthesis operator is not defined in Krein spaces. It means that we cannot conclude from the behavior of the synthesis operator that the frame has proper bounds. The absence of a generalized *-inverse for the synthesis operator complicates matters because it implies that even having upper and lower bounds for the synthesis operator, does not ensure that the corresponding properties will hold as they do in Hilbert spaces.

Lemma 4.3. *Suppose that $\{f_k\}_{k=1}^{\infty}$ constitutes a frame for K with frame bounds A and B, then the subsequent two conditions hold true:*

(i) *The sequence $\{f_k\}_{k=1}^{\infty}$ forms a complete set within the space K.*
(ii) *The synthesis operator T is established as well-defined on $\pounds^2(\mathbb{N})$, where $\pounds^2(\mathbb{N})$ denotes the Krein space defined in the Examples*

2.1 and 3.4 and Definition 3.10

(4.1)
$$A\,\|\{c_k\}_{k=1}^{\infty}\|_{\tilde{J}}^2 \le \|T\{c_k\}_{k=1}^{\infty}\|_{\tilde{J}}^2 \le B\,\|\{c_k\}_{k=1}^{\infty}\|_{\tilde{J}}^2, \quad \{c_k\}_{k=1}^{\infty} \in N(T)^{[\perp]}.$$

Proof. If $\{f_k\}_{k=1}$ represents a Bessel sequence, then the right-side inequality in (4.1) is fulfilled. Now suppose that $\{f_k\}_{k=1}^{\infty}$ meets the lower frame condition with bound A, then (i) satisfies. Since $R\left(T^{\sharp}\right)$ is closed, so is $R(T)$ by Theorems 3.10, 2.11 and Proposition 2.6. So $N(T)^{[\perp]} = R\left(T^{\sharp}\right)$ by Proposition 2.10, i.e. $N(T)^{[\perp]}$ includes all sequences represented by $\tilde{J}\{[f, f_k]\}_{k=1}^{\infty}$, $f \in K$. For a particular $f \in K$

$$\begin{aligned}\left(\sum_{k=1}^{\infty} |[f, f_k]|^2\right)^2 &= |[Sf, f]|^2 \\ &= |\langle Sf, Jf\rangle|^2 \\ &\le \|Sf\|_J^2\,\|f\|_J^2 \\ &\le \|Sf\|_J^2\,\frac{1}{A}\sum_{k=1}^{\infty} |[f, f_k]|^2.\end{aligned}$$

This causes, for $\{c_k\}_{k=1}^{\infty} \in N(T)^{[\perp]}$

$$\begin{aligned}A\,\|\{c_k\}_{k=1}^{\infty}\|_{\tilde{J}}^2 &= A\left\|\tilde{J}\{[f, f_k]\}_{k=1}^{\infty}\right\|_{\tilde{J}}^2 \\ &= A\sum_{k=1}^{\infty} |[f, f_k]|^2 \\ &\le \|Sf\|_J^2 \\ &= \langle Sf, Sf, \rangle \\ &= \|T\{[f, f_k]\}_{k=1}^{\infty}\|_J^2.\end{aligned}$$

□

5. Riesz Bases in Krein Spaces

As shown in (3.13) a frame $\{f_k\}_{k=1}^{\infty}$ in a Krein space K possesses one of the key characteristics of a basis. All Riesz bases are bases, and this creates a natural framework to explore the connection between frames and Riesz bases. The findings in this section show that every Riesz basis is a frame and define the specific conditions that characterize frames that truly qualify as Riesz bases.

Definition 5.1. A Riesz basis associated with the Krein space K is a family of the type $\{Ue_k\}_{k=1}^{\infty}$ where $\{e_k\}_{k=1}^{\infty}$ serves as a J-orthonormal basis for the space K and U is a bijective operator in $B(K)$.

A frame that is not a Riesz base is called overcomplete. The subsequent theorem demonstrates the existence of a certain dependency among the frame elements when $\{f_k\}_{k=1}^{\infty}$ is overcomplete.

Theorem 5.2. *Consider $\{f_k\}_{k=1}^{\infty}$ as a frame for the Krein space K. Consequently, the statements listed below are equivalent:*

(i) *$\{f_k\}_{k=1}^{\infty}$ constitutes a Riesz basis for the Krein space K.*

(ii) *If $\sum_{k=1}^{\infty} c_k f_k = 0$ holds for $\{c_k\}_{k=1}^{\infty} \in \pounds^2(\mathbb{N})$ then it follows that $c_k = 0$ for every $k \in \mathbb{N}$.*

Proof. $(i) \to (ii)$ Let us consider that $\{f_k\}_{k=1}^{\infty}$ forms a Riesz basis for the krein space K, and that $\sum_{k=1}^{\infty} c_k f_k = 0$ holds for the sequence $\{c_k\}_{k=1}^{\infty} \in \pounds^2(\mathbb{N})$. Let $f_k = Ue_k$ for every $k \in \mathbb{N}$, where U is a bijective operator in $B(K)$ and $\{e_k\}_{k=1}^{\infty}$ represents a J-orthonormal basis for the krein space K. Then, if $\sum_{k=1}^{\infty} c_k f_k = 0$, therefore, $U\left(\sum_{k=1}^{\infty} c_k e_k\right) = 0$. Given that U is injective, it follows that $c_k = 0$ for every $k \in \mathbb{N}$.

$(ii) \to (i)$ The assumption guarantees that the synthesis operator T is both injective and surjective by Theorem 4.1. Since $f_k = T\delta_k$ for each $k \in \mathbb{N}$, in which $\{\delta_k\}_{k=1}^{\infty}$ is a canonical J-orthonormal basis for the Krein space $\pounds^2(\mathbb{N})$, $\{f_k\}_{k=1}^{\infty}$ constitutes a Riesz basis for K. □

Example 5.3. Consider $\left(\mathbb{C}^2, [.,.]\right)$ be a Krein space with the indefinite inner product

$$[(x_1, x_2), (y_1, y_2)] = x_1\overline{y_1} - x_2\overline{y_2}, \quad (x_1, x_2),\ (y_1, y_2) \in \mathbb{C}^2,$$

and the fundamental symmetry J as:

$$J(x_1, x_2) = (x_1, -x_2), \quad (x_1, x_2) \in \mathbb{C}^2.$$

Consider $e_1 = (1, 0)$, $e_2 = (0, 1)$ and $U = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}$. Since U is a bijective operator, and $\{Ue_1, Ue_2\}$ is a frame for $\mathbb{C}^2$, with upper and lower bounds, $B = 3 + 2\sqrt{2}$ and $A = 3 - 2\sqrt{2}$, respectively. Therefore, by Definition 5.1 and Theorem 5.2, $\{Ue_1, Ue_2\}$ is a Riesz basis for $\mathbb{C}^2$.

Definition 5.4. The sequence $\{f_k\}_{k=1}^{\infty}$ is considered minimal in the space K if for each $j \in \mathbb{N}$, $f_j \notin \overline{span}\{f_k\}_{k \neq j}$. An exact frame is defined as a frame that loses its status as a frame when any specific element is taken away.

Lemma 5.5. *Consider a sequence $\{f_k\}_{k=1}^{\infty}$ that belongs to the space K. Then the sequence $\{f_k\}_{k=1}^{\infty}$ possesses a unique J-biorthogonal sequence $\{g_k\}_{k=1}^{\infty}$ if and only if $\{f_k\}_{k=1}^{\infty}$ is both minimal and complete.*

Proof. Initially, let us assume that the sequence $\{f_k\}_{k=1}^{\infty}$ has a J-biorthogonal sequence represented as $\{g_k\}_{k=1}^{\infty}$, consequently, for every $j \in \mathbb{N}$, it holds that $f_j \notin \overline{span}\{f_k\}_{k\neq j}$. Because if it did, then $[f_j, g_j] = 0$ for each $j \in \mathbb{N}$, which is a contradiction. So $\{f_k\}_{k=1}^{\infty}$ is minimal. Now, suppose that the sequence $\{f_k\}_{k=1}^{\infty}$ has a unique J-biorthogonal sequence denoted as $\{g_k\}_{k=1}^{\infty}$. Consider $K_0 = \overline{span}\{f_k\}_{k=1}^{\infty}$. Then the Krein space K can be written as

$$K = \overline{K_0 + (K_0)^{[\perp]} \left[\dot{+}\right] J((K_0)^{\circ})}.$$

For some $h_k \in K_0^{[\perp]} - J(K_0^{\circ})$, substitute $\{g_k\}_{k=1}^{\infty}$ with $\{g_k + h_k\}_{k=1}^{\infty}$, thereby obtaining an alternative J-biorthogonal system for $\{f_k\}_{k=1}^{\infty}$.

Conversely, assume that the sequence $\{f_k\}_{k=1}^{\infty}$ is both minimal and complete. Consider $M_j = \overline{span}\{f_k\}_{k\neq j}$, then

$$K = \overline{M_j + M_j^{[\perp]}} \left[\dot{+}\right] J(M_j^{\circ}).$$

Choose $g_j \in M_j^{[\perp]}$, such a vector g_j obviously exists and

$$[f_j, g_j] = 1 \ \forall j \in \mathbb{N}, \quad [f_k, g_j] = 0, k \neq j.$$

So $\{g_k\}_{k=1}^{\infty}$ is J-biorthogonal to $\{f_k\}_{k=1}^{\infty}$. Now if the J-biorthogonal sequence is not unique, and $\{h_k\}$ is another such sequence, then for each $k \in \mathbb{N}$

$$[f_k, g_k - h_k] = 0,$$

which is a contradiction with completeness of $\{f_k\}_{k=1}^{\infty}$ in K. □

According to Lemma 5.5, the sequence $\{f_k\}_{k=1}^{\infty}$ is minimal if and only if it possesses a J-biorthogonal sequence. The following theorem presents several useful equivalent criteria for a frame to be minimal, a Riesz basis, or exact.

Theorem 5.6. *Consider the sequence $\{f_k\}_{k=1}^{\infty}$ as a frame for the space K. Then the subsequent statements are equivalent:*

(i) *The sequence $\{f_k\}_{k=1}^{\infty}$ constitutes a Riesz basis for K.*
(ii) *If the series $\sum_{k=1}^{\infty} c_k f_k = 0$ for some sequence $\{c_k\}_{k=1}^{\infty}$ belonging to $\pounds^2(\mathbb{N})$, then it follows that $c_k = 0$ for every $k \in \mathbb{N}$.*
(iii) *The sequence $\{f_k\}_{k=1}^{\infty}$ is an exact frame.*
(iv) *The sequences $\{f_k\}_{k=1}^{\infty}$ and $\left\{S^{-1}f_k\right\}_{k=1}^{\infty}$ are J-biorthogonal.*
(v) *The sequence $\{f_k\}_{k=1}^{\infty}$ possesses a J-biorthogonal sequence.*
(vi) *The sequence $\{f_k\}_{k=1}^{\infty}$ is minimal.*

(vii) *The sequence* $\{f_k\}_{k=1}^{\infty}$ *is a basis.*

Proof. $(i) \Leftrightarrow (ii)$. It is obtained by Theorem 5.2.

$(i) \to (iii)$. Assume that the sequence $\{f_k\}_{k=1}^{\infty}$ is a Riesz basis. If a random element is taken out, then the residual set is incomplete, thus, it is an exact frame.

$(iii) \to (iv)$. Let j be a fixed element in $\mathbb{N}$, the sequence $\{f_k\}_{k\neq j}$ does not constitute a frame for K. In the base of the proof of Lemma 3.17

$$a_j := \left[f_j, S^{-1}f_j\right] = 1.$$

Also, similar to the demonstration provided for Lemma 3.17, the condition $a_j = 1$ is enough to infer that

$$a_k = \left[S^{-1}f_j, f_k\right] = 0, \quad j \neq k.$$

So the sequences $\{f_k\}_{k=1}^{\infty}$ and $\left\{S^{-1}f_k\right\}_{k=1}^{\infty}$ are J-biorthogonal.

$(iv) \to (v)$. It is evident.

$(v) \to (vi)$. It is demonstrated in Lemma 5.5.

$(vi) \to (iii)$. Fix $j \in \mathbb{N}$, then the family $\{f_k\}_{k\neq j}$ is not complete in K, and hence $\{f_k\}_{k\neq j}$ does not constitute a frame for K.

$(iv) \to (vii)$. Based on frame decomposition, for every $f \in K$, $f = \sum_{k=1}^{\infty} \left[f, S^{-1}f_k\right] f_k$. To demonstrate that $\{f_k\}_{k=1}^{\infty}$ forms a basis, it suffices to prove that the above decomposition is unique. Suppose that f can be expressed as $f = \sum_{k=1}^{\infty} b_k f_k$ for certain coefficients b_k, consequently,

$$\begin{aligned}\left[f, S^{-1}f_k\right] &= \left[\sum_{j=1}^{\infty} b_j f_j, S^{-1}f_k\right] \\ &= \sum_{j=1}^{\infty} b_j \left[f_j, S^{-1}f_k\right] \\ &= b_k.\end{aligned}$$

$(vii) \to (ii)$. It is evident. □

6. Dual Frames

A key result in the theory of frames is the "Frame Decomposition Theorem", which describes how a vector in a Krein space K can be expressed as a linear combination of the elements of the frame. In this section, all dual frames are characterized and an explicit construction is proposed for general duals.

Two Bessel sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ serve as dual frames within the Krein space K if

$$f = \sum_{k=1}^{\infty} [f, g_k] f_k, \quad f \in K. \tag{6.1}$$

Taking T and U to be the synthesis operators for $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$, respectively, (6.1) means that $T\tilde{J}U^{\sharp} = I$.

The following lemmas demonstrate that the role of $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ can be swapped, and the lower frame condition applies to the Bessel sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ if condition (6.1) is satisfied.

Example 6.1. Consider $\left(\mathbb{C}^2, [.,.]\right)$ be a Krein space as defined in Example 5.3. Then $\{e_1, e_2\}$ and $\{e_1, -e_2\}$ are dual frames in $\mathbb{C}^2$.

Lemma 6.2. *Consider $\{e_k\}_{k=1}^{\infty}$ to be a J-orthonormal basis for the space K such that $Je_k = \sigma_k e_k$ for each $k \in \mathbb{N}$. Let us suppose that $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are Bessel sequences within the space K. Consequently, the subsequent conditions are equal:*

(i) $f = \sum_{k=1}^{\infty} [f, g_k] f_k, \quad f \in K$

(ii) $f = \sum_{k=1}^{\infty} [f, f_k] g_k, \quad f \in K.$

(iii) $[f, g] = \sum_{k=1}^{\infty} [f, f_k][g_k, g], \quad f, g \in K.$

(iv) $\|f\|_J^2 = [f, Jf] = \sum_{k=1}^{\infty} [f, f_k][Jg_k, f], \quad f \in K.$

(v) *For each J-orthonormal basis $\{e_i\}_{i=1}^{\infty}$ for K*

$$\sum_{k=1}^{\infty} [e_i, f_k][g_k, e_j] = [e_i, e_j].$$

Proof. Suppose that T and U serve as the synthesis operators for the sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$, respectively.

- $(i) \Leftrightarrow (ii)$. (i) means that $T\tilde{J}U^{\sharp} = I$, by Proposition 2.7, this is the same as stating that $U\tilde{J}T^{\sharp} = I$, which corresponds to (ii). Similarly (ii) is equivalent to (i).

- $(ii) \to (iii)$. For each $f, g \in K$,

$$[f, g] = \left[\sum_{k=1}^{\infty} [f, f_k] g_k, g\right] = \sum_{k=1}^{\infty} [f, f_k][g_k, g].$$

$(iii) \to (ii)$. Let f be a fixed element in K, $\sum_{k=1}^{\infty}[f,f_k]g_k$ converges in the space K, since the sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are Bessel. At this point (iii) implies that

$$\left[f-\sum_{k=1}^{\infty}[f,f_k]g_k,g\right]=0,\quad g\in K.$$

Therefore,

$$\left\langle f-\sum_{k=1}^{\infty}[f,f_k]g_k,g\right\rangle=0,\quad g\in K.$$

So (ii) follows.

$(iii) \to (iv)$. It is evident.

$(iv) \to (ii)$. For each $f \in K$

$$\langle f,f\rangle=[f,Jf]=\sum_{k=1}^{\infty}[f,f_k][Jg_k,f]=\left\langle U\tilde{J}T^{\sharp}f,f\right\rangle,$$

So, $f=U\tilde{J}T^{\sharp}f$.

$(iii) \to (v)$. It is evident.

$(v) \to (iii)$. For each $f,g\in K$

$$\begin{aligned}
&\sum_{k=1}^{\infty}[f,f_k][g_k,g]\\
&\quad=\langle\{[f,f_k]\}_{k=1}^{\infty},\{[g,g_k]\}_{k=1}^{\infty}\rangle\\
&\quad=\left\langle\left\{\left[f,\sum_{i=1}^{\infty}\sigma_i[f_k,e_i]e_i\right]\right\}_{k=1}^{\infty},\left\{\left[g,\sum_{j=1}^{\infty}\sigma_j[g_k,e_j]e_j\right]\right\}_{k=1}^{\infty}\right\rangle\\
&\quad=\left\langle\left\{\sum_{i=1}^{\infty}[f,\sigma_i[f_k,e_i]e_i]\right\}_{k=1}^{\infty},\left\{\sum_{j=1}^{\infty}[g,\sigma j[g_k,e_j]e_j]\right\}_{k=1}^{\infty}\right\rangle\\
&\quad=\sum_{i,j=1}^{\infty}\langle\{[f,\sigma_i[f_k,e_i]e_i]\}_{k=1}^{\infty},\{[g,\sigma_j[g_k,e_j]e_j]\}_{k=1}^{\infty}\rangle\\
&\quad=\sum_{i,j=1}^{\infty}\sum_{k=1}^{\infty}[f,\sigma_i[f_k,e_i]e_i]\,[\sigma_j[g_k,e_j]e_j,g]\\
&\quad=\sum_{i,j=1}^{\infty}\sigma_i\sigma_j[f,e_i][e_j,g][e_i,e_j]
\end{aligned}$$

$$= \sum_{i=1}^{\infty} \sigma_i [f, e_i][e_i, g]$$
$$= [f, g]. \qquad \square$$

Definition 6.3. Persume that the sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are Bessel sequences within the space K. Then the pair $(\{f_k\}_{k=1}^{\infty}, \{g_k\}_{k=1}^{\infty})$ forms a dual pair if at least one of the statements in Lemma 6.2 holds true.

Lemma 6.4. *Suppose that the pair $(\{f_k\}_{k=1}^{\infty}, \{g_k\}_{k=1}^{\infty})$ constitutes a dual pair and $Je_k = \sigma_k e_k$ for each $k \in \mathbb{N}$, then the sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are frames.*

Proof. Assume that the sequence $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence in the krein space K with bound B. For each $f \in K$

$$\begin{aligned} \|f\|_J^2 &= \left| \sum_{k=1}^{\infty} [g_k, f][Jf, f_k] \right| \\ &\leq \left(\sum_{k=1}^{\infty} |[f, g_k]|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} |[f_k, Jf]|^2 \right)^{\frac{1}{2}} \\ &\leq \sqrt{B} \, \|f\|_J \left(\sum_{k=1}^{\infty} |[f, g_k]|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

So

$$\frac{\|f\|_J^2}{B} \leq \sum_{k=1}^{\infty} |[f, g_k]|^2. \qquad \square$$

Similary it can proved that $\{f_k\}_{k=1}^{\infty}$ serves as a frame for K.

Corollary 6.5. *Consider $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are Bessel sequences for the space K and $Je_k = \sigma_k e_k$ for each $k \in \mathbb{N}$. Then the subsequent statements are equal:*

(i) *$(\{f_k\}_{k=1}^{\infty}, \{g_k\}_{k=1}^{\infty})$ is a dual pair for K.*
(ii) *$(\{f_k\}_{k=1}^{\infty}, \{Jg_k\}_{k=1}^{\infty})$ forms a dual pair for the associated Hilbert space $(K, [.,.]_J)$.*
(iii) *$(\{Jf_k\}_{k=1}^{\infty}, \{g_k\}_{k=1}^{\infty})$ constitutes a dual pair for the associated Hilbert space $(K, [.,.]_J)$.*

Based on characterization of duals of frames in Hilbert spaces, the following lemma gives an explicit structure for duals of frames in Krein spaces:

Lemma 6.6. *Consider $\{f_k\}_{k=1}^{\infty}$ as a frame for the space K. The dual frames for $\{f_k\}_{k=1}^{\infty}$ are exactly the families $\{g_k\} = \{JV\delta_k\}_{k=1}^{\infty}$, such that $V : \left(\pounds^2(\mathbb{N}), [.,.]_{\tilde{J}}\right) \to (K, [.,.]_J)$ is a bounded left inverse of T_0^* and $\{\delta_k\}_{k=1}^{\infty}$ represents a canonical orthonormal basis for $\left(\pounds^2(\mathbb{N}), [.,.]_{\tilde{J}}\right)$.*

Proof. The Proof is obtained by Corollary 6.5 and [11, Lemma 5.7.2]. □

Corollary 6.7. *Allow $\{f_k\}_{k=1}^{\infty}$ represent a frame for the space K. The dual frames of $\{f_k\}_{k=1}^{\infty}$ are exactly the families*

$$\{g_k\}_{k=1}^{\infty} = \left\{ S^{-1}f_k + Jh_k - \sum_{j=1}^{\infty} \left[S^{-1}f_k, f_j\right] Jh_j \right\}_{k=1}^{\infty}$$

in provided that $\{h_k\}_{k=1}^{\infty}$ constitutes a Bessel sequence for K.

Proof. The proof is obtained from Lemma 6.6 and [11, Lemma 5.7.3]. □

7. CONCLUSION

Inspired by the definition of Esmeral, Ferrer, and Wanger, similar outcomes on frames in Hilbert spaces can be extended to Krein spaces.

Initially, it is shown that under the conditions where $U \in B(K)$ is a regular operator, both $\overline{N(U)}$ and $\overline{R(U)}$ are non-degenerate, and it follows that $D(U) \subseteq N(U) \left[\dot{+}\right] N(U)^{[\perp]}$, consequently, $\{Uf_k\}_{k=1}^{\infty}$ is also a frame when $\{f_k\}_{k=1}^{\infty}$ is a frame. Next, the importance of the value of $a_j = \left[f_j, S^{-1}f_j\right]$ is recognized to conclude that $\{f_k\}_{k=1}^{\infty}$ constitutes a frame or incomplete set. Subsequently, a clear structure for frames and Riesz bases in Krein spaces are presented, and conditions that are equivalent for a frame to be minimal and to be Riesz basis, or an exact frame are established, accordingly. Eventually, the dual frames of $\{f_k\}_{k=1}^{\infty}$ are characterized, and based on the obtained result, it is shown that the dual frames of $\{f_k\}_{k=1}^{\infty}$ are exactly the families $\{JV\delta_k\}_{k=1}^{\infty}$ in such a way that $V : (\pounds^2(\mathbb{N}, [.,.]_{\tilde{J}}) \to (K, [.,.]_J)$ is a bounded left inverse of T_0^* and J is a fundamental symmetry in K. Finally, we emphasize that, since the spaces $(K^+, [.,.])$ and $(K^-, -[.,.])$ defined by the norms $\|x\| = [x,x]^{\frac{1}{2}} (x \in K^+)$, $\|x\| = -[x,x]^{\frac{1}{2}} (x \in K^-)$ are Hilbert spaces, all theorems and results previously obtained for frames in Hilbert spaces are also true for these spaces as Hilbert spaces and can be concluded.

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