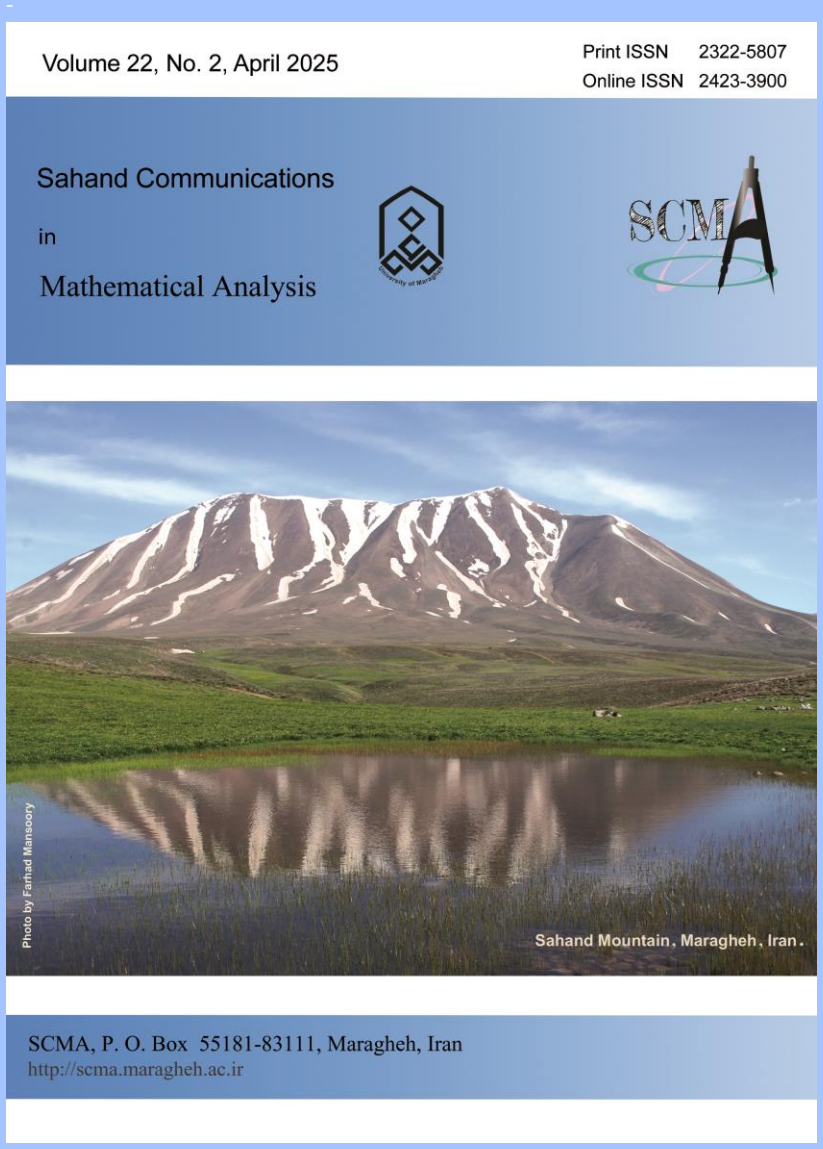
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Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 22
Number: 2
Pages: 307-332

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2024.2023382.1633



Sahand Communications in Mathematical Analysis (SCMA) Vol. 22 No. 2 (2025), 307-332
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2024.2023382.1633

Some New Improvements of Hermite-Hadamard Type Inequalities Using Strongly (s, m)-Convex Function with Applications

Arslan Munir[1] , Hüseyin Budak[2], Artion Kashuri[3], Irza Faiz[4], Hasan Kara[5] and Ather Qayyum[6*]

Abstract. The trapezoidal-type inequalities are discovered in this study using the fractional operator, which produces powerful results. We established a general identity for Caputo-Fabrizio integral operators and the second derivative function. Using this identity new error bounds and estimates for strongly (s, m)-convex functions are obtained. Moreover, some novel trapezoidal-type inequalities are offered taking this identity into account using the known inequalities like Young, Jensen, Hölder and power-mean inequalities. Finally, we present some applications for matrix inequality, estimation error regarding trapezoidal formulas and special means for real numbers.

1. Introduction

Fractional calculus, also known as the calculus of variations, is indeed a fascinating and important area of mathematics and science. Fractional calculus has gained significant recognition and practical applications in a wide range of scientific disciplines. The versatility of fractional calculus in addressing complex, real-world problems has led to its adoption in various fields, for example geophysics [11], bioengineering and biology [7, 17], demography [14], and signal processing. The growing interest and focus on fractional calculus over the last three decades reflect the increasing recognition of its potential in addressing real-world problems

2020 *Mathematics Subject Classification.* 26D07, 26D10, 26D15.
Key words and phrases. Trapezoidal-type inequality, Strongly (s, m)-convex function, Young's inequality, Jensen inequality.
Received: 2024-02-20, Accepted: 2024-07-17.
* Corresponding author.

and advancing scientific understanding. This surge in interest has resulted in a significant body of research and applications in the field of fractional calculus. The development of new fractional operators with distinct singular and non-singular kernels is a key strategy for advancing the field of fractional calculus and to provide effective tools for systematizing and solving real-world problems of engineering and science [9, 16, 21, 27]. These operators enhance the flexibility, accuracy, and applicability of fractional calculus across a wide range of disciplines. One of the significant limitations of integer-order operators in traditional calculus is that they are local operators. They are not equipped to effectively capture the effects of neighbors or long-range dependencies in situations where heavy tails occur. Fractional-order operators, in contrast to classical integer-order operators, are non-local in nature, which makes them well-suited to address sensitive situations and problems where classical operators may exhibit singularities or fail to provide accurate solutions.

Fractional operators play a significant role in integral inequalities, which is a fascinating area of mathematical research with various practical implications. These integral inequalities involve fractional derivatives and integrals and have applications in different fields. Convex function plays an important role in the study of integral inequalities, particularly in the context of establishing and analyzing various types of inequalities. The Hermite-Hadamard inequality for convex functions is a powerful tool in mathematical analysis and plays a crucial role in various fields, and its original formed is followed as:

$$\Lambda\left(\frac{\Omega+\Pi}{2}\right) \leq \frac{1}{\Pi-\Omega}\int_{\Omega}^{\Pi} \Lambda(x)\,dx \leq \frac{\Lambda(\Omega)+\Lambda(\Pi)}{2}.$$

The findings and principles behind the Hermite-Hadamard inequality which deals with integral inequalities for convex functions, can be extended and applied to fractional integral inequalities as well as the accompanying optimization problems. Several types of inequalities have been extensively studied, new proofs, generalizations, improvements, refinements and numerical applications, and illustrations have been discussed. Many authors have concentrated on the Hermite-Hadamard type inequalities for various classes of convex functions and mappings, such as s-convex function [25], quasi-convex function [26], strongly-convex function [8], m-convex function [2], m and (κ, m)-convex function [22]. In [15] Qaisar et al established the new error bound for Caputo-Fabrizio and three time differentiable functions. In [19] P. O. Mohammed et al introduced new Hermite-Hadamard type inequalities for differentiable function involving the incomplete gamma functions. In [29] Wang et al. proved the Hermite-Hadamard type inequalities using the Caputo-Fabrizio fractional operator and h-convex function. In [5]

Butt et al. developed several novel Hermite-Hadamard type inequalities for s and (s,m)-convex function and obtained new bounds for Caputao-Fabrizio derivatives. In [18] Khaled derived a new Hermite-Hadamard inequality and provided applications to midpoint formula, and Modified Bessel functions. For more information on fractional operators and convexity, see [20], [1, 12, 28]. This study is inspired by recent worked on Hermite-Hadamard type inequalities for different fractional operator and convexity.

2. Preliminaries

Definition 2.1 ([13])**.** The mapping of $\Lambda : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex, if

$$\Lambda\left(\Phi\Omega + (1-\Phi)\,\Pi\right) \leq \Phi\Lambda\left(\Omega\right) + (1-\Phi)\,\Lambda\left(\Pi\right),$$

for all $\Omega, \Pi \in I$ and $\Phi \in [0,1]$.

Definition 2.2 ([10])**.** The function $\Lambda : I \subseteq \mathbb{R} \to \mathbb{R}_0 := [0,\infty)$ is said to be s-convex, if

$$\Lambda\left(\Phi\Omega + (1-\Phi)\,\Pi\right) \leq \Phi^s\Lambda\left(\Omega\right) + (1-\Phi)^s\,\Lambda\left(\Pi\right),$$

for all $\Omega, \Pi \in I$ and $\Phi \in [0,1]$.

Definition 2.3 ([4])**.** The function $\Lambda : I \subseteq \mathbb{R} \to \mathbb{R}_0$ is said to be strongly (s,m)-convex, if

$$\Lambda\left(\Phi\Omega + (1-\Phi)\,\Pi\right) \leq \Phi^s\Lambda\left(\Omega\right) + m\,(1-\Phi)^s\,\Lambda\left(\Pi\right) - \mu\Phi\,(1-\Phi)\,(\Omega-\Pi)^2,$$

with modulus $\mu \geq 0$ and for all $\Omega, \Pi \in I$, $\Phi \in [0,1]$ and $(s,m) \in [0,1]^2$.

Definition 2.4 ([6])**.** Let $H^1(\Omega,\Pi)$ be the Sobolev space of order one defined as:

$$H^1(\Omega,\Pi) := \left\{g \in L^2(\Omega,\Pi) : g' \in L^2(\Omega,\Pi)\right\},$$

where

$$L^2(\Omega,\Pi) := \left\{g(z) : \left(\int_\Omega^\Pi g^2(z)\,dz\right)^{\frac{1}{2}} < \infty\right\}.$$

Let $\Lambda \in H^1(\Omega,\Pi)$, $\Omega < \Pi$, $\kappa \in (0,1)$, then the notion of left derivative in the sense of Caputo-Fabrizio is defined as:

$$\left({}^{CFD}_{\Omega}D^{\kappa}\Lambda\right)(x) = \frac{\beta(\kappa)}{1-\kappa}\int_\Omega^x \Lambda'(\Phi)\,e^{\frac{-\kappa(x-\Phi)^\kappa}{1-\kappa}}d\Phi,$$

$x > \Omega$ and the associated integral operator is

$$\left({}^{CF}_{\Omega}I^{\kappa}\Lambda\right)(x) = \frac{1-\kappa}{\beta(\kappa)}\Lambda(x) + \frac{\kappa}{\beta(\kappa)}\int_\Omega^x \Lambda(\Phi)\,d\Phi,$$

where $\beta(\kappa) > 0$ is the normalization function satisfying the condition $\beta(0) = \beta(1) = 1$. For $\kappa = 0$ and $\kappa = 1$, the left derivative is defined as follows, respectively

$$\left({}^{CFD}_{\Omega}D^{0}\Lambda\right)(x) = \Lambda'(x)$$
$$\left({}^{CFD}_{\Omega}D^{1}\Lambda\right)(x) = \Lambda(x) - \Lambda(\Omega).$$

For the right derivative operator

$$\left({}^{CFD}_{\Pi}D^{\kappa}\Lambda\right)(x) = \frac{\beta(\kappa)}{1-\kappa}\int_{x}^{\Pi}\Lambda'(\Phi)\, e^{\frac{-\kappa(\Phi-x)^{\kappa}}{1-\kappa}}d\Phi,$$

$x < \Pi$ and the associated integral operator is

$$\left({}^{CF}I^{\kappa}_{\Pi}\Lambda\right)(x) = \frac{1-\kappa}{\beta(\kappa)}\Lambda(x) + \frac{\kappa}{\beta(\kappa)}\int_{x}^{\Pi}\Lambda(\Phi)\, d\Phi.$$

Hong et al. [30] presented the Hermite-Hadamard type new inequalities via Caputo-Fabrizio integral operator is followed as:

Theorem 2.5. *Let $\Lambda : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I^{o} (the interior of I), $\Omega, \Pi \in I^{o}$ with $\Omega < \Pi$. If $\Lambda'' \in L[\Omega, \Pi]$ is strongly (s,m)-convex with modulus $\mu \geq 0$ for $(s,m) \in (0,1] \times (0,1]$, then following inequality holds:*

$$\begin{aligned}
&\left|\Lambda\left(\frac{\Omega+\Pi}{2}\right) + \frac{4(1-\kappa)}{\beta(\kappa)} - \frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\left[\left({}^{CF}_{\Omega}I^{\kappa}\Lambda\right)(x) + \left({}^{CF}_{\frac{\Omega+\Pi}{2}}I^{\kappa}\Lambda\right)(x)\right.\right.\\
&\qquad \left.\left. + \left({}^{CF}I^{\kappa}_{\frac{\Omega+\Pi}{2}}\Lambda\right)(x) + \left({}^{CF}I^{\kappa}_{\Pi}\Lambda\right)(x)\right]\right|\\
&\leq \frac{(\Pi-\Omega)^{2}}{2^{4+s}}\left[\frac{\left(2^{s+4} - 14 - s(7+s)\right)}{(s+1)(s+2)(s+3)}\left(\left|\Lambda''(\Omega)\right| + \left|\Lambda''(\Pi)\right|\right)\right.\\
&\qquad \left. + m\left(\frac{1}{s+1}\right)\left(\left|\Lambda''\left(\frac{\Omega}{m}\right)\right| + \left|\Lambda''\left(\frac{\Pi}{m}\right)\right|\right)\right].
\end{aligned}$$

Proposition 2.6. *If we put $s = m = 1$ in Theorem 2.5, then we have*

$$\begin{aligned}
&\left|\Lambda\left(\frac{\Omega+\Pi}{2}\right) + \frac{4(1-\kappa)}{\beta(\kappa)} - \frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\left[\left(\left({}^{CF}_{\Omega}I^{\kappa}\Lambda\right)(x) + \left({}^{CF}_{\frac{\Omega+\Pi}{2}}I^{\kappa}\Lambda\right)(x)\right)\right.\right.\\
&\qquad \left.\left. + \left(\left({}^{CF}I^{\kappa}_{\frac{\Omega+\Pi}{2}}\Lambda\right)(x) + \left({}^{CF}I^{\kappa}_{\Pi}\Lambda\right)(x)\right)\right]\right|\\
&\leq \frac{(\Pi-\Omega)^{2}}{24}\left(\frac{|\Lambda''(\Omega)| + |\Lambda''(\Pi)|}{2}\right).
\end{aligned}$$

The main motivation of this article is to prove a fractional integral identity involving the Caputo-Fabrizio integral operator and to derive some new trapezoidal type inequalities for convex and concave function with the help of known inequalities like Young, Jensen, Hölder and

power-mean inequalities. Some applications, including special means of real numbers, error estimation for trapezoidal formulas and matrix inequality will be discussed in detail.

3. Main Results

In this section we have derived a new identity for twice differentiable convex functions, which is required to obtain our main results.

Lemma 3.1. *Let $\Lambda : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ be a twice differentiable mapping on I^0 and $\Omega, \Pi \in I^0$ with $\Omega < \Pi$. If $\Lambda'' \in L[\Omega, \Pi]$ and $\kappa \in [0,1]$ for all $x \in [\Omega, \Pi]$, then the following fractional equality holds:*

$$\begin{aligned}
&\frac{\left[(x-\Omega)^2 - (\Pi - x)^2\right]\Lambda'(x) + 2(\Pi - x)\Lambda(\Pi) + 2(x-\Omega)\Lambda(\Omega)}{2(\Pi-\Omega)} \\
&\quad - \frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\left[\left({}^{CF}_{\Omega}I^{\kappa}\Lambda\right)(x) + \left({}^{CF}I^{\kappa}_{\Pi}\Lambda\right)(x)\right] + \frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x) \\
&= \frac{(x-\Omega)^3}{2(\Pi-\Omega)}\int_0^1 \left(1-\Phi^2\right)\Lambda''(\Phi\Omega + (1-\Phi)x)\,d\Phi \\
&\quad + \frac{(\Pi - x)^3}{2(\Pi-\Omega)}\int_0^1 \left(1-\Phi^2\right)\Lambda''(\Phi\Pi + (1-\Phi)x)\,d\Phi.
\end{aligned}$$

Proof. Let

$$\begin{aligned}
I &= \frac{(x-\Omega)^3}{2(\Pi-\Omega)}\int_0^1 \left(1-\Phi^2\right)\Lambda''(\Phi\Omega + (1-\Phi)x)\,d\Phi \\
&\quad + \frac{(\Pi - x)^3}{2(\Pi-\Omega)}\int_0^1 \left(1-\Phi^2\right)\Lambda''(\Phi\Pi + (1-\Phi)x)\,d\Phi \\
&= \frac{(x-\Omega)^3}{2(\Pi-\Omega)}I_1 + \frac{(\Pi-x)^3}{2(\Pi-\Omega)}I_2.
\end{aligned}$$

Using the integration by parts, we have

(3.1)

$$\begin{aligned}
I_1 &= \int_0^1 \left(1-\Phi^2\right)\Lambda''(\Phi\Omega + (1-\Phi)x)\,d\Phi \\
&= \frac{\left(1-\Phi^2\right)\Lambda'(\Phi\Omega + (1-\Phi)x)}{\Omega - x}\Bigg|_0^1 + \frac{2}{\Omega - x}\int_0^1 \Phi\Lambda'(\Phi\Omega + (1-\Phi)x)\,d\Phi \\
&= \frac{\Lambda'(x)}{(x-\Omega)} - \frac{2}{(x-\Omega)}\Lambda(\Omega)\int_0^1 \Phi\Lambda'(\Phi\Omega + (1-\Phi)x)\,d\Phi
\end{aligned}$$

$$= \frac{\Lambda'(x)}{(x-\Omega)} + \frac{2}{(x-\Omega)^2}\Lambda(\Omega) - \frac{2}{(x-\Omega)^3}\int_{\Omega}^{x} \Lambda(\Phi\Omega + (1-\Phi)x)\, d\Phi.$$

Multiply the equality (3.1) with $\frac{(x-\Omega)^3}{2}$, we get

$$\frac{(x-\Omega)^3}{2} I_1 = \frac{(x-\Omega)^3}{2}\frac{\Lambda'(x)}{(x-\Omega)} + \frac{2}{(x-\Omega)^2}\Lambda(\Omega)\frac{(x-\Omega)^3}{2} - \int_{\Omega}^{x}\Lambda(u)\, du.$$

Thus, we have

$$\frac{(x-\Omega)^3}{2(\Pi-\Omega)}\int_0^1 \left(1-\Phi^2\right)\Lambda''(\Phi\Omega + (1-\Phi)x)\, d\Phi = \frac{(x-\Omega)^2}{2}\Lambda'(x) + \frac{2(x-\Omega)\Lambda(\Omega)}{2} - \int_{\Omega}^{x}\Lambda(u)\, du. \tag{3.2}$$

Similarly, we get

$$I_2 = \int_0^1 \left(1-\Phi^2\right)\Lambda''(\Phi\Pi + (1-\Phi)x)\, d\Phi = -\frac{\Lambda'(x)}{(\Pi-x)} + \frac{2}{(\Pi-x)^2}\Lambda(\Pi) - \frac{2}{(\Pi-x)^3}\int_x^{\Pi}\Lambda(u)\, du. \tag{3.3}$$

Multiply the equality (3.3) with $\frac{(\Pi-x)^3}{2}$, we get

$$\frac{(\Pi-x)^3}{2} I_2 = -\frac{\Lambda'(x)}{(\Pi-x)}\frac{(\Pi-x)^3}{2} + \frac{2}{(\Pi-x)^2}\Lambda(\Pi)\frac{(\Pi-x)^3}{2} - \int_x^{\Pi}\Lambda(u)\, du.$$

Therefore, we get

$$\frac{(\Pi-x)^3}{2}\int_0^1 \left(1-\Phi^2\right)\Lambda''(\Phi\Pi + (1-\Phi)x)\, d\Phi = -\frac{(\Pi-x)^2}{2}\Lambda'(x) + \frac{2(\Pi-x)\Lambda(\Pi)}{2} - \int_x^{\Pi}\Lambda(u)\, du \tag{3.4}$$

and

$$\frac{(\Pi-x)^3}{2} I_2 = -\frac{\Lambda'(x)}{(\Pi-x)}\frac{(\Pi-x)^3}{2} + \frac{2}{(\Pi-x)^2}\Lambda(\Pi)\frac{(\Pi-x)^3}{2} - \int_x^{\Pi}\Lambda(u)\, du.$$

Adding the equalities (3.2) and (3.4), we get

$$
\begin{aligned}
(3.5)\qquad & \frac{(x-\Omega)^3}{2(\Pi-\Omega)}\int_0^1\left(1-\Phi^2\right)\Lambda''\left(\Phi\Omega+(1-\Phi)x\right)d\Phi \\
&\quad+\frac{(\Pi-x)^3}{2}\int_0^1\left(1-\Phi^2\right)\Lambda''\left(\Phi\Pi+(1-\Phi)x\right)d\Phi \\
&=\frac{(x-\Omega)^2}{2}\Lambda'(x)+\frac{2(x-\Omega)\Lambda(\Omega)}{2}-\int_\Omega^x\Lambda(u)\,du \\
&\quad-\frac{(\Pi-x)^2}{2}\Lambda'(x)+\frac{2(\Pi-x)\Lambda(\Pi)}{2}-\int_x^\Pi\Lambda(u)\,du \\
&=\frac{(x-\Omega)^2}{2}\Lambda'(x)-\frac{(\Pi-x)^2}{2}\Lambda'(x)+\frac{2(x-\Omega)\Lambda(\Omega)}{2} \\
&\quad+\frac{2(\Pi-x)\Lambda(\Pi)}{2}-\int_\Omega^\Pi\Lambda(u)\,du.
\end{aligned}
$$

Divide $(\Pi-\Omega)$ with equality (3.5), we have

$$
\begin{aligned}
(3.6)\qquad & \frac{(x-\Omega)^3}{2(\Pi-\Omega)}\int_0^1\left(1-\Phi^2\right)\Lambda''\left(\Phi\Omega+(1-\Phi)x\right)d\Phi \\
&\quad+\frac{(\Pi-x)^3}{2}\int_0^1\left(1-\Phi^2\right)\Lambda''\left(\Phi\Pi+(1-\Phi)x\right)d\Phi \\
&=\frac{(x-\Omega)^2}{2(\Pi-\Omega)}\Lambda'(x)-\frac{(\Pi-x)^2}{2(\Pi-\Omega)}\Lambda'(x)+\frac{2(x-\Omega)\Lambda(\Omega)}{2(\Pi-\Omega)} \\
&\quad+\frac{2(\Pi-x)\Lambda(\Pi)}{2(\Pi-\Omega)}-\frac{1}{(\Pi-\Omega)}\int_\Omega^\Pi\Lambda(u)\,du.
\end{aligned}
$$

Subtracting $\frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)$ with equality (3.6), we have

$$
\begin{aligned}
&\frac{(x-\Omega)^3}{2(\Pi-\Omega)}\int_0^1\left(1-\Phi^2\right)\Lambda''\left(\Phi\Omega+(1-\Phi)x\right)d\Phi \\
&\quad+\frac{(\Pi-x)^3}{2}\int_0^1\left(1-\Phi^2\right)\Lambda''\left(\Phi\Pi+(1-\Phi)x\right)d\Phi-\frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x) \\
&=\frac{(x-\Omega)^2}{2(\Pi-\Omega)}\Lambda'(x)-\frac{(\Pi-x)^2}{2(\Pi-\Omega)}\Lambda'(x)+\frac{2(x-\Omega)\Lambda(\Omega)}{2(\Pi-\Omega)} \\
&\quad+\frac{2(\Pi-x)\Lambda(\Pi)}{2(\Pi-\Omega)}-\frac{1}{(\Pi-\Omega)}\int_\Omega^\Pi\Lambda(u)\,du-\frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x) \\
&=\frac{(x-\Omega)^2}{2(\Pi-\Omega)}\Lambda'(x)-\frac{(\Pi-x)^2}{2(\Pi-\Omega)}\Lambda'(x)+\frac{2(x-\Omega)\Lambda(\Omega)}{2(\Pi-\Omega)}
\end{aligned}
$$

$$+\frac{2(\Pi-x)\Lambda(\Pi)}{2(\Pi-\Omega)}-\frac{\beta(\kappa)\kappa}{\kappa(\Pi-\Omega)\beta(\kappa)}$$
$$\times\left(\int_{\Omega}^{x}\Lambda(u)\,du-\frac{(1-\kappa)}{\beta(\kappa)}\Lambda(x)+\int_{x}^{\Pi}\Lambda(u)\,du-\frac{(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right).$$

Thus, we have

$$\begin{aligned}&\frac{\left[(x-\Omega)^2-(\Pi-x)^2\right]\Lambda'(x)+2(\Pi-x)\Lambda(\Pi)+2(x-\Omega)\Lambda(\Omega)}{2(\Pi-\Omega)}\\&\quad-\frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\left[\left({}_{\Omega}^{CF}I^{\kappa}\Lambda\right)(x)+\left({}^{CF}I_{\Pi}^{\kappa}\Lambda\right)(x)\right]+\frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\\&=\frac{(x-\Omega)^3}{2(\Pi-\Omega)}\int_0^1\left(1-\Phi^2\right)\Lambda''(\Phi\Omega+(1-\Phi)x)\,d\Phi\\&\quad+\frac{(\Pi-x)^3}{2}\int_0^1\left(1-\Phi^2\right)\Lambda''(\Phi\Pi+(1-\Phi)x)\,d\Phi.\end{aligned}$$

This completes the proof. □

Theorem 3.2. *Under the assumptions of Lemma 3.1 holds. If* $\Lambda''\in L[\Omega,\Pi]$ *and* $|\Lambda''|$ *is strongly* (s,m)*-convex with modulus* $\mu\geq 0$, $(s,m)\in(0,1]\times(0,1]$, *and* $\kappa\in[0,1]$ *for all* $x\in[\Omega,\Pi]$, *then the following fractional inequality:*

$$\begin{aligned}&\left|\frac{\left[(x-\Omega)^2-(\Pi-x)^2\right]\Lambda'(x)+2(\Pi-x)\Lambda(\Pi)+2(x-\Omega)\Lambda(\Omega)}{2(\Pi-\Omega)}\right.\\&\quad\left.-\frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\left[\left({}_{\Omega}^{CF}I^{\kappa}\Lambda\right)(x)+\left({}^{CF}I_{\Pi}^{\kappa}\Lambda\right)(x)\right]+\frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right|\\&\leq\frac{(x-\Omega)^3}{2(\Pi-\Omega)}\left[\frac{2}{3+4s+s^2}\left|\Lambda''(\Omega)\right|+m\left(\frac{4+s}{6+5s+s^2}\right)\left|\Lambda''\left(\frac{x}{m}\right)\right|\right.\\&\quad\left.-\frac{7\mu m}{60}\left(\Omega-\frac{x}{m}\right)^2\right]\\&\quad+\frac{(\Pi-x)^3}{2(\Pi-\Omega)}\left[\frac{2}{3+4s+s^2}\left|\Lambda''(\Pi)\right|+m\left(\frac{4+s}{6+5s+s^2}\right)\left|\Lambda''\left(\frac{x}{m}\right)\right|\right.\\&\quad\left.-\frac{7\mu m}{60}\left(\Pi-\frac{x}{m}\right)^2\right].\end{aligned}$$

Proof. By using the Lemma 3.1, and the strongly-(s,m)-convexity of $|\Lambda''|$, we have

$$\begin{aligned}
&\left|\frac{\left[(x-\Omega)^2-(\Pi-x)^2\right]\Lambda'(x)+2(\Pi-x)\Lambda(\Pi)+2(x-\Omega)\Lambda(\Omega)}{2(\Pi-\Omega)}\right.\\
&\quad\left.-\frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\left[\left({}_{\Omega}^{CF}I^{\kappa}\Lambda\right)(x)+\left({}^{CF}I_{\Pi}^{\kappa}\Lambda\right)(x)\right]+\frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right|\\
&\leq\frac{(x-\Omega)^3}{2(\Pi-\Omega)}\int_0^1\left(1-\Phi^2\right)\Lambda''(\Phi\Omega+(1-\Phi)x)\,d\Phi\\
&\quad+\frac{(\Pi-x)^3}{2}\int_0^1\left(1-\Phi^2\right)\Lambda''(\Phi\Pi+(1-\Phi)x)\,d\Phi\\
&\leq\frac{(x-\Omega)^3}{2(\Pi-\Omega)}\int_0^1\left(1-\Phi^2\right)\\
&\quad\times\left(\Phi^s|\Lambda''(\Omega)|+m(1-\Phi)^s\left|\Lambda''\left(\frac{x}{m}\right)\right|-m\mu\Phi(1-\Phi)\left(\Omega-\frac{x}{m}\right)^2\right)d\Phi\\
&\quad+\frac{(\Pi-x)^3}{2(\Pi-\Omega)}\int_0^1\left(1-\Phi^2\right)\\
&\times\left(\Phi^s|\Lambda''(\Pi)|+m(1-\Phi)^s\left|\Lambda''\left(\frac{x}{m}\right)\right|-m\mu\Phi(1-\Phi)\left(\Pi-\frac{x}{m}\right)^2\right)d\Phi\\
&=\frac{(x-\Omega)^3}{2(\Pi-\Omega)}\left[\frac{2}{3+4s+s^2}|\Lambda''(\Omega)|+m\left(\frac{4+s}{6+5s+s^2}\right)\left|\Lambda''\left(\frac{x}{m}\right)\right|\right.\\
&\quad\left.-\frac{7\mu m}{60}\left(\Omega-\frac{x}{m}\right)^2\right]+\frac{(\Pi-x)^3}{2(\Pi-\Omega)}\left[\frac{2}{3+4s+s^2}|\Lambda''(\Pi)|\right.\\
&\quad\left.+m\left(\frac{4+s}{6+5s+s^2}\right)\left|\Lambda''\left(\frac{x}{m}\right)\right|-\frac{7\mu m}{60}\left(\Pi-\frac{x}{m}\right)^2\right].
\end{aligned}$$

The proof of Theorem 3.2 is completed. □

Corollary 3.3. *If we choose $x=\frac{\Omega+\Pi}{2}$ in Theorem 3.2, then we have the following trapezoidal-type inequality:*

$$\begin{aligned}
&\left|\frac{\Lambda(\Omega)+\Lambda(\Pi)}{2}-\frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\left[\left({}_{\Omega}^{CF}I^{\kappa}\Lambda\right)(x)+\left({}^{CF}I_{\Pi}^{\kappa}\Lambda\right)(x)\right]\right.\\
&\quad\left.+\frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right|\\
&\leq\frac{(\Pi-\Omega)^2}{16}\left[\frac{4}{3+4s+s^2}\left(\left|\Lambda''(\Omega)\right|+\left|\Lambda''(\Pi)\right|\right)\right.\\
&\quad\left.+2m\left(\frac{4+s}{6+5s+s^2}\right)\left|\Lambda''\left(\frac{\Omega+\Pi}{2m}\right)\right|\right]
\end{aligned}$$

$$-\frac{7\mu m}{60}\left(\left(\Omega-\frac{\Omega+\Pi}{2m}\right)^2+\left(\Pi-\frac{\Omega+\Pi}{2m}\right)^2\right).$$

Corollary 3.4. *If we choose $\mu=0$ in Corollary 3.3, then we get*

$$\begin{aligned}&\left|\frac{\Lambda(\Omega)+\Lambda(\Pi)}{2}-\frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\left[\left({}_{\Omega}^{CF}I^{\kappa}\Lambda\right)(x)+\left({}^{CF}I_{\Pi}^{\kappa}\Lambda\right)(x)\right]\right.\\&\left.+\frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right|\\&\leq\frac{(\Pi-\Omega)^2}{16}\left[\frac{4}{3+4s+s^2}\left(\left|\Lambda''(\Omega)\right|+\left|\Lambda''(\Pi)\right|\right)\right.\\&\left.+2m\left(\frac{4+s}{6+5s+s^2}\right)\left|\Lambda''\left(\frac{\Omega+\Pi}{2m}\right)\right|\right].\end{aligned}$$

Corollary 3.5. *If we choose $m=1$ in Corollary 3.4, then we obtain*

$$\begin{aligned}&\left|\frac{\Lambda(\Omega)+\Lambda(\Pi)}{2}-\frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\left[\left({}_{\Omega}^{CF}I^{\kappa}\Lambda\right)(x)+\left({}^{CF}I_{\Pi}^{\kappa}\Lambda\right)(x)\right]\right.\\&\left.+\frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right|\\&\leq\frac{(\Pi-\Omega)^2}{16}\left[\frac{4}{3+4s+s^2}\left(\left|\Lambda''(\Omega)\right|+\left|\Lambda''(\Pi)\right|\right)\right.\\&\left.+\frac{2(4+s)}{6+5s+s^2}\left|\Lambda''\left(\frac{\Omega+\Pi}{2}\right)\right|\right].\end{aligned}$$

Corollary 3.6. *If we choose $s=1$ in Corollary 3.5, then we have*

$$\begin{aligned}&\left|\frac{\Lambda(\Omega)+\Lambda(\Pi)}{2}-\frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\left[\left({}_{\Omega}^{CF}I^{\kappa}\Lambda\right)(x)+\left({}^{CF}I_{\Pi}^{\kappa}\Lambda\right)(x)\right]\right.\\&\left.+\frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right|\\&\leq\frac{(\Pi-\Omega)^2}{48}\left[\left(\left|\Lambda''(\Omega)\right|+\left|\Lambda''(\Pi)\right|\right)\right].\end{aligned}$$

Remark 3.7. If we choose $\kappa=1$ and $\beta(0)=\beta(1)=1$ in Corollary 3.6, then we get

$$\begin{aligned}&\left|\frac{\Lambda(\Omega)+\Lambda(\Pi)}{2}-\frac{1}{(\Pi-\Omega)}\int_{\Omega}^{\Pi}\Lambda(x)\,dx\right|\\&\leq\frac{(\Pi-\Omega)^2}{48}\left(\left|\Lambda''(\Omega)\right|+\left|\Lambda''(\Pi)\right|\right).\end{aligned}\tag{3.7}$$

We emphasize the fact that our obtained inequality (3.7) improved the error bound given by Sarikaya et al. in [24, Proposition 2].

Corollary 3.8. *If we choose $s=0$, $m=1$ and $x=\frac{\Omega+\Pi}{2}$ in Theorem 3.2, then we have*

$$\begin{aligned}&\left|\frac{\Lambda(\Omega)+\Lambda(\Pi)}{2}-\frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\left[\left({}_{\Omega}^{CF}I^{\kappa}\Lambda\right)(x)+\left({}^{CF}I_{\Pi}^{\kappa}\Lambda\right)(x)\right]\right.\\&\quad\left.+\frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right|\\&\leq\frac{(\Pi-\Omega)^2}{16}\left[\frac{4}{3}\left(\left|\Lambda''(\Omega)\right|+\left|\Lambda''(\Pi)\right|\right)+\frac{4}{6}\left|\Lambda''\left(\frac{\Omega+\Pi}{2}\right)\right|\right.\\&\quad\left.-\frac{7\mu}{60}\left(\left(\Omega-\frac{\Omega+\Pi}{2}\right)^2+\left(\Pi-\frac{\Omega+\Pi}{2}\right)^2\right)\right].\end{aligned}$$

Corollary 3.9. *If we choose $s=1$, $m=1$ and $x=\frac{\Omega+\Pi}{2}$ in Theorem 3.2, then we get*

$$\begin{aligned}&\left|\frac{\Lambda(\Omega)+\Lambda(\Pi)}{2}-\frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\left[\left({}_{\Omega}^{CF}I^{\kappa}\Lambda\right)(x)+\left({}^{CF}I_{\Pi}^{\kappa}\Lambda\right)(x)\right]\right.\\&\quad\left.+\frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right|\\&\leq\frac{(\Pi-\Omega)^2}{96}\left[\left(\left|\Lambda''(\Omega)\right|+\left|\Lambda''(\Pi)\right|\right)\right.\\&\quad\left.-\frac{7\mu}{60}\left(\left(\Omega-\frac{\Omega+\Pi}{2}\right)^2+\left(\Pi-\frac{\Omega+\Pi}{2}\right)^2\right)\right].\end{aligned}$$

Theorem 3.10. *Give the validity of Lemma 3.1, considering that the $|\Lambda''|^q$ and $q>1$, is strongly (s,m)-convex with modulus $\mu\geq 0$, $(s,m)\in(0,1]\times(0,1]$, and $\kappa\in[0,1]$ for all $x\in[\Omega,\Pi]$, with $\frac{1}{p}+\frac{1}{q}=1$, then the following fractional inequality:*

$$\begin{aligned}&\left|\frac{\left[(x-\Omega)^2-(\Pi-x)^2\right]\Lambda'(x)+2(\Pi-x)\Lambda(\Pi)+2(x-\Omega)\Lambda(\Omega)}{2(\Pi-\Omega)}\right.\\&\quad\left.-\frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\left[\left({}_{\Omega}^{CF}I^{\kappa}\Lambda\right)(x)+\left({}^{CF}I_{\Pi}^{\kappa}\Lambda\right)(x)\right]+\frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right|\\&\leq\frac{(x-\Omega)^3}{2(\Pi-\Omega)}\left(\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(p+1)}{2\Gamma\left(p+\frac{3}{2}\right)}\right)^{\frac{1}{p}}\\&\quad\times\left[\frac{1}{s+1}\left|\Lambda''(\Omega)\right|^q+\frac{m}{s+1}\left|\Lambda''\left(\frac{x}{m}\right)\right|^q-\frac{\mu m}{6}\left(\Omega-\frac{x}{m}\right)^2\right]^{\frac{1}{q}}\end{aligned}$$

$$+\frac{(\Pi-x)^3}{2(\Pi-\Omega)}\left(\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(p+1)}{2\Gamma\left(p+\frac{3}{2}\right)}\right)^{\frac{1}{p}}$$
$$\times\left[\frac{1}{s+1}\left|\Lambda''(\Pi)\right|^q+\frac{m}{s+1}\left|\Lambda''\left(\frac{x}{m}\right)\right|^q-\frac{\mu m}{6}\left(\Pi-\frac{x}{m}\right)^2\right]^{\frac{1}{q}}.$$

Proof. From Lemma 3.1, using the Hölder's inequality and strongly (s,m)-convexity of $|\Lambda''|^q$, we have

$$\begin{aligned}
&\left|\frac{\left[(x-\Omega)^2-(\Pi-x)^2\right]\Lambda'(x)+2(\Pi-x)\Lambda(\Pi)+2(x-\Omega)\Lambda(\Omega)}{2(\Pi-\Omega)}\right.\\
&\quad\left.-\frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\left[\left({}_{\Omega}^{CF}I^{\kappa}\Lambda\right)(x)+\left({}^{CF}I_{\Pi}^{\kappa}\Lambda\right)(x)\right]+\frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right|\\
&\le\frac{(x-\Omega)^3}{2(\Pi-\Omega)}\int_0^1\left(1-\Phi^2\right)\Lambda''(\Phi\Omega+(1-\Phi)x)\,d\Phi\\
&\quad+\frac{(\Pi-x)^3}{2}\int_0^1\left(1-\Phi^2\right)\Lambda''(\Phi\Pi+(1-\Phi)x)\,d\Phi\\
&\le\frac{(x-\Omega)^3}{2(\Pi-\Omega)}\left[\left(\int_0^1\left(1-\Phi^2\right)^p d\Phi\right)^{\frac{1}{p}}\left(\int_0^1\left|\Lambda''(\Phi\Omega+(1-\Phi)x)\right|^q d\Phi\right)^{\frac{1}{q}}\right]\\
&\quad+\frac{(\Pi-x)^3}{2(\Pi-\Omega)}\left[\left(\int_0^1\left(1-\Phi^2\right)^p d\Phi\right)^{\frac{1}{p}}\left(\int_0^1\left|\Lambda''(\Phi\Pi+(1-\Phi)x)\right|^q d\Phi\right)^{\frac{1}{q}}\right]\\
&\le\frac{(x-\Omega)^3}{2(\Pi-\Omega)}\left[\left(\int_0^1\left(1-\Phi^2\right)^p d\Phi\right)^{\frac{1}{p}}\int_0^1\left(\Phi^s\left|\Lambda''(\Omega)\right|^q+m(1-\Phi)^s\left|\Lambda''\left(\frac{x}{m}\right)\right|^q\right.\right.\\
&\quad\left.\left.-m\mu\Phi(1-\Phi)\left(\Omega-\frac{x}{m}\right)^2\right)^{\frac{1}{q}}\right]+\frac{(\Pi-x)^3}{2(\Pi-\Omega)}\left[\left(\int_0^1\left(1-\Phi^2\right)^p d\Phi\right)^{\frac{1}{p}}\right.\\
&\quad\times\left[\left(\left(\int_0^1\Phi^s\left|\Lambda''(\Pi)\right|^q+m(1-\Phi)^s\left|\Lambda''\left(\frac{x}{m}\right)\right|^q\right.\right.\right.\\
&\quad\left.\left.\left.-m\mu\Phi(1-\Phi)\left(\Omega-\frac{x}{m}\right)^2\right)\right)^{\frac{1}{q}}\right]\\
&=\frac{(x-\Omega)^3}{2(\Pi-\Omega)}\left(\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(p+1)}{2\Gamma\left(p+\frac{3}{2}\right)}\right)^{\frac{1}{p}}\\
&\quad\times\left[\frac{1}{s+1}\left|\Lambda''(\Omega)\right|^q+\frac{m}{s+1}\left|\Lambda''\left(\frac{x}{m}\right)\right|^q-\frac{\mu m}{6}\left(\Omega-\frac{x}{m}\right)^2\right]^{\frac{1}{q}}\\
&\quad+\frac{(\Pi-x)^3}{2(\Pi-\Omega)}\left(\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(p+1)}{2\Gamma\left(p+\frac{3}{2}\right)}\right)^{\frac{1}{p}}
\end{aligned}$$

$$\times \left[\frac{1}{s+1}\left|\Lambda''(\Pi)\right|^q + \frac{m}{s+1}\left|\Lambda''\left(\frac{x}{m}\right)\right|^q - \frac{\mu m}{6}\left(\Pi - \frac{x}{m}\right)^2\right]^{\frac{1}{q}}. \qquad \square$$

The proof of Theorem 3.10 is completed.

Corollary 3.11. *If we choose $x = \frac{\Omega+\Pi}{2}$ in Theorem 3.10, then we have the following trapezoidal-type inequality:*

$$\begin{aligned}
&\left|\frac{\Lambda(\Omega)+\Lambda(\Pi)}{2} - \frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\right. \\
&\qquad \left.\times\left[\left({}^{CF}_{\Omega}I^{\kappa}\Lambda\right)(x) + \left({}^{CF}I^{\kappa}_{\Pi}\Lambda\right)(x)\right] + \frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right| \\
&\leq \frac{(\Pi-\Omega)^2}{16}\left(\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(p+1)}{2\Gamma\left(p+\frac{3}{2}\right)}\right)^{\frac{1}{p}} \\
&\qquad \times\left[\frac{1}{s+1}\left(\left|\Lambda''(\Omega)\right|^q + \left|\Lambda''(\Pi)\right|^q\right) + \frac{2m}{s+1}\left|\Lambda''\left(\frac{\Omega+\Pi}{2m}\right)\right|^q\right. \\
&\qquad \left.-\frac{\mu m}{6}\left(\left(\Omega - \frac{\Omega+\Pi}{2m}\right)^2 + \left(\Pi - \frac{\Omega+\Pi}{2m}\right)^2\right)\right]^{\frac{1}{q}}.
\end{aligned}$$

Corollary 3.12. *If we choose $\mu = 0$ in Corollary 3.11, then we get*

$$\begin{aligned}
&\left|\frac{\Lambda(\Omega)+\Lambda(\Pi)}{2} - \frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\right. \\
&\qquad \left.\times\left[\left({}^{CF}_{\Omega}I^{\kappa}\Lambda\right)(x) + \left({}^{CF}I^{\kappa}_{\Pi}\Lambda\right)(x)\right] + \frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right| \\
&\leq \frac{(\Pi-\Omega)^2}{16}\left(\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(p+1)}{2\Gamma\left(p+\frac{3}{2}\right)}\right)^{\frac{1}{p}} \\
&\qquad \times\left[\frac{1}{s+1}\left(\left|\Lambda''(\Omega)\right|^q + \left|\Lambda''(\Pi)\right|^q\right) + \frac{2m}{s+1}\left|\Lambda''\left(\frac{\Omega+\Pi}{2m}\right)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Corollary 3.13. *If we choose $m = 1$ in Corollary 3.12, then we obtain*

$$\begin{aligned}
&\left|\frac{\Lambda(\Omega)+\Lambda(\Pi)}{2} - \frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\right. \\
&\qquad \left.\times\left[\left({}^{CF}_{\Omega}I^{\kappa}\Lambda\right)(x) + \left({}^{CF}I^{\kappa}_{\Pi}\Lambda\right)(x)\right] + \frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right| \\
&\leq \frac{(\Pi-\Omega)^2}{16}\left(\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(p+1)}{2\Gamma\left(p+\frac{3}{2}\right)}\right)^{\frac{1}{p}} \\
&\qquad \times\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left[\left(\left|\Lambda''(\Omega)\right| + \left|\Lambda''(\Pi)\right|\right)\right].
\end{aligned}$$

Corollary 3.14. *If we choose $s = 1$ in Corollary 3.13, then we have*

$$\begin{aligned}&\left|\frac{\Lambda(\Omega)+\Lambda(\Pi)}{2}-\frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\right.\\&\qquad\left.\times\left[\left({}^{CF}_{\Omega}I^{\kappa}\Lambda\right)(x)+\left({}^{CF}I^{\kappa}_{\Pi}\Lambda\right)(x)\right]+\frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right|\\&\leq\frac{(\Pi-\Omega)^2}{16}\left(\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(p+1)}{2\Gamma\left(p+\frac{3}{2}\right)}\right)^{\frac{1}{p}}\left[\left(\left|\Lambda''(\Omega)\right|+\left|\Lambda''(\Pi)\right|\right)\right].\end{aligned}$$

Corollary 3.15. *If we choose $x=\frac{\Omega+\Pi}{2}$ and $s=m=1$ in Theorem 3.10, then we get*

$$\begin{aligned}&\left|\frac{\Lambda(\Omega)+\Lambda(\Pi)}{2}-\frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\right.\\&\qquad\left.\times\left[\left({}^{CF}_{\Omega}I^{\kappa}\Lambda\right)(x)+\left({}^{CF}I^{\kappa}_{\Pi}\Lambda\right)(x)\right]+\frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right|\\&\leq\frac{(\Pi-\Omega)^2}{16}\left(\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(p+1)}{2\Gamma\left(p+\frac{3}{2}\right)}\right)^{\frac{1}{p}}\left[\left(\left|\Lambda''(\Omega)\right|^q+\left|\Lambda''(\Pi)\right|^q\right)\right.\\&\qquad\left.-\frac{\mu}{6}\left(\left(\Omega-\frac{\Omega+\Pi}{2}\right)^2+\left(\Pi-\frac{\Omega+\Pi}{2}\right)^2\right)\right]^{\frac{1}{q}}.\end{aligned}$$

Corollary 3.16. *If we choose $x=\frac{\Omega+\Pi}{2}$, $s=0$ and $m=1$ in Corollary 3.15, then we obtain*

$$\begin{aligned}&\left|\frac{\Lambda(\Omega)+\Lambda(\Pi)}{2}-\frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\right.\\&\qquad\left.\times\left[\left({}^{CF}_{\Omega}I^{\kappa}\Lambda\right)(x)+\left({}^{CF}I^{\kappa}_{\Pi}\Lambda\right)(x)\right]+\frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right|\\&\leq\frac{(\Pi-\Omega)^2}{16}\left(\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(p+1)}{2\Gamma\left(p+\frac{3}{2}\right)}\right)^{\frac{1}{p}}\\&\qquad\times\left[\left(\left|\Lambda''(\Omega)\right|^q+\left|\Lambda''(\Pi)\right|^q\right)-\frac{\mu}{6}\left((\Omega+\Pi)^2+(\Pi-\Omega)^2\right)\right]^{\frac{1}{q}}.\end{aligned}$$

Theorem 3.17. *Under the assumptions of Lemma 3.1, the following holds: If $\Lambda''\in L[\Omega,\Pi]$ and $|\Lambda''|^q$ is strongly (s,m)-convex with modulus $\mu\geq 0$, $(s,m)\in(0,1]\times(0,1]$, and $\kappa\in[0,1]$ for all $x\in[\Omega,\Pi]$ and $q\geq 1$, then the following fractional inequality holds:*

$$\begin{aligned}
&\left|\frac{\left[(x-\Omega)^2-(\Pi-x)^2\right]\Lambda'(x)+2(\Pi-x)\Lambda(\Pi)+2(x-\Omega)\Lambda(\Omega)}{2(\Pi-\Omega)}\right.\\
&\quad\left.-\frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\left[\left({}_{\Omega}^{CF}I^{\kappa}\Lambda\right)(x)+\left({}^{CF}I_{\Pi}^{\kappa}\Lambda\right)(x)\right]+\frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right|\\
&\leq\frac{(x-\Omega)^3}{2(\Pi-\Omega)}\left(\frac{2}{3}\right)^{1-\frac{1}{q}}\left[\frac{2}{3+4s+s^2}\left|\Lambda''(\Omega)\right|^q\right.\\
&\quad\left.+m\left(\frac{4+s}{6+5s+s^2}\right)\left|\Lambda''\left(\frac{x}{m}\right)\right|^q-\frac{7\mu m}{60}\left(\Omega-\frac{x}{m}\right)^2\right]^{\frac{1}{q}}\\
&\quad+\frac{(\Pi-x)^3}{2(\Pi-\Omega)}\left(\frac{2}{3}\right)^{1-\frac{1}{q}}\left[\frac{2}{3+4s+s^2}\left|\Lambda''(\Pi)\right|^q\right.\\
&\quad\left.+m\left(\frac{4+s}{6+5s+s^2}\right)\left|\Lambda''\left(\frac{x}{m}\right)\right|^q-\frac{7\mu m}{60}\left(\Pi-\frac{x}{m}\right)^2\right]^{\frac{1}{q}}.
\end{aligned}$$

Proof. From Lemma 3.1, applying the power-mean inequality and strongly (s,m)-convexity of $|\Lambda''|^q$, we have

$$\begin{aligned}
&\left|\frac{\left[(x-\Omega)^2-(\Pi-x)^2\right]\Lambda'(x)+2(\Pi-x)\Lambda(\Pi)+2(x-\Omega)\Lambda(\Omega)}{2(\Pi-\Omega)}\right.\\
&\quad\left.-\frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\left[\left({}_{\Omega}^{CF}I^{\kappa}\Lambda\right)(x)+\left({}^{CF}I_{\Pi}^{\kappa}\Lambda\right)(x)\right]+\frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right|\\
&\leq\frac{(x-\Omega)^3}{2(\Pi-\Omega)}\int_0^1\left(1-\Phi^2\right)\Lambda''(\Phi\Omega+(1-\Phi)x)\,d\Phi\\
&\quad+\frac{(\Pi-x)^3}{2}\int_0^1\left(1-\Phi^2\right)\Lambda''(\Phi\Pi+(1-\Phi)x)\,d\Phi\\
&\leq\frac{(x-\Omega)^3}{2(\Pi-\Omega)}\left[\left(\int_0^1\left(1-\Phi^2\right)\right)^{1-\frac{1}{q}}\right.\\
&\quad\left.\times\left(\int_0^1\left(1-\Phi^2\right)\left|\Lambda''(\Phi\Omega+(1-\Phi)x)\right|^q d\Phi\right)^{\frac{1}{q}}\right]\\
&\quad+\frac{(\Pi-x)^3}{2(\Pi-\Omega)}\left[\left(\int_0^1\left(1-\Phi^2\right)\right)^{1-\frac{1}{q}}\right.\\
&\quad\left.\times\left(\int_0^1\left(1-\Phi^2\right)\left|\Lambda''(\Phi\Pi+(1-\Phi)x)\right|^q d\Phi\right)^{\frac{1}{q}}\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(x-\Omega)^3}{2(\Pi-\Omega)}\left[\left(\int_0^1\left(1-\Phi^2\right)\right)^{1-\frac{1}{q}}\right.\\
&\quad\times\left(\int_0^1\left(1-\Phi^2\right)\left(\Phi^s\left|\Lambda''(\Omega)\right|^q+m(1-\Phi)^s\left|\Lambda''\left(\frac{x}{m}\right)\right|^q\right.\right.\\
&\quad\left.\left.\left.-\mu\Phi m(1-\Phi)\left(\Omega-\frac{x}{m}\right)^2\right)^{\frac{1}{q}}d\Phi\right)\right]\\
&\quad+\frac{(\Pi-x)^3}{2(\Pi-\Omega)}\left[\left(\int_0^1\left(1-\Phi^2\right)\right)^{1-\frac{1}{q}}\right.\\
&\quad\times\int_0^1\left(1-\Phi^2\right)\left(\Phi^s\left|\Lambda''(\Pi)\right|^q+m(1-\Phi)^s\left|\Lambda''\left(\frac{x}{m}\right)\right|^q\right.\\
&\quad\left.\left.-\mu\Phi m(1-\Phi)\left(\Pi-\frac{x}{m}\right)^2\right)\right)^{\frac{1}{q}}d\Phi\Bigg]\\
&=\frac{(x-\Omega)^3}{2(\Pi-\Omega)}\left(\frac{2}{3}\right)^{1-\frac{1}{q}}\left[\frac{2}{3+4s+s^2}\left|\Lambda''(\Omega)\right|^q\right.\\
&\quad\left.+m\left(\frac{4+s}{6+5s+s^2}\right)\left|\Lambda''\left(\frac{x}{m}\right)\right|^q-\frac{7\mu m}{60}\left(\Omega-\frac{x}{m}\right)^2\right]^{\frac{1}{q}}\\
&\quad+\frac{(\Pi-x)^3}{2(\Pi-\Omega)}\left(\frac{2}{3}\right)^{1-\frac{1}{q}}\left[\frac{2}{3+4s+s^2}\left|\Lambda''(\Pi)\right|^q\right.\\
&\quad\left.+m\left(\frac{4+s}{6+5s+s^2}\right)\left|\Lambda''\left(\frac{x}{m}\right)\right|^q-\frac{7\mu}{60}\left(\Pi-\frac{x}{m}\right)^2\right]^{\frac{1}{q}}.
\end{aligned}
$$

The proof of Theorem 3.17 is completed. □

Corollary 3.18. *If we choose $x=\frac{\Omega+\Pi}{2}$ in Theorem 3.17, then we have the following trapezoidal-type inequality:*

$$
\begin{aligned}
&\left|\frac{\Lambda(\Omega)+\Lambda(\Pi)}{2}-\frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\left[\left({}^{CF}_{\Omega}I^{\kappa}\Lambda\right)(x)+\left({}^{CF}I^{\kappa}_{\Pi}\Lambda\right)(x)\right]+\frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right|\\
&\leq\frac{(\Pi-\Omega)^2}{16}\left(\frac{2}{3}\right)^{1-\frac{1}{q}}\left[\frac{4}{3+4s+s^2}\left(\left|\Lambda''(\Omega)\right|^q+\left|\Lambda''(\Pi)\right|^q\right)\right.\\
&\quad+m\left(\frac{4+s}{6+5s+s^2}\right)\left|\Lambda''\left(\frac{\Omega+\Pi}{2m}\right)\right|^q\\
&\quad\left.-\frac{7\mu m}{60}\left(\left(\Omega-\frac{\Omega+\Pi}{2m}\right)^2+\left(\Pi-\frac{\Omega+\Pi}{2m}\right)^2\right)\right]^{\frac{1}{q}}.
\end{aligned}
$$

Corollary 3.19. *If we choose $\mu = 0$ in Corollary 3.18, then we get*

$$\begin{aligned}
&\left|\frac{\Lambda(\Omega)+\Lambda(\Pi)}{2}-\frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\left[\left({}^{CF}_{\Omega}I^{\kappa}\Lambda\right)(x)+\left({}^{CF}I^{\kappa}_{\Pi}\Lambda\right)(x)\right]+\frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right| \\
&\leq \frac{(\Pi-\Omega)^2}{16}\left(\frac{2}{3}\right)^{1-\frac{1}{q}}\left[\frac{4}{3+4s+s^2}\left(|\Lambda''(\Omega)|^q+|\Lambda''(\Pi)|^q\right)\right. \\
&\quad \left.+m\left(\frac{4+s}{6+5s+s^2}\right)\left|\Lambda''\left(\frac{\Omega+\Pi}{2m}\right)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Corollary 3.20. *If we choose $m = 1$ in Corollary 3.19, then we obtain*

$$\begin{aligned}
&\left|\frac{\Lambda(\Omega)+\Lambda(\Pi)}{2}-\frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\left[\left({}^{CF}_{\Omega}I^{\kappa}\Lambda\right)(x)+\left({}^{CF}I^{\kappa}_{\Pi}\Lambda\right)(x)\right]+\frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right| \\
&\leq \frac{(\Pi-\Omega)^2}{16}\left(\frac{2}{3}\right)^{1-\frac{1}{q}}\left[\frac{4}{3+4s+s^2}\left(|\Lambda''(\Omega)|^q+|\Lambda''(\Pi)|^q\right)\right. \\
&\quad \left.+\left(\frac{4+s}{6+5s+s^2}\right)\left|\Lambda''\left(\frac{\Omega+\Pi}{2}\right)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Corollary 3.21. *If we choose $s = 1$ in Corollary 3.20, then we get*

$$\begin{aligned}
&\left|\frac{\Lambda(\Omega)+\Lambda(\Pi)}{2}-\frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\left[\left({}^{CF}_{\Omega}I^{\kappa}\Lambda\right)(x)+\left({}^{CF}I^{\kappa}_{\Pi}\Lambda\right)(x)\right]\right. \\
&\quad \left.+\frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right| \\
&\leq \frac{17(\Pi-\Omega)^2}{384}\left(\frac{2}{3}\right)^{1-\frac{1}{q}}\left[\left(\left|\Lambda''(\Omega)\right|+\left|\Lambda''(\Pi)\right|\right)\right].
\end{aligned}$$

Corollary 3.22. *If we choose $x = \frac{\Omega+\Pi}{2}$ and $s = m = 1$ in Theorem 3.17, then we obtain*

$$\begin{aligned}
&\left|\frac{\Lambda(\Omega)+\Lambda(\Pi)}{2}-\frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\left[\left({}^{CF}_{\Omega}I^{\kappa}\Lambda\right)(x)+\left({}^{CF}I^{\kappa}_{\Pi}\Lambda\right)(x)\right]\right. \\
&\quad \left.+\frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right| \\
&\leq \frac{17(\Pi-\Omega)^2}{384}\left(\frac{2}{3}\right)^{1-\frac{1}{q}} \\
&\quad \times\left[\left(\left|\Lambda''(\Omega)\right|+\left|\Lambda''(\Pi)\right|\right)-\frac{7\mu}{60}\left((\Omega-\Pi)^2+(\Pi-\Omega)^2\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Corollary 3.23. *If we choose* $x = \frac{\Omega+\Pi}{2}$, $s = 0$ *and* $m = 1$ *in Theorem 3.17, then we have*

$$\begin{aligned}
&\left|\frac{\Lambda(\Omega)+\Lambda(\Pi)}{2} - \frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\left[\left({}^{CF}_{\Omega}I^{\kappa}\Lambda\right)(x) + \left({}^{CF}I^{\kappa}_{\Pi}\Lambda\right)(x)\right]\right. \\
&\qquad \left. + \frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right| \\
&\leq \frac{(\Pi-\Omega)^2}{16}\left(\frac{2}{3}\right)^{1-\frac{1}{q}} \\
&\qquad \times \left[\frac{5}{3}\left(\left|\Lambda''(\Omega)\right| + \left|\Lambda''(\Pi)\right|\right) - \frac{7\mu}{60}\left((\Omega-\Pi)^2 + (\Pi-\Omega)^2\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Theorem 3.24. *Under the assumptions of Lemma 3.1 holds. If* $\Lambda'' \in L[\Omega,\Pi]$ *and* $|\Lambda''|$ *is concave on* $[\Omega,\Pi]$, *then the following fractional inequality:*

$$\begin{aligned}
&\left|\frac{\left[(x-\Omega)^2 - (\Pi-x)^2\right]\Lambda'(x) + 2(\Pi-x)\Lambda(\Pi) + 2(x-\Omega)\Lambda(\Omega)}{2(\Pi-\Omega)}\right. \\
&\qquad \left. - \frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\left[\left({}^{CF}_{\Omega}I^{\kappa}\Lambda\right)(x) + \left({}^{CF}I^{\kappa}_{\Pi}\Lambda\right)(x)\right] + \frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right| \\
&\leq \frac{(x-\Omega)^3}{2(\Pi-\Omega)}\left|\Lambda''\left(\frac{\Omega}{4}+\frac{5x}{12}\right)\right| + \frac{(\Pi-x)^3}{2(\Pi-\Omega)}\left|\Lambda''\left(\frac{\Pi}{4}+\frac{5x}{12}\right)\right|.
\end{aligned}$$

Proof. From Lemma 3.1, using the Jensen inequality, we obtain

$$\begin{aligned}
&\left|\frac{\left[(x-\Omega)^2 - (\Pi-x)^2\right]\Lambda'(x) + 2(\Pi-x)\Lambda(\Pi) + 2(x-\Omega)\Lambda(\Omega)}{2(\Pi-\Omega)}\right. \\
&\qquad \left. - \frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\left[\left({}^{CF}_{\Omega}I^{\kappa}\Lambda\right)(x) + \left({}^{CF}I^{\kappa}_{\Pi}\Lambda\right)(x)\right] + \frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right| \\
&\leq \frac{(x-\Omega)^3}{2(\Pi-\Omega)}\int_0^1 \left(1-\Phi^2\right)\Lambda''(\Phi\Omega + (1-\Phi)x)\,d\Phi \\
&\qquad + \frac{(\Pi-x)^3}{2}\int_0^1 \left(1-\Phi^2\right)\Lambda''(\Phi\Pi + (1-\Phi)x)\,d\Phi \\
&\leq \frac{(x-\Omega)^3}{2(\Pi-\Omega)}\left(\int_0^1 \left(1-\Phi^2\right)d\Phi\right) \\
&\qquad \times \left|\Lambda''\left(\frac{\int_0^1 \left(1-\Phi^2\right)(\Phi\Omega + (1-\Phi)x)\,d\Phi}{\int_0^1 (1-\Phi^2)\,d\Phi}\right)\right|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(\Pi-x)^3}{2(\Pi-\Omega)}\left(\int_0^1\left(1-\Phi^2\right)d\Phi\right)\\
&\times\left|\Lambda''\left(\frac{\int_0^1\left(1-\Phi^2\right)(\Phi\Pi+(1-\Phi)x)\,d\Phi}{\int_0^1\left(1-\Phi^2\right)d\Phi}\right)\right|\\
&=\frac{(x-\Omega)^3}{2(\Pi-\Omega)}\left|\Lambda''\left(\frac{\Omega}{4}+\frac{5x}{12}\right)\right|+\frac{(\Pi-x)^3}{2(\Pi-\Omega)}\left|\Lambda''\left(\frac{\Pi}{4}+\frac{5x}{12}\right)\right|.
\end{aligned}
$$

The proof of Theorem 3.24 is completed. □

Corollary 3.25. *If we select $x=\frac{\Omega+\Pi}{2}$ in Theorem 3.24, then we have the following trapezoidal-type inequality:*

$$
\begin{aligned}
&\left|\frac{\Lambda(\Omega)+\Lambda(\Pi)}{2}-\frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\left[\left({}_{\Omega}^{CF}I^{\kappa}\Lambda\right)(x)+\left({}^{CF}I_{\Pi}^{\kappa}\Lambda\right)(x)\right]\right.\\
&\qquad\left.+\frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right|\\
&\quad\leq\frac{(\Pi-\Omega)^2}{16}\left[\left|\Lambda''\left(\frac{11\Omega+5\Pi}{24}\right)\right|+\left|\Lambda''\left(\frac{11\Pi+5\Omega}{24}\right)\right|\right].
\end{aligned}
$$

Theorem 3.26. *Under the assumptions of Lemma 3.1 holds. If $\Lambda''\in L[\Omega,\Pi]$ and $|\Lambda''|^q$ is s-convex on $[\Omega,\Pi]$ for some fixed $s\in(0,1]$ and $q>1$, then the following fractional inequality:*

$$
\begin{aligned}
&\left|\frac{\left[(x-\Omega)^2-(\Pi-x)^2\right]\Lambda'(x)+2(\Pi-x)\Lambda(\Pi)+2(x-\Omega)\Lambda(\Omega)}{2(\Pi-\Omega)}\right.\\
&\qquad\left.-\frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\left[\left({}_{\Omega}^{CF}I^{\kappa}\Lambda\right)(x)+\left({}^{CF}I_{\Pi}^{\kappa}\Lambda\right)(x)\right]+\frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right|\\
&\leq\frac{(x-\Omega)^3}{2(\Pi-\Omega)}\\
&\quad\times\left[\frac{1}{p}\left(\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(p+1)}{2\Gamma\left(p+\frac{3}{2}\right)}\right)+\frac{1}{q(s+1)}\left(\left|\Lambda''(\Omega)\right|^q+\left|\Lambda''(x)\right|^q\right)\right]\\
&\quad+\frac{(\Pi-x)^3}{2(\Pi-\Omega)}\left[\frac{1}{p}\left(\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(p+1)}{2\Gamma\left(p+\frac{3}{2}\right)}\right)\right.\\
&\quad\left.+\frac{1}{q(s+1)}\left(\left|\Lambda''(\Pi)\right|^q+\left|\Lambda''(x)\right|^q\right)\right].
\end{aligned}
$$

Proof. From Lemma 3.1, we have

$$\begin{aligned}
&\left|\frac{\left[(x-\Omega)^2-(\Pi-x)^2\right]\Lambda'(x)+2(\Pi-x)\Lambda(\Pi)+2(x-\Omega)\Lambda(\Omega)}{2(\Pi-\Omega)}\right.\\
&\quad\left.-\frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\left[\left({}^{CF}_{\Omega}I^{\kappa}\Lambda\right)(x)+\left({}^{CF}I^{\kappa}_{\Pi}\Lambda\right)(x)\right]+\frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right|\\
&\leq\frac{(x-\Omega)^3}{2(\Pi-\Omega)}\int_0^1\left(1-\Phi^2\right)\Lambda''(\Phi\Omega+(1-\Phi)x)\,d\Phi\\
&\quad+\frac{(\Pi-x)^3}{2}\int_0^1\left(1-\Phi^2\right)\Lambda''(\Phi\Pi+(1-\Phi)x)\,d\Phi.
\end{aligned}$$

By using the Young inequality

$$\Omega\Pi\leq\frac{1}{p}\Omega^p+\frac{1}{q}\Pi^q,$$

we have

$$\begin{aligned}
&\left|\frac{\left[(x-\Omega)^2-(\Pi-x)^2\right]\Lambda'(x)+2(\Pi-x)\Lambda(\Pi)+2(x-\Omega)\Lambda(\Omega)}{2(\Pi-\Omega)}\right.\\
&\quad\left.-\frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\left[\left({}^{CF}_{\Omega}I^{\kappa}\Lambda\right)(x)+\left({}^{CF}I^{\kappa}_{\Pi}\Lambda\right)(x)\right]+\frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right|\\
&\leq\frac{(x-\Omega)^3}{2(\Pi-\Omega)}\int_0^1\left(1-\Phi^2\right)\Lambda''(\Phi\Omega+(1-\Phi)x)\,d\Phi\\
&\quad+\frac{(\Pi-x)^3}{2}\int_0^1\left(1-\Phi^2\right)\Lambda''(\Phi\Pi+(1-\Phi)x)\,d\Phi\\
&\leq\frac{(x-\Omega)^3}{2(\Pi-\Omega)}\left[\frac{1}{p}\left(\int_0^1\left(1-\Phi^2\right)^p d\Phi\right)\right.\\
&\quad\left.+\frac{1}{q}\left(\int_0^1\Phi^s\left|\Lambda''(\Omega)\right|^q d\Phi+\int_0^1(1-\Phi)^s\left|\Lambda''(x)\right|^q d\Phi\right)\right]\\
&\quad+\frac{(\Pi-x)^3}{2(\Pi-\Omega)}\left[\frac{1}{p}\left(\int_0^1\left(1-\Phi^2\right)^p d\Phi\right)\right.\\
&\quad\left.+\frac{1}{q}\left(\int_0^1\Phi^s\left|\Lambda''(\Pi)\right|^q d\Phi+\int_0^1(1-\Phi)^s\left|\Lambda''(x)\right|^q d\Phi\right)\right]\\
&=\frac{(x-\Omega)^3}{2(\Pi-\Omega)}\left[\frac{1}{p}\left(\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(p+1)}{2\Gamma\left(p+\frac{3}{2}\right)}\right)+\frac{1}{q(s+1)}\left(\left|\Lambda''(\Omega)\right|^q+\left|\Lambda''(x)\right|^q\right)\right]\\
&\quad+\frac{(\Pi-x)^3}{2(\Pi-\Omega)}\left[\frac{1}{p}\left(\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(p+1)}{2\Gamma\left(p+\frac{3}{2}\right)}\right)+\frac{1}{q(s+1)}\left(\left|\Lambda''(\Pi)\right|^q+\left|\Lambda''(x)\right|^q\right)\right].
\end{aligned}$$

The proof of Theorem 3.26 is completed. □

Corollary 3.27. *If we select* $x = \frac{\Omega+\Pi}{2}$ *in Theorem 3.17, then we have the following trapezoidal-type inequality:*

$$\begin{aligned}
&\left|\frac{\Lambda(\Omega)+\Lambda(\Pi)}{2} - \frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\left[\left({}^{CF}_{\Omega}I^{\kappa}\Lambda\right)(x) + \left({}^{CF}I^{\kappa}_{\Pi}\Lambda\right)(x)\right]\right. \\
&\quad \left. + \frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right| \\
&\leq \frac{(\Pi-\Omega)^2}{16}\left[\frac{1}{p}\left(\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(p+1)}{2\Gamma\left(p+\frac{3}{2}\right)}\right)\right. \\
&\quad \left. + \frac{1}{q}\left(\frac{2}{s+1}\left(\left|\Lambda''(\Omega)\right|^q + \left|\Lambda''(\Pi)\right|^q\right) + \frac{1}{s+1}\left|\Lambda''\left(\frac{\Omega+\Pi}{2}\right)\right|^q\right)\right].
\end{aligned}$$

Corollary 3.28. *If we choose* $s = 1$ *in Corollary 3.27, then we get*

$$\begin{aligned}
&\left|\frac{\Lambda(\Omega)+\Lambda(\Pi)}{2} - \frac{\beta(\kappa)}{\kappa(\Pi-\Omega)} - \frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\left[\left({}^{CF}_{\Omega}I^{\kappa}\Lambda\right)(x) + \left({}^{CF}I^{\kappa}_{\Pi}\Lambda\right)(x)\right]\right. \\
&\quad \left. + \frac{2(1-\kappa)}{\beta(\kappa)}\Lambda(x)\right| \\
&\leq \frac{(\Pi-\Omega)^2}{16}\left[\frac{1}{p}\left(\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(p+1)}{2\Gamma\left(p+\frac{3}{2}\right)}\right) + \frac{5}{4q}\left(\left|\Lambda''(\Omega)\right|^q + \left|\Lambda''(\Pi)\right|^q\right)\right].
\end{aligned}$$

4. Applications

4.1. **Matrix Inequality.** In [23] Sababheh proved that the function $\psi(\theta) := \left\|A^{\theta}XB^{1-\theta} + A^{1-\theta}XB^{\theta}\right\|$, $\Omega, \Pi \in M_n^+$, $X \in M_n$ is convex for all $\theta \in [0,1]$. Then by using Corollary 3.6 for $0 \leq \Omega \leq x \leq \Pi \leq 1$, we have

$$\begin{aligned}
&\frac{1}{2}\left[\left\|A^{\Omega}XB^{1-\Omega} + A^{1-\Omega}XB^{\Omega}\right\| + \left\|A^{\Pi}XB^{1-\Pi} + A^{1-\Pi}XB^{\Pi}\right\|\right] \\
&\quad - \frac{\beta(\kappa)}{\kappa(\Pi-\Omega)}\left[{}^{CF}_{\Omega}I^{\kappa}\left\|A^{x}XB^{1-x} + A^{1-x}XB^{x}\right\|\right. \\
&\quad \left. + {}^{CF}I^{\kappa}_{\Pi}\left\|A^{x}XB^{1-x} + A^{1-x}XB^{x}\right\|\right] \\
&\quad + \frac{2(1-\kappa)}{\kappa(\Pi-\Omega)}\left\|A^{x}XB^{1-x} + A^{1-x}XB^{x}\right\| \\
&\leq \frac{(\Pi-\Omega)^2}{48}\left[\left\|A^{\Omega}XB^{1-\Omega} + A^{1-\Omega}XB^{\Omega}\right\|\right. \\
&\quad \left. + \left\|A^{\Pi}XB^{1-\Pi} + A^{1-\Pi}XB^{\Pi}\right\|\right].
\end{aligned}$$

4.2. **Special means.** We shall consider the following special means:
(a) The arithmetic mean:

$$A = A(\Omega, \Pi) := \frac{\Omega + \Pi}{2}.$$

(b) The Logarithmic Mean:

$$L(\Omega, \Pi) := \frac{\Pi - \Omega}{\ln \Pi - \ln \Omega}, \quad \Omega, \Pi > 0, \Omega \neq \Pi.$$

(c) The Generalized Logarithmic Mean:

$$L_r = L_r(\Omega, \Pi) := \left[\frac{\Pi^{r+1} - \Omega^{r+1}}{(r+1)(\Pi - \Omega)}\right]^{\frac{1}{r}}, \quad r \in \mathbb{R} \setminus \{-1, 0\}.$$

Proposition 4.1. *For $\Omega, \Pi \in \mathbb{R}$ where $0 \leq \Omega < \Pi$ and $n \geq 2$, we have*

$$|L_n^n(\Omega, \Pi) - A^n(\Omega, \Pi)| \leq \frac{n(n-1)(\Pi - \Omega)^2}{48}\left[\Omega^{n-2} + \Pi^{n-2}\right].$$

Proof. The assertion follows from Corollary 3.6, employing the $\Lambda(x) = x^n$, $\kappa = 1$ and $\beta(0) = \beta(1) = 1$. □

Proposition 4.2. *For $\Omega, \Pi \in \mathbb{R}$ where $0 \leq \Omega < \Pi$ and $n \geq 2$, we obtain*

$$\begin{aligned}
&|L_n^n(\Omega, \Pi) - A^n(\Omega, \Pi)| \\
&\leq \frac{n(n-1)(\Pi - \Omega)^2}{16} \\
&\quad \times \left[\left(\frac{11\Omega^{(n-2)} + 5\Pi^{(n-2)}}{24}\right) + \frac{5\Omega^{(n-2)} + 11\Pi^{(n-2)}}{24}\right].
\end{aligned}$$

Proof. The assertion follows from Corollary 3.25, employing the $\Lambda(x) = x^n$, $\kappa = 1$ and $\beta(0) = \beta(1) = 1$. □

4.3. **Error Estimation for Trapezoidal Formula.** Let d is the partition of the interval $[\Omega, \Pi]$, $d : \Omega = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = \Pi$, then the quadrature formula is defined by

$$\int_\Omega^\Pi \Lambda(x)\, dx := \Phi(\Lambda, d) + E(\Lambda, d),$$

where the trapezoidal version is as follows:

$$\Phi(\Lambda, d) := \sum_{i=0}^{n-1} \frac{\Lambda(x_i) + \Lambda(x_{i+1})}{2}(x_{i+1} - x_i)$$

and $E(\Lambda, d)$ give the associated error estimation.

Theorem 4.3. *Under the assumption of Corollary 3.6, we have the following bound for error estimation:*

$$|E(\Lambda, d)| \leq \sum_{i=0}^{n-1} \frac{(x_{i+1}-x_i)^3}{48} \left(\left|\Lambda''(x_i)\right| + \left|\Lambda''(x_{i+1})\right|\right).$$

Proof. Applying the Corollary 3.6 on the interval $[x_i, x_{i+1}]$ and $\kappa = 1$, $\beta(0) = \beta(1) = 1$ and $i \in \{0, 1, 2, \ldots, n-1\}$, we have

$$\left|\frac{\Lambda(x_i)+\Lambda(x_{i+1})}{2} - \frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} \Lambda(x)\,dx\right| \leq \frac{(x_{i+1}-x_i)^2}{48}\left(\left|\Lambda''(x_i)\right| + \left|\Lambda''(x_{i+1})\right|\right).$$

Therefore, we have

$$\begin{aligned}
|E(\Lambda, d)| &= \left|\int_{\Omega}^{\Pi} \Lambda(x)\,dx - \Phi(\Lambda, d)\right| \\
&= \left|\sum_{i=0}^{n-1}\left[\int_{x_i}^{x_{i+1}} \Lambda(x)\,dx - \frac{\Lambda(x_i)+\Lambda(x_{i+1})}{2}(x_{i+1}-x_i)\right]\right| \\
&\leq \sum_{i=0}^{n-1}\left|\left[\int_{x_i}^{x_{i+1}} \Lambda(x)\,dx - \frac{\Lambda(x_i)+\Lambda(x_{i+1})}{2}(x_{i+1}-x_i)\right]\right| \\
&\leq \sum_{i=0}^{n-1} \frac{(x_{i+1}-x_i)^3}{48}\left(\left|\Lambda''(x_i)\right| + \left|\Lambda''(x_{i+1})\right|\right).
\end{aligned}$$

□

5. Conclusion

Numerous authors continue to work on various fractional operators to study the integral inequalities. In this paper, we derive trapezoidal-type inequalities using the fractional operators yielding powerful results. We obtain a general identity for Caputo-Fabrizio integral operators and second derivative function. By applying this identity, new error bounds and estimates for strongly (s, m)-convex functions are given. Moreover, some novel trapezoidal-type inequalities are offered taking this identity into account via the known inequalities like Young, Jensen, Hölder and power-mean inequalities. Finally, we provide meaningful applications to matrix inequality theory, estimation error bound regarding trapezoidal formulas and finally special means for real numbers. The authors hope that the ideas and techniques of this paper will inspire interested readers working in this field to find new results for different classes of convexity. The authors might use the novel methods and practical concepts of convexity presented in this study for various fractional integral operators in their future works.

Acknowledgment. N/A

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[1] School of Mathematical Sciences, University of Science and Technology in Hefei, China.
Email address: munirarslan999@gmail.com

[2] Department of Mathematics, Düzce University, Düzce, Turkey.
Email address: hsyn.budak@gmail.com

[3] Department of Mathematical Engineering, Polytechnic University of Tirana, Tirana 1001, Albania.
Email address: a.kashuri@fimif.edu.al

[4] Department of Mathematics, Government College University Faisalabed Sahiwal Campus, Pakistan.
Email address: faizirza48@gmail.com

[5] Department of Mathematics, Düzce University, Düzce, Turkey.
Email address: hasan64kara@gmail.com

[6] QFBA-Northumbria University Doha Qatar and Gulf University Bahrain.
Email address: atherqayyum@gmail.com