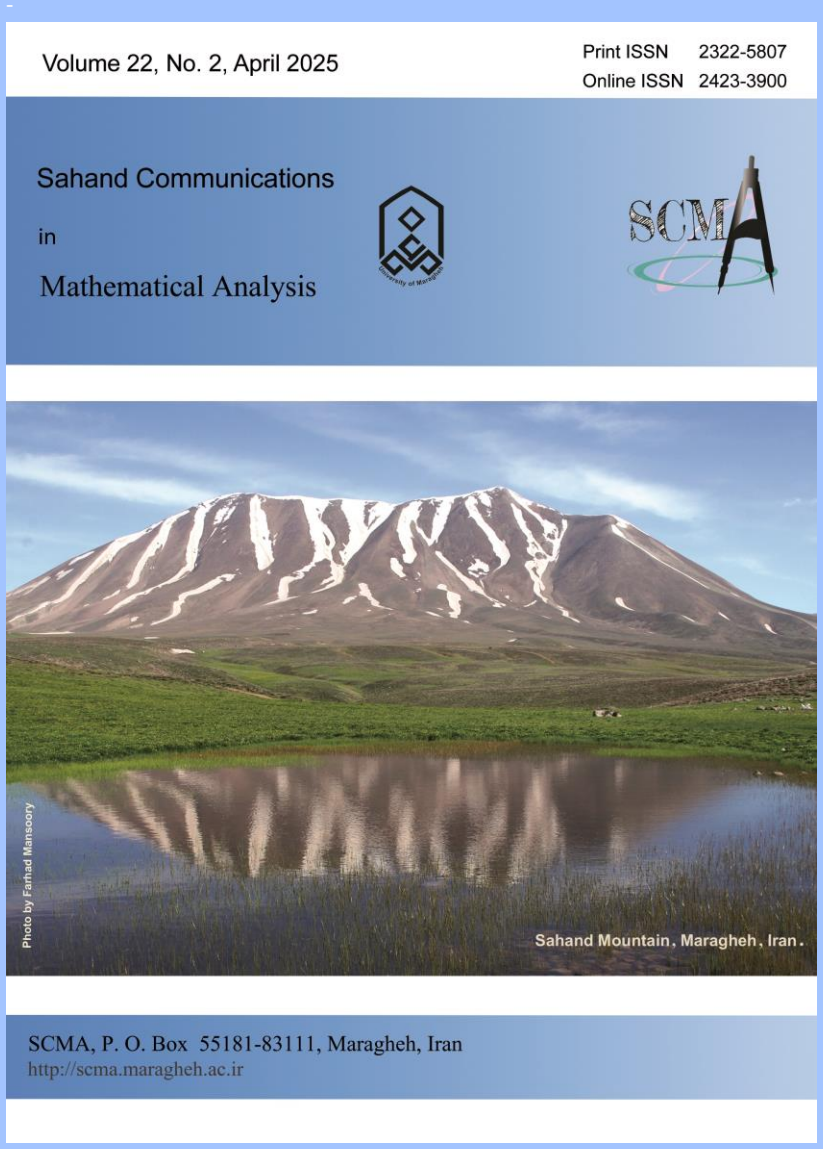
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Bellman Type Inequality for Sugeno Integrals

Bayaz Daraby[1*] and Ramin Mosalman[2]

Abstract. In this paper, we prove Bellman type inequality for Sugeno integral and generalize it for pseudo-integrals. Furthermore, we generalize the Bellman's inequality in abstract spaces. The strengthened version of this inequality is demonstrated in pseudo-integrals and abstract spaces. Also, we illustrate theorems with some examples.

1. Introduction and Preliminaries

The theory of fuzzy measures and fuzzy integrals was presented by Sugeno ([25]) as a tool for modelling non-deterministic problems [26]. Fuzzy integral is a non-linear functional dependent on fuzzy measures for which there has been very research from a mathematical point of view. The study of inequalities for Sugeno integral was started by Roman-Flores et al.([24]), and was later continued by other authors ([2, 15, 17]).

Recently, B. Daraby et al. generalized several well-known integral inequalities for fuzzy integrals ([7–9]).

Pseudo-analysis is a generalization of the classical analysis, where instead of the field of real numbers a semiring is taken on a real interval $[t, z] \subset [-\infty, \infty]$ endowed with pseudo-addition $\oplus$ and with pseudo-multiplication $\odot$ ([1, 3, 15, 19, 20, 25]). The classical Bellman's integral inequality is stated and proved in ([4]) as follows: If a and b are non-negative and concave on [0,1], then the inequality

$$\int_0^1 ab \geq \frac{1}{2}\left(\int_0^1 a^2\right)^{\frac{1}{2}}\left(\int_0^1 b^2\right)^{\frac{1}{2}}$$

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holds.

In the first step, we present several basic properties of the fuzzy and it's generalization for pseudo-integrals and abstract spaces.

Assume that η is a $\sigma - algebra$ of subsets of X and let $\mu : \eta \to [0, \infty]$ be a non-negative, extended real-valued set function. Then μ is a fuzzy measure if it satisfies:

(1) $\mu(\emptyset) = 0$,
(2) $T, K \in \eta$ and $T \subset K$ imply $\mu(T) \leq \mu(K)$ (monotonicity).
(3) $\{T_n\} \subset \eta, T_1 \subset T_2 \subset \ldots.$ imply $\lim_{n\to\infty} \mu(T_n) = \mu\left(\bigcup_{n=1}^{\infty} T_n\right)$ (continuity from below).
(4) $\{T_n\} \subset \eta, T_1 \supset T_2 \supset \ldots, \mu(T_1) < \infty$, imply $\lim_{n\to\infty} \mu(T_n) = \mu\left(\bigcap_{n=1}^{\infty} T_n\right)$ (continuity from above).

The difference between the definition of fuzzy measure and classic measure is in the countable additivity property. This means that if (T_i) is a sequence of disjoint sets in Σ (i.e., $T_i \in \Sigma$) for each $i = 1, 2, 3, \ldots$ and $T_i \cap T_j = \emptyset$ for every $i \neq j$, then

$$\mu\left(\bigcup_{i=1}^{\infty} T_i\right) = \sum_{i=1}^{\infty} \mu\left(T_i\right).$$

So, in comparison to the classical measure, the fuzzy measure abandons the additivity.

(X, η, μ) is called a fuzzy measure space. If f is a non-negative real-valued function defined on X , we will denote by $L_\alpha a = \{x \in X : a(x) \geq \alpha\} = \{a \geq \alpha\}$ the $\alpha - level$ of a, for $\alpha > 0$. Note that if $\alpha \leq \beta$, then $\{a \geq \beta\} \subset \{a \geq \alpha\}$.

Definition 1.1. A real-valued function $f : X \to (-\infty, \infty)$ is F-measurable iff $f^{-1}(B) = \{x/f(x) \in B\} \in \eta$, for any Borel set $B \in \mathcal{B}$.

Let (X, η, μ) be a fuzzy measure space, by $F_+(X)$, we denote the set of all non-negative measurable functions with respect to η.

Definition 1.2. Let (X, η, μ) be a fuzzy measure space, $a \in F_+(X)$ and $A \in \eta$; the fuzzy integral (or Sugeno integral) of a on A with respect to the fuzzy measure μ is defined as

$$\oint_A a d\mu = \bigvee_{\alpha \geq 0} [\alpha \wedge \mu(A \cap \{a \geq \alpha\})],$$

where $\vee$ and $\wedge$ denote the operations sup and inf on $[0, \infty)$, respectively.

Next, we state some of important cases of the fuzzy integral, which were proved in ([26]).

Proposition 1.3. *Let (X, η, μ) be a fuzzy measure space, $A, B \in \eta$ and $a, b \in f^{\mu}(X)$, therefore:*

(1) $\int_A a d\mu \leq \mu(A)$,
(2) $\int_A k d\mu = k \wedge \mu(A)$, *where k is a non-negative constant,*
(3) *if $a \leq b$ on A, then $\int_A a d\mu \leq \int_A b d\mu$,*
(4) *if $A \subset B$, then $\int_A a d\mu \leq \int_B a d\mu$,*
(5) *if $\mu(A) < \infty$, then $\mu(A \cap a \geq \alpha) \geq \alpha \quad \Rightarrow \quad \int_A a d\mu \geq \alpha$,*
(6) *if $\mu(A) < \infty$, then $\mu(A \cap a \geq \alpha) \leq \alpha \quad \Rightarrow \quad \int_A a d\mu \leq \alpha$,*

Remark 1.4. Let K be the function defined by $K(\alpha) = \mu(A \cap \{a \geq \alpha\})$; then, from parts (5) and (6) of Proposition 1.3,

$$K(\alpha) = \alpha \quad \Rightarrow \quad \int_A a d\mu = \alpha.$$

Hence, from a numerical point of view, the Sugeno integral can be calculated by solving the equation $K(\alpha) = \alpha$.

Theorem 1.5. *Let μ be the Lebesgue measure on $\mathbb{R}$ and let $b : [0, \infty) \to [0, \infty)$ be a continuous and strictly increasing function. If $\int_0^t b d\mu = m$, then*

$$b(t - m) \geq \int_0^t b d\mu = m, \quad \forall t \geq 0. \tag{1.1}$$

Also, if $0 < m < t$, we have $b(t - m) = m$.

Theorem 1.6 ([26])**.** *Let $\int_A a d\mu = m < \infty$. Then for all $s \geq m$, $\int_A a d\mu = \int_0^s (\mu(A \cap K_\alpha) dl$, wherever l is the Lebesgue measure.*

Theorem 1.7 ([13])**.** *Assume that a is a monotone and nonnegative concave function on $[0, 1]$, and that μ is the Lebesgue measure on $\mathbb{R}$. Then for $m > 1$ the inequality*

$$\left(m^{\frac{1}{1-m}} - m^{\frac{m}{1-m}} + 1\right) \left(\int_0^1 a(x) d\mu\right) \geq \left(\int_0^1 (a(x))^m d\mu\right)^{\frac{1}{m}}$$

holds.

Definition 1.8. Let $[t, z]$ be a closed and subinterval of $[-\infty, \infty]$. The full order on $[t, z]$ will be denoted by $\preceq$.

The operation $\oplus$ (pseudo-addition) is a function $\oplus : [t, z] \times [t, z] \to [t, z]$ which is associative, commutative, nondecreasing (with respect to $\preceq$) and with a zero (neutral) element denoted by $\mathbf{0}$, i.e., for every $x \in [t, z], \mathbf{0} \oplus x = x$ holds. And let $[t, z]_+ = \{x | x \in [t, z], \mathbf{0} \preceq x\}$.

The operation $\odot$ (pseudo-multiplication) is a function $\odot : [t,z] \times [t,z] \to [t,z]$ which is positively, commutative, non-decreasing and associative. We suppose too $\mathbf{0} \odot x = \mathbf{0}$ that $\odot$ is a distributive pseudo-multiplication with respect to $\oplus$. The structure $([t,z], \oplus, \odot)$ is a semiring ([14, 24]). We ought to consider the semiring $([t,z], \oplus, \odot)$, when pseudo-operations are generated by a monotone and continuous function $g : [t,z] \to [0,\infty]$, i.e., pseudo-operations are given with $x \oplus y = g^{-1}(g(x) + g(y))$ and $x \odot y = g^{-1}(g(x)g(y))$.

Therefore, the pseudo-integral for a function $a : [e,j] \to [t,z]$ reduces on the g-integral ([20]),

$$\int_{[e,j]}^{\oplus} a(x)dx = g^{-1}\left(\int_e^j g(a(x))dx\right). \tag{1.2}$$

Now when $x \oplus y = \max(x,y)$ and $x \odot y = g^{-1}(g(x)g(y))$, the pseudo-integral for a function $f : \mathbb{R} \to [t,z]$ is given by

$$\int_{\mathbb{R}}^{\oplus} a \odot dl = \sup\left(a(x) \odot \psi(x)\right),$$

where function ψ defines sup-measure l.

Theorem 1.9 ([16]). *Let l be a* sup*-measure on $([0,\infty], \mathbb{B}[0,\infty])$, where $\mathbb{B}([0,\infty])$ is the Borel σ-algebra on $[0,\infty]$, $l(A) = ess\sup(x|x \in A) = \sup\{t|\mu(x|x \in A, x > t) > 0\} = ess\sup_\mu(\psi(x)|x \in A)$, and $\psi : [0,\infty] \to [0,\infty]$ is a continuouse density. Therefore for each pseudo-addition $\oplus$ with a generator g there exists a family, l_λ of $\oplus_\lambda$-measure on $([0,\infty], \mathbb{B})$, where $\oplus_\lambda$ is a generated by g^λ (the function g of the power λ), $\lambda \in (0,\infty)$, considering that $\lim_{\lambda\to\infty} l_\lambda = l$.*

Theorem 1.10 ([16]). *Let $([0,\infty], \sup, \odot)$ be a semiring, when $\odot$ is a generated with g, i.e., we have $x \odot y = g^{-1}(g(x)g(y))$ for each $x, y \in (0,\infty)$. Let l be the same as in Theorem 1.9, so there exists a family $\{l_\lambda\}$ of $\oplus_\lambda$ -measures, where $\oplus_\lambda$ is a generated by $g^\lambda, \lambda \in (0,\infty)$ considering that for each continuous function $a : [0,\infty] \to [0,\infty]$,*

$$\begin{aligned}\int^{\sup} a \odot dl &= \lim_{\lambda\to\infty} \int^{\oplus_\lambda} a \odot dl_\lambda \\ &= \lim_{\lambda\to\infty} \left(g^\lambda\right)^{-1}\left(\int g^\lambda(a(x))dx\right).\end{aligned}$$

Definition 1.11. Functions $a, b : X \to \mathbb{R}$ are said to be comonotone if for all $x, y \in X$,

$$(a(x) - a(y))(b(x) - b(y)) \geq 0.$$

Theorem 1.12 ([18]). *Let $a, b \in F_+(X)$ and μ be an arbitrary fuzzy measure considering that both $\int_A ad\mu$ and $\int_A bd\mu$ are finite. And let*

$\bigstar : [0,\infty)^2 \to [0,\infty)$ be continuos and nondecreasing in both arguments and bounded from above by minimum. If a, b are comonotone, then the inequality

$$\int_A a \bigstar b d\mu \geq \left(\int_A a d\mu\right) \bigstar \left(\int_A b d\mu\right)$$

holds.
It is of special absorption to determine the operator $\bigstar$ considering that

$$\int_A a \bigstar b d\mu = \left(\int_A a d\mu\right) \bigstar \left(\int_A b d\mu\right)$$

holds for any comonotone function a, b, and for any fuzzy measure μ and any measurable set A.

Corollary 1.13 ([18])**.** *Let $a, b \in F_+(X)$ and μ be an arbitrary fuzzy measure considering that both $\int_A a d\mu$ and $\int_A b d\mu$ are finite. And let $\bigstar : [0,\infty)^2 \to [0,\infty)$ be continuos and nondecreasing in both arguments and blong to minimum operator. If a, b are comonotone, then the inequality*

$$\left(\int_A (a \bigstar b)^s d\mu\right)^{\frac{1}{s}} = \left(\int_A a^s d\mu\right)^{\frac{1}{s}} \bigstar \left(\int_A b^s d\mu\right)^{\frac{1}{s}}$$

holds for all $0 < s < \infty$.

Now, in the Main Results, firstly, we generalize the Bellman type inequality for Sugeno integral, later on, we generalize the Bellman type inequality for pseudo-integrals and finally, we generalize the Bellman type inequality in abstract spaces.

2. Main Results

Theorem 2.1. *(Fuzzy Bellman's inequality). Let $a, b : [0,1] \to \mathbb{R}$ be two real-valued functions and let μ be the Lebesgue measure on $\mathbb{R}$. If a and b are both measurable, non-negative and concave functions on [0,1], therefore the inequality*

$$\int_0^1 ab d\mu \geq \frac{1}{2} \left(\int_0^1 a^2 d\mu\right)^{\frac{1}{2}} \left(\int_0^1 b^2 d\mu\right)^{\frac{1}{2}}$$

holds.

Proof. Let $\left(\int_0^1 a^2 d\mu\right)^{\frac{1}{2}} = m$, $\left(\int_0^1 b^2 d\mu\right)^{\frac{1}{2}} = n$ and $\int_0^1 ab d\mu = s$. Since $\int_A a d\mu \leq \mu(A)$, so we have $m, n \in [0,1]$. Moreover, by assume that $m, n \in (0,1]$. Then we will be separate in two case:
Case (1): suppose that $m, n \in (0,1)$. We have $0 < s < 1$ and from Theorem 1.5. We have

$$ab(1-s) = s. \tag{2.1}$$

So,

$$mn < m \quad \text{and} \quad mn < n, \tag{2.2}$$

therefor

$$1 - mn \geq 1 - m \quad \text{and} \quad 1 - mn \geq 1 - n. \tag{2.3}$$

So, since a and b are increasing functions, therefore by Theorem 1.5 and (2.3) we attain that

$$a(1 - mn) \geq a(1 - m) \geq m \tag{2.4}$$

and

$$b(1 - mn) \geq b(1 - n) \geq n. \tag{2.5}$$

So, if we assume that $s < mn$, then $1 - s \geq 1 - mn$ and, therefore

$$a(1 - s) \geq a(1 - mn) \geq a(1 - m) \geq m \tag{2.6}$$

and

$$b(1 - s) \geq b(1 - mn) \geq b(1 - n) \geq n. \tag{2.7}$$

Finally, from (2.6) and (2.7) we obtain that

$$ab(1 - s) = a(1 - s)b(1 - s) > mn. \tag{2.8}$$

This is a contradiction with (2.1). It follows that $s \geq mn$.
Case (2): $m = 1$ or $n = 1$. Without lossing of generality we can assume that $m = 1$ and, in this case, we attain that

$$mn = n. \tag{2.9}$$

Therefor, by Theorem 1.5, we have

$$a(1 - m) \geq m \quad \Rightarrow \quad a(0) \geq 1. \tag{2.10}$$

Since a is strictly increasing, we have $a \geq 1$ on [0,1] and, by (2.10) we attain

$$ab \geq b. \tag{2.11}$$

So, from (2.9), (2.11) and (2.11) we have

$$\begin{aligned} \fint_0^1 abd\mu &\geq \fint_0^1 bd\mu \\ &\geq \frac{1}{2}\left(\fint_0^1 b^2 d\mu\right)^{\frac{1}{2}} \\ &= \frac{1}{2}n \\ &= \frac{1}{2}mn \end{aligned}$$

$$= \frac{1}{2}\left(\oint_0^1 a^2 d\mu\right)^{\frac{1}{2}}\left(\oint_0^1 b^2 d\mu\right)^{\frac{1}{2}}.$$

So, the proof is complete. □

Example 2.2. Let $a(x) = \sqrt{\frac{x}{4}}$ and $b(x) = \sqrt{x}$. By Definition 1.2, we have

$$(i) \quad \oint_0^1 abd\mu = \oint_0^1 \frac{1}{2}xd\mu = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \mu(fg \geq \alpha)] \cong 0.33,$$

$$(ii) \quad \oint_0^1 a^2 d\mu = \oint_0^1 \frac{x}{4}d\mu = 0.2, \quad \text{then} \quad \left(\oint_0^1 \frac{x}{4}d\mu\right)^{\frac{1}{2}} \cong 0.44,$$

$$(iii) \quad \oint_0^1 b^2 d\mu = \oint_0^1 xd\mu = 0.5, \quad \text{then} \quad \left(\oint_0^1 xd\mu\right)^{\frac{1}{2}} \cong 0.7.$$

Therefore

$$\begin{aligned} 0.33 &= \oint_0^1 abd\mu \\ &\geq \frac{1}{2}\left(\oint_0^1 a^2 d\mu\right)^{\frac{1}{2}}\left(\oint_0^1 b^2 d\mu\right)^{\frac{1}{2}} \\ &= 0.15. \end{aligned}$$

In the continue, we generalize the Bellman type inequality for pseudo-integrals when pseudo-operation $\oplus$ and $\odot$ are generated from an increasing continuous function.

Theorem 2.3 (Bellman's inequality for pseudo-integrals)**.** *Let $h, k : [0,1] \to [t,z]$ be two measurable functions and let a generator $g : [t,z] \to [0,\infty)$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing and non-negative function. If h and k are comonotone, then the inequality*

$$\left(\int_{[0,1]}^{\oplus} (h \odot k)dx\right) \geq \frac{1}{2} \odot \left(\int_{[0,1]}^{\oplus} h^2 dx\right)^{\frac{1}{2}} \odot \left(\int_{[0,1]}^{\oplus} k^2 dx\right)^{\frac{1}{2}}$$

holds.

Proof. We know that

$$\begin{aligned} \int_{[0,1]}^{\oplus} (h \odot k)dx &= g^{-1}\int_0^1 g\left(g^{-1}\left(g(h).g(k)\right)\right)dx \\ &= g^{-1}\int_0^1 g(h).g(k)dx. \end{aligned}$$

Then using the Bellman integral inequality and because g is an increasing function, it's inverse function g^{-1} is also increasing function, so

$$\begin{aligned}
\int_{[0,1]}^{\oplus}(h\odot k)dx &\geq g^{-1}\left(\frac{1}{2}\left(\int_0^1(g(h))^2dx\right)^{\frac{1}{2}}.\left(\int_0^1(g(k))^2dx\right)^{\frac{1}{2}}\right)\\
&= g^{-1}\left(\frac{1}{2}\left(gg^{-1}\int_0^1(g(h))^2dx\right)^{\frac{1}{2}}.\left(gg^{-1}\int_0^1(g(k))^2dx\right)^{\frac{1}{2}}\right)\\
&= g^{-1}\left(\frac{1}{2}\left(gg^{-1}\int_0^1 gg^{-1}(g(h).g(h))dx\right)^{\frac{1}{2}}\right.\\
&\quad\left.\cdot\left(gg^{-1}\int_0^1 gg^{-1}(g(k).g(k))dx\right)^{\frac{1}{2}}\right)\\
&= g^{-1}\left(\frac{1}{2}\left(g\int_{[0,1]}^{\oplus}h^2dx.g\int_{[0,1]}^{\oplus}k^2dx\right)^{\frac{1}{2}}\right)\\
&= g^{-1}\left(\frac{1}{2}\left(gg^{-1}\left(g\int_{[0,1]}^{\oplus}h^2dx.g\int_{[0,1]}^{\oplus}k^2dx\right)^{\frac{1}{2}}\right)\right)\\
&= g^{-1}\left(\frac{1}{2}\left(g(\int_{[0,1]}^{\oplus}h^2dx\odot\int_{[0,1]}^{\oplus}k^2dx)^{\frac{1}{2}}\right)\right)\\
&= g^{-1}\left(\frac{1}{2}\left(g\left(\left(\int_{[0,1]}^{\oplus}h^2dx\right)^{\frac{1}{2}}\odot\left(\int_{[0,1]}^{\oplus}k^2dx\right)^{\frac{1}{2}}\right)\right)\right)\\
&= \frac{1}{2}\odot\left(\int_{[0,1]}^{\oplus}h^2dx\right)^{\frac{1}{2}}\odot\left(\int_{[0,1]}^{\oplus}k^2dx\right)^{\frac{1}{2}}.
\end{aligned}$$

Which complete the proof. □

Example 2.4. Let $g(x)=x^2$. The correlative pseudo-operations are $x\oplus y=(x^2+y^2)^{\frac{1}{2}}$ and $x\odot y=xy$. If a,b are comonotone, therefore it holds

$$\left(\int_0^1(ab)^2d\mu\right)^{\frac{1}{2}}\geq\frac{1}{2}\left(\int_0^1a^4d\mu\right)^{\frac{1}{4}}\left(\int_0^1b^4d\mu\right)^{\frac{1}{4}}.$$

In the continue, we suppose that the semiring $([0,1],sup,\odot)$, when the operation $\odot$ is generated by a generator, and prove Bellman's inequality in this case.

Theorem 2.5. *Let $h,k:[0,1]\to[t,z]$ be two continuous functions and $\odot$ is exhibit by an increasing multiplicative generator g and l be the same*

as in Theorem 1.9. If h and k are comonotone, therefore the inequality

$$\int_{[0,1]}^{\sup}(h\odot k\odot dl)\geq\frac{1}{2}\odot\left(\int_{[0,1]}^{\sup}h^2\odot dl\right)^{\frac{1}{2}}\odot\left(\int_{[0,1]}^{\sup}k^2\odot dl\right)^{\frac{1}{2}}$$

holds.

Proof. Utilizing the same arguments in Theorem 2.3, the proof is easy. □

In the following, we prove a strengthened version of Bellman's inequality for pseudo-integrals.

Theorem 2.6. *Let $h,k:[0,1]\to[t,z]$ be two measurable functions and let a generator $g:[a,b]\to[0,\infty)$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function and $\varphi:[t,z]\to[t,z]$ be a continuous and strictly increasing function considering that φ commutes with $\odot$. If h and k are comonotone, so the inequality*

$$\int_{[0,1]}^{\oplus}\varphi(h\odot k)dx\geq\frac{1}{2}\odot\left[\left(\int_{[0,1]}^{\oplus}(\varphi(h))^2dx\right)^{\frac{1}{2}}\odot\left(\int_{[0,1]}^{\oplus}(\varphi(k))^2dx\right)^{\frac{1}{2}}\right]$$

holds.

Proof. because φ commutes with $\odot$, therefore

$$\int_{[0,1]}^{\oplus}\varphi(h\odot k)dx=\int_{[0,1]}^{\oplus}(\varphi(h)\odot\varphi(k))dx. \tag{2.12}$$

If h and k are comonotone functions and φ is a continuous and strictly increasing function, so $\varphi(h)$ and $\varphi(k)$ are too comonotone. From (2.12) and utilizing Theorem 2.3, we have

$$\int_{[0,1]}^{\oplus}\varphi(h\odot k)dx\geq\frac{1}{2}\odot\left[\left(\int_{[0,1]}^{\oplus}(\varphi(h))^2dx\right)^{\frac{1}{2}}\odot\left(\int_{[0,1]}^{\oplus}(\varphi(k))^2dx\right)^{\frac{1}{2}}\right]$$

Where φ commutes with $\odot$. □

In the continue, we suppose that the semiring $([0,1],sup,\odot)$, when the operation $\odot$ is generated by a generator.

Theorem 2.7. *Let $h,k:[0,1]\to[t,z]$ be two continuous functions and $\odot$ is exposed by an increasing multiplicative generator and $\varphi:[t,z]\to[t,z]$ be a continuous and strictly increasing function considering that φ commutes with $\odot$ and l be the same as in Theorem 1.9, if h,k are comonotone, then the inequality*

$$\int_{[0,1]}^{sup} \varphi(h\odot k)\odot dl \geq \frac{1}{2}\odot\left[\left(\int_{[0,1]}^{sup}(\varphi(h))^2\odot dl\right)^{\frac{1}{2}}\odot\left(\int_{[0,1]}^{sup}(\varphi(k))^2\odot dl\right)^{\frac{1}{2}}\right]$$

holds.

Proof. Using from Theorem 1.10 and the same arguments in Theorem 2.6, the proof is easy. □

In the following Theorem, we state and generalize the Bellman type inequality in abstract spaces, but before of it, we prove following Lemma.

Lemma 2.8. *Let $a, b \in F_+(X)$ and μ be an arbitrary fuzzy measure considering that both $\oint_A a d\mu$ and $\oint_A b d\mu$ are finite. And let $\bigstar : [0,\infty)^2 \to [0,\infty)$ be continuous and nondecreasing in both arguments and bounded from above by minimum. If a, b are comonotone, then the inequality*

$$\oint_A (a\bigstar b)d\mu \geq \frac{1}{2}\left(\oint_A (a\bigstar b)^2 d\mu\right)^{\frac{1}{2}}$$

holds.

Proof. By Theorems 1.7, 1.12 and Corollary 1.13, the proof is easy. □

Theorem 2.9 (Generalization of Bellman's inequality in abstract spaces)**.** *Let $a, b \in F_+(X)$ and μ be an arbitrary fuzzy measure considering that both $\oint_A a d\mu$ and $\oint_A b d\mu$ are finite. And let $\bigstar : [0,\infty)^2 \to [0,\infty)$ be continuous and nondecreasing in both arguments and bounded from above by minimum. If a, b are comonotone, then the inequality*

$$\oint_A (a\bigstar b)d\mu \geq \frac{1}{2}\left(\oint_A a^2 d\mu\right)^{\frac{1}{2}} \bigstar \left(\oint_A b^2 d\mu\right)^{\frac{1}{2}}$$

holds.

Proof. Notice that the finiteness of $\oint_A a d\mu \oint_A b d\mu$ implies that of finiteness $\left(\oint_A a^2 d\mu\right)^{\frac{1}{2}}\left(\oint_A b^2 d\mu\right)^{\frac{1}{2}}$. Put $\oint_A a^2 d\mu = m$, $\oint_A b^2 d\mu = n$ and $s > max\{m, n\}$. undoubtedly, if $m^{\frac{1}{2}}\bigstar n^{\frac{1}{2}} = 0$. Then the theorem holds , therefor we necessity lonely to prove the induction for
$m^{\frac{1}{2}}\bigstar n^{\frac{1}{2}} > 0$. Now whereas $\bigstar$ is bounded from above by minimum, we have that $0 < m, n < s$. Denote

$$A(\alpha) = \mu(A \cap \{x : a^2(x) \geq \alpha\}), \qquad B(\alpha) = \mu(A \cap \{x : b^2(x) \geq \alpha\}),$$

therefor, by using Theorem 1.6, we have

$$\oint_0^s A(\alpha)dl = m \quad \text{and} \quad \oint_0^s B(\alpha)dl = n,$$

where l is the Lebesgue measure. Then we necessity lonely to prove $\oint_A (a \bigstar b) d\mu \geq \frac{1}{2}\left(m^{\frac{1}{2}} \bigstar n^{\frac{1}{2}}\right)$. There fore we have

$$\mu\left(A \cap K_{(m-\epsilon)^{\frac{1}{2}}}\right) = A(m-\epsilon) \geq m$$

and

$$\mu\left(A \cap D_{(n-\epsilon)^{\frac{1}{2}}}\right) = B(n-\epsilon) \geq n.$$

By the monotonicity of $\bigstar$ and the comonotonicity of a, b, we have

$$\begin{aligned}
\mu\left(A \cap H_{(m-\epsilon)^{\frac{1}{2}} \star (n-\epsilon)^{\frac{1}{2}}}\right) &\geq \mu\left(A \cap K_{(m-\epsilon)^{\frac{1}{2}}} \cap D_{(n-\epsilon)^{\frac{1}{2}}}\right) \\
&= \min\left(\mu\left(A \cap K_{(m-\epsilon)^{\frac{1}{2}}}\right), \mu\left(A \cap D_{(n-\epsilon)^{\frac{1}{2}}}\right)\right) \\
&= \min\left(A\left(m-\epsilon\right), B\left(n-\epsilon\right)\right) \\
&\geq \min(m, n) \\
&\geq \left(m^{\frac{1}{2}} \bigstar n^{\frac{1}{2}}\right)^2.
\end{aligned}$$

Where $H_\alpha = \{x : a(x) \bigstar b(x) \geq \alpha\}$. Notation that if we denote

$$W(\alpha) = \mu\left(A \cap \{x : (a(x) \bigstar b(x)) \geq \alpha\}\right).$$

Then

$$\begin{aligned}
W\left(((m-\epsilon) \bigstar (n-\epsilon))\right) &= \mu\left(A \cap H_{(m-\epsilon)^{\frac{1}{2}} \star (n-\epsilon)^{\frac{1}{2}}}\right) \\
&\geq \left(m^{\frac{1}{2}} \bigstar n^{\frac{1}{2}}\right)^2
\end{aligned}$$

holds for every ϵ. Letting $\epsilon \to 0$, by the continuity of $\bigstar$ we attain that

$$W\left(\left(m^{\frac{1}{2}} \bigstar n^{\frac{1}{2}}\right) -\right) \geq \left(m^{\frac{1}{2}} \bigstar n^{\frac{1}{2}}\right).$$

Thus we have $\oint_A (a \bigstar b)^2 d\mu \geq \left(m^{\frac{1}{2}} \bigstar n^{\frac{1}{2}}\right)^2$ and which implies that

$$\begin{aligned}
\oint_A (a \bigstar b)^2 d\mu &\geq \left(m^{\frac{1}{2}} \bigstar n^{\frac{1}{2}}\right)^2 \\
&= \left(\left(\oint_A a^2 d\mu\right)^{\frac{1}{2}} \bigstar \left(\oint_A b^2 d\mu\right)^{\frac{1}{2}}\right)^2.
\end{aligned}$$

Therefore

$$\left(\oint_A (a \bigstar b)^2 d\mu\right)^{\frac{1}{2}} \geq \left(\left(\oint_A a^2 d\mu\right)^{\frac{1}{2}} \bigstar \left(\oint_A b^2 d\mu\right)^{\frac{1}{2}}\right)$$

now by using Lemma 2.8, since $\oint_A (a\bigstar b)d\mu \geq \frac{1}{2}\left(\oint_A (a\bigstar b)^2 d\mu\right)^{\frac{1}{2}}$, so

$$\begin{aligned}\oint_A (a\bigstar b)d\mu &\geq \frac{1}{2}\left(\oint_A (a\bigstar b)^2 d\mu\right)^{\frac{1}{2}} \\ &\geq \frac{1}{2}\left(\left(\oint_A a^2 d\mu\right)^{\frac{1}{2}} \bigstar \left(\oint_A b^2 d\mu\right)^{\frac{1}{2}}\right).\end{aligned}$$

Then

$$\oint_A (a\bigstar b)d\mu \geq \frac{1}{2}\left(\oint_A a^2 d\mu\right)^{\frac{1}{2}} \bigstar \left(\oint_A b^2 d\mu\right)^{\frac{1}{2}}.$$

This completes the proof. □

Example 2.10. Let $a(x) = x^{\frac{1}{2}}$, $b(x) = 1$, $A = [0,1]$, and $\bigstar = \min$. Then we have

$$\begin{aligned}\oint_0^1 a d\mu &= \oint_0^1 x^{\frac{1}{2}} d\mu \cong 0.618, \\ \oint_0^1 b d\mu &= \oint_0^1 1 d\mu = 1, \\ \oint_0^1 a^2 d\mu &= \oint_0^1 x^{\frac{1}{2}^2} d\mu = 0.5.\end{aligned}$$

It follows that:

$$\left(\oint_0^1 a^2 d\mu\right)^{\frac{1}{2}} \cong 0.707, \qquad \oint_0^1 b^2 d\mu = \oint_0^1 (1)^2 d\mu = 1,$$

then

$$\begin{aligned}&\left(\oint_0^1 b^2 d\mu\right)^{\frac{1}{2}} = 1, \\ &\oint_0^1 (a\bigstar b)d\mu = \oint_0^1 (x^{\frac{1}{2}})\bigstar 1 d\mu = \oint_0^1 x^{\frac{1}{2}} d\mu \cong 0.618, \\ &\left(\oint_0^1 a^2 d\mu\right)^{\frac{1}{2}} \bigstar \left(\oint_0^1 b^2 d\mu\right)^{\frac{1}{2}} = (0.707)\bigstar(1) = 0.707.\end{aligned}$$

Hence

$$0.618 \cong \oint_0^1 (a\bigstar b)d\mu \geq \frac{1}{2}\left(\oint_0^1 a^2 d\mu\right)^{\frac{1}{2}} \bigstar \left(\oint_0^1 b^2 d\mu\right)^{\frac{1}{2}} \cong 0.353.$$

Theorem 2.11. *Let $a, b \in F_+(X)$ and μ be arbitrary fuzzy measure considering that $\oint_A \varphi(a)d\mu$ and $\oint_A \varphi(b)d\mu$ are finite. And let $\bigstar : [0,\infty)^2 \to [0,\infty)$ be continuous and nondecreasing in both arguments and bounded from above by minimum and $\varphi : [0,\infty) \to [0,\infty)$ be a continuous ,*

strictly increasing and convex function. If a, b are comonotone, then the inequality

$$\varphi^{-1}\left(\int_A \varphi(a\bigstar b)d\mu\right) \geq \frac{1}{2}\left(\varphi^{-1}\int_A (\varphi(a))^2\, d\mu\right)^{\frac{1}{2}} \bigstar \left(\varphi^{-1}\int_A (\varphi(b))^2\, d\mu\right)^{\frac{1}{2}}$$

holds.

Proof. Let $\left(\int_A (\varphi(a))^2\, d\mu\right)^{\frac{1}{2}} = m < \infty$ and $\left(\int_A (\varphi(b))^2\, d\mu\right)^{\frac{1}{2}} = n < \infty$. Part(5) of Proposition 1.3, implies that

$$\begin{aligned} &\left(\int_A (\varphi(a))^2\, d\mu\right)^{\frac{1}{2}} = m \\ \Rightarrow\quad &\left(\mu\left(A \cap (\varphi(a))^2\right)^{\frac{1}{2}} \geq m\right) \geq m \\ \Rightarrow\quad &\mu\left(A \cap F_{\varphi^{-1}(p)}\right) \geq p \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} &\left(\int_A (\varphi(g))^2\, d\mu\right)^{\frac{1}{2}} = q \\ \Rightarrow\quad &\left(\mu\left(A \cap (\varphi(g))^2\right)^{\frac{1}{2}} \geq q\right) \geq q \\ \Rightarrow\quad &\mu\left(A \cap K_{\varphi^{-1}(n)}\right) \geq n, \end{aligned} \tag{2.14}$$

where $K_\alpha = \{x : (a(x))^2 \geq \alpha\}$ and $D_\alpha = \{x : (b(x))^2 \geq \alpha\}$. Since $\bigstar : [0,\infty)^2 \to [0,\infty)$ is a continuous, nondecreasing and bounded from above by minimum and $\varphi : [0,\infty) \to [0,\infty)$ is a continuous, strictly increasing and convex function, so we have

$$\varphi^{-1}\left(\min\{m,n\}\right) = \min\left\{\varphi^{-1}(n), \varphi^{-1}(m)\right\} \geq \varphi^{-1}(m)\bigstar\varphi^{-1}(n),$$

then

$$\min\{m,n\} \geq \varphi\left(\varphi^{-1}(m)\bigstar\varphi^{-1}(n)\right). \tag{2.15}$$

Therefore (2.13), (2.14) and (2.15) imply that

$$\begin{aligned} \mu\left(A \cap H_{\varphi^{-1}(m)\star\varphi^{-1}(n)}\right) &\geq \mu\left(A \cap (K_{\varphi^{-1}(m)} \cap D_{\varphi^{-1}(n)})\right) \\ &= \min\{\mu(A \cap K_{\varphi^{-1}(m)}), \mu(A \cap D_{\varphi^{-1}(n)})\} \\ &\geq \min\{m,n\} \\ &\geq \varphi\left(\varphi^{-1}(m)\bigstar\varphi^{-1}(n)\right), \end{aligned}$$

where $H_\alpha = \{x : a(x)\bigstar b(x) \geq \alpha\}$. Denote $W_\alpha = \{x : \varphi(a(x)\bigstar b(x) \geq \alpha)\}$, then

$$\mu\left(A \cap W_{\varphi(\varphi^{-1}(m)\star\varphi^{-1}(n))}\right) = \mu\left(A \cap H_{\varphi^{-1}(m)\star\varphi^{-1}(n)}\right) \tag{2.16}$$

$$\geq \varphi\left(\varphi^{-1}(m)\bigstar\varphi^{-1}(n)\right)$$

and, consequently, from (2.16) and part (5) of Proposition 1.3, we obtain

$$\fint_A \varphi(a\bigstar b)d\mu \geq \varphi\left(\varphi^{-1}(m)\bigstar\varphi^{-1}(n)\right)$$

then

$$\varphi^{-1}\left(\fint_A \varphi(a\bigstar b)d\mu\right) \geq \varphi^{-1}(m)\bigstar\varphi^{-1}(n),$$

which completes the proof. □

3. Conclusion

In this paper, we have studied a general type of Bellman's inequality for Sugeno integral, pseudo-integral and abstract spaces. In future research, we will present a general $(\oplus,\otimes)$-convex function based on semiring $([a,b],\oplus,\otimes)$ with pseudo-addition $\oplus$ and pseudo-multiplication $\otimes$. Also we will prove a general Bellman's inequality for the pseudo-integral.

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