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Exploring the Inclusion Properties of Integral Operators Related to the Pascal Distribution Series in Certain Subclasses of Univalent Functions

Manas Kumar Giri[1] and Raghavendar Kondooru[2*]

Abstract. This study investigates the geometric characteristics of integral operators connected with the Pascal distribution series. The objective of this article is to establish conditions on the parameters of these operators by utilizing coefficient bounds from class $\mathcal{R}^{\tau}_{\wp,v}(\gamma)$. Also, the study analyzes the inclusion outcomes of integral operators related to the Pascal distribution series within various subclasses of analytic functions. The findings presented here incorporated previously published results as specific instances.

1. Introduction

In recent times, there has been increasing interest in using special functions. Researchers have developed criteria for the parameters of these special functions using various methodologies, thus classifying them into different subclasses of normalized analytic functions. Let us consider a function $h(z)$ defined by

$$h(z) = z + \sum_{j=2}^{\infty} x_j z^j. \tag{1.1}$$

Let $\mathcal{A}$ be the class of analytic functions within the open unit disk $\mathbb{D}$ ($\mathbb{D} = \{z : |z| < 1\}$), given by equation (1.1). Additionally, the classes $\mathcal{S}$, $\mathcal{S}^*$, and $\mathcal{K}$ (see [4]) represent the categories of univalent, starlike, and

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convex functions, respectively defined as:

$$\mathcal{S} := \{h \in \mathcal{A} : h \text{ is univalent in } \mathbb{D}\}.$$

$$\mathcal{S}^* := \{h \in \mathcal{S} : h \text{ is starlike in } \mathbb{D}\}.$$
$$\mathcal{K} := \{h \in \mathcal{S} : h \text{ is convex in } \mathbb{D}\}.$$

Let us establish some fundamental definitions to frame our work. A function $h \in \mathcal{A}$ belongs to $k-\mathcal{UCV}(\sigma)$ (see [33]), the class of $k-$uniformly convex of order σ, if and only if

$$\Re\left(1 + \frac{zh''}{h'}\right) \geq k\left|\frac{zh''}{h'}\right| + \sigma, \quad \forall z \in \mathbb{D}$$

where $0 \leq k < \infty$ and $0 \leq \sigma < 1$. A corresponding class $k - \mathcal{S}_p(\sigma)$ is derived using the Alexander transform, where $g \in k - \mathcal{UCV}(\sigma)$ if and only if $zg' \in k - \mathcal{S}_p(\sigma)$.

Note that, for $\sigma = 0$ (see [8–11, 27])

$$k - \mathcal{UCV}(\sigma) \equiv k - \mathcal{UCV}$$
$$k - \mathcal{S}_p(\sigma) \equiv k - \mathcal{ST}$$

and for $k = 1$

$$k - \mathcal{UCV}(\sigma) \equiv \mathcal{UCV}(\sigma).$$

Consider $h \in \mathcal{A}$ with $0 < \alpha < \infty$. Then $h(z) \in \mathcal{CP}(\alpha)$ (see [2]), if and only if

$$\Re\left(1 + \frac{zh''}{h'}\right) + \alpha \geq \left|\frac{zh''}{h'} + 1 - \alpha\right|, \quad \forall z \in \mathbb{D}.$$

For $\lambda > 0$, the classes $\mathcal{S}^*_\lambda$ and $\mathcal{K}_\lambda$ are defined as follows: [6]

$$\mathcal{S}^*_\lambda = \left\{h \in \mathcal{A} : \left|\frac{zh'}{h} - 1\right| < \lambda, \quad \forall z \in \mathbb{D}\right\}.$$
$$\mathcal{K}_\lambda = \left\{h \in \mathcal{A} : \left|\frac{zh''}{h'}\right| < \lambda, \quad \forall z \in \mathbb{D}\right\}.$$

A function $h \in \mathcal{S}$ is spiral-like (refer to [29]) if

$$\Re\left(e^{-i\vartheta}\frac{zh'}{h}\right) > 0$$

where $|\vartheta| < \frac{\pi}{2}$ and this holds for all $z \in \mathbb{D}$.

Definition 1.1 ([16, 17])**.** For $0 \leq \nu < 1$ and $0 \leq \varrho < 1$, the classes $\mathcal{S}(\vartheta, \nu, \varrho)$ and $\mathcal{K}(\vartheta, \nu, \varrho)$ are defined accordingly:

$$\mathcal{S}(\vartheta, \nu, \varrho) := \left\{h \in \mathcal{S} : \Re\left(e^{i\vartheta}\frac{zh'}{(1-\varrho)h + \varrho zh'}\right) > \nu\cos\vartheta, \quad \forall z \in \mathbb{D}\right\}.$$

$$\mathcal{K}(\vartheta,\nu,\varrho) := \left\{ h \in \mathcal{S} : \Re\left(e^{i\vartheta} \frac{zh'' + h'}{h' + \varrho z h''} \right) > \nu \cos\vartheta, \quad \forall z \in \mathbb{D} \right\}.$$

For $\varrho = 0$, (see [13])

$$\mathcal{S}(\vartheta,\nu,\varrho) \equiv \mathcal{SP}(\vartheta,\nu),$$
$$\mathcal{K}(\vartheta,\nu,\varrho) \equiv \mathcal{KP}(\vartheta,\nu).$$

Definition 1.2 ([28])**.** For $0 \leq \delta < 1$ and $\sigma \leq 1$, the classes $\mathcal{UCSP}(\sigma,\delta)$ and $\mathcal{SP}_p(\sigma,\delta)$ are defined in the following manner:

$$\mathcal{UCSP}(\sigma,\delta) = \left\{ h \in \mathcal{A} : \Re\left\{ e^{-i\sigma} \left(1 + \frac{zh''}{h'} \right) \right\} \geq \left| \frac{zh''}{h'} \right| + \delta, \quad \forall z \in \mathbb{D} \right\}$$

and

$$\mathcal{SP}_p(\sigma,\delta) = \left\{ h \in \mathcal{A} : \Re\left\{ e^{-i\sigma} \frac{zh'}{h} \right\} \geq \left| \frac{zh'}{h} - 1 \right| + \delta, \quad \forall z \in \mathbb{D} \right\}.$$

For $\delta = 0$, (see [26])

$$\mathcal{UCSP}(\sigma,\delta) \equiv \mathcal{UCSP}(\sigma),$$
$$\mathcal{SP}_p(\sigma,\delta) \equiv \mathcal{SP}_p(\sigma).$$

A function $h \in \mathcal{A}$, with $0 \leq \lambda < 1$ and $0 \leq \beta < 1$, belongs to $\mathcal{S}(\lambda,\beta)$ (see [32]), if and only if

$$\Re\left(\frac{zh' + \lambda z^2 h''}{h} \right) > \beta, \quad \forall z \in \mathbb{D}.$$

Definition 1.3 ([33])**.** If $h \in \mathcal{A}$ and is a member of $\mathcal{R}^{\tau}_{\wp,\upsilon}(\gamma)$, where $\tau \in \mathbb{C} \setminus \{0\}$, $0 \leq \wp < 1$, $0 \leq \upsilon \leq 1$ and $\gamma < 1$, then it meets the condition expressed by the following inequality:

$$\left| \frac{(1-\upsilon+2\wp)\frac{h}{z} + (\upsilon - 2\wp)h' + \wp z h'' - 1}{2\tau(1-\gamma) + (1-\upsilon+2\wp)\frac{h}{z} + (\upsilon-2\wp)h' + \wp z h'' - 1} \right| < 1, \quad \forall z \in \mathbb{D}.$$

The boundary for the coefficient of the function $h(z)$ belonging to the class $\mathcal{R}^{\tau}_{\wp,\upsilon}(\gamma)$ is determined by:

$$|x_j| \leq \frac{2|\tau|(1-\gamma)}{1 + (j-1)(\upsilon - 2\wp + j\wp)}, \quad j = 2,3,\ldots. \tag{1.2}$$

Since $1+(j-1)(\upsilon-2\wp+j\wp) = (1-\upsilon)+\wp(2+j^2)+j(\upsilon-3\wp) \geq j(\upsilon-3\wp)$, the inequality (1.2) can be rewritten as (see [33])

$$|x_j| \leq \frac{2|\tau|(1-\gamma)}{j(\upsilon - 3\wp)}, \quad j = 2,3,\ldots. \tag{1.3}$$

Porwal, in [23], introduced the Poisson distribution series. He showed how it could be excitingly used with univalent functions. Later on,

many researchers introduced series such as the hypergeometric distribution (refer to [20]), confluent hypergeometric distribution (refer to [21]), Binomial distribution (refer to [18]), Pascal distribution (refer to [5, 16]), and generalized distribution (refer to [19]). They also established conditions necessary and sufficient for specific classes of univalent functions. Recently, El-Deeb et al. (see [5]) introduced a power series in which the coefficients correspond to probabilities derived from the Pascal distribution. They analyzed how this series relates to geometric characteristics in univalent functions. Now, let us remember the definition of the Pascal distribution, which is given by:

A random variable y follows a Pascal distribution if it assumes values $0, 1, 2, 3, \ldots$ with corresponding probabilities

$$(1-\mu)^b, \frac{\mu b(1-\mu)^b}{1!}, \frac{\mu^2 b(b+1)(1-\mu)^b}{2!}, \frac{\mu^3 b(b+1)(b+2)(1-\mu)^b}{3!}, \ldots$$

respectively, here μ and b are termed as the parameters.

In [5], the Pascal distribution series defined as

$$\mathcal{P}_{\mu}^{b}(z) = z + \sum_{j=2}^{\infty} \binom{j+b-2}{r-1} \mu^{(j-1)}(1-\mu)^b z^j, \quad \forall z \in \mathbb{D},$$

where $b \geq 1$ and $0 \leq \mu \leq 1$.

If $f, g \in \mathcal{A}$, then the Hadamard product of f and g is defined as:

$$(f * g)(z) = z + \sum_{j=2}^{\infty} x_j y_j z^j,$$

where $f(z) = z + \sum_{j=2}^{\infty} x_j z^j$ and $g(z) = z + \sum_{j=2}^{\infty} y_j z^j$.

The linear operator $\mathcal{P}_{\mu}^{b}(h, z)$ is defined as (see [5])

$$\begin{aligned} \mathcal{P}_{\mu}^{b}(h, z) &= \mathcal{P}_{\mu}^{b}(z) * h(z) \\ &= z + \sum_{j=2}^{\infty} \binom{j+b-2}{b-1} \mu^{(j-1)}(1-\mu)^b x_j z^j, \end{aligned}$$

where $*$ is the Hadamard product. Equivalently we can write as

$$\mathcal{P}_{\mu}^{b}(h, z) = z + \sum_{j=2}^{\infty} \frac{(b)_{j-1}}{(1)_{j-1}} \mu^{(j-1)}(1-\mu)^b x_j z^j,$$

where $b \geq 1$, $0 \leq \mu \leq 1$ and

$$X_j = \frac{(b)_{j-1}}{(1)_{j-1}} \mu^{(j-1)}(1-\mu)^b x_j. \tag{1.4}$$

Drawing inspiration from findings on the interconnections among various subclasses of analytic univalent functions by the use of hypergeometric functions (see [7, 12, 24, 30, 31]), confluent hypergeometric functions (refer to [3, 14, 15]), Wright's hypergeometric functions (see [25]) and generalized Bessel functions (see [1, 15, 22]), we establish specific inclusion relation between $\mathcal{R}^{\tau}_{\wp,\upsilon}(\gamma)$ and various categories of analytic univalent functions employing integral operators $\mathcal{P}^{b}_{\mu}(h,z)$ and $\mathcal{T}^{b}_{\mu}(h,z)$.

Now, we outline the conditions that are necessary and sufficient for the function $h(z) \in \mathcal{A}$ within the above classes, which are relevant to our present study.

2. Preliminary Results

Lemma 2.1 ([17]). *Let $h \in \mathcal{A}$ and belongs to $\mathcal{S}(\vartheta,\nu,\varrho)$ if the following inequality holds true:*

$$\sum_{j=2}^{\infty}\left[((1-\varrho)\sec\vartheta+\varrho(1-\nu))j+(1-\varrho)(1-\nu-\sec\vartheta)\right]|x_j|\leq 1-\nu,$$

under the conditions that $|\vartheta|<\frac{\pi}{2}$, $0\leq\nu<1$ and $0\leq\varrho<1$.

Lemma 2.2 ([16]). *Let $h \in \mathcal{A}$ and belongs to $\mathcal{K}(\vartheta,\nu,\varrho)$ if the following inequality holds true:*

$$\sum_{j=2}^{\infty} j\left[((1-\varrho)\sec\vartheta+\varrho(1-\nu))j+(1-\varrho)(1-\nu-\sec\vartheta)\right]|x_j|\leq 1-\nu,$$

under the conditions that $|\vartheta|<\frac{\pi}{2}$, $0\leq\nu<1$ and $0\leq\varrho<1$.

Lemma 2.3 ([28]). *Let $h \in \mathcal{A}$ and belongs to $\mathcal{SP}_p(\sigma,\delta)$ if the following inequality holds true:*

$$\sum_{j=2}^{\infty}(2j-\cos\sigma-\delta)\,|x_j|\leq\cos\sigma-\delta,$$

under the conditions that $\sigma\leq 1$, $0\leq\delta<1$.

Lemma 2.4 ([28]). *Let $h \in \mathcal{A}$ and belongs to $\mathcal{UCSP}(\sigma,\delta)$ if the following inequality holds true:*

$$\sum_{j=2}^{\infty} j\,(2j-\cos\sigma-\delta)\,|x_j|\leq\cos\sigma-\delta,$$

under the conditions that $\sigma\leq 1$, $0\leq\delta<1$.

Lemma 2.5 ([33]). *Let $h \in \mathcal{A}$ and belongs to $k - \mathcal{S}_p(\sigma)$ if the following inequality holds true:*

$$\sum_{j=2}^{\infty} (j(1+k) - (k+\sigma)) |x_j| \leq 1 - \sigma,$$

under the conditions that $0 \leq \sigma < 1$ and $0 \leq k < \infty$.

Lemma 2.6 ([33]). *Let $h \in \mathcal{A}$ and belongs to $k - \mathcal{UCV}(\sigma)$ if the following inequality holds true:*

$$\sum_{j=2}^{\infty} j\,(j(1+k) - (k+\sigma)) |x_j| \leq 1 - \sigma,$$

under the conditions that $0 \leq \sigma < 1$ and $0 \leq k < \infty$.

Lemma 2.7 ([2]). *Let $h \in \mathcal{A}$ and belongs to $\mathcal{CP}(\alpha)$ if the following inequality holds true:*

$$\sum_{j=2}^{\infty} (j + 2(\alpha - 1))\, j|x_j| \leq 2\alpha - 1,$$

under the conditions that $0 < \alpha < \infty$.

Lemma 2.8 ([6]). *Let $h \in \mathcal{A}$ and belongs to $\mathcal{S}^*_\lambda$ if the following inequality holds true:*

$$\sum_{j=2}^{\infty} (\lambda + j - 1)|x_j| \leq \lambda,$$

under the condition that $\lambda > 0$.

Lemma 2.9 ([6]). *Let $h \in \mathcal{A}$ and belongs to $\mathcal{K}_\lambda$ if the following inequality holds true:*

$$\sum_{j=2}^{\infty} j(\lambda + j - 1)|x_j| \leq \lambda,$$

under the condition that $\lambda > 0$.

Lemma 2.10 ([32]). *Let $h \in \mathcal{A}$ and belongs to $\mathcal{S}(\lambda, \beta)$ if the following inequality holds true:*

$$\sum_{j=2}^{\infty} (j + \lambda j(j-1) - \beta) |x_j| \leq 1 - \beta,$$

under the conditions that $0 \leq \lambda < 1$ and $0 \leq \beta < 1$.

In the next section, we provide the main results on the inclusion properties of certain operators, establishing conditions for which they will be in various subclasses of univalent functions. The following theorem specifies the key conditions under which the function $h \in \mathcal{R}^{\tau}_{\wp,\upsilon}(\gamma)$.

3. Main Results

In the following theorem, we derive the condition for the operator $\mathcal{P}_{\mu}^{b}(h,z)$ to be in the class $\mathcal{S}(\vartheta,\nu,\varrho)$.

Theorem 3.1. *Let h belong to $\mathcal{A}$ as specified by form (1.1). Assuming $h(z) \in \mathcal{R}_{\wp,\upsilon}^{\tau}(\gamma)$ and satisfies the following inequality, let*

$$\begin{aligned}&((1-\varrho)\sec\vartheta+\varrho(1-\nu))\left(1-(1-\mu)^{b}\right)\\&\quad+\frac{(1-\varrho)(1-\nu-\sec\vartheta)}{\mu(b-1)}\left(1-\mu-(1+\mu(b-1))(1-\mu)^{b}\right)\\&\leq\frac{(1-\nu)(\upsilon-3\wp)}{2|\tau|(1-\gamma)},\end{aligned}$$

where $|\vartheta|<\frac{\pi}{2}$, $0\leq\nu<1$ and $0\leq\varrho<1$. Then, for $b>0$ and $0\leq\mu\leq1$, $\mathcal{P}_{\mu}^{b}(h,z)\in\mathcal{S}(\vartheta,\nu,\varrho)$.

Proof. The condition that is enough to ensure that $h\in\mathcal{S}(\vartheta,\nu,\varrho)$ as stated in Lemma 2.1 is:

(3.1)
$$\sum_{j=2}^{\infty}\left[((1-\varrho)\sec\vartheta+\varrho(1-\nu))j+(1-\varrho)(1-\nu-\sec\vartheta)\right]|X_j|\leq1-\nu.$$

By substituting (1.4) into equation (3.1), we obtain

$$\begin{aligned}\sum_{j=2}^{\infty}&\big[((1-\varrho)\sec\vartheta+\varrho(1-\nu))j\\&+(1-\varrho)(1-\nu-\sec\vartheta)\big]\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}(1-\mu)^{b}|x_j|\\&\leq1-\nu.\end{aligned}\tag{3.2}$$

Applying (1.3) to the left side of equation (3.2), we proceed with

(3.3)
$$\begin{aligned}&\sum_{j=2}^{\infty}\big[((1-\varrho)\sec\vartheta+\varrho(1-\nu))j\\&\quad+(1-\varrho)(1-\nu-\sec\vartheta)\big]\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}(1-\mu)^{b}|x_j|\\&\leq\sum_{j=2}^{\infty}\big[(1-\varrho)\sec\vartheta+\varrho(1-\nu)\big]j\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}(1-\mu)^{b}\frac{2|\tau|(1-\gamma)}{j(\upsilon-3\wp)}\\&\quad+\sum_{j=2}^{\infty}(1-\varrho)(1-\nu-\sec\vartheta)\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}(1-\mu)^{b}\frac{2|\tau|(1-\gamma)}{j(\upsilon-3\wp)}\end{aligned}$$

$$= \frac{2|\tau|(1-\gamma)}{(\upsilon - 3\wp)}(1-\mu)^b \Bigg(((1-\varrho)\sec\vartheta + \varrho(1-\nu)) \sum_{j=2}^{\infty} \frac{(b)_{j-1}}{(1)_{j-1}} \mu^{(j-1)}$$

$$+ (1-\varrho)(1-\nu-\sec\vartheta) \sum_{j=2}^{\infty} \frac{(b)_{j-1}}{(1)_j} \mu^{(j-1)} \Bigg)$$

$$= \frac{2|\tau|(1-\gamma)}{(\upsilon - 3\wp)}(1-\mu)^b \Bigg(((1-\varrho)\sec\vartheta + \varrho(1-\nu)) \left((1-\mu)^{-b} - 1 \right)$$

$$+ (1-\varrho)(1-\nu-\sec\vartheta) \frac{1}{\mu(b-1)} \left((1-\mu)^{-(b-1)} - 1 - \mu(b-1) \right) \Bigg)$$

$$= \frac{2|\tau|(1-\gamma)}{(\upsilon - 3\wp)} \Bigg(((1-\varrho)\sec\vartheta + \varrho(1-\nu)) \left(1 - (1-\mu)^b \right)$$

$$+ (1-\varrho)(1-\nu-\sec\vartheta) \frac{(1-\mu) - (1+\mu(b-1))(1-\mu)^b}{\mu(b-1)} \Bigg).$$

Using (3.3) in equation (3.1), we obtain

$$\begin{aligned} (3.4) \quad & ((1-\varrho)\sec\vartheta + \varrho(1-\nu)) \left(1 - (1-\mu)^b \right) \\ & + \frac{(1-\varrho)(1-\nu-\sec\vartheta)}{\mu(b-1)} \left((1-\mu) - (1+\mu(b-1))(1-\mu)^b \right) \\ & \leq \frac{(1-\nu)(\upsilon - 3\wp)}{2|\tau|(1-\gamma)}. \end{aligned}$$

Utilizing the hypothesis of the theorem along with equation (3.4) will complete the proof. □

Remark 3.2. Note that when $\wp = 0$ in Theorem 3.1, it is equivalent to Theorem 3.1 of [16].

In the following theorem, we derive the condition for the operator $\mathcal{P}_{\mu}^{b}(h,z)$ to be in the class $\mathcal{K}(\vartheta,\nu,\varrho)$.

Theorem 3.3. *Let h belong to $\mathcal{A}$ as specified by form* (1.1)*. Assuming $h(z) \in \mathcal{R}_{\wp,\upsilon}^{\tau}(\gamma)$ and satisfies the following inequality, let*

$$\begin{aligned} & ((1-\varrho)\sec\vartheta + \varrho(1-\nu))\, b\mu(1-\mu)^{-1} \\ & \leq (1-\nu) \left((1-\mu)^b - 1 + \frac{(\upsilon - 3\wp)}{2|\tau|(1-\gamma)} \right), \end{aligned}$$

where $|\vartheta| < \frac{\pi}{2}$, $0 \leq \nu < 1$ and $0 \leq \varrho < 1$. Then, for $b > 0$ and $0 \leq t \leq 1$, $\mathcal{P}_{\mu}^{b}(h,z) \in \mathcal{K}(\vartheta,\nu,\varrho)$.

Proof. The condition that is enough to ensure that $h \in \mathcal{K}(\vartheta, \nu, \varrho)$ as stated in Lemma 2.2 is:

(3.5)
$$\sum_{j=2}^{\infty} j\left[((1-\varrho)\sec\vartheta + \varrho(1-\nu))j + (1-\varrho)(1-\nu-\sec\vartheta)\right]|X_j| \leq 1-\nu.$$

By substituting (1.4) into equation (3.5), we obtain

$$\begin{aligned}(3.6)\qquad &\sum_{j=2}^{\infty} j\big[((1-\varrho)\sec\vartheta + \varrho(1-\nu))j \\ &\quad + (1-\varrho)(1-\nu-\sec\vartheta)\big]\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}(1-\mu)^b|x_j| \\ &\quad \leq 1-\nu.\end{aligned}$$

Applying (1.3) to the left side of equation (3.6), we proceed with

$$\begin{aligned}&\sum_{j=2}^{\infty} j\big[((1-\varrho)\sec\vartheta + \varrho(1-\nu))j \\ &\quad + (1-\varrho)(1-\nu-\sec\vartheta)\big]\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}(1-\mu)^b|x_j| \\ &\leq \sum_{j=2}^{\infty} j\big[((1-\varrho)\sec\vartheta + \varrho(1-\nu))j \\ &\quad + (1-\varrho)(1-\nu-\sec\vartheta)\big]\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}(1-\mu)^b\frac{2|\tau|(1-\gamma)}{j(\upsilon-3\wp)}.\end{aligned}$$

Let $((1-\varrho)\sec\vartheta + \varrho(1-\nu))j + (1-\varrho)(1-\nu-\sec\vartheta) = ((1-\varrho)\sec\vartheta + \varrho(1-\nu))(j-1) + 1 - \nu$. Then

(3.7)
$$\begin{aligned}&\frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)}(1-\mu)^b\Bigg(((1-\varrho)\sec\vartheta + \varrho(1-\nu))\sum_{j=2}^{\infty}\frac{(b)_{j-1}}{(1)_{j-2}}\mu^{(j-1)} \\ &\quad + (1-\nu)\sum_{j=2}^{\infty}\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}\Bigg) \\ &= \frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)}(1-\mu)^b\Big(((1-\varrho)\sec\vartheta + \varrho(1-\nu))\,b\mu(1-\mu)^{-(b+1)} \\ &\quad + (1-\nu)\Big((1-\mu)^{-b}-1\Big)\Big) \\ &= \frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)}\Big(((1-\varrho)\sec\vartheta + \varrho(1-\nu))\,b\mu(1-\mu)^{-1}\end{aligned}$$

$$+ (1-\nu)\left(1-(1-\mu)^b\right)\Big).$$

Using (3.7) in equation (3.5), we obtain

$$\begin{aligned}(3.8)\qquad &((1-\varrho)\sec\vartheta + \varrho(1-\nu))\, b\mu(1-\mu)^{-1} \\ &\leq (1-\nu)\left((1-\mu)^b - 1 + \frac{(\upsilon - 3\wp)}{2|\tau|(1-\gamma)}\right).\end{aligned}$$

Utilizing the hypothesis of the theorem along with equation (3.8) will complete the proof. □

Remark 3.4. Note that when $\wp = 0$ in Theorem 3.3, it is equivalent to Theorem 3.2 of [16].

Corollary 3.5. *Let h belong to $\mathcal{A}$ as specified by form* (1.1)*. Assuming $h(z) \in \mathcal{R}^{\tau}_{\wp,\upsilon}(\gamma)$ and satisfies the following inequality, let*

$$\begin{aligned}&\sec\vartheta\left(1-(1-\mu)^b\right) + \frac{(1-\nu-\sec\vartheta)}{\mu(b-1)}\left((1-\mu)-(1+\mu(b-1))(1-\mu)^b\right) \\ &\leq \frac{(1-\nu)(\upsilon-3\wp)}{2|\tau|(1-\gamma)},\end{aligned}$$

where $|\vartheta| < \frac{\pi}{2}$ and $0 \leq \nu < 1$. Then, for $b > 0$ and $0 \leq \mu \leq 1$, $\mathcal{P}^b_\mu(h,z) \in \mathcal{SP}(\vartheta,\nu)$.

Corollary 3.6. *Let h belong to $\mathcal{A}$ as specified by form* (1.1)*. Assuming $h(z) \in \mathcal{R}^{\tau}_{\wp,\upsilon}(\gamma)$ and satisfies the following inequality, let*

$$b\mu(1-\mu)^{-1}\sec\vartheta - (1-\nu)(1-\mu)^b \leq (1-\nu)\left(\frac{(\upsilon-3\wp)}{2|\tau|(1-\gamma)} - 1\right).$$

where $|\vartheta| < \frac{\pi}{2}$ and $0 \leq \nu < 1$. Then, for $b > 0$ and $0 \leq \mu \leq 1$, $\mathcal{P}^b_\mu(h,z) \in \mathcal{KP}(\vartheta,\nu)$.

The following theorem provides the condition for $\mathcal{P}^b_\mu(h,z)$ to be in $\mathcal{SP}_p(\sigma,\delta)$.

Theorem 3.7. *Let h belong to $\mathcal{A}$ as specified by form* (1.1)*. Assuming $h(z) \in \mathcal{R}^{\tau}_{\wp,\upsilon}(\gamma)$ and satisfies the following inequality, let*

$$\begin{aligned}&2\left(1-(1-\mu)^b\right) - \frac{(\cos\sigma+\delta)}{\mu(b-1)}\left(1-\mu-(1+\mu(b-1))(1-\mu)^b\right) \\ &\leq \frac{(\upsilon-3\wp)(\cos\sigma-\delta)}{2|\tau|(1-\gamma)},\end{aligned}$$

where $\sigma \leq 1$ and $0 \leq \delta < 1$. Then, for $b > 0$ and $0 \leq \mu \leq 1$, $\mathcal{P}^b_\mu(h,z) \in \mathcal{SP}_p(\sigma,\delta)$.

Proof. The condition that is enough to ensure that $h \in \mathcal{SP}_p(\sigma,\delta)$ as stated in Lemma 2.3 is:

$$\sum_{j=2}^{\infty}(2j-\cos\sigma-\delta)\,|X_j| \le \cos\sigma-\delta. \tag{3.9}$$

By substituting (1.4) into equation (3.9), we obtain

$$\sum_{j=2}^{\infty}(2j-\cos\sigma-\delta)\,\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}(1-\mu)^b|x_j| \le \cos\sigma-\delta. \tag{3.10}$$

Applying (1.3) to the left side of equation (3.10), we proceed with

$$\begin{aligned}
&\sum_{j=2}^{\infty}(2j-\cos\sigma-\delta)\,\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}(1-\mu)^b|x_j| \qquad (3.11)\\
&\quad\le \sum_{j=2}^{\infty}(2j-\cos\sigma-\delta)\,\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}(1-\mu)^b\frac{2|\tau|(1-\gamma)}{j(\upsilon-3\wp)}\\
&\quad= \frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)}(1-\mu)^b\Bigg(\sum_{j=2}^{\infty}2\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}\\
&\qquad-(\cos\sigma+\delta)\sum_{j=2}^{\infty}\frac{(b)_{j-1}}{(1)_j}\mu^{(j-1)}\Bigg)\\
&\quad= \frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)}(1-\mu)^b\Bigg(2\left((1-\mu)^{-b}-1\right)\\
&\qquad-\frac{(\cos\sigma+\delta)}{\mu(b-1)}\left((1-\mu)^{-(b-1)}-1-\mu(b-1)\right)\Bigg)\\
&\quad= \frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)}\Bigg(2\left(1-(1-\mu)^b\right)\\
&\qquad-\frac{(\cos\sigma+\delta)}{\mu(b-1)}\left((1-\mu)-(1+\mu(b-1))(1-\mu)^b\right)\Bigg).
\end{aligned}$$

Using (3.11) in equation (3.9), we obtain

$$\begin{aligned}
&2\left(1-(1-\mu)^b\right)-\frac{(\cos\sigma+\delta)}{\mu(b-1)}\left(1-\mu-(1+\mu(b-1))(1-\mu)^b\right) \qquad (3.12)\\
&\qquad\le \frac{(\upsilon-3\wp)(\cos\sigma-\delta)}{2|\tau|(1-\gamma)}.
\end{aligned}$$

Utilizing the hypothesis of the theorem along with equation (3.12) will complete the proof. □

If $\delta = 0$ then $\mathcal{SP}_p(\sigma,0) = \mathcal{SP}_p(\sigma)$.

Corollary 3.8. *Let h belong to $\mathcal{A}$ as specified by form (1.1). Assuming $h(z) \in \mathcal{R}^{\tau}_{\wp,\upsilon}(\gamma)$ and satisfies the following inequality, let*

$$2\left(1-(1-\mu)^b\right) - \frac{\cos\sigma}{\mu(b-1)}\left(1-\mu-(1+\mu(b-1))(1-\mu)^b\right) \leq \frac{(\upsilon-3\wp)\cos\sigma}{2|\tau|(1-\gamma)},$$

where $\sigma \leq 1$. Then, for $b>0$ and $0 \leq \mu \leq 1$, $\mathcal{P}^b_\mu(h,z) \in \mathcal{SP}_p(\sigma)$.

The conditions for $\mathcal{P}^b_\mu(h,z)$ to be in $\mathcal{UCSP}(\sigma,\delta)$ are outlined in the following theorem.

Theorem 3.9. *Let h belong to $\mathcal{A}$ as specified by form (1.1). Assuming $h(z) \in \mathcal{R}^{\tau}_{\wp,\upsilon}(\gamma)$ and satisfies the following inequality, let*

$$2b\mu(1-\mu)^{-1} + (2-\cos\sigma-\delta)\left(1-(1-\mu)^b\right) \leq (\cos\sigma-\delta)\frac{(\upsilon-3\wp)}{2|\tau|(1-\gamma)},$$

where $\sigma \leq 1$ and $0 \leq \delta < 1$. Then, for $b>0$ and $0 \leq \mu \leq 1$, $\mathcal{P}^b_\mu(h,z) \in \mathcal{UCSP}(\sigma,\delta)$.

Proof. The condition that is enough to ensure that $h \in \mathcal{UCSP}(\sigma,\delta)$ as stated in Lemma 2.4 is:

$$\sum_{j=2}^{\infty} j\,(2j-\cos\sigma-\delta)\,|X_j| \leq \cos\sigma-\delta. \tag{3.13}$$

By substituting (1.4) into equation (3.13), we obtain

$$\sum_{j=2}^{\infty} j\,(2j-\cos\sigma-\delta)\,\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}(1-\mu)^b|x_j| \leq \cos\sigma-\delta. \tag{3.14}$$

Applying (1.3) to the left side of equation (3.14), we proceed with

$$\begin{aligned}\sum_{j=2}^{\infty} j\,(2j-\cos\sigma-\delta)\,\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}(1-\mu)^b|x_j| &\leq \sum_{j=2}^{\infty} j\,(2j-\cos\sigma-\delta)\,\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}(1-\mu)^b\frac{2|\tau|(1-\gamma)}{j(\upsilon-3\wp)}\\ &= \frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)}(1-\mu)^b\Bigg(\sum_{j=2}^{\infty}2\frac{(b)_{j-1}}{(1)_{j-2}}\mu^{(j-1)}\\ &\quad + (2-\cos\sigma-\delta)\sum_{j=2}^{\infty}\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}\Bigg)\end{aligned} \tag{3.15}$$

$$= \frac{2|\tau|(1-\gamma)}{(\upsilon - 3\wp)}(1-\mu)^b \Bigg(2b\mu \sum_{j=0}^{\infty} \frac{(b+1)_j}{(1)_j}\mu^{(j)} + (2-\cos\sigma-\delta)\sum_{j=1}^{\infty}\frac{(b)_j}{(1)_j}\mu^{(j)} \Bigg)$$

$$= \frac{2|\tau|(1-\gamma)}{(\upsilon - 3\wp)}(1-\mu)^b \Bigg(2b\mu(1-\mu)^{-(b+1)} + (2-\cos\sigma-\delta)((1-\mu)^{-b}-1) \Bigg).$$

Using (3.15) in equation (3.13), we obtain

$$(1-\mu)^b \left(2b\mu(1-\mu)^{-(b+1)} + (2-\cos\sigma-\delta)((1-\mu)^{-b}-1) \right) \leq (\cos\sigma-\delta)\frac{(\upsilon-3\wp)}{2|\tau|(1-\gamma)}. \tag{3.16}$$

Utilizing the hypothesis of the theorem along with equation (3.16) will complete the proof. □

The following theorem provides the condition for $\mathcal{P}_\mu^b(h,z)$ to be in $k-\mathcal{S}_p(\sigma)$.

Theorem 3.10. *Let h belong to $\mathcal{A}$ as specified by form (1.1). Assuming $h(z) \in \mathcal{R}^{\tau}_{\wp,\upsilon}(\gamma)$ and satisfies the following inequality, let*

$$(1+k)\left(1-(1-\mu)^b\right) - \frac{(k+\sigma)}{\mu(b-1)}\left(1-\mu-(1+\mu(b-1))(1-\mu)^b\right) \leq \frac{(\upsilon-3\wp)(1-\sigma)}{2|\tau|(1-\gamma)},$$

where $0 \leq \sigma < 1$ and $0 \leq k < \infty$. Then, for $b > 0$ and $0 \leq \mu \leq 1$, $\mathcal{P}_\mu^b(h,z) \in k-\mathcal{S}_p(\sigma)$.

Proof. The condition that is enough to ensure that $h \in k-\mathcal{S}_p(\sigma)$ as stated in Lemma 2.5 is:

$$\sum_{j=2}^{\infty}(j(1+k)-(k+\sigma))\,|X_j| \leq 1-\sigma. \tag{3.17}$$

By substituting (1.4) into equation (3.17), we obtain

$$\sum_{j=2}^{\infty}(j(1+k)-(k+\sigma))\,\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}(1-\mu)^b|x_j| \leq 1-\sigma. \tag{3.18}$$

Applying (1.3) to the left side of equation (3.18), we proceed with

$$
\begin{aligned}
(3.19)\quad & \sum_{j=2}^{\infty} (j(1+k)-(k+\sigma)) \frac{(b)_{j-1}}{(1)_{j-1}} \mu^{(j-1)} (1-\mu)^b |x_j| \\
&\leq \sum_{j=2}^{\infty} (j(1+k)-(k+\sigma)) \frac{(b)_{j-1}}{(1)_{j-1}} \mu^{(j-1)} (1-\mu)^b \frac{2|\tau|(1-\gamma)}{j(\upsilon-3\wp)} \\
&= \frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)} (1-\mu)^b \Bigg((1+k) \sum_{j=2}^{\infty} \frac{(b)_{j-1}}{(1)_{j-1}} \mu^{(j-1)} \\
&\quad - (k+\sigma) \sum_{j=2}^{\infty} \frac{(b)_{j-1}}{(1)_j} \mu^{(j-1)} \Bigg) \\
&= \frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)} (1-\mu)^b \Bigg((1+k) \left((1-\mu)^{-b} - 1 \right) \\
&\quad - \frac{(k+\sigma)}{\mu(b-1)} \left((1-\mu)^{-(b-1)} - 1 - (b-1)\mu \right) \Bigg) \\
&= \frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)} \Bigg((1+k) \left(1-(1-\mu)^b \right) \\
&\quad - \frac{(k+\sigma)}{\mu(b-1)} \left(1-\mu-(1+\mu(b-1)(1-\mu)^b \right) \Bigg).
\end{aligned}
$$

Using (3.19) in equation (3.17), we obtain

(3.20)

$$
\begin{aligned}
&(1+k) \left(1-(1-\mu)^b \right) - \frac{(k+\sigma)}{\mu(b-1)} \left(1-\mu-(1+\mu(b-1))(1-\mu)^b \right) \\
&\quad \leq (1-\sigma) \frac{(\upsilon-3\wp)}{2|\tau|(1-\gamma)}.
\end{aligned}
$$

Utilizing the hypothesis of the theorem along with equation (3.20) will complete the proof. □

The following theorem provides the condition under which the operator $\mathcal{P}_{\mu}^{b}(h,z)$ to be in $k-\mathcal{UCV}(\sigma)$.

Theorem 3.11. *Let h belong to $\mathcal{A}$ as specified by form (1.1). Assuming $h(z) \in \mathcal{R}_{\wp,\upsilon}^{\tau}(\gamma)$ and satisfies the following inequality, let*

$$
\frac{b\mu(1+k)}{1-\mu} - (1-\sigma)(1-\mu)^b \leq (1-\sigma) \left(\frac{\upsilon-3\wp}{2|\tau|(1-\gamma)} - 1 \right),
$$

where $0 \leq \sigma < 1$ and $0 \leq k < \infty$. Then, for $b > 0$ and $0 \leq \mu \leq 1$, $\mathcal{P}_{\mu}^{b}(h,z) \in k-\mathcal{UCV}(\sigma)$.

Proof. The condition that is enough to ensure that $h \in k - \mathcal{UCV}(\sigma)$ as stated in Lemma 2.6 is:

$$\sum_{j=2}^{\infty} j\left(j(1+k)-(k+\sigma)\right)|X_j| \leq 1-\sigma. \tag{3.21}$$

By substituting (1.4) into equation (3.21), we obtain

$$\sum_{j=2}^{\infty} j\left(j(1+k)-(k+\sigma)\right)\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}(1-\mu)^b|x_j| \leq 1-\sigma. \tag{3.22}$$

Applying (1.3) to the left side of equation (3.22), we proceed with

(3.23)

$$\begin{aligned}
&\sum_{j=2}^{\infty} j\left(j(1+k)-(k+\sigma)\right)\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}(1-\mu)^b|x_j| \\
&\quad\leq \sum_{j=2}^{\infty} j\left(j(1+k)-(k+\sigma)\right)\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}(1-\mu)^b\frac{2|\tau|(1-\gamma)}{j(\upsilon-3\wp)} \\
&\quad= \frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)}(1-\mu)^b\Bigg((1+k)b\mu\sum_{j=2}^{\infty}\frac{(b+1)_{j-2}}{(1)_{j-2}}\mu^{(j-2)} \\
&\qquad+(1-\sigma)\sum_{j=2}^{\infty}\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}\Bigg) \\
&\quad= \frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)}(1-\mu)^b\Bigg((1+k)b\mu\left((1-\mu)^{-(b+1)}\right) \\
&\quad+(1-\sigma)\left((1-\mu)^{-b}-1\right)\Bigg) \\
&\quad= \frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)}\left((1+k)b\mu(1-\mu)^{-1}+(1-\sigma)\left(1-(1-\mu)^b\right)\right).
\end{aligned}$$

Using (3.23) in equation (3.21), we obtain

(3.24)

$$(1+k)b\mu(1-\mu)^{-1}+(1-\sigma)-(1-\sigma)(1-\mu)^b \leq (1-\sigma)\frac{(\upsilon-3\wp)}{2|\tau|(1-\gamma)}.$$

Utilizing the hypothesis of the theorem along with equation (3.24) will complete the proof. □

If $\sigma = 0$ then $k - \mathcal{UCV}(0) \equiv k - \mathcal{UCV}$ and $k - \mathcal{S}_p(o) \equiv k - \mathcal{ST}$, then the following results directly.

Corollary 3.12. *Let h belong to $\mathcal{A}$ as specified by form (1.1). Assuming $h(z) \in \mathcal{R}^{\tau}_{\wp,\upsilon}(\gamma)$ and satisfies the following inequality, let*

$$(1+k)\left(1-(1-\mu)^b\right) - \frac{k}{\mu(b-1)}\left(1-\mu-(1+\mu(b-1))(1-\mu)^b\right) < \frac{(\upsilon-3\wp)}{2|\tau|(1-\gamma)},$$

where $0 \leq k < \infty$. Then, for $b > 0$ and $0 \leq \mu \leq 1$, $\mathcal{P}^b_\mu(h,z) \in k - \mathcal{ST}$.

Corollary 3.13. *Let h belong to $\mathcal{A}$ as specified by form (1.1). Assuming $h(z) \in \mathcal{R}^{\tau}_{\wp,\upsilon}(\gamma)$ and satisfies the following inequality, let*

$$\frac{b\mu(1+k)}{1-\mu} - (1-\mu)^b + 1 \leq \frac{\upsilon-3\wp}{2|\tau|(1-\gamma)},$$

where $0 \leq k < \infty$. Then, for $b > 0$ and $0 \leq \mu \leq 1$, $\mathcal{P}^b_\mu(h,z) \in k - \mathcal{UCV}$.

In the next theorem, we establish the condition for $\mathcal{P}^b_\mu(h,z)$ to be in $\mathcal{CP}(\alpha)$.

Theorem 3.14. *Let h belong to $\mathcal{A}$ as specified by form (1.1). Assuming $h(z) \in \mathcal{R}^{\tau}_{\wp,\upsilon}(\gamma)$ and satisfies the following inequality, let*

$$\frac{b\mu}{\mu-1} - (2\alpha-1)(1-\mu)^b \leq (2\alpha-1)\left(\frac{\upsilon-3\wp}{2|\tau|(1-\gamma)} - 1\right),$$

where $0 < \alpha \leq \infty$. Then, for $b > 0$ and $0 \leq \mu \leq 1$, $\mathcal{P}^b_\mu(h,z) \in \mathcal{CP}(\alpha)$.

Proof. The condition that is enough to ensure that $h \in \mathcal{CP}(\alpha)$ as stated in Lemma 2.7 is:

$$\sum_{j=2}^{\infty} (j+2(\alpha-1))\, j|X_j| \leq 2\alpha - 1. \tag{3.25}$$

By substituting (1.4) into equation (3.25), we obtain

$$\sum_{j=2}^{\infty} (j+2(\alpha-1))\, j\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}(1-\mu)^b|x_j| \leq 2\alpha - 1. \tag{3.26}$$

Applying (1.3) to the left side of equation (3.26), we proceed with

(3.27)

$$\sum_{j=2}^{\infty} (j+2(\alpha-1))\, j\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}(1-\mu)^b|x_j| \leq \sum_{j=2}^{\infty} (j+2(\alpha-1))\, j\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}(1-\mu)^b\frac{2|\tau|(1-\gamma)}{j(\upsilon-3\wp)}$$

$$= \frac{2|\tau|(1-\gamma)}{(\upsilon - 3\wp)}(1-\mu)^b \Bigg(b\mu \sum_{j=2}^{\infty} \frac{(b+1)_{j-2}}{(1)_{j-2}} \mu^{(j-2)} + (2\alpha - 1) \sum_{j=2}^{\infty} \frac{(b)_{j-1}}{(1)_{j-1}} \mu^{(j-1)} \Bigg)$$

$$= \frac{2|\tau|(1-\gamma)}{(\upsilon - 3\wp)}(1-\mu)^b \left(b\mu(1-\mu)^{-(b+1)} + (2\alpha - 1) \left((1-\mu)^{-b} - 1 \right) \right)$$

$$= \frac{2|\tau|(1-\gamma)}{(\upsilon - 3\wp)} \left(b\mu(1-\mu)^{-1} + (2\alpha - 1) \left(1 - (1-\mu)^b \right) \right).$$

Using (3.27) in equation (3.25), we obtain

(3.28)

$$b\mu(1-\mu)^{-1} + (2\alpha - 1) - (2\alpha - 1)(1-\mu)^b \leq (2\alpha - 1)\frac{(\upsilon - 3\wp)}{2|\tau|(1-\gamma)}.$$

Utilizing the hypothesis of the theorem along with equation (3.28) will complete the proof. □

In the following theorem, we derive the condition for the operator $\mathcal{P}_{\mu}^{b}(h,z)$ to be in $\mathcal{S}_{\lambda}^{*}$.

Theorem 3.15. *Let h belong to $\mathcal{A}$ as specified by form (1.1). Assuming $h(z) \in \mathcal{R}_{\wp,\upsilon}^{\tau}(\gamma)$ and satisfies the following inequality, let*

$$\frac{(\lambda - 1)}{\mu(b-1)} \left(1 - \mu - (1 + (b-1)\mu)(1-\mu)^b \right) - (1-\mu)^b \leq \lambda \frac{(\upsilon - 3\wp)}{2|\tau|(1-\gamma)} - 1,$$

where $\lambda > 0$. Then, for $b > 0$ and $0 \leq \mu \leq 1$, $\mathcal{P}_{\mu}^{b}(h,z) \in \mathcal{S}_{\lambda}^{*}$.

Proof. The condition that is enough to ensure that $h \in \mathcal{S}_{\lambda}^{*}$ as stated in Lemma 2.8 is:

$$\sum_{j=2}^{\infty} (\lambda + j - 1)|X_j| \leq \lambda. \tag{3.29}$$

By substituting (1.4) into equation (3.29), we obtain

$$\sum_{j=2}^{\infty} (\lambda + j - 1)\frac{(b)_{j-1}}{(1)_{j-1}} \mu^{(j-1)}(1-\mu)^b |x_j| \leq \lambda. \tag{3.30}$$

Applying (1.3) to the left side of equation (3.30), we proceed with

(3.31)

$$\sum_{j=2}^{\infty} (\lambda + j - 1)\frac{(b)_{j-1}}{(1)_{j-1}} \mu^{(j-1)}(1-\mu)^b |x_j|$$

$$
\begin{aligned}
&\leq \sum_{j=2}^{\infty}(\lambda+j-1)\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}(1-\mu)^b\frac{2|\tau|(1-\gamma)}{j(\upsilon-3\wp)}\\
&= \frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)}(1-\mu)^b\left(\sum_{j=2}^{\infty}\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}+(\lambda-1)\sum_{j=2}^{\infty}\frac{(b)_{j-1}}{(1)_j}\mu^{(j-1)}\right)\\
&= \frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)}(1-\mu)^b\Bigg(((1-\mu)^{-b}-1)\\
&\quad+\frac{(\lambda-1)}{\mu(b-1)}\left((1-\mu)^{-(b-1)}-1-(b-1)\mu\right)\Bigg)\\
&= \frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)}\Bigg(\left(1-(1-\mu)^b\right)\\
&\quad+\frac{(\lambda-1)}{\mu(b-1)}\left(1-\mu-(1+(b-1)\mu)(1-\mu)^b\right)\Bigg).
\end{aligned}
$$

Using (3.31) in equation (3.29), we obtain

$$
\begin{aligned}
&\left(1-(1-\mu)^b\right)+\frac{(\lambda-1)}{\mu(b-1)}\left(1-\mu-(1+(b-1)\mu)(1-\mu)^b\right)\\
&\qquad\leq \lambda\frac{(\upsilon-3\wp)}{2|\tau|(1-\gamma)}.
\end{aligned}
\tag{3.32}
$$

Utilizing the hypothesis of the theorem along with equation (3.32) will complete the proof. □

In the next theorem, we derive the condition for $\mathcal{P}_{\mu}^{b}(h,z)$ to be in $\mathcal{K}_{\lambda}$.

Theorem 3.16. *Let h belong to $\mathcal{A}$ as specified by form* (1.1)*. Assuming $h(z)\in\mathcal{R}_{\wp,\upsilon}^{\tau}(\gamma)$ and satisfies the following inequality, let*

$$
\frac{b\mu}{1-\mu}-\lambda(1-\mu)^b\leq\lambda\left(\frac{(\upsilon-3\wp)}{2|\tau|(1-\gamma)}-1\right),
$$

where $\lambda>0$. Then, for $b>0$ and $0\leq\mu\leq 1$, $\mathcal{P}_{\mu}^{b}(h,z)\in\mathcal{K}_{\lambda}$.

Proof. The condition that is enough to ensure that $h\in\mathcal{K}_{\lambda}$ as stated in Lemma 2.9 is:

$$
\sum_{j=2}^{\infty}j(\lambda+j-1)|X_j|\leq\lambda. \tag{3.33}
$$

By substituting (1.4) into equation (3.33), we obtain

$$
\sum_{j=2}^{\infty}j(\lambda+j-1)\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}(1-\mu)^b|x_j|\leq\lambda. \tag{3.34}
$$

Applying (1.3) to the left side of equation (3.34), we proceed with (3.35)

$$\begin{aligned}
&\sum_{j=2}^{\infty} j(\lambda+j-1)\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}(1-\mu)^b|x_j| \\
&\quad\leq \sum_{j=2}^{\infty} j(\lambda+j-1)\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}(1-\mu)^b\frac{2|\tau|(1-\gamma)}{j(\upsilon-3\wp)} \\
&\quad= \frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)}(1-\mu)^b\left(\lambda\sum_{j=2}^{\infty}\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}+\sum_{j=2}^{\infty}\frac{(r)_{j-1}}{(1)_{j-2}}\mu^{(j-1)}\right) \\
&\quad= \frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)}(1-\mu)^b\left(\lambda\sum_{j=1}^{\infty}\frac{(b)_{j}}{(1)_{j}}\mu^{(j)}+b\mu\sum_{j=2}^{\infty}\frac{(b+1)_{j-2}}{(1)_{j-2}}\mu^{(j-2)}\right) \\
&\quad= \frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)}(1-\mu)^b\left[\lambda\left((1-\mu)^{-b}-1\right)+b\mu(1-\mu)^{-(b+1)}\right] \\
&\quad= \frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)}\left[\lambda\left(1-(1-\mu)^b\right)+b\mu(1-\mu)^{-1}\right].
\end{aligned}$$

Using (3.35) in equation (3.33), we obtain

$$\lambda\left(1-(1-\mu)^b\right)+b\mu(1-\mu)^{-1}\leq\lambda\frac{(\upsilon-3\wp)}{2|\tau|(1-\gamma)}. \tag{3.36}$$

Utilizing the hypothesis of the theorem along with equation (3.36) will complete the proof. □

In the next theorem, we determine the condition for $\mathcal{P}_{\mu}^{b}(h,z)$ to belong to the class $\mathcal{S}(\lambda,\beta)$.

Theorem 3.17. *Let h belong to $\mathcal{A}$ as specified by form (1.1). Assuming $h(z)\in\mathcal{R}_{\wp,\upsilon}^{\tau}(\gamma)$ and satisfies the following inequality, let*

$$\begin{aligned}
&\lambda b\mu(1-\mu)^{-1}-(1-\mu)^b-\frac{\beta}{(b-1)\mu}\left[(1-\mu)-(1+\mu(b-1))(1-\mu)^b\right]+1 \\
&\quad\leq(1-\beta)\frac{(\upsilon-3\wp)}{2|\tau|(1-\gamma)},
\end{aligned}$$

where $\lambda>0$ and $0\leq\beta<1$. Then, for $b>0$ and $0\leq\mu\leq1$, $\mathcal{P}_{\mu}^{b}(h,z)\in\mathcal{S}(\lambda,\beta)$.

Proof. The condition that is enough to ensure that $h\in\mathcal{S}(\lambda,\beta)$ as stated in Lemma 2.10 is:

$$\sum_{j=2}^{\infty}\left(j+\lambda j(j-1)-\beta\right)|X_j|\leq1-\beta. \tag{3.37}$$

By substituting (1.4) into equation (3.37), we obtain

$$\sum_{j=2}^{\infty}(j+\lambda j(j-1)-\beta)\,\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}(1-\mu)^{b}|x_j|\leq 1-\beta. \tag{3.38}$$

Applying (1.3) to the left side of equation (3.38), we proceed with

$$\begin{aligned}
&\sum_{j=2}^{\infty}(j+\lambda j(j-1)-\beta)\,\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}(1-\mu)^{b}|x_j| \qquad (3.39)\\
&\quad\leq\sum_{j=2}^{\infty}(j+\lambda j(j-1)-\beta)\,\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}(1-\mu)^{b}\frac{2|\tau|(1-\gamma)}{j(\upsilon-3\wp)}\\
&\quad=\frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)}(1-\mu)^{b}\Bigg(\sum_{j=2}^{\infty}\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}\\
&\qquad+\lambda\sum_{j=2}^{\infty}\frac{(b)_{j-1}}{(1)_{j-2}}\mu^{(j-1)}-\beta\sum_{j=2}^{\infty}\frac{(b)_{j-1}}{(1)_{j}}\mu^{(j-1)}\Bigg)\\
&\quad=\frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)}(1-\mu)^{b}\Bigg(\sum_{j=1}^{\infty}\frac{(b)_{j}}{(1)_{j}}\mu^{(j)}\\
&\qquad+\lambda b\mu\sum_{j=2}^{\infty}\frac{(b+1)_{j-2}}{(1)_{j-2}}\mu^{(j-2)}-\frac{\beta}{(b-1)\mu}\sum_{j=2}^{\infty}\frac{(b-1)_{j}}{(1)_{j}}\mu^{(j)}\Bigg)\\
&\quad=\frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)}(1-\mu)^{b}\Bigg(((1-\mu)^{-b}-1)+\lambda b\mu(1-\mu)^{-(b+1)}\\
&\qquad-\frac{\beta}{(b-1)\mu}((1-\mu)^{-(b-1)}-1-\mu(b-1))\Bigg)\\
&\quad=\frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)}\Bigg(1-(1-\mu)^{b}+\lambda b\mu(1-\mu)^{-1}\\
&\qquad-\frac{\beta}{(b-1)\mu}\left[(1-\mu)-(1+\mu(b-1))(1-\mu)^{b}\right]\Bigg).
\end{aligned}$$

Using (3.39) in equation (3.37), we obtain

$$\begin{aligned}
&\lambda b\mu(1-\mu)^{-1}-(1-\mu)^{b}-\frac{\beta\left[(1-\mu)-(1+\mu(b-1))(1-\mu)^{b}\right]}{(b-1)\mu}+1 \qquad (3.40)\\
&\quad\leq(1-\beta)\frac{(\upsilon-3\wp)}{2|\tau|(1-\gamma)}.
\end{aligned}$$

Utilizing the hypothesis of the theorem along with equation (3.40) will complete the proof. □

Let us consider the linear operator $\mathcal{T}_{\mu}^{b}(z)$ defined as

$$\begin{aligned}\mathcal{T}_{\mu}^{b}(z) &= \int_{0}^{z} \frac{\mathcal{P}_{\mu}^{b}(\theta)}{\theta} d\theta \\ &= z + \sum_{j=2}^{\infty} \frac{(b)_{j-1}}{(1)_{j}} \mu^{(j-1)} (1-\mu)^{b} z^{j}.\end{aligned}$$

Let $h(z) \in \mathcal{A}$ then

$$\begin{aligned}\mathcal{T}_{\mu}^{b}(h,z) &= z + \sum_{j=2}^{\infty} \frac{(b)_{j-1}}{(1)_{j}} \mu^{(j-1)} (1-\mu)^{b} x_{j} z^{j} \\ &= z + \sum_{j=2}^{\infty} Y_{j} z^{j}\end{aligned}$$

here

$$Y_{j} = \frac{(b)_{j-1}}{(1)_{j}} \mu^{(j-1)} (1-\mu)^{b} x_{j}. \tag{3.41}$$

In the following theorem, we establish the condition for the integral operator $\mathcal{T}_{\mu}^{b}(h,z)$ to belong to the class $\mathcal{K}(\vartheta,\nu,\varrho)$.

Theorem 3.18. *Let h belong to $\mathcal{A}$ as specified by form* (1.1)*. Assuming $h(z) \in \mathcal{R}_{\wp,\upsilon}^{\tau}(\gamma)$ and satisfies the following inequality, let*

$$\begin{aligned}&((1-\varrho)\sec\vartheta + \varrho(1-\nu))\left(1-(1-\mu)^{b}\right) \\ &\quad + \frac{(1-\varrho)(1-\nu-\sec\vartheta)}{\mu(b-1)}(1-\mu-(1+\mu(b-1))(1-\mu)^{b}) \\ &\leq \frac{(1-\nu)(\upsilon-3\wp)}{2|\tau|(1-\gamma)},\end{aligned}$$

where $|\vartheta| < \frac{\pi}{2}$, $0 \leq \nu < 1$ and $0 \leq \varrho < 1$. Then, for $b > 0$ and $0 \leq \mu \leq 1$, $\mathcal{T}_{\mu}^{b}(h,z) \in \mathcal{K}(\vartheta,\nu,\varrho)$.

Proof. The condition that is enough to ensure that $h \in \mathcal{K}(\vartheta,\nu,\varrho)$ as stated in Lemma 2.2 is:

$$\sum_{j=2}^{\infty} j\left[((1-\varrho)\sec\vartheta + \varrho(1-\nu))j + (1-\varrho)(1-\nu-\sec\vartheta)\right]|Y_{j}| \leq 1-\nu. \tag{3.42}$$

By substituting (3.41) into equation (3.42), we obtain

$$\sum_{j=2}^{\infty} j\big[((1-\varrho)\sec\vartheta + \varrho(1-\nu))j \tag{3.43}$$

$$+ (1-\varrho)(1-\nu-\sec\vartheta)\big]\frac{(b)_{j-1}}{(1)_j}\mu^{(j-1)}(1-\mu)^b|x_j|$$
$$\leq 1-\nu.$$

Applying (1.3) to the left side of equation (3.43), we proceed with

(3.44)
$$\sum_{j=2}^{\infty} j\big[((1-\varrho)\sec\vartheta+\varrho(1-\nu))j$$
$$+ (1-\varrho)(1-\nu-\sec\vartheta)\big]\frac{(b)_{j-1}}{(1)_j}\mu^{(j-1)}(1-\mu)^b|x_j|$$
$$\leq \sum_{j=2}^{\infty} j\big[((1-\varrho)\sec\vartheta+\varrho(1-\nu))j$$
$$+ (1-\varrho)(1-\nu-\sec\vartheta)\big]\frac{(b)_{j-1}}{(1)_j}\mu^{(j-1)}(1-\mu)^b\frac{2|\tau|(1-\gamma)}{j(\upsilon-3\wp)}$$
$$= \frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)}(1-\mu)^b\Bigg(((1-\varrho)\sec\vartheta+\varrho(1-\nu))\sum_{j=2}^{\infty}\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}$$
$$+ \frac{(1-\varrho)(1-\nu-\sec\vartheta)}{\mu(b-1)}\sum_{j=2}^{\infty}\frac{(b-1)_j}{(1)_j}\mu^{(j)}\Bigg)$$
$$= \frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)}(1-\mu)^b\Bigg(((1-\varrho)\sec\vartheta+\varrho(1-\nu))((1-\mu)^{-b}-1)$$
$$+ \frac{(1-\varrho)(1-\nu-\sec\vartheta)}{\mu(b-1)}((1-\mu)^{-(b-1)}-1-\mu(b-1))\Bigg)$$
$$= \frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)}\Bigg(\left((1-\varrho)\sec\vartheta+\varrho(1-\nu)\right)\left(1-(1-\mu)^b\right)$$
$$+ \frac{(1-\varrho)(1-\nu-\sec\vartheta)}{\mu(b-1)}\left(1-\mu-(1+\mu(b-1))\,(1-\mu)^b\right)\Bigg).$$

Using (3.44) in equation (3.42), we obtain

(3.45)
$$\left((1-\varrho)\sec\vartheta+\varrho(1-\nu)\right)\left(1-(1-\mu)^b\right)$$
$$+ \frac{(1-\varrho)(1-\nu-\sec\vartheta)}{\mu(b-1)}\left(1-\mu-(1+\mu(b-1))\,(1-\mu)^b\right)$$
$$\leq \frac{(1-\nu)(\upsilon-3\wp)}{2|\tau|(1-\gamma)}.$$

Utilizing the hypothesis of the theorem along with equation (3.45) will complete the proof. □

Remark 3.19. Note that the conditions described in Theorem 3.18, for the linear operator $\mathcal{T}_{\mu}^{b}(h,z)$ to belong to the class $\mathcal{K}(\vartheta,\nu,\varrho)$, correspond precisely with conditions in Theorem 3.1, ensuring that the linear operator $\mathcal{P}_{\mu}^{b}(h,z)$ is included in $\mathcal{S}(\vartheta,\nu,\varrho)$.

The next theorem presents the condition for the integral operator $\mathcal{T}_{\mu}^{b}(h,z)$ to be in the class $\mathcal{UCSP}(\sigma,\delta)$.

Theorem 3.20. *Let h belong to $\mathcal{A}$ as specified by form (1.1). Assuming $h(z) \in \mathcal{R}_{\wp,\upsilon}^{\tau}(\gamma)$ and satisfies the following inequality, let*

$$2\left(1-(1-\mu)^{b}\right)-\frac{(\cos\sigma+\delta)}{\mu(b-1)}\left(1-\mu-(1+\mu(b-1))(1-\mu)^{b}\right) \leq (\cos\sigma-\delta)\frac{(\upsilon-3\wp)}{2|\tau|(1-\gamma)},$$

where $\sigma \leq 1$ and $0 \leq \delta < 1$. Then, for $b > 0$ and $0 \leq \mu \leq 1$, $\mathcal{T}_{\mu}^{b}(h,z) \in \mathcal{UCSP}(\sigma,\delta)$.

Proof. The condition that is enough to ensure that $h \in \mathcal{UCSP}(\sigma,\delta)$ as stated in Lemma 2.4 is:

$$\sum_{j=2}^{\infty} j\,(2j-\cos\sigma-\delta)\,|Y_j| \leq \cos\sigma-\delta. \tag{3.46}$$

By substituting (3.41) into equation (3.46), we obtain

$$\sum_{j=2}^{\infty} j\,(2j-\cos\sigma-\delta)\,\frac{(b)_{j-1}}{(1)_j}\mu^{(j-1)}(1-\mu)^{b}|x_j| \leq \cos\sigma-\delta. \tag{3.47}$$

Applying (1.3) to the left side of equation (3.47), we proceed with

$$\begin{aligned}\sum_{j=2}^{\infty} j\,(2j-\cos\sigma-\delta)\,\frac{(b)_{j-1}}{(1)_j}\mu^{(j-1)}(1-\mu)^{b}|x_j| &\leq \sum_{j=2}^{\infty} j\,(2j-\cos\sigma-\delta)\,\frac{(b)_{j-1}}{(1)_j}\mu^{(j-1)}(1-\mu)^{b}\frac{2|\tau|(1-\gamma)}{j(\upsilon-3\wp)}\\ &= \frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)}(1-\mu)^{b}\left(\sum_{j=2}^{\infty}2\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)}\right.\\ &\quad\left.-(\cos\sigma+\delta)\sum_{j=2}^{\infty}\frac{(b)_{j-1}}{(1)_j}\mu^{(j-1)}\right)\end{aligned} \tag{3.48}$$

$$= \frac{2|\tau|(1-\gamma)}{(\upsilon - 3\wp)}(1-\mu)^b \Bigg(2((1-\mu)^{-b} - 1) - \frac{(\cos\sigma + \delta)}{\mu(b-1)} \Big((1-\mu)^{-(b-1)} - 1 - \mu(b-1) \Big) \Bigg)$$

$$= \frac{2|\tau|(1-\gamma)}{(\upsilon - 3\wp)} \Bigg(2 \Big(1 - (1-\mu)^b \Big) - \frac{(\cos\sigma + \delta)}{\mu(b-1)} \Big(1 - \mu - (1 + \mu(b-1))(1-\mu)^b \Big) \Bigg).$$

Using (3.48) in equation (3.46), we obtain

$$2 \Big(1 - (1-\mu)^b \Big) - \frac{(\cos\sigma + \delta)}{\mu(b-1)} \Big(1 - \mu - (1 + \mu(b-1))(1-\mu)^b \Big) \leq (\cos\sigma - \delta) \frac{(\upsilon - 3\wp)}{2|\tau|(1-\gamma)}. \tag{3.49}$$

Utilizing the hypothesis of the theorem along with equation (3.49) will complete the proof. □

Remark 3.21. Note that the conditions described in Theorem 3.20, for the linear operator $\mathcal{T}_\mu^b(h,z)$ to belong to the class $\mathcal{UCSP}(\sigma,\delta)$, correspond precisely with conditions in Theorem 3.7, ensuring that the linear operator $\mathcal{P}_\mu^b(h,z)$ is included in $\mathcal{SP}_p(\sigma,\delta)$.

The following theorem provides the condition under which the integral operator $\mathcal{T}_\mu^b(h,z)$ to be in $k - \mathcal{UCV}(\sigma)$.

Theorem 3.22. *Let h belong to $\mathcal{A}$ as specified by form* (1.1)*. Assuming $h(z) \in \mathcal{R}_{\wp,\upsilon}^{\tau}(\gamma)$ and satisfies the following inequality, let*

$$(1+k) \Big(1 - (1-\mu)^b \Big) - \frac{(k+\sigma)}{\mu(b-1)} \Big(1 - \mu - (1 + \mu(b-1))(1-\mu)^b \Big) \leq \frac{(\upsilon - 3\wp)(1-\sigma)}{2|\tau|(1-\gamma)},$$

where $0 \leq \sigma < 1$ and $0 \leq k < \infty$. Then, for $b > 0$ and $0 \leq \mu \leq 1$, $\mathcal{T}_\mu^b(h,z) \in k - \mathcal{UCV}(\sigma)$.

Proof. The condition that is enough to ensure that $h \in k - \mathcal{UCV}(\sigma)$ as stated in Lemma 2.6 is:

$$\sum_{j=2}^{\infty} j \left(j(1+k) - (k+\sigma) \right) |Y_j| \leq 1 - \sigma. \tag{3.50}$$

By substituting (3.41) into equation (3.50), we obtain

$$\sum_{j=2}^{\infty} j\left(j(1+k)-(k+\sigma)\right)\frac{(b)_{j-1}}{(1)_j}\mu^{(j-1)}(1-\mu)^b|x_j| \leq 1-\sigma. \tag{3.51}$$

Applying (1.3) to the left side of equation (3.51), we proceed with

$$\begin{aligned}
&\sum_{j=2}^{\infty} j\left(j(1+k)-(k+\sigma)\right)\frac{(b)_{j-1}}{(1)_j}\mu^{(j-1)}(1-\mu)^b|x_j| \\
&\quad\leq \sum_{j=2}^{\infty} j\left(j(1+k)-(k+\sigma)\right)\frac{(b)_{j-1}}{(1)_j}\mu^{(j-1)}(1-\mu)^b\frac{2|\tau|(1-\gamma)}{j(\upsilon-3\wp)} \\
&\quad= \frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)}(1-\mu)^b\Bigg((1+k)\sum_{j=2}^{\infty}\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)} \\
&\qquad-(k+\sigma)\sum_{j=2}^{\infty}\frac{(b)_{j-1}}{(1)_j}\mu^{(j-1)}\Bigg) \\
&\quad= \frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)}(1-\mu)^b\Bigg((1+k)\left((1-\mu)^{-b}-1\right) \\
&\qquad-\frac{(k+\sigma)}{\mu(b-1)}\left((1-\mu)^{-(b-1)}-1-(b-1)\mu\right)\Bigg) \\
&\quad= \frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)}\Bigg((1+k)\left(1-(1-\mu)^b\right) \\
&\qquad-\frac{(k+\sigma)}{\mu(b-1)}\left(1-\mu-(1+(b-1)\mu)(1-\mu)^b\right)\Bigg).
\end{aligned} \tag{3.52}$$

Using (3.52) in equation (3.50), we obtain

$$\begin{aligned}
&(1+k)\left(1-(1-\mu)^b\right)-\frac{(k+\sigma)}{\mu(b-1)}\left(1-\mu-(1+(b-1)\mu)(1-\mu)^b\right) \\
&\quad\leq (1-\sigma)\frac{(\upsilon-3\wp)}{2|\tau|(1-\gamma)}.
\end{aligned} \tag{3.53}$$

Utilizing the hypothesis of the theorem along with equation (3.53) will complete the proof. □

Remark 3.23. Note that the conditions described in Theorem 3.22, for the linear operator $\mathcal{T}_{\mu}^{b}(h,z)$ to belong to the class $k-\mathcal{UCV}(\sigma)$, correspond precisely with conditions in Theorem 3.10, ensuring that the linear operator $\mathcal{P}_{\mu}^{b}(h,z)$ is included in $k-\mathcal{S}_p(\sigma)$.

The next theorem provides the condition for the integral operator $\mathcal{T}_{\mu}^{b}(h,z)$ to be in $\mathcal{K}_{\lambda}$.

Theorem 3.24. *Let h belong to $\mathcal{A}$ as specified by form (1.1). Assuming $h(z) \in \mathcal{R}^{\tau}_{\wp,\upsilon}(\gamma)$ and satisfies the following inequality, let*

$$\frac{(\lambda-1)}{\mu(b-1)}\left[1-\mu-(1+(b-1)\mu)(1-\mu)^b\right]-(1-\mu)^b \leq \lambda\frac{(\upsilon-3\wp)}{2|\tau|(1-\gamma)}-1,$$

where $\lambda > 0$. Then, for $b > 0$ and $0 \leq \mu \leq 1$, $\mathcal{T}^b_\mu(h,z) \in \mathcal{K}_\lambda$.

Proof. The condition that is enough to ensure that $h \in \mathcal{K}_\lambda$ as stated in Lemma 2.9 is:

$$\sum_{j=2}^{\infty} j(\lambda+j-1)|Y_j| \leq \lambda. \tag{3.54}$$

By substituting (3.41) into equation (3.54), we obtain

$$\sum_{j=2}^{\infty} j(\lambda+j-1)\frac{(b)_{j-1}}{(1)_j}\mu^{(j-1)}(1-\mu)^b|x_j| \leq \lambda. \tag{3.55}$$

Applying (1.3) to the left side of equation (3.55), we proceed with

$$\begin{aligned}
&\sum_{j=2}^{\infty} j(\lambda+j-1)\frac{(b)_{j-1}}{(1)_j}\mu^{(j-1)}(1-\mu)^b|x_j| \\
&\quad\leq \sum_{j=2}^{\infty} j(\lambda+j-1)\frac{(b)_{j-1}}{(1)_j}\mu^{(j-1)}(1-\mu)^b\frac{2|\tau|(1-\gamma)}{j(\upsilon-3\wp)} \\
&\quad= \frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)}(1-\mu)^b\Bigg(\sum_{j=2}^{\infty}\frac{(b)_{j-1}}{(1)_{j-1}}\mu^{(j-1)} \\
&\qquad+(\lambda-1)\sum_{j=2}^{\infty}\frac{(b)_{j-1}}{(1)_j}\mu^{(j-1)}\Bigg) \\
&\quad= \frac{2|\tau|(1-\gamma)}{(\upsilon-3\wp)}(1-\mu)^b\Bigg(((1-\mu)^{-b}-1) \\
&\qquad+\frac{(\lambda-1)}{\mu(b-1)}\left((1-\mu)^{-(b-1)}-1-(b-1)\mu\right)\Bigg).
\end{aligned} \tag{3.56}$$

Using (3.56) in equation (3.54), we obtain

$$\begin{aligned}
&\frac{(\lambda-1)}{\mu(b-1)}\left(1-\mu-(1+(b-1)\mu)(1-\mu)^b\right)-(1-\mu)^b \\
&\quad\leq \lambda\frac{(\upsilon-3\wp)}{2|\tau|(1-\gamma)}-1.
\end{aligned} \tag{3.57}$$

Utilizing the hypothesis of the theorem along with equation (3.57) will complete the proof. □

Remark 3.25. Note that the conditions described in Theorem 3.24, for the linear operator $\mathcal{T}_{\mu}^{b}(h,z)$ to belong to the class $\mathcal{K}_{\lambda}$, correspond precisely with conditions in Theorem 3.15, ensuring that the linear operator $\mathcal{P}_{\mu}^{b}(h,z)$ is included in $\mathcal{S}_{\lambda}^{*}$.

4. Conclusion

This study utilizes a Pascal distribution series within integral operators to explore the geometric properties of the class $\mathcal{R}_{\wp,\upsilon}^{\tau}(\gamma)$. Certain conditions are set on the parameters to ensure the integral operators fall within certain subclasses of univalent functions. By utilizing the coefficient bound of the class, the geometric features of integral operators with a Pascal distribution series are determined. Additionally, the findings included some previously published special cases.

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