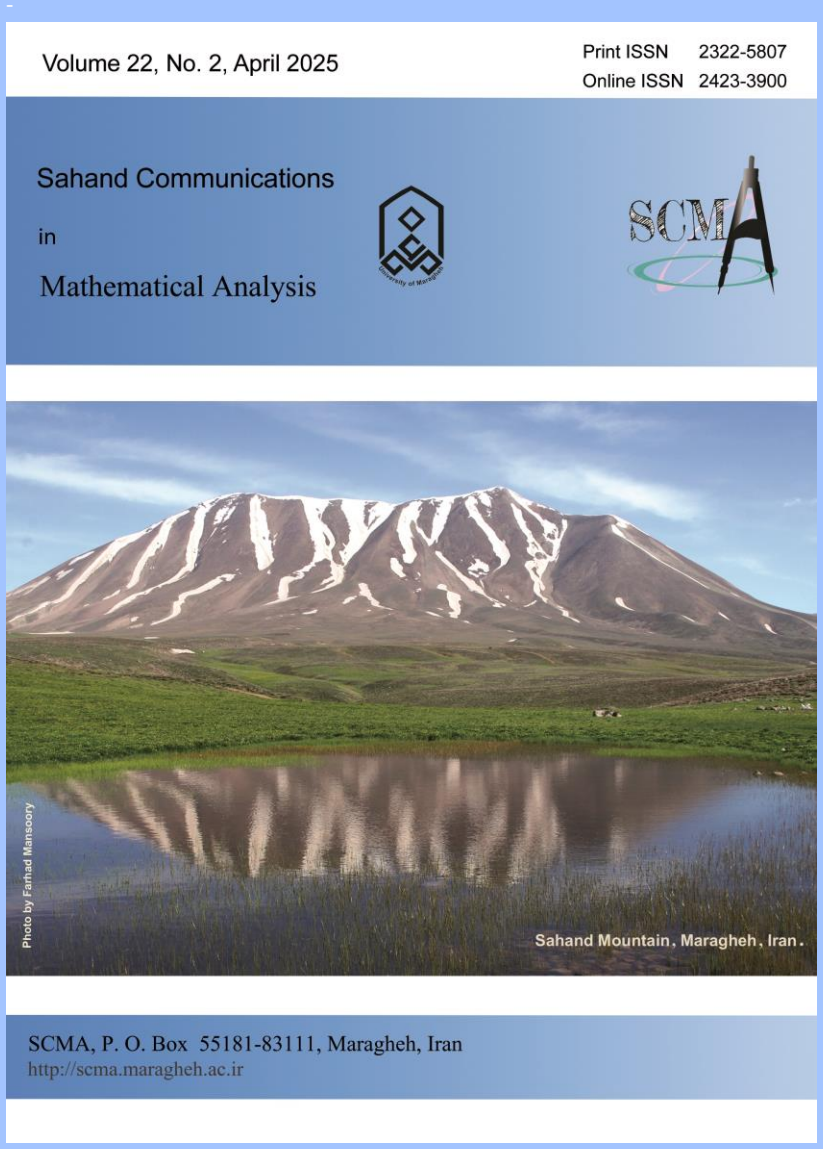
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Analysis of P-Superquadraticity and Related Integer and Fractional Order Inequalities with Applications

Dawood Khan[1] and Saad Ihsan Butt[2*]

Abstract. In this manuscript, we consider the concept of P- superquadratic functions and explore their key properties. From these properties, we can establish Jensen's, Mercer-Jensen's, and Hermite-Hadamard (H.H) type inequalities for this class of functions. The results are confirmed by the numerical computations and graphical illustration with several examples. Besides, a comparison is performed between the inequalities obtained through P-superquadratic functions and the ones acquired through P-convex functions. The analysis reflects that inequalities associated with P-superquadratic functions are far more refined than those deduced from P-convex functions. Additionally, we generalize the work with some fractional versions of the inequalities in question of H.H-type inequality involving Riemann-Liouville (R.L) fractional integral operators of P-superquadratic functions. We present several applications in the study involving type-1 modified Bessel functions, moments of random variables and special means. The novel results provided here not only advance the literature but significantly add new perspectives to it.

1. Introduction

In the applied sciences, mathematical inequalities are regarded as crucial instruments for obtaining both numerical and descriptive insights. Stated otherwise, inequalities offer a systematic approach to measuring, contrasting, and examining data or systems, assisting researchers in comprehending both specific numerical correlations and more general patterns or trends within a certain sector. They serve as a foundational

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framework for understanding a wide range of scientific phenomena. It is well acknowledged that inequalities have played a significant role in the growth and expansion of mathematics. Convex functions have the potential to be used in many fascinating and challenging study domains. They also play a fantastic role in a number of subjects, including design, inequality theory, material science, optimisation, coding hypothesis, and data hypothesis. In order to find and examine a variety of discoveries that are useful and important for applications, mathematicians constantly invest their time and energy in this field. In science and engineering today, inequality has undergone a remarkable theoretical and practical evolution [4, 13, 14, 19, 29]. One of the primary applications of the convex function is the construction of inequalities. multiple domains have seen the construction and application of multiple outstanding inequalities of different classes associated with convex functions.

Fractional calculus is a useful tool in many disciplines, including computer science [7], physics [21], engineering [30] and biological sciences [33]. It extends classical calculus to non-integer order derivatives and integrals. It has been more and more well-liked in recent years due to its capacity to simulate systems with memory and genetic features. Apart from its use in these fields, fractional calculus has contributed significantly to the field of inequality theory, specifically in the progression of fractional inequalities. Mathematicians have spent the last ten years examining various types of fractional integral inequalities. A significant advancement occurred in 2013 when Sarikaya et al. [32] presented the H.H inequality in relation to fractional integrals of R.L. New avenues for fractional inequality research have been made possible by this extension of the traditional H.H inequality, which gives constraints for the integral of convex functions.

Scholars have made significant progress in investigating H.H type inequalities for a range of fractional integrals since the advent of the H.H inequality for R.L fractional integrals. Among them are the following: Sarikaya fractional integrals [22], generalised proportional fractional integrals [28], Katugampola fractional integrals [18], generalised R.L fractional integrals [20], ψ-R.L fractional integrals [27], and conformable fractional integrals [12]. Readers are referred to recent studies [6, 8, 9, 31] and the sources listed therein for the most recent advances in fractional integral inequalities.

Superquadratic function is also a generalization of a convex function, which was first proposed by Abramovich et al. in [1]. They mentioned that integral inequalities derived for this function are more refined than

those integral inequalities derived from a convex function. So such a function is defined as follows:

Definition 1.1. A superquadratic function $\mathrm{f} : [0, \infty) \longrightarrow \Re$, must fulfils the condition (1.1), $\forall\ \varkappa \geq 0$,

$$\mathrm{f}(\mathrm{y}) \geq \mathrm{f}(\varkappa) + \mathfrak{C}_{\varkappa}(\mathrm{y} - \varkappa) + \mathrm{f}(|\mathrm{y} - \varkappa|), \tag{1.1}$$

$\forall\ \mathrm{y} \geq 0$, where $\mathfrak{C}_{\varkappa} \in \mathrm{R}$ is a constant.

Remark 1.2. If the inequality "$\leq$" in (1.1) flips then f is called subquadratic. It implies that if "$-\mathrm{f}$" is superquadratic then f is subquadratic. However, for (1.1) to hold, the superquadratic function must be non-negative.

To illustrate, consider the function $\mathrm{f}(\varkappa) = \varkappa^{\mathfrak{p}}$. So for $\varkappa \geq 0$ and $\mathfrak{p} \geq 2$, the aforementioned function is termed as superquadratic but for $0 \leq \mathfrak{p} < 2$ it is subquadratic. For this function, we can write the constant $\mathfrak{C}_{\varkappa}$ as $\mathfrak{C}_{\varkappa} = \mathrm{f}^{'}(\varkappa)$. Additionally, equality in (1.1) holds when $\mathfrak{p} = 2$, as this corresponds to the boundary between subquadratic and superquadratic behavior.

Superquadratic functions are defined and fundamentally described by the scholars Abramovich, Jameson, and Sinnamon in [1] and [2]. In contrast to the convexity criteria, condition (1.1) seems to be more powerful. According to [2], the aforementioned assertion, i.e., (1.1), is true for superquadratic functions as long as they are non-negative. To be more specific, the following axioms are satisfied by a randomly chosen superquadratic function:

(1) $\mathrm{f}(0) \leq 0$.
(2) $\mathrm{f}'(\varkappa) = \mathfrak{C}_{\varkappa}$, provided f is differentiable at $\varkappa > 0$ and $\mathrm{f}(0) = \mathrm{f}'(0) = 0$.
(3) If f is non-negative and $\mathrm{f}(0) = \mathrm{f}'(0) = 0$ then the function f is convex.

The two important inequalities which broaden and deepen this concept are H.H's and Jensen's inequalities. In the view of superquadratic functions, the following inequalities are of central importance because they not only encapsulate the very essence of superquadraticity but find extensive application in several mathematical analyses. In particular, the classical form of Jensen's inequality, discussed below, offers an important insight into what characterizes a superquadratic function. It provides a criterion that distinguishes these functions, helping to define their unique properties and behaviors:

Theorem 1.3 ([2])**.** *If* f *is superquadratic then the inequality* (1.2)

$$\sum_{i=1}^{n}\beta_i \mathrm{f}(\varkappa_i) \geq \mathrm{f}(\overline{\varkappa}) + \sum_{i=1}^{n}\beta_i \mathrm{f}(|\varkappa_i - \overline{\varkappa}|), \tag{1.2}$$

holds, $\forall\ \beta_i \geq 0, \varkappa_i \geq 0$ *and* $0 \leq \beta_i \leq 1$, *where* $\overline{\varkappa} = \sum_{i=1}^{n}\beta_i\varkappa_i$ *and* $\sum_{i=1}^{n}\beta_i$=1,

Banić and colleagues developed H.H type inequalities specifically with in the context of superquadratic functions, as discussed in their work [10].

Theorem 1.4. *If* $\mathrm{f}: I \subset [0,\infty[\longrightarrow \Re$ *is assumed to be a superquadratic on* I=$[\mathfrak{a}_o, \mathfrak{b}_o]$, *where* $0 \leq \mathfrak{a}_o < \mathfrak{b}_o$, *then*

$$\begin{aligned}
&\mathrm{f}\left(\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}\right) + \frac{1}{\mathfrak{b}_o-\mathfrak{a}_o}\int_{\mathfrak{a}_o}^{\mathfrak{b}_o}\mathrm{f}\left(\left|\varkappa - \frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}\right|\right)d\varkappa \\
&\quad\leq \frac{1}{\mathfrak{b}_o-\mathfrak{a}_o}\int_{\mathfrak{a}_o}^{\mathfrak{b}_o}\mathrm{f}(\varkappa)d\varkappa \\
&\quad\leq \frac{\mathrm{f}(\mathfrak{a}_o)+\mathrm{f}(\mathfrak{b}_o)}{2} - \frac{1}{(\mathfrak{b}_o-\mathfrak{a}_o)^2}\int_{\mathfrak{a}_o}^{\mathfrak{b}_o}[(\mathfrak{b}_o-\varkappa)\mathrm{f}(\varkappa-\mathfrak{a}_o) + (\varkappa-\mathfrak{a}_o)\mathrm{f}(\mathfrak{b}_o-\varkappa)]d\varkappa.
\end{aligned}$$

Another definition for a non-negative function f to be superquadratic is obtained by setting n=2 in Jensen's inequality which is as follows.

Definition 1.5 ([1])**.** If f is superquadratic then f must holds the condition (1.3).

(1.3)

$$\begin{aligned}
\mathrm{f}((1-\beta)\varkappa_1 + \beta\varkappa_2) \leq (1-\beta)\mathrm{f}(\varkappa_1) + \beta\mathrm{f}(\varkappa_2) - \beta\mathrm{f}((1-\beta)\,|\varkappa_1-\varkappa_2|) \\
- (1-\beta)\mathrm{f}(\beta\,|\varkappa_1-\varkappa_2|),
\end{aligned}$$

$\forall\ \varkappa_1, \varkappa_2 \geq 0$ and $0 < \beta < 1$.

Remark 1.6. If we reverse the "$\leq$" in this inequality then f is instead classified as a subquadratic function. This reversal changes the growth characteristics of the function, distinguishing subquadratic behavior from superquadratic.

Fractional H.H's type inequality via R.L fractional operators proved by Li and Chen in [26] is stated as follows:

Theorem 1.7. *If* $\mathrm{f}: I \subset [0,\infty) \longrightarrow \mathrm{R}$, *is superquadratic and integrable on* I=$[\mathfrak{a}_o, \mathfrak{b}_o]$, *with* $0 < \mathfrak{a}_o < \mathfrak{b}_o$, *then*

$$\begin{aligned}
&\mathrm{f}\left(\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}\right) + \frac{\lambda_o}{2\,(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}}\int_{\mathfrak{a}_o}^{\mathfrak{b}_o}\mathrm{f}\left(\left|\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2} - \varkappa\right|\right) \\
&\qquad\times\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1} + (\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right)d\varkappa
\end{aligned}$$

$$\leq \frac{\Gamma(1+\lambda_o)}{2(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}}\left(\mathfrak{J}^{\lambda_o}_{\mathfrak{a}_o+}\mathrm{f}(\mathfrak{b}_o)+\mathfrak{J}^{\lambda_o}_{\mathfrak{b}_o-}\mathrm{f}(\mathfrak{a}_o)\right)$$
$$\leq \frac{\mathrm{f}(\mathfrak{a}_o)+\mathrm{f}(\mathfrak{b}_o)}{2}-\frac{\lambda_o}{2(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}}$$
$$\times\int_{\mathfrak{a}_o}^{\mathfrak{b}_o}\left(\left(\frac{\varkappa-\mathfrak{a}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)\mathrm{f}(\mathfrak{b}_o-\varkappa)+\left(\frac{\mathfrak{b}_o-\varkappa}{\mathfrak{b}_o-\mathfrak{a}_o}\right)\mathrm{f}(\varkappa-\mathfrak{a}_o)\right)$$
$$\times\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1}+(\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right)d\varkappa.$$

The notations $\mathfrak{J}^{\beta}_{\mathfrak{a}_o+}\mathrm{f}(\varkappa)$, and $\mathfrak{J}^{\beta}_{\mathfrak{b}_o-}\mathrm{f}(\varkappa)$, represent the left and right sided R.L operators respectively, with fractional order $\beta\geq 0$, and $\mathfrak{a}_o\geq 0$, which are stated as follows:

$$\mathfrak{J}^{\lambda_o}_{\mathfrak{a}_o^+}\mathrm{f}(\varkappa)=\frac{1}{\Gamma(\lambda_o)}\int_{\mathfrak{a}_o}^{\varkappa}(\varkappa-\mathrm{y})^{\lambda_o-1}\mathrm{f}(\mathrm{y})\,d\mathrm{y},\quad(\varkappa>\mathfrak{a}_o). \tag{1.4}$$

$$\mathfrak{J}^{\lambda_o}_{\mathfrak{b}_o^-}\mathrm{f}(\varkappa)=\frac{1}{\Gamma(\lambda_o)}\int_{\varkappa}^{\mathfrak{b}_o}(\mathrm{y}-\varkappa)^{\lambda_o-1}\mathrm{f}(\mathrm{y})\,d\mathrm{y},\quad(\varkappa<\mathfrak{b}_o). \tag{1.5}$$

where $\Gamma(\lambda_o)=\int_0^{\infty}\mathrm{y}^{\lambda_o-1}e^{-\mathrm{y}}d\mathrm{y}$, is termed as the gamma function.

As reported in [1] and [2], Abramovich, Jameson and Sinnamon have made substantial contributions to the definition and development of a fundamental knowledge of superquadratic functions. The condition they developed in their study is a potent criteria for recognizing superquadratic functions. A more rigorous test for determining if a function is superquadratic is provided by this condition, which is stronger than the standard convexity requirement. The researchers showed this requirement in detail in [2].

Different subclasses of superquadratic functions have been treated by various researchers up to this time. Each has opened different insights and applications into this mathematical structure. For example, in [5], the authors extend the definition of superquadratic functions by introducing the h-superquadratic function as a generalization yielding other kinds of superquadratic behaviors. They discuss several fundamental properties of h-superquadratic functions by defining the conditions under which these types of functions yield the described behavior. This gave a wider overview of superquadratic functions in the more specialized settings. Along this line, Mario Krnić and his colleagues contributed to the same field of study by adding to the literature the notion of a multiplicatively superquadratic function in [24]. The concept of a center-radius or cr-order interval valued superquadratic function, including its fractional variation, was further explored by Khan and Butt, as described in [23]. In addition to introducing this novel kind of superquadratic

function, their study illustrated its uses, demonstrating how this sophisticated idea might be applied to increasingly intricate mathematical models. In his contribution, they specified conditions that must necessarily be satisfied by a function for it to be identified as logarithmically superquadratic. Butt and Khan have further expanded the concept of h-superquadraticity by developing both simple and fractional form of Fejér and Hermite-Hadamard type inequalities, as presented in [15]. These extended inequalities provide a framework for analyzing h-superquadratic functions in greater depth, while their applications demonstrate the utility of these functions in practical scenarios. In [16], Butt et al. presented the fractional H.H's type inequalities via Atangana-Baleanu's fractional operators. Moreover they introduced another novel class of superquadraticity known as interval valued superquadratic function along with its applications in information theory [17]. Through the addition of new structural constraints and characteristics, these innovative classes expand upon the conventional understanding of superquadratic functions. They enhanced the theory of superquadraticity with these contributions, opening the door for more uses and understanding of this branch of mathematics. For additional results on superquadraticity, readers are encouraged to consult [3, 10, 26]. We extend previous work related to convexity, superquadraticity and fractional calculus. The rationale behind our paper was in finding the gap: so far, no proper inequalities and features related to P-superquadraticity have been constructed. Since superquadratic functions and their associated results are the modifications of convex functions and their results, our proposed concept is the modification of the concept of P-convex functions and their associated results. For the first time, we provided several examples of this new class of functions. As a modified extension of P-convexity, these examples are also applicable to P-convex functions. Moreover, we explored its applications in areas such as special functions, special means and moments of random variables domains that remain unexplored even for P-convex functions.

In the past several years, fractional calculus has been used much more frequently with integral inequalities. The ideas of fractional calculus in P-superquadraticity must be incorporated in order to go down this path and reach new conclusions in the theory of inequalities. As previously stated, there has never been any prior research on P-superquadraticity, its inequalities, or its fractional form. Our goal is to use fractional calculus, time-scale calculus, fuzzy calculus and quantum calculus to develop and enhance H.H type inequalities in order to close this gap in the literature. We will begin our investigation in this work by combining fractional calculus with P-superquadraticity in order to illustrate our

primary discoveries. In subsequent articles, these ideas will be discussed in the previously mentioned areas.

The organization of the paper is outlined in the following paragraph: We provide concepts of convexity, fractional calculus, superquadraticity and their associated results in Section 1. In Section 2, we study P-superquadratic functions and their properties, and more significantly, we prove new inequalities. Section 3 consists of comparative analysis between integral inequalities for functions possessing P-superquadraticity and P-convexity. The fractional formulations for H.H's type inequality for P-superquadratic functions are then obtained in Section 4. Graphical behaviours and examples of what was discovered are also taken into consideration in order to assess whether the results are beneficial. Applicability of the established results for special functions, moment of random variables and special means is given in Part 5. Section 6, the last section, looks at a brief conclusion and potential directions for future research based on the work's findings.

2. P-Superquadratic Functions and Their Properties

In this section, we provide the properties, examples and inequalities associated to P-superquadratic functions:

Definition 2.1 ([5])**.** A function $\mathrm{f} : I \subset [0,\infty) \longrightarrow [0,\infty)$, exhibits P-superquadraticity on I, if the subsequent inequality

$$\begin{aligned}\mathrm{f}\left(\lambda_o \mathfrak{a}_o + (1-\lambda_o)\,\mathfrak{b}_o\right) \leq {} & \left[\mathrm{f}\left(\mathfrak{a}_o\right) - \mathrm{f}\left((1-\lambda_o)\left|\mathfrak{a}_o - \mathfrak{b}_o\right|\right)\right] \\ & + \left[\mathrm{f}\left(\mathfrak{b}_o\right) - \mathrm{f}\left(\lambda_o\left|\mathfrak{a}_o - \mathfrak{b}_o\right|\right)\right],\end{aligned} \tag{2.1}$$

holds for any $\mathfrak{a}_o, \mathfrak{b}_o \in I$, where $\mathfrak{a}_o < \mathfrak{b}_o$ and $\lambda_o \in [0,1]$.

Remark 2.2. The function f exhibits P-subquadraticity if the symbol $\leq$ reverses in (2.1).

Here we consider some basic examples of P-superquadratic functions.

Example 2.3. Consider the Mittag-Leffler function $E_{\alpha_o}(\varkappa)$.

$$E_{\alpha_o}(\varkappa) = \sum_{k=0}^{\infty} \frac{\varkappa^k}{\Gamma(1+k\alpha_o)}, \quad \varkappa \in \Re,$$

for all $\varkappa \geq 3$, it is P-superquadratic on $I = [\mathfrak{a}_o, \mathfrak{b}_o]$ if $\mathfrak{b}_o - \mathfrak{a}_o = 1$ and $0 < \alpha_o \leq 2$.

The Graph 1 confirms the P-superquadraticity of the aforementioned function via Definition 2.1 for $\lambda_o \in [0,1]$, $\mathfrak{a}_o = 4, \mathfrak{b}_o = 5$ and, $1.5 \leq \alpha_o \leq 2$.

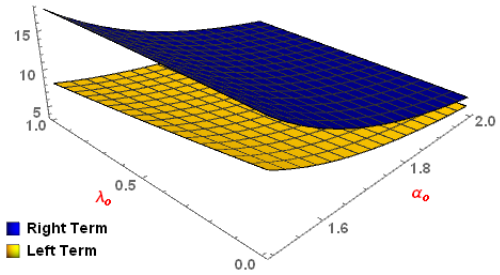
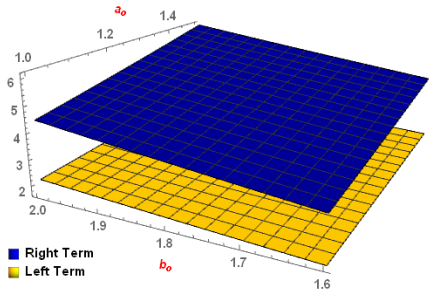
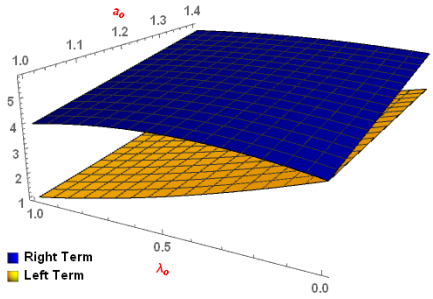


FIGURE 1. Graphical illustration of Example 2.3.

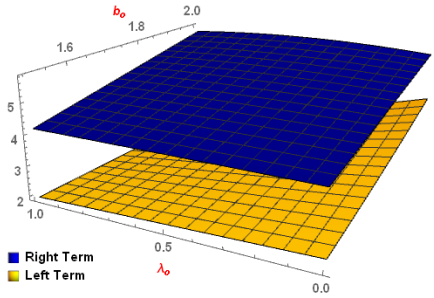
Example 2.4. $\mathrm{f}(\varkappa) = \varkappa^{\mathfrak{p}}$ exhibits P-superquadraticity on $[\mathfrak{a}_o, \mathfrak{b}_o]$ provided $\mathfrak{b}_o - \mathfrak{a}_o = 1$, $\forall\ \varkappa \geq 1$ and $\mathfrak{p} \geq 2$. This claim is confirmed by the graphs given by Figure 2, for $\mathfrak{p} = 2$ and for various values of $\mathfrak{a}_o$, $\mathfrak{b}_o$ and λ_o via Definition 2.1.



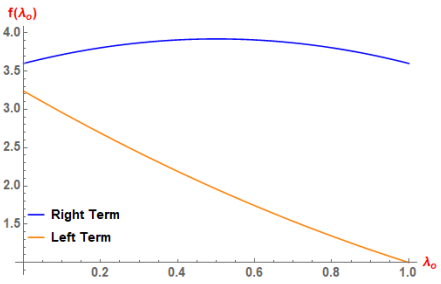
(A) λ_o is fixed while $\mathfrak{a}_o$ and $\mathfrak{b}_o$ vary such that $\lambda_o = 0.5$, $\mathfrak{a}_o \in [1, 1.5]$ and $\mathfrak{b}_o \in [1.6, 2]$.



(B) $\mathfrak{b}_o$ is fixed while $\mathfrak{a}_o$ and λ_o varies such that $\mathfrak{b}_o = 2$, $\mathfrak{a}_o \in [1, 1.4]$ and $\lambda_o \in [0, 1]$.



(C) $\mathfrak{a}_o$ is fixed while $\mathfrak{b}_o$ and λ_o varies such that $\mathfrak{a}_o = 1.4$, $\mathfrak{b}_o \in [1.5, 2]$ and $\lambda_o \in [0, 1]$.



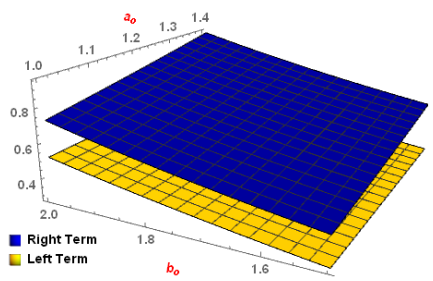
(D) $\mathfrak{a}_o$ and $\mathfrak{b}_o$ are fixed while λ_o varies such that $\mathfrak{a}_o = 1$, $\mathfrak{b}_o = 1.8$ and $\lambda_o \in [0, 1]$.

FIGURE 2. Graphical illustration of Example 2.4.

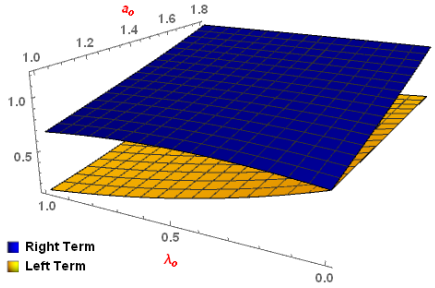
Example 2.5. Consider the type one modified Bessel function $\mathfrak{J}_{\mathfrak{p}}$ [34].

$$\mathfrak{J}_{\mathfrak{p}}(\varkappa) = \sum_{\mathfrak{n}=0}^{\infty} \frac{\left(\frac{\varkappa}{2}\right)^{\mathfrak{p}+2\mathfrak{n}}}{\mathfrak{n}!\Gamma(\mathfrak{p}+\mathfrak{n}+1)}, \quad \text{where } \varkappa \in \Re. \tag{2.2}$$

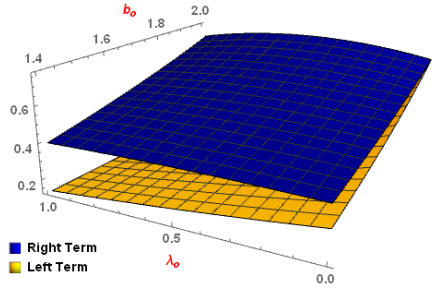
So $\mathfrak{J}_{\mathfrak{p}}(\varkappa)$ is P-superquadratic on $[\mathfrak{a}_o, \mathfrak{b}_o]$, $\forall \quad \mathfrak{p} \geq 2$ and $\varkappa \geq 1$ and this claim is confirmed by the Graphs given by Figure 3 for $\mathfrak{p} = 2$ via Definition 2.1 for various values of $\lambda_o, \mathfrak{a}_o$ and $\mathfrak{b}_o$.



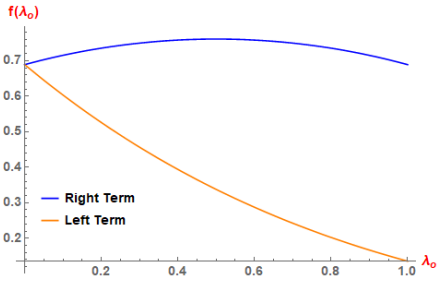
(A) λ_o is fixed while $\mathfrak{a}_o$ and $\mathfrak{b}_o$ vary such that $\lambda_o = 0.2$, $\mathfrak{a}_o \in [1, 1.4]$ and $\mathfrak{b}_o \in [1.5, 2]$.



(B) $\mathfrak{b}_o$ is fixed while $\mathfrak{a}_o$ and λ_o varies such that $\mathfrak{b}_o = 2$, $\mathfrak{a}_o \in [1, 1.8]$ and $\lambda_o \in [0, 1]$.



(C) $\mathfrak{a}_o$ is fixed while $\mathfrak{b}_o$ and λ_o varies such that $\mathfrak{a}_o = 1$, $\mathfrak{b}_o \in [1.4, 2]$ and $\lambda_o \in [0, 1]$.



(D) $\mathfrak{a}_o$ and $\mathfrak{b}_o$ are fixed while λ_o varies such that $\mathfrak{a}_o = 1$, $\mathfrak{b}_o = 2$ and $\lambda_o \in [0, 1]$.

FIGURE 3. Graphical illustration of Example 2.5.

We shall direct our concentration on P-superquadraticity since a function $\mathfrak{f}(\varkappa)$ is P-subquadratic only in the case - $\mathfrak{f}$ is P-superquadratic. Thus, every superquadratic findings may be simply reformulated in the context of subquadratic functions.

In the subsequent, we derive the properties of the P-superquadratic function.

Proposition 2.6. *If* $\mathfrak{f}$ *and* $\mathfrak{g}$ *are P-superquadratic functions then* $\mathrm{K}\mathfrak{f}$, *where* $\mathrm{K} > 0$ *and* $\mathfrak{f} + \mathfrak{g}$ *are P-superquadratic.*

Proof. Suppose that the functions $\mathfrak{g}$ and f are P-superquadratic i.e., $\forall \quad \mathfrak{a}_o, \mathfrak{b}_o \in I$ and $\lambda_o \in (0,1)$ we attain

$$\begin{aligned}&\mathrm{f}\left(\lambda_o\mathfrak{a}_o+(1-\lambda_o)\mathfrak{b}_o\right)\\&\quad\leq\left(\mathrm{f}\left(\mathfrak{a}_o\right)-\mathrm{f}\left(\left(1-\lambda_o\right)\left|\mathfrak{a}_o-\mathfrak{b}_o\right|\right)\right)+\left(\mathrm{f}\left(\mathfrak{b}_o\right)-\mathrm{f}\left(\lambda_o|\mathfrak{a}_o-\mathfrak{b}_o|\right)\right),\end{aligned}$$

and

$$\begin{aligned}&\mathfrak{g}\left(\lambda_o\mathfrak{a}_o+(1-\lambda_o)\mathfrak{b}_o\right)\\&\quad\leq\left(\mathfrak{g}(\mathfrak{a}_o)-\mathfrak{g}((1-\lambda_o)|\mathfrak{a}_o-\mathfrak{b}_o|)\right)+\left(\mathfrak{g}(\mathfrak{b}_o)-\mathfrak{g}(\lambda_o|\mathfrak{a}_o-\mathfrak{b}_o|)\right),\end{aligned}$$

now

$$\begin{aligned}&(\mathrm{f}+\mathfrak{g})\left(\lambda_o\mathfrak{a}_o+(1-\lambda_o)\mathfrak{b}_o\right)\\&\quad=\mathrm{f}\left(\lambda_o\mathfrak{a}_o+(1-\lambda_o)\mathfrak{b}_o\right)+\mathfrak{g}\left(\lambda_o\mathfrak{a}_o+(1-\lambda_o)\mathfrak{b}_o\right)\\&\quad\leq\left(\mathrm{f}\left(\mathfrak{a}_o\right)-\mathrm{f}\left(\left(1-\lambda_o\right)|\mathfrak{a}_o-\mathfrak{b}_o|\right)\right)+\left(\mathrm{f}\left(\mathfrak{b}_o\right)-\mathrm{f}\left(\lambda_o|\mathfrak{a}_o-\mathfrak{b}_o|\right)\right)\\&\qquad+\left(\mathfrak{g}(\mathfrak{a}_o)-\mathfrak{g}((1-\lambda_o)|\mathfrak{a}_o-\mathfrak{b}_o|)\right)+\left(\mathfrak{g}(\mathfrak{b}_o)-\mathfrak{g}(\lambda_o|\mathfrak{a}_o-\mathfrak{b}_o|)\right)\\&\quad=\left((\mathrm{f}+\mathfrak{g})(\mathfrak{a}_o)-(\mathrm{f}+\mathfrak{g})((1-\lambda_o)|\mathfrak{a}_o-\mathfrak{b}_o|)\right)\\&\qquad+\left((\mathrm{f}+\mathfrak{g})(\mathfrak{b}_o)-(\mathrm{f}+\mathfrak{g})(\lambda_o|\mathfrak{a}_o-\mathfrak{b}_o|)\right),\end{aligned}$$

thus

(2.3)

$$\begin{aligned}(\mathrm{f}+\mathfrak{g})\left(\lambda_o\mathfrak{a}_o+(1-\lambda_o)\mathfrak{b}_o\right)\leq&\left((\mathrm{f}+\mathfrak{g})(\mathfrak{a}_o)-(\mathrm{f}+\mathfrak{g})((1-\lambda_o)|\mathfrak{a}_o-\mathfrak{b}_o|)\right)\\&+\left((\mathrm{f}+\mathfrak{g})(\mathfrak{b}_o)-(\mathrm{f}+\mathfrak{g})(\lambda_o|\mathfrak{a}_o-\mathfrak{b}_o|)\right),\end{aligned}$$

similarly

(2.4)

$$\begin{aligned}&\mathrm{Kf}\left(\lambda_o\mathfrak{a}_o+(1-\lambda_o)\mathfrak{b}_o\right)\\&\quad\leq\mathrm{K}\left(\left(\mathrm{f}\left(\mathfrak{a}_o\right)-\mathrm{f}((1-\lambda_o)|\mathfrak{a}_o-\mathfrak{b}_o|)\right)+\left(\mathrm{f}\left(\mathfrak{b}_o\right)-\mathrm{f}(\lambda_o|\mathfrak{a}_o-\mathfrak{b}_o|)\right)\right)\\&\quad=\left(\mathrm{Kf}\left(\mathfrak{a}_o\right)-\mathrm{Kf}((1-\lambda_o)|\mathfrak{a}_o-\mathfrak{b}_o|)\right)+\left(\mathrm{Kf}\left(\mathfrak{b}_o\right)-\mathrm{Kf}(\lambda_o|\mathfrak{a}_o-\mathfrak{b}_o|)\right),\end{aligned}$$

thus

(2.5)

$$\begin{aligned}&\mathrm{Kf}\left(\lambda_o\mathfrak{a}_o+(1-\lambda_o)\mathfrak{b}_o\right)\\&\quad\leq\left(\mathrm{Kf}\left(\mathfrak{a}_o\right)-\mathrm{Kf}((1-\lambda_o)|\mathfrak{a}_o-\mathfrak{b}_o|)\right)+\left(\mathrm{Kf}\left(\mathfrak{b}_o\right)-\mathrm{Kf}(\lambda_o|\mathfrak{a}_o-\mathfrak{b}_o|)\right).\end{aligned}$$

Hence from (2.3) and (2.5), we conclude that the functions Kf, and $\mathrm{f}+\mathfrak{g}$, are P-superquadratic. □

Proposition 2.7. *If both the functions* f *and* $\mathfrak{g}$ *are P-superquadratic and convex respectively on I, then* $\mathrm{f}\circ\mathfrak{g}$ *also exhibits P-superquadraticity.*

Proof. Let $\mathfrak{a}_o, \mathfrak{b}_o \in I$ and $\lambda_o \in (0,1)$, then one attain

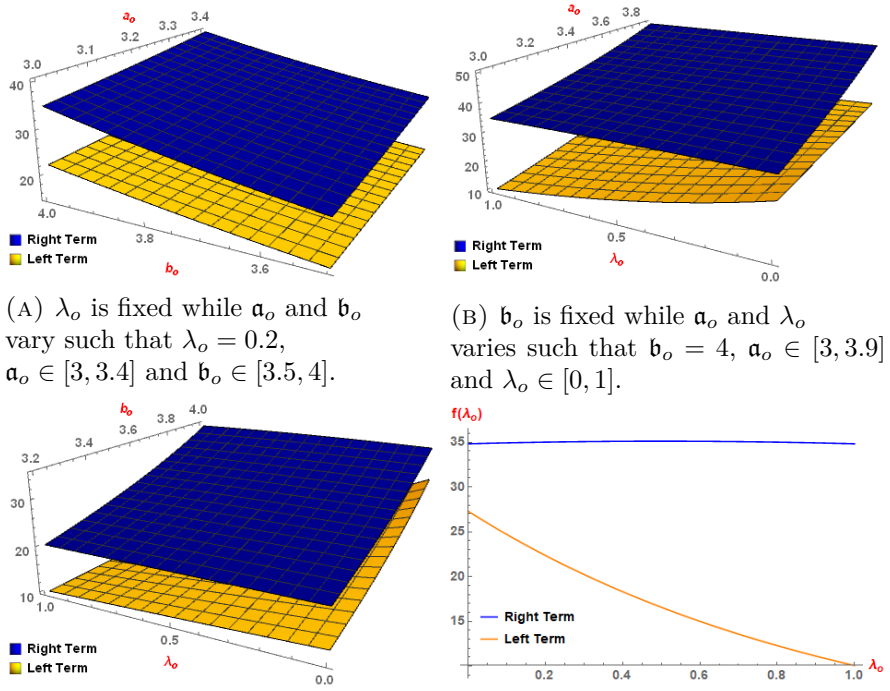
$$\begin{aligned}(\mathrm{f}\circ\mathfrak{g})\left(\lambda_o\mathfrak{a}_o+(1-\lambda_o)\mathfrak{b}_o\right)&=\mathrm{f}\left(\mathfrak{g}\left(\lambda_o\mathfrak{a}_o+(1-\lambda_o)\mathfrak{b}_o\right)\right)\\&\leq\mathrm{f}\left(\lambda_o\mathfrak{g}\left(\mathfrak{a}_o\right)+(1-\lambda_o)\,\mathfrak{g}\left(\mathfrak{b}_o\right)\right)\end{aligned}$$

$$\leq \left(\mathrm{f}\left(\mathfrak{g}\left(\mathfrak{a}_o\right)\right) - \mathrm{f}\left((1-\lambda_o)|\mathfrak{g}\left(\mathfrak{a}_o\right) - \mathfrak{g}\left(\mathfrak{b}_o\right)|\right)\right)$$
$$+ \left(\mathrm{f}\left(\mathfrak{g}\left(\mathfrak{b}_o\right)\right) - \mathrm{f}\left(\lambda_o|\mathfrak{g}\left(\mathfrak{a}_o\right) - \mathfrak{g}\left(\mathfrak{b}_o\right)|\right)\right),$$

this demonstrates that the function $\mathrm{f} \circ \mathfrak{g}$ is P-superquadratic. □

To evaluate the efficacy of Proposition 2.7, consider the following example.

Example 2.8. It is to be noted that if we pick $\alpha_o = 2$ then $E_2(\varkappa) = \cosh\sqrt{\varkappa}$ respectively. It implies that $\mathfrak{g}(\varkappa) = \cosh\sqrt{\varkappa}$ is a P- superquadratic on [3, 4], while $\mathrm{f}(\varkappa) = \varkappa^2$ is a convex funtion on $[3,4]$ then $(\mathrm{f} \circ \mathfrak{g})(\varkappa) = \cosh^2 \varkappa$ for various values of $\mathfrak{a}_o, \mathfrak{b}_o$ and λ_o satisfying Definition 2.1. The graphs given in Figure 4 confirm the claim of Proposition 2.7.



(A) λ_o is fixed while $\mathfrak{a}_o$ and $\mathfrak{b}_o$ vary such that $\lambda_o = 0.2$, $\mathfrak{a}_o \in [3, 3.4]$ and $\mathfrak{b}_o \in [3.5, 4]$.

(B) $\mathfrak{b}_o$ is fixed while $\mathfrak{a}_o$ and λ_o varies such that $\mathfrak{b}_o = 4$, $\mathfrak{a}_o \in [3, 3.9]$ and $\lambda_o \in [0, 1]$.

(C) $\mathfrak{a}_o$ is fixed while $\mathfrak{b}_o$ and λ_o varies such that $\mathfrak{a}_o = 3$, $\mathfrak{b}_o \in [3.2, 4]$ and $\lambda_o \in [0, 1]$.

(D) $\mathfrak{a}_o$ and $\mathfrak{b}_o$ are fixed while λ_o varies such that $\mathfrak{a}_o = 3$, $\mathfrak{b}_o = 4$ and $\lambda_o \in [0, 1]$.

FIGURE 4. Graphical illustration of Proposition 2.7.

Theorem 2.9. *Let* f *is a P-superquadratic then*

(i) $\mathrm{f}(0) \leq 0$.

(ii) *If* $\mathrm{f}(\varkappa) \geq 0, \quad \varkappa \geq 0$ *then* f *is a P-convex and* $\mathrm{f}(0) = 0$.

Proof. (i) Considering the definition of P-superquadratic functions.

$$\begin{aligned} \mathrm{f}\left(\lambda_o\varkappa+(1-\lambda_o)\mathrm{y}\right) \leq{}& \mathrm{f}\left(\varkappa\right)-\mathrm{f}\left(\left(1-\lambda_o\right)|\varkappa-\mathrm{y}|\right) \\ &+\left(\mathrm{f}\left(\mathrm{y}\right)-\mathrm{f}\left(\lambda_o|\varkappa-\mathrm{y}|\right)\right). \end{aligned} \tag{2.6}$$

Setting $\varkappa=\mathrm{y}$ in (2.6) we get

$$\begin{aligned} \mathrm{f}\left(\varkappa\right) &\leq \left(\mathrm{f}\left(\varkappa\right)-\mathrm{f}(0)\right)+\left(\mathrm{f}\left(\varkappa\right)-\mathrm{f}(0)\right) \\ &= \left(\mathrm{f}\left(\varkappa\right)-\mathrm{f}(0)\right)+\left(\mathrm{f}\left(\varkappa\right)-\mathrm{f}(0)\right) \\ &= \mathrm{f}\left(\varkappa\right)-\mathrm{f}(0) \end{aligned} \tag{2.7}$$

It implies that

$$\mathrm{f}(0)\leq 0 \tag{2.8}$$

□

Proof. (ii) Let f is a non-negative P-superquadratic function then

$$\begin{aligned} \mathrm{f}\left(\lambda_o\varkappa+(1-\lambda_o)\mathrm{y}\right) \leq{}& \left(\mathrm{f}\left(\varkappa\right)-\mathrm{f}\left(\left(1-\lambda_o\right)|\varkappa-\mathrm{y}|\right)\right) \\ &+\left(\mathrm{f}\left(\mathrm{y}\right)-\mathrm{f}\left(\lambda_o|\varkappa-\mathrm{y}|\right)\right) \leq \mathrm{f}\left(\varkappa\right)+\mathrm{f}\left(\mathrm{y}\right). \end{aligned}$$

This suggests that f is a P-convex function. To prove the 2nd part, we go over (2.8).

$$\mathrm{f}\left(0\right)\leq 0. \tag{2.9}$$

It is given that f is a non-negative. It implies that

$$\mathrm{f}(0)\geq 0. \tag{2.10}$$

From (2.9) and (2.10) we have

$$\mathrm{f}(0)=0. \tag{2.11}$$

Hence the proof. □

Theorem 2.10. *Let* f *be a function for all points* $\varkappa_1,\varkappa_2,\varkappa_3\in I$ *such that* $\varkappa_1<\varkappa_2<\varkappa_3$. *If* f *is a P-superquadratic then the subsequent result*

$$\begin{vmatrix} \mathrm{f}(\varkappa_2) & 0 \\ 0 & 1 \end{vmatrix} \leq \begin{vmatrix} \mathrm{f}(\varkappa_1) & 1 \\ \mathrm{f}(\varkappa_2-\varkappa_1) & 1 \end{vmatrix} + \begin{vmatrix} \mathrm{f}(\varkappa_3) & 1 \\ \mathrm{f}(\varkappa_3-\varkappa_2) & 1 \end{vmatrix} \tag{2.12}$$

holds.

Proof. Since f exhibits P-superquadraticity, then

$$\begin{aligned} &\mathrm{f}\left(\lambda_o\varkappa+(1-\lambda_o)\mathrm{y}\right) \\ &\quad\leq \left(\mathrm{f}\left(\varkappa\right)-\mathrm{f}((1-\lambda_o)|\varkappa-\mathrm{y}|)\right)+\left(\mathrm{f}(\mathrm{y})-\mathrm{f}(\lambda_o|\varkappa-\mathrm{y}|)\right). \end{aligned} \tag{2.13}$$

Setting

$$\varkappa_2=\lambda_o\varkappa+(1-\lambda_o)\mathrm{y}. \tag{2.14}$$

Next substituting $\varkappa = \varkappa_1$ and $\mathrm{y} = \varkappa_3$ in (2.14) we attain

$$\varkappa_2 = \lambda_o \varkappa_1 + (1-\lambda_o)\varkappa_3 \quad \Rightarrow \quad \lambda_o = \frac{\varkappa_3 - \varkappa_2}{\varkappa_3 - \varkappa_1}.$$

Next substituting $\varkappa = \varkappa_1$, $\mathrm{y} = \varkappa_3$ and $\lambda_o = \frac{\varkappa_3 - \varkappa_2}{\varkappa_3 - \varkappa_1}$ in (2.13) we get

$$\mathrm{f}(\varkappa_2) \leq (\mathrm{f}(\varkappa_1) - \mathrm{f}(|\varkappa_2 - \varkappa_1|)) + (\mathrm{f}(\varkappa_3) - \mathrm{f}(\varkappa_3 - \varkappa_2)),$$

therefore, we have

$$(\mathrm{f}(\varkappa_1) - \mathrm{f}(|\varkappa_2 - \varkappa_1|)) + (\mathrm{f}(\varkappa_3) - \mathrm{f}(\varkappa_3 - \varkappa_2)) - \mathrm{f}(\varkappa_2) \geq 0.$$

Hence we have

$$\begin{vmatrix} \mathrm{f}(\varkappa_1) & 1 \\ \mathrm{f}(\varkappa_2 - \varkappa_1) & 1 \end{vmatrix} + \begin{vmatrix} \mathrm{f}(\varkappa_3) & 1 \\ \mathrm{f}(\varkappa_3 - \varkappa_2) & 1 \end{vmatrix} \geq \begin{vmatrix} \mathrm{f}(\varkappa_2) & 0 \\ 0 & 1 \end{vmatrix} \tag{2.15}$$

□

We consider the following example to know how the statement (2.12) works for conforming the definition of P-superquadraticity of a function.

Example 2.11. Let we have a P-superquadratic function $\mathrm{f}(\varkappa) = \sqrt{\varkappa}$ on $I = [\mathfrak{a}_o, \mathfrak{b}_o]$ such that $\mathfrak{b}_o - \mathfrak{a}_o = 1, \quad \forall \quad \varkappa \geq 3$.

Suppose that $\varkappa_1 = \mathfrak{a}_o$, $\varkappa_2 = \frac{\mathfrak{a}_o + \mathfrak{b}_o}{2}$ and $\varkappa_3 = \mathfrak{b}_o$ then we have

$$\begin{vmatrix} \mathrm{f}(\varkappa_1) & 1 \\ \mathrm{f}(\varkappa_2 - \varkappa_1) & 1 \end{vmatrix} + \begin{vmatrix} \mathrm{f}(\varkappa_3) & 1 \\ \mathrm{f}(\varkappa_3 - \varkappa_2) & 1 \end{vmatrix} = \sqrt{\mathfrak{a}_o} + \sqrt{\mathfrak{b}_o} - 2\sqrt{\frac{\mathfrak{b}_o - \mathfrak{a}_o}{2}} \tag{2.16}$$

while

$$\begin{vmatrix} \mathrm{f}(\varkappa_2) & 0 \\ 0 & 1 \end{vmatrix} = \sqrt{\mathfrak{b}_o}. \tag{2.17}$$

The subsequent graph confirm the validity of the statement of Theorem 2.10.

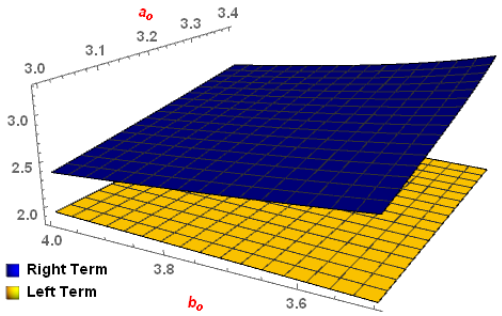


FIGURE 5. Graphical illustration of Theorem 2.10 for a function $\mathrm{f}(\varkappa) = \sqrt{\varkappa}$ for $\mathfrak{a}_o \in [3, 3.4]$ and $\mathfrak{b}_o \in [3.5, 4]$.

Theorem 2.12 (Jensen inequality)**.** *If the map* f *is P-superquadratic function on* $\boldsymbol{I}$*,* $\forall\ \lambda_{oi}, \varkappa_i \geq 0$*, where* $i = 1, \ldots, n$*, then*

$$\mathrm{f}\left(\sum_{i=1}^{n} \lambda_{oi}\varkappa_i\right) \leq \sum_{i=1}^{n} \mathrm{f}\left(\varkappa_i\right) - \sum_{i=1}^{n} \mathrm{f}\left(\left|\varkappa_i - \sum_{i=1}^{n} \lambda_{oi}\varkappa_i\right|\right). \tag{2.18}$$

Where $\sum\limits_{i=1}^{n} \lambda_{oi} = 1$.

Proof. We will prove this statement on the basis of mathematical induction. Since f is P-superquadratic, therefore for $\mathfrak{n} = 2$, we attain

$$\mathrm{f}\left(\lambda_{o1}\varkappa_1 + \lambda_{o2}\varkappa_2\right) \leq \mathrm{f}(\varkappa_1) + \mathrm{f}(\varkappa_2) - \mathrm{f}(\lambda_{o2}\left|\varkappa_1 - \varkappa_2\right|) - \mathrm{f}\left(\lambda_{o1}\left|\varkappa_1 - \varkappa_2\right|\right),$$

which is a definition of P-superquadratic function. Thus, the statement (2.18) is true for $\mathfrak{n} = 2$. Let the statement (2.18) is true for $\mathfrak{n} - 1$ then we get

$$\mathrm{f}\left(\sum_{i=1}^{n-1} \lambda_{oi}\varkappa_i\right) \leq \sum_{i=1}^{n-1} \mathrm{f}(\varkappa_i) - \sum_{i=1}^{n-1} \mathrm{f}\left(\left|\varkappa_i - \sum_{i=1}^{n-1} \lambda_{oi}\varkappa_i\right|\right). \tag{2.19}$$

Now let us prove that (2.18) is valid for $\mathfrak{n}$.

$$\begin{aligned}\mathrm{f}\left(\sum_{i=1}^{\mathfrak{n}} \lambda_{oi}\varkappa_i\right) &= \mathrm{f}\left(\lambda_{on}\varkappa_n + \sum_{i=1}^{\mathfrak{n}-1} \lambda_{oi}\varkappa_i\right)\\ &= \mathrm{f}\left(\lambda_{on}\varkappa_n + \left(\frac{\lambda_o}{\lambda_o}\right)\sum_{i=1}^{\mathfrak{n}-1} \lambda_{oi}\varkappa_i\right).\end{aligned}$$

Where $\lambda_o = \sum\limits_{i=1}^{\mathfrak{n}-1} \lambda_{oi}$

$$\begin{aligned}\mathrm{f}\left(\sum_{i=1}^{\mathfrak{n}} \lambda_{oi}\varkappa_i\right) \leq\ & \mathrm{f}(\varkappa_n) + \mathrm{f}\left(\sum_{i=1}^{\mathfrak{n}-1}\left(\frac{\lambda_{oi}\varkappa_i}{\lambda_o}\right)\right)\\ &- \mathrm{f}\left(\lambda_o\left|\varkappa_n - \sum_{i=1}^{\mathfrak{n}-1}\left(\frac{\lambda_{oi}\varkappa_i}{\lambda_o}\right)\right|\right)\\ &- \mathrm{f}\left(\lambda_{on}\left|\varkappa_n - \sum_{i=1}^{\mathfrak{n}-1}\left(\frac{\lambda_{oi}\varkappa_i}{\lambda_o}\right)\right|\right).\end{aligned} \tag{2.20}$$

Using (2.19) in (2.20) then we obtain

$$\mathrm{f}\left(\sum_{i=1}^{\mathfrak{n}} \lambda_{oi}\varkappa_i\right)$$

$$
\begin{aligned}
&\leq \mathrm{f}(\varkappa_n) + \sum_{i=1}^{\mathfrak{n}-1} \mathrm{f}(\varkappa_i) - \sum_{i=1}^{\mathfrak{n}-1} \mathrm{f}\left(\left|\varkappa_i - \sum_{i=1}^{\mathfrak{n}-1} \lambda_{oi}\varkappa_i\right|\right) \\
&\quad - \mathrm{f}\left(\lambda_o\left|\varkappa_n - \sum_{i=1}^{\mathfrak{n}-1}\left(\frac{\lambda_{oi}\varkappa_i}{\lambda_o}\right)\right|\right) - \mathrm{f}\left(\lambda_{on}\left|\varkappa_n - \sum_{i=1}^{\mathfrak{n}-1}\left(\frac{\lambda_{oi}\varkappa_i}{\lambda_o}\right)\right|\right) \\
&= \mathrm{f}(\varkappa_n) + \sum_{i=1}^{\mathfrak{n}-1} \mathrm{f}(\varkappa_i) - \sum_{i=1}^{\mathfrak{n}-1} \mathrm{f}\left(\left|\varkappa_i - \sum_{i=1}^{\mathfrak{n}-1} \lambda_{oi}\varkappa_i\right|\right) \\
&\quad - \mathrm{f}\left(\left|\lambda_o\varkappa_n - \sum_{i=1}^{\mathfrak{n}-1}\lambda_{oi}\varkappa_i\right|\right) - \mathrm{f}\left(\lambda_{on}\left|\varkappa_n - \sum_{i=1}^{\mathfrak{n}-1}\left(\frac{\lambda_{oi}\varkappa_i}{\lambda_o}\right)\right|\right) \\
&= \sum_{i=1}^{\mathfrak{n}} \mathrm{f}(\varkappa_i) - \sum_{i=1}^{\mathfrak{n}-1} \mathrm{f}\left(\left|\varkappa_i - \sum_{i=1}^{\mathfrak{n}-1} \lambda_{oi}\varkappa_i\right|\right) \\
&\quad - \mathrm{f}\left(\left|(1-\lambda_{on})\varkappa_n - \sum_{i=1}^{\mathfrak{n}-1}\lambda_{oi}\varkappa_i\right|\right) - \mathrm{f}\left(\lambda_{on}\left|\varkappa_n - \sum_{i=1}^{\mathfrak{n}-1}\left(\frac{\lambda_{oi}\varkappa_i}{\lambda_o}\right)\right|\right) \\
&= \sum_{i=1}^{\mathfrak{n}} \mathrm{f}(\varkappa_i) - \sum_{i=1}^{\mathfrak{n}-1} \mathrm{f}\left(\left|\varkappa_i - \sum_{i=1}^{\mathfrak{n}-1} \lambda_{oi}\varkappa_i\right|\right) \\
&\quad - \mathrm{f}\left(\left|\varkappa_n - \lambda_{on}\varkappa_n - \sum_{i=1}^{\mathfrak{n}-1}\lambda_{oi}\varkappa_i\right|\right) - \mathrm{f}\left(\lambda_{on}\left|\varkappa_n - \sum_{i=1}^{\mathfrak{n}-1}\left(\frac{\lambda_{oi}\varkappa_i}{\lambda_o}\right)\right|\right).
\end{aligned}
$$

It implies that

$$
\begin{aligned}
\mathrm{f}\left(\sum_{i=1}^{\mathfrak{n}} \lambda_{oi}\varkappa_i\right) &\leq \sum_{i=1}^{\mathfrak{n}} \mathrm{f}(\varkappa_i) - \sum_{i=1}^{\mathfrak{n}-1} \mathrm{f}\left(\left|\varkappa_i - \sum_{i=1}^{\mathfrak{n}-1} \lambda_{oi}\varkappa_i\right|\right) \\
&\quad - \mathrm{f}\left(\left|\varkappa_n - \sum_{i=1}^{\mathfrak{n}} \lambda_{oi}\varkappa_i\right|\right) \\
&= \sum_{i=1}^{\mathfrak{n}} \mathrm{f}(\varkappa_i) - \sum_{i=1}^{\mathfrak{n}} \mathrm{f}\left(\left|\varkappa_i - \sum_{i=1}^{\mathfrak{n}} \lambda_{oi}\varkappa_i\right|\right).
\end{aligned}
$$

Hence the proof. □

Theorem 2.13. *Let* f *is a P-superquadratic function and* $(\mathrm{m}_o, \mathrm{M}_o)$ $\subseteq I$ *then for* $\varkappa_1, \varkappa_2, \ldots, \varkappa_n \in (\mathrm{m}_o, \mathrm{M}_o)$ *we attain*

$$
\sum_{i=1}^{\mathfrak{n}} \mathrm{f}(\varkappa_i) \leq \mathrm{f}(\mathrm{m}_o) + \mathrm{f}(\mathrm{M}_o) - \sum_{i=1}^{\mathfrak{n}} \left[\mathrm{f}(\varkappa_i - \mathrm{m}_o) + \mathrm{f}(\mathrm{M}_o - \varkappa_i)\right].
$$

Proof. It can be proved by substituting $\varkappa_1 = \mathrm{m}_o, \varkappa_2 = \varkappa_i$ and $\varkappa_3 = \mathrm{M}_o$ in (2.12), such that

$$\begin{vmatrix} \mathrm{f}(\varkappa_i) & 0 \\ 0 & 1 \end{vmatrix} \leq \begin{vmatrix} \mathrm{f}(\mathrm{m}_o) & 1 \\ \mathrm{f}(\varkappa_i - \mathrm{m}_o) & 1 \end{vmatrix} + \begin{vmatrix} \mathrm{f}(\mathrm{M}_o) & 1 \\ \mathrm{f}(\mathrm{M}_o - \varkappa_i) & 1 \end{vmatrix} \tag{2.21}$$

It implies that

$$\mathrm{f}(\varkappa_i) \leq \mathrm{f}(\mathrm{m}_o) + \mathrm{f}(\mathrm{M}_o) - \mathrm{f}(\varkappa_i - \mathrm{m}_o) - \mathrm{f}(\mathrm{M}_o - \varkappa_i).$$

Summing it up to n we get the required result. □

In the following we will develop a variation of Jensen's inequality, often termed as Mercer-Jensen's inequality.

Theorem 2.14 (Mercer-Jensen's inequality)**.** *Let* f *be a P-superquadratic function defined on a real interval* $I = [\mathfrak{a}_o, \mathfrak{b}_o]$ *and finite positive increasing sequence* $\{\varkappa_\mathrm{k}\}_{\mathrm{k}=1}^{\mathrm{n}} \in I$ *where* $1 \leq \mathrm{k} \leq \mathrm{n}$*, we have*

$$\begin{aligned} \mathrm{f}\left(\varkappa_1 + \varkappa_\mathrm{n} - \sum_{\mathrm{k}=1}^{\mathrm{n}} \lambda_{o\mathrm{k}} \varkappa_\mathrm{k}\right) \leq{}& \mathrm{f}(\varkappa_1) + \mathrm{f}(\varkappa_n) \\ &- \sum_{\mathrm{k}=1}^{\mathrm{n}} \left[\mathrm{f}(\varkappa_\mathrm{k} - \varkappa_1) + \mathrm{f}(\varkappa_n - \varkappa_\mathrm{k})\right] \\ &- \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{f}\left(\left|\varkappa_\mathrm{k} - \sum_{\mathrm{j}=1}^{\mathrm{n}} \lambda_{o\mathrm{j}} \varkappa_\mathrm{j}\right|\right), \end{aligned}$$

$\forall \lambda_{o\mathrm{k}} \in [0,1]$ *such that* $\sum_{\mathrm{k}=1}^{\mathrm{n}} \lambda_{o\mathrm{k}} = 1$.

Proof. Consider the L.H.S of the aforementioned result, we get

$$\mathrm{f}\left(\varkappa_1 + \varkappa_\mathrm{n} - \sum_{\mathrm{k}=1}^{\mathrm{n}} \lambda_{o\mathrm{k}} \varkappa_\mathrm{k}\right) = \mathrm{f}\left(\sum_{\mathrm{k}=1}^{\mathrm{n}} \lambda_{o\mathrm{k}} (\varkappa_1 + \varkappa_\mathrm{n} - \varkappa_\mathrm{k})\right),$$

using the result of Theorem 2.12 we get

$$\begin{aligned} &\mathrm{f}\left(\varkappa_1 + \varkappa_\mathrm{n} - \sum_{\mathrm{k}=1}^{\mathrm{n}} \lambda_{o\mathrm{k}} \varkappa_\mathrm{k}\right) \\ &\quad = \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{f}(\varkappa_1 + \varkappa_\mathrm{n} - \varkappa_\mathrm{k}) \\ &\qquad - \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{f}\left(\left|(\varkappa_1 + \varkappa_\mathrm{n} - \varkappa_\mathrm{k}) - \sum_{\mathrm{j}=1}^{\mathrm{n}} \lambda_{o\mathrm{j}} (\varkappa_1 + \varkappa_\mathrm{n} - \varkappa_\mathrm{j})\right|\right) \end{aligned} \tag{2.22}$$

Using $\varkappa_{\mathrm{k}} = \alpha_{\mathrm{k}}\varkappa_1 + \beta_{\mathrm{k}}\varkappa_n$, where $\alpha_{\mathrm{k}} + \beta_{\mathrm{k}} = 1$ in the first term on the R.H.S of (2.22), we get

$$
\begin{aligned}
(2.23)\quad & \mathrm{f}\left(\varkappa_1 + \varkappa_{\mathrm{n}} - \sum_{\mathrm{k}=1}^{\mathrm{n}} \lambda_{o\mathrm{k}}\varkappa_{\mathrm{k}}\right) \\
&\leq \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{f}\left(\alpha_{\mathrm{k}}\varkappa_n + \beta_{\mathrm{k}}\varkappa_1\right) - \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{f}\left(\left|\varkappa_{\mathrm{k}} - \sum_{\mathrm{j}=1}^{\mathrm{n}} \lambda_{o\mathrm{j}}\varkappa_{\mathrm{j}}\right|\right) \\
&= \sum_{\mathrm{k}=1}^{\mathrm{n}} \left[\mathrm{f}\left(\varkappa_n\right) + \mathrm{f}\left(\varkappa_1\right) - \mathrm{f}\left(\alpha_{\mathrm{k}}\left|\varkappa_1 - \varkappa_n\right|\right) - \mathrm{f}\left(\beta_{\mathrm{k}}\left|\varkappa_1 - \varkappa_n\right|\right)\right] \\
&\quad - \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{f}\left(\left|\varkappa_{\mathrm{k}} - \sum_{\mathrm{j}=1}^{\mathrm{n}} \lambda_{o\mathrm{j}}\varkappa_{\mathrm{j}}\right|\right) \\
&= \mathrm{f}(\varkappa_n) + \mathrm{f}(\varkappa_1) - \sum_{\mathrm{k}=1}^{\mathrm{n}} \left[\mathrm{f}\left(\alpha_{\mathrm{k}}\left|\varkappa_1 - \varkappa_n\right|\right) + \mathrm{f}\left(\beta_{\mathrm{k}}\left|\varkappa_1 - \varkappa_n\right|\right)\right] \\
&\quad - \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{f}\left(\left|\varkappa_{\mathrm{k}} - \sum_{\mathrm{j}=1}^{\mathrm{n}} \lambda_{o\mathrm{j}}\varkappa_{\mathrm{j}}\right|\right),
\end{aligned}
$$

setting $\alpha_{\mathrm{k}} = \frac{\varkappa_{\mathrm{k}} - \varkappa_1}{\varkappa_{\mathrm{n}} - \varkappa_1}$ and $\beta_{\mathrm{k}} = \frac{\varkappa_{\mathrm{n}} - \varkappa_{\mathrm{k}}}{\varkappa_{\mathrm{n}} - \varkappa_1}$ in (2.23) we get

$$
\begin{aligned}
\mathrm{f}\left(\varkappa_1 + \varkappa_{\mathrm{n}} - \sum_{\mathrm{k}=1}^{\mathrm{n}} \lambda_{o\mathrm{k}}\varkappa_{\mathrm{k}}\right) &\leq \mathrm{f}\left(\varkappa_1\right) + \mathrm{f}\left(\varkappa_n\right) \\
&\quad - \sum_{\mathrm{k}=1}^{\mathrm{n}} \left(\mathrm{f}\left(\varkappa_{\mathrm{k}} - \varkappa_1\right) + \mathrm{f}\left(\varkappa_n - \varkappa_{\mathrm{k}}\right)\right) \\
&\quad - \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{f}\left(\left|\varkappa_{\mathrm{k}} - \sum_{\mathrm{j}=1}^{\mathrm{n}} \lambda_{o\mathrm{j}}\varkappa_{\mathrm{j}}\right|\right).
\end{aligned}
$$

Hence the proof. □

Remark 2.15. If the inequality in Theorem 2.14 is reversed, then we get Mercer-Jensen's inequality for P-subquadratic function.

Theorem 2.16 (Hermite Hadamard's inequality)**.** *Let the map* f *is a P-superquadratic on* $[\mathfrak{a}_o, \mathfrak{b}_o]$ *with* $\mathfrak{a}_o < \mathfrak{b}_o$ *then the subsequent inequality*

$$
\begin{aligned}
(2.24)\quad & \frac{1}{2}\mathrm{f}\left(\frac{\mathfrak{a}_o + \mathfrak{b}_o}{2}\right) + \frac{1}{\mathfrak{b}_o - \mathfrak{a}_o}\int_{\mathfrak{a}_o}^{\mathfrak{b}_o} \mathrm{f}\left(\left|\frac{\mathfrak{a}_o + \mathfrak{b}_o}{2} - \varkappa\right|\right) d\varkappa \\
&\leq \frac{1}{\mathfrak{b}_o - \mathfrak{a}_o}\int_{\mathfrak{a}_o}^{\mathfrak{b}_o} \mathrm{f}\left(\varkappa\right) d\varkappa
\end{aligned}
$$

$$\leq \mathrm{f}\left(\mathfrak{a}_o\right)+\mathrm{f}\left(\mathfrak{b}_o\right)-\frac{1}{\mathfrak{b}_o-\mathfrak{a}_o}\int_{\mathfrak{a}_o}^{\mathfrak{b}_o}\left(\mathrm{f}(\mathfrak{b}_o-\varkappa)+\mathrm{f}(\varkappa-\mathfrak{a}_o)\right)d\varkappa.$$

holds.

Proof. Since f is a P-superquadratic, therefore it must satisfy the following inequality

$$\begin{aligned}\mathrm{f}\left(\lambda_o\varkappa+(1-\lambda_o)\mathrm{y}\right)\leq{}&\left(\mathrm{f}\left(\varkappa\right)-\mathrm{f}((1-\lambda_o)|\varkappa-\mathrm{y}|)\right)\\&+\left(\mathrm{f}(\mathrm{y})-\mathrm{f}(\lambda_o|\varkappa-\mathrm{y}|)\right).\end{aligned}\tag{2.25}$$

Putting $\lambda_o=\frac{1}{2}$ in (2.25) we get

$$\mathrm{f}\left(\frac{\varkappa+\mathrm{y}}{2}\right)\leq\mathrm{f}\left(\varkappa\right)+\mathrm{f}(\mathrm{y})-2\mathrm{f}\left(\frac{|\varkappa-\mathrm{y}|}{2}\right).\tag{2.26}$$

Changing $\varkappa$ by $\lambda_o\mathfrak{a}_o+(1-\lambda_o)\mathfrak{b}_o$ and y by $\lambda_o\mathfrak{b}_o+(1-\lambda_o)\mathfrak{a}_o$ in (2.26), we attain

$$\begin{aligned}\mathrm{f}\left(\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}\right)\leq{}&\mathrm{f}(\lambda_o\mathfrak{a}_o+(1-\lambda_o)\mathfrak{b}_o)+\mathrm{f}(\lambda_o\mathfrak{b}_o+(1-\lambda_o)\mathfrak{a}_o)\\&-2\mathrm{f}\left(\frac{(1-2\lambda_o)\left|\mathfrak{b}_o-\mathfrak{a}_o\right|}{2}\right).\end{aligned}\tag{2.27}$$

Integrating 2.27 via λ_o over $[0,1]$. we attain

$$\begin{aligned}\mathrm{f}\left(\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}\right)\leq{}&\int_0^1\mathrm{f}(\lambda_o\mathfrak{a}_o+(1-\lambda_o)\mathfrak{b}_o)d\lambda_o\\&+\int_0^1\mathrm{f}(\lambda_o\mathfrak{b}_o+(1-\lambda_o)\mathfrak{a}_o)d\lambda_o\\&-2\int_0^1\mathrm{f}\left(\frac{(1-2\lambda_o)\left|\mathfrak{b}_o-\mathfrak{a}_o\right|}{2}\right)d\lambda_o.\end{aligned}\tag{2.28}$$

It is to be noted that

$$\int_0^1\mathrm{f}(\lambda_o\mathfrak{b}_o+(1-\lambda_o)\mathfrak{a}_o)d\lambda_o=\int_0^1\mathrm{f}(\lambda_o\mathfrak{a}_o+(1-\lambda_o)\mathfrak{b}_o)d\lambda_o,$$

then (2.28) becomes

$$\begin{aligned}\mathrm{f}\left(\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}\right)\leq{}&2\int_0^1\mathrm{f}(\lambda_o\mathfrak{a}_o+(1-\lambda_o)\mathfrak{b}_o)d\lambda_o\\&-2\int_0^1\mathrm{f}\left(\frac{(1-2\lambda_o)\left|\mathfrak{b}_o-\mathfrak{a}_o\right|}{2}\right)d\lambda_o.\end{aligned}\tag{2.29}$$

Setting $\lambda_o\mathfrak{a}_o+(1-\lambda_o)\mathfrak{b}_o=\varkappa$ and $\lambda_o=\frac{\mathfrak{b}_o-\varkappa}{\mathfrak{b}_o-\mathfrak{a}_o}$ in (2.29) then we obtain

$$\frac{1}{2}\mathrm{f}\left(\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}\right)+\frac{1}{\mathfrak{b}_o-\mathfrak{a}_o}\int_{\mathfrak{a}_o}^{\mathfrak{b}_o}\mathrm{f}\left(\left|\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}-\varkappa\right|\right)d\varkappa\tag{2.30}$$

$$\leq \frac{1}{\mathfrak{b}_o - \mathfrak{a}_o} \int_{\mathfrak{a}_o}^{\mathfrak{b}_o} \mathrm{f}\left(\varkappa\right) d\varkappa.$$

Again as f is possessing P-superquadraticity therefore we have

$$\mathrm{f}\left(\lambda_o \mathfrak{a}_o + (1-\lambda_o)\mathfrak{b}_o\right) \leq \left(\mathrm{f}\left(\mathfrak{a}_o\right) - \mathrm{f}((1-\lambda_o)|\mathfrak{b}_o - \mathfrak{a}_o|)\right) + \left(\mathrm{f}\left(\mathfrak{b}_o\right) - \mathrm{f}\left(\lambda_o|\mathfrak{b}_o - \mathfrak{a}_o|\right)\right), \tag{2.31}$$

and

$$\mathrm{f}\left(\lambda_o \mathfrak{b}_o + (1-\lambda_o)\mathfrak{a}_o\right) \leq \left(\mathrm{f}\left(\mathfrak{b}_o\right) - \mathrm{f}((1-\lambda_o)|\mathfrak{b}_o - \mathfrak{a}_o|)\right) + \left(\mathrm{f}\left(\mathfrak{a}_o\right) - \mathrm{f}\left(\lambda_o|\mathfrak{b}_o - \mathfrak{a}_o|\right)\right), \tag{2.32}$$

adding (2.31) and (2.32) we get

$$\begin{aligned} &\mathrm{f}\left(\lambda_o \mathfrak{a}_o + (1-\lambda_o)\mathfrak{b}_o\right) + \mathrm{f}\left(\lambda_o \mathfrak{b}_o + (1-\lambda_o)\mathfrak{a}_o\right) \\ &\quad \leq 2\left(\mathrm{f}\left(\mathfrak{a}_o\right) + \mathrm{f}\left(\mathfrak{b}_o\right) - \mathrm{f}\left((1-\lambda_o)|\mathfrak{b}_o - \mathfrak{a}_o|\right)\right) - \mathrm{f}(\lambda_o|\mathfrak{b}_o - \mathfrak{a}_o|). \end{aligned} \tag{2.33}$$

If we set $\varkappa = \lambda_o \mathfrak{a}_o + (1-\lambda_o)\mathfrak{b}_o$ then $\lambda_o \mathfrak{b}_o + (1-\lambda_o)\mathfrak{a}_o = \mathfrak{a}_o + \mathfrak{b}_o - \varkappa$, $(1-\lambda_o) = \frac{\varkappa - \mathfrak{a}_o}{\mathfrak{b}_o - \mathfrak{a}_o}$ and $\lambda_o = \frac{\mathfrak{b}_o - \varkappa}{\mathfrak{b}_o - \mathfrak{a}_o}$. So using these values in (2.33) we get.

$$\mathrm{f}\left(\varkappa\right) + \mathrm{f}\left(\mathfrak{a}_o + \mathfrak{b}_o - \varkappa\right) \leq 2\left(\mathrm{f}\left(\mathfrak{a}_o\right) + \mathrm{f}\left(\mathfrak{b}_o\right) - \mathrm{f}\left(\mathfrak{b}_o - \varkappa\right) - \mathrm{f}\left(\varkappa - \mathfrak{a}_o\right)\right). \tag{2.34}$$

Now consider

$$\begin{aligned} \int_{\mathfrak{a}_o}^{\mathfrak{b}_o} \mathrm{f}\left(\varkappa\right) d\varkappa &= \int_{\mathfrak{a}_o}^{\frac{\mathfrak{a}_o + \mathfrak{b}_o}{2}} \mathrm{f}\left(\varkappa\right) d\varkappa + \int_{\frac{\mathfrak{a}_o + \mathfrak{b}_o}{2}}^{\mathfrak{b}_o} \mathrm{f}\left(\varkappa\right) d\varkappa \\ &= \int_{\frac{\mathfrak{a}_o + \mathfrak{b}_o}{2}}^{\mathfrak{b}_o} \left(\mathrm{f}\left(\varkappa\right) + \mathrm{f}\left(\mathfrak{a}_o + \mathfrak{b}_o - \varkappa\right)\right) d\varkappa. \end{aligned} \tag{2.35}$$

Using the value of (2.34) in (2.35) we get

$$\begin{aligned} \int_{\mathfrak{a}_o}^{\mathfrak{b}_o} \mathrm{f}\left(\varkappa\right) d\varkappa &\leq \int_{\frac{\mathfrak{a}_o + \mathfrak{b}_o}{2}}^{\mathfrak{b}_o} \left(2\mathrm{f}\left(\mathfrak{a}_o\right) + 2\mathrm{f}\left(\mathfrak{b}_o\right) - 2\mathrm{f}(\mathfrak{b}_o - \varkappa) - 2\mathrm{f}(\varkappa - \mathfrak{a}_o)\right) d\varkappa \\ &= \left(\mathrm{f}\left(\mathfrak{a}_o\right) + \mathrm{f}\left(\mathfrak{b}_o\right)\right)(\mathfrak{b}_o - \mathfrak{a}_o) \\ &\quad - 2\int_{\frac{\mathfrak{a}_o + \mathfrak{b}_o}{2}}^{\mathfrak{b}_o} \left[\mathrm{f}(\mathfrak{b}_o - \varkappa) + \mathrm{f}(\varkappa - \mathfrak{a}_o)\right] d\varkappa \\ &= \left(\mathrm{f}\left(\mathfrak{a}_o\right) + \mathrm{f}\left(\mathfrak{b}_o\right)\right)(\mathfrak{b}_o - \mathfrak{a}_o) \\ &\quad - \int_{\mathfrak{a}_o}^{\mathfrak{b}_o} \left[\mathrm{f}(\mathfrak{b}_o - \varkappa) + \mathrm{f}(\varkappa - \mathfrak{a}_o)\right] d\varkappa. \end{aligned}$$

It implies that

$$\frac{1}{\mathfrak{b}_o - \mathfrak{a}_o} \int_{\mathfrak{a}_o}^{\mathfrak{b}_o} \mathrm{f}\left(\varkappa\right) d\varkappa \leq \left(\mathrm{f}\left(\mathfrak{a}_o\right) + \mathrm{f}\left(\mathfrak{b}_o\right)\right) \tag{2.36}$$

$$- \frac{1}{\mathfrak{b}_o - \mathfrak{a}_o} \int_{\mathfrak{a}_o}^{\mathfrak{b}_o} [\mathrm{f}(\mathfrak{b}_o - \varkappa) + \mathrm{f}(\varkappa - \mathfrak{a}_o)]\, d\varkappa.$$

From (2.30) and (2.36) we get

$$\begin{aligned} (2.37)\quad & \frac{1}{2}\mathrm{f}\left(\frac{\mathfrak{a}_o + \mathfrak{b}_o}{2}\right) + \frac{1}{\mathfrak{b}_o - \mathfrak{a}_o} \int_{\mathfrak{a}_o}^{\mathfrak{b}_o} \mathrm{f}\left(\left|\frac{\mathfrak{a}_o + \mathfrak{b}_o}{2} - \varkappa\right|\right) d\varkappa \\ & \leq \frac{1}{\mathfrak{b}_o - \mathfrak{a}_o} \int_{\mathfrak{a}_o}^{\mathfrak{b}_o} \mathrm{f}(\varkappa)\, d\varkappa \\ & \leq (\mathrm{f}(\mathfrak{a}_o) + \mathrm{f}(\mathfrak{b}_o)) - \frac{1}{\mathfrak{b}_o - \mathfrak{a}_o} \int_{\mathfrak{a}_o}^{\mathfrak{b}_o} [\mathrm{f}(\mathfrak{b}_o - \varkappa) + \mathrm{f}(\varkappa - \mathfrak{a}_o)]\, d\varkappa. \end{aligned}$$

Hence the proof. □

Remark 2.17. If the inequality in (2.24) is reversed, then we attain the inequality of H.H type for P-subquadratic function.

Example 2.18. Since we know that $\cosh(\varkappa)$ is possessing P-superquadraticity on $I = [\mathfrak{a}_o, \mathfrak{b}_o]$ then we have the following Right, Left, and Middle terms for the Theorem 2.16.

$$\begin{aligned} \text{Left Term} &= \frac{1}{2}\cosh\left(\frac{\mathfrak{a}_o + \mathfrak{b}_o}{2}\right) + \frac{2}{\mathfrak{b}_o - \mathfrak{a}_o}\sinh\left(\frac{\mathfrak{b}_o - \mathfrak{a}_o}{2}\right), \\ \text{Middle Term} &= \frac{\sinh(\mathfrak{b}_o) - \sinh(\mathfrak{a}_o)}{\mathfrak{b}_o - \mathfrak{a}_o}, \\ \text{Right Term} &= \cosh(\mathfrak{a}_o) + \cosh(\mathfrak{b}_o) - \frac{2\sinh(\mathfrak{b}_o - \mathfrak{a}_o)}{\mathfrak{b}_o - \mathfrak{a}_o}. \end{aligned}$$

The Graph 6 clearly confirms the veracity of Theorem 2.16.

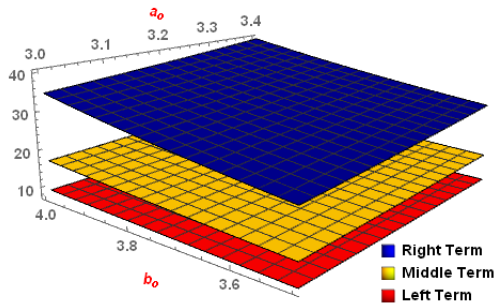


FIGURE 6. Graphical illustration of Right, Middle and Left terms for $\mathfrak{a}_o \in [3, 3.4]$ and $\mathfrak{b}_o \in [3.5, 4]$.

3. Comparative Analysis Between Integral Inequalities for P-superquadraticity and P-convexity

We have seen that the concept of P-convex function is introduced by Dragomir et al. in [19]. Prominent inequalities of such a concept are the Jensen and H.H type inequalities are as follows:

Jensen's inequality

$$\mathrm{f}\left(\sum_{i=1}^{\mathfrak{n}}\lambda_{oi}\varkappa_i\right)\leq\sum_{i=1}^{\mathfrak{n}}\mathrm{f}\left(\varkappa_i\right),\quad\forall\lambda_{oi}\in[0,1]\quad\text{and}\quad\sum_{i=1}^{\mathfrak{n}}\lambda_{oi}=1. \tag{3.1}$$

H.H's inequality

$$\frac{1}{2}\mathrm{f}\left(\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}\right)\leq\frac{1}{\mathfrak{b}_o-\mathfrak{a}_o}\int_{\mathfrak{a}_o}^{\mathfrak{b}_o}\mathrm{f}\left(\varkappa\right)d\varkappa\leq\left(\mathrm{f}\left(\mathfrak{a}_o\right)+\mathrm{f}\left(\mathfrak{b}_o\right)\right). \tag{3.2}$$

So in this section, we present the comparison between the integral inequalities such that Jensen's and H.H's type integral inequalities for P-superquadratict functions and for P-convex functions. We will see how integral inequalities of functions which are P-superquadratic are more sophisticated than the inequalities of P-convexity.

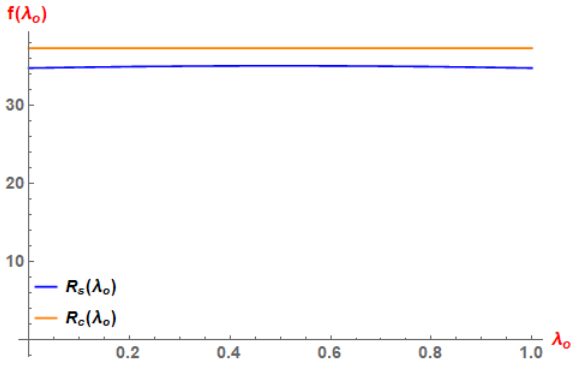
Example 3.1. Let us choose the function $\cosh\left(\varkappa\right)$, which is a P- superquadraticity as well as P-convex function on $I=[\mathfrak{a}_o,\mathfrak{b}_o]$ then we get the following R.H.Ss for the Jensen,s inequalities for "n=2" for P-superquadratic which we denote by R_s and for P-convex function which we denote by R_c are as follow. Here we assume the interval $\mathfrak{a}_o=3$ and $\mathfrak{b}_o=4$.

$$\begin{aligned}&\mathrm{R}_s(\lambda_o)=(\cosh(3)-\cosh(1-\lambda_o))+(\cosh(4)-\cosh(\lambda_o)),\\&\mathrm{R}_c(\lambda_o)=\cosh(3)+\cosh(4).\end{aligned}$$

Consider the below graph and table, which display that P-superquadratic function provides a refined Jensen inequality than the P-convex function.

Values of λ_o	$\mathrm{R}_c(\lambda_o)$	$\mathrm{R}_s(\lambda_o)$
0.1	37.3758	34.9378
0.2	37.3758	35.0184
0.3	37.3758	35.0754
0.4	37.3758	35.1094
0.5	37.3758	35.1206
0.6	37.3758	35.1094
0.7	37.3758	35.0754
0.8	37.3758	35.0184
0.9	37.3758	34.9378
1.0	37.3758	34.8328

(A) Numerical illustration of $\mathrm{R}_s(\lambda_o)$ and $\mathrm{R}_c(\lambda_o)$.



(B) Graphical illustration of $\mathrm{R}_s(\lambda_o)$ and $\mathrm{R}_c(\lambda_o)$

FIGURE 7. Numerical and Graphical illustration of $\mathrm{R}_s(\lambda_o)$ and $\mathrm{R}_c(\lambda_o)$.

Example 3.2. Let us consider the same function as in Example 3.1 for $[3,4]$, we get the following L.H.Ss and R.H.Ss for the H.H inequality for P-superquadratic function which we denote by L_s and R_s and for P-convex function which we denote by L_c and R_c respectively are as follows. We do not take the M.H.Ss (Middle Sides) of the H.H inequalities under consideration because it is the same for both P-superquadratic and P-convex functions.

$$\mathrm{L}_s(\mathfrak{a}_o,\mathfrak{b}_o)=\frac{1}{2}\cosh\left(\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}\right)+\frac{2}{\mathfrak{b}_o-\mathfrak{a}_o}\sinh\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{2}\right),$$

$$\mathrm{L}_c(\mathfrak{a}_o,\mathfrak{b}_o)=\frac{1}{2}\cosh\left(\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}\right),$$

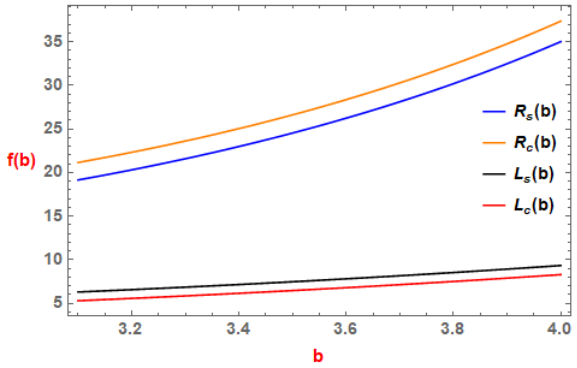
$$\mathrm{R}_s(\mathfrak{a}_o,\mathfrak{b}_o)=\cosh(\mathfrak{a}_o)+\cosh(\mathfrak{b}_o)-\frac{2\sinh(\mathfrak{b}_o-\mathfrak{a}_o)}{\mathfrak{b}_o-\mathfrak{a}_o},$$

$$\mathrm{R}_c(\mathfrak{a}_o, \mathfrak{b}_o) = \cosh(\mathfrak{a}_o) + \cosh(\mathfrak{b}_o).$$

Look at to the following graphs where R_s and L_s graphs are between the graphs R_c and L_c similarly the tabulated values of P-superquadraticity lie between the tabulated values of P-convexity which display that H.H inequality for P- superquadratic function provides sharp bounds than the H.H inequality for P-convex function.

$\mathfrak{b}$	$\mathrm{R}_c(\mathfrak{b})$	$\mathrm{R}_s(\mathfrak{b})$	$\mathrm{L}_s(\mathfrak{b})$	$\mathrm{L}_c(\mathfrak{b})$
3.1	21.14	19.14	6.29	5.29
3.2	22.31	20.30	6.56	5.56
3.3	23.60	21.57	6.84	5.84
3.4	25.03	22.97	7.15	6.14
3.5	26.61	24.52	7.46	6.45
3.6	28.35	26.23	7.80	6.78
3.7	30.27	28.11	8.15	7.13
3.8	32.40	30.18	8.52	7.49
3.9	34.75	32.47	8.91	7.88
4.0	37.35	35.00	9.32	8.28

(A) Numerical illustration of $\mathrm{R}_s(\mathfrak{b})$, $\mathrm{R}_c(\mathfrak{b})$, $\mathrm{L}_s(\mathfrak{b})$ and $\mathrm{L}_c(\mathfrak{b})$



(B) Graphical illustration of $\mathrm{R}_s(\mathfrak{b})$, $\mathrm{R}_c(\mathfrak{b})$, $\mathrm{L}_s(\mathfrak{b})$ and $\mathrm{L}_c(\mathfrak{b})$

FIGURE 8. Numerical and Graphical illustration of $\mathrm{R}_s(\mathfrak{b})$, $\mathrm{R}_c(\mathfrak{b})$, $\mathrm{L}_s(\mathfrak{b})$ and $\mathrm{L}_c(\mathfrak{b})$ for $\mathfrak{b} \in [3.1, 4]$ and $\mathfrak{a} = 3$.

4. Fractional H.H Type Inequalities for P-Superquadratic Functions

The purpose of the results that come next is to examine how fractional operators can be used to apply P-superquadratic functions to integral inequalities.

Lemma 4.1. *Let* f *is a P-superquadratic function on* $[\mathfrak{a}_o, \mathfrak{b}_o]$ *then*

$$\mathrm{f}\left(\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}\right) \leq \mathrm{f}\left(\varkappa\right) + \mathrm{f}\left(\mathfrak{a}_o+\mathfrak{b}_o-\varkappa\right) - 2\mathrm{f}\left(\left|\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}-\varkappa\right|\right),$$

holds, $\forall\ \varkappa \in [\mathfrak{a}_o, \mathfrak{b}_o]$.

Proof. Since the function $\mathrm{f}\colon [\mathfrak{a}_o, \mathfrak{b}_o] \longrightarrow \Re$, is a P-superquadratic on $[\mathfrak{a}_o, \mathfrak{b}_o]$, therefore by Definition (2.1) we attain

$$\begin{aligned}(4.1)\quad &\mathrm{f}\left((1-\lambda_o)\varkappa_2+\lambda_o\varkappa_1\right)\\ &\leq \mathrm{f}(\varkappa_2)+\mathrm{f}(\varkappa_1)-\mathrm{f}\left((1-\lambda_o)\left|\varkappa_2-\varkappa_1\right|\right)-\mathrm{f}\left(\lambda_o\left|\varkappa_2-\varkappa_1\right|\right).\end{aligned}$$

Setting $\lambda_o = \frac{1}{2}$ in (4.1) we get

$$\begin{aligned}(4.2)\qquad \mathrm{f}\left(\frac{\varkappa_1+\varkappa_2}{2}\right) &\leq \mathrm{f}(\varkappa_2)+\mathrm{f}(\varkappa_1)-\mathrm{f}\left(\left|\varkappa_2-\left(\frac{\varkappa_2+\varkappa_1}{2}\right)\right|\right)\\ &\quad-\mathrm{f}\left(\left|\varkappa_1-\left(\frac{\varkappa_2+\varkappa_1}{2}\right)\right|\right).\end{aligned}$$

Suppose that $\mathfrak{a}_o \leq \varkappa \leq \frac{\mathfrak{a}_o+\mathfrak{b}_o}{2} \leq \mathfrak{a}_o+\mathfrak{b}_o-\varkappa \leq \mathfrak{b}_o$.
Changing $\varkappa_2$ by $\varkappa$ and $\varkappa_1$ by $\mathfrak{a}_o+\mathfrak{b}_o-\varkappa$ in 4.2 we get.

$$(4.3)\quad \mathrm{f}\left(\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}\right) \leq \mathrm{f}\left(\varkappa\right)+\mathrm{f}(\mathfrak{a}_o+\mathfrak{b}_o-\varkappa)-2\mathrm{f}\left(\left|\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}-\varkappa\right|\right).$$

Hence the proof. □

Theorem 4.2. *Let* f *is a P-superquadratic function on* $[\mathfrak{a}_o, \mathfrak{b}_o]$ *then*

$$\begin{aligned}&\mathrm{f}\left(\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}\right)+\frac{\lambda_o}{(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}}\int_{\mathfrak{a}_o}^{\mathfrak{b}_o}\mathrm{f}\left(\left|\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}-x\right|\right)\\ &\qquad\times\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1}+(\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right)d\varkappa\\ &\leq\frac{\Gamma(1+\lambda_o)}{(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}}\left(\mathfrak{J}^{\lambda_o}_{\mathfrak{a}_o+}\mathrm{f}(\mathfrak{b}_o)+\mathfrak{J}^{\lambda_o}_{\mathfrak{b}_o-}\mathrm{f}(\mathfrak{a}_o)\right).\end{aligned}$$

Proof. Let us consider

(4.4)

$$\begin{aligned}&\mathrm{f}\left(\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}\right)\\ &=\frac{\lambda_o\mathrm{f}\left(\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}\right)}{2(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}}\int_{\mathfrak{a}_o}^{\mathfrak{b}_o}\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1}+(\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right)d\varkappa\\ &=\frac{\lambda_o}{(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}}\int_{\mathfrak{a}_o}^{\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}}\mathrm{f}\left(\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}\right)\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1}+(\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right)d\varkappa.\end{aligned}$$

Using Lemma 4.1 in (4.4) we get

(4.5)

$$
\begin{aligned}
&\mathrm{f}\left(\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}\right)\\
&\quad\leq \frac{\lambda_o}{(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}}\int_{\mathfrak{a}_o}^{\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}}\left(\mathrm{f}\left(\varkappa\right)+\mathrm{f}(\mathfrak{a}_o+\mathfrak{b}_o-\varkappa)-2\mathrm{f}\left(\left|\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}-\varkappa\right|\right)\right.\\
&\qquad\left.\times\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1}+(\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right)\right)d\varkappa\\
&\quad=\frac{\lambda_o}{(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}}\left(\int_{\mathfrak{a}_o}^{\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}}\left(\mathrm{f}\left(\varkappa\right)+\mathrm{f}(\mathfrak{a}_o+\mathfrak{b}_o-\varkappa)\right)\right.\\
&\qquad\left.\times\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1}+(\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right)d\varkappa\right)\\
&\qquad-2\int_{\mathfrak{a}_o}^{\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}}\mathrm{f}\left(\left|\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}-\varkappa\right|\right)\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1}+(\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right)d\varkappa\\
&\quad=\frac{\lambda_o}{(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}}\left(\int_{\mathfrak{a}_o}^{\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}}\left(\mathrm{f}\left(\varkappa\right)\right)\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1}+(\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right)\right)d\varkappa\\
&\qquad+\int_{\mathfrak{a}_o}^{\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}}\left(\mathrm{f}(\mathfrak{a}_o+\mathfrak{b}_o-\varkappa)\right)\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1}+(\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right)d\varkappa\\
&\qquad-\int_{\mathfrak{a}_o}^{\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}}\mathrm{f}\left(\left|\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}-\varkappa\right|\right)\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1}+(\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right)d\varkappa\\
&\qquad-\int_{\mathfrak{a}_o}^{\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}}\mathrm{f}\left(\left|\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}-\varkappa\right|\right)\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1}+(\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right)d\varkappa\\
&\quad=\frac{\lambda_o}{(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}}\left(\int_{\mathfrak{a}_o}^{\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}}\left(\mathrm{f}\left(\varkappa\right)\right)\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1}+(\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right)\right)d\varkappa\\
&\qquad+\int_{\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}}^{\mathfrak{b}_o}\left(\mathrm{f}(\mathfrak{a}_o+\mathfrak{b}_o-\varkappa)\right)\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1}+(\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right)d\varkappa\\
&\qquad-\int_{\mathfrak{a}_o}^{\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}}\mathrm{f}\left(\left|\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}-\varkappa\right|\right)\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1}+(\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right)d\varkappa\\
&\qquad-\int_{\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}}^{\mathfrak{b}_o}\mathrm{f}\left(\left|\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}-\varkappa\right|\right)\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1}+(\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right)d\varkappa\\
&\quad=\frac{\lambda_o}{(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}}\left(\int_{\mathfrak{a}_o}^{\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}}\left(\mathrm{f}\left(\varkappa\right)\right)\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1}+(\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right)\right)d\varkappa
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}}^{\mathfrak{b}_o}(\mathrm{f}(\varkappa))\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1}+(\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right)d\varkappa\\
&-\int_{\mathfrak{a}_o}^{\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}}\mathrm{f}\left(\left|\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}-\varkappa\right|\right)\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1}+(\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right)d\varkappa\\
&-\int_{\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}}^{\mathfrak{b}_o}\mathrm{f}\left(\left|\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}-\varkappa\right|\right)\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1}+(\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right)d\varkappa\\
=&\frac{\lambda_o}{(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}}\left(\int_{\mathfrak{a}_o}^{\mathfrak{b}_o}(\mathrm{f}(\varkappa))\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1}+(\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right)\right)d\varkappa\\
&-\int_{\mathfrak{a}_o}^{\mathfrak{b}_o}\mathrm{f}\left(\left|\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}-\varkappa\right|\right)\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1}+(\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right)d\varkappa\\
=&\frac{\lambda_o}{(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}}\Bigg(\Gamma(\lambda_o)\left(\mathfrak{J}^{\lambda_o}_{\mathfrak{a}_o+}\mathrm{f}(\mathfrak{b}_o)+\mathfrak{J}^{\lambda_o}_{\mathfrak{b}_o-}\mathrm{f}(\mathfrak{a}_o)\right)\\
&-\int_{\mathfrak{a}_o}^{\mathfrak{b}_o}\mathrm{f}\left(\left|\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}-x\right|\right)\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1}+(\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right)d\varkappa\Bigg).
\end{aligned}
$$

It implies that

$$
\begin{aligned}
&\mathrm{f}\left(\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}\right)+\frac{\lambda_o}{(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}}\int_{\mathfrak{a}_o}^{\mathfrak{b}_o}\mathrm{f}\left(\left|\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}-x\right|\right)\\
&\qquad\times\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1}+(\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right)d\varkappa\\
&\leq\frac{\Gamma(1+\lambda_o)}{(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}}\left(\mathfrak{J}^{\lambda_o}_{\mathfrak{a}_o+}\mathrm{f}(\mathfrak{b}_o)+\mathfrak{J}^{\lambda_o}_{\mathfrak{b}_o-}\mathrm{f}(\mathfrak{a}_o)\right).
\end{aligned}
$$

This concludes the proof. □

Lemma 4.3. *Let* f *exhibits P-superquadraticity function on* $[\mathfrak{a}_o,\mathfrak{b}_o]$*, then*

$$\mathrm{f}(\varkappa)+\mathrm{f}(\mathfrak{a}_o+\mathfrak{b}_o-\varkappa)\leq 2\left[\mathrm{f}(\mathfrak{a}_o)+\mathrm{f}(\mathfrak{b}_o)-\mathrm{f}(\mathfrak{b}_o-\varkappa)-\mathrm{f}(\varkappa-\mathfrak{a}_o)\right],$$

$\forall\varkappa\in[\mathfrak{a}_o,\mathfrak{b}_o]$.

Proof. Since $\mathrm{f}\colon[\mathfrak{a}_o,\mathfrak{b}_o]\longrightarrow\Re$, be a P-Superquadratic function on $[\mathfrak{a}_o,\mathfrak{b}_o]$ then by Definition 2.1, $\forall\quad\varkappa_1,\varkappa_2\in[\mathfrak{a}_o,\mathfrak{b}_o]$, we have

$$
\begin{aligned}
&\mathrm{f}\left((1-\lambda_o)\varkappa_2+\lambda_o\varkappa_1\right)\\
&\qquad\leq\mathrm{f}(\varkappa_2)+\mathrm{f}(\varkappa_1)-\mathrm{f}\left((1-\lambda_o)|\varkappa_2-\varkappa_1|\right)-\mathrm{f}\left(\lambda_o|\varkappa_2-\varkappa_1|\right).
\end{aligned}\tag{4.6}
$$

Suppose that $\mathfrak{a}_o\leq\varkappa\leq\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}\leq\mathfrak{a}_o+\mathfrak{b}_o-\varkappa\leq\mathfrak{b}_o$. Let us set $\varkappa_1=\mathfrak{a}_o$ and $\varkappa_2=\mathfrak{b}_o$ in (4.6) we get

$$
\begin{aligned}
&\mathrm{f}\left((1-\lambda_o)\mathfrak{b}_o+\lambda_o\mathfrak{a}_o\right)\\
&\qquad\leq\mathrm{f}(\mathfrak{b}_o)+\mathrm{f}(\mathfrak{a}_o)-\mathrm{f}\left((1-\lambda_o)\left|\mathfrak{b}_o-\mathfrak{a}_o\right|\right)-\mathrm{f}\left(\lambda_o\left|\mathfrak{b}_o-\mathfrak{a}_o\right|\right).
\end{aligned}\tag{4.7}
$$

Next choosing $\varkappa = (1-\lambda_o)\mathfrak{b}_o+\lambda_o\mathfrak{a}_o$ then solving for λ_o we get $\lambda_o = \frac{\mathfrak{b}_o-\varkappa}{\mathfrak{b}_o-\mathfrak{a}_o}$ and substituting it in (4.7) we obtain

$$\mathrm{f}(\varkappa) \leq \mathrm{f}(\mathfrak{a}_o) + \mathrm{f}(\mathfrak{b}_o) - \mathrm{f}(\varkappa - \mathfrak{a}_o)) - \mathrm{f}(\mathfrak{b}_o - \varkappa). \tag{4.8}$$

Changing $\varkappa$ by $\mathfrak{a}_o + \mathfrak{b}_o - \varkappa$ in 4.8 we get

$$\mathrm{f}(\mathfrak{a}_o + \mathfrak{b}_o - \varkappa) \leq \mathrm{f}(\mathfrak{a}_o) + \mathrm{f}(\mathfrak{b}_o) - \mathrm{f}(\mathfrak{b}_o - \varkappa) - \mathrm{f}(\varkappa - \mathfrak{a}_o). \tag{4.9}$$

Adding 4.8 and 4.9 we get.

$$\mathrm{f}(\varkappa) + \mathrm{f}(\mathfrak{a}_o + \mathfrak{b}_o - \varkappa) \leq 2\left[\mathrm{f}(\mathfrak{a}_o) + \mathrm{f}(\mathfrak{b}_o) - \mathrm{f}(\mathfrak{b}_o - \varkappa) - \mathrm{f}(\varkappa - \mathfrak{a}_o)\right]. \tag{4.10}$$

Hence the proof. □

Theorem 4.4. *Let the function* f *exhibits P-superquadraticity on* $[\mathfrak{a}_o, \mathfrak{b}_o]$, *then* $\forall \varkappa \in [\mathfrak{a}_o, \mathfrak{b}_o]$, *the subsequent inequality holds*

(4.11)

$$\begin{aligned}
&\frac{\Gamma(1+\lambda_o)}{(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}}\left(\mathfrak{J}^{\lambda_o}_{\mathfrak{a}_o+}\mathrm{f}(\mathfrak{b}_o) + \mathfrak{J}^{\lambda_o}_{\mathfrak{b}_o-}\mathrm{f}(\mathfrak{a}_o)\right)\\
&\quad\leq 2\left(\mathrm{f}(\mathfrak{a}_o) + \mathrm{f}(\mathfrak{b}_o)\right) - \frac{\lambda_o}{(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}}\int_{\mathfrak{a}_o}^{\mathfrak{b}_o}\left(\mathrm{f}\left(\mathfrak{b}_o-\varkappa\right) + \mathrm{f}\left(\varkappa-\mathfrak{a}_o\right)\right)\\
&\qquad\times\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1} + (\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right)d\varkappa.
\end{aligned}$$

Proof. Let us take the left hand term of (4.11) into account

(4.12)

$$\begin{aligned}
&\frac{\Gamma(1+\lambda_o)}{(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}}\left(\mathfrak{J}^{\lambda_o}_{\mathfrak{a}_o+}\mathrm{f}(\mathfrak{b}_o) + \mathfrak{J}^{\lambda_o}_{\mathfrak{b}_o-}\mathrm{f}(\mathfrak{a}_o)\right)\\
&\quad= \frac{\lambda_o}{(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}}\int_{\mathfrak{a}_o}^{\mathfrak{b}_o}\mathrm{f}\left(\varkappa\right)\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1} + (\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right)d\varkappa\\
&\quad= \frac{\lambda_o}{(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}}\int_{\mathfrak{a}_o}^{\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}}\mathrm{f}\left(\varkappa\right)\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1} + (\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right)d\varkappa\\
&\qquad+ \frac{\lambda_o}{(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}}\int_{\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}}^{\mathfrak{b}_o}\mathrm{f}\left(\varkappa\right)\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1} + (\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right)d\varkappa\\
&\quad= \frac{\lambda_o}{(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}}\int_{\mathfrak{a}_o}^{\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}}\mathrm{f}\left(\varkappa\right)\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1} + (\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right)d\varkappa\\
&\qquad+ \frac{\lambda_o}{(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}}\int_{\mathfrak{a}_o}^{\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}}\mathrm{f}\left(\mathfrak{a}_o+\mathfrak{b}_o-\varkappa\right)\\
&\qquad\times\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1} + (\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right)d\varkappa
\end{aligned}$$

$$= \frac{\lambda_o}{(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}} \int_{\mathfrak{a}_o}^{\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}} \left(\mathrm{f}\left(\varkappa\right)+\mathrm{f}\left(\mathfrak{a}_o+\mathfrak{b}_o-\varkappa\right)\right) \times \left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1}+(\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right) d\varkappa,$$

employing Lemma 4.3 in (4.12), we attain

$$\begin{aligned}
&\frac{\Gamma(1+\lambda_o)}{(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}} \left(\mathfrak{J}^{\lambda_o}_{\mathfrak{a}_o+}\mathrm{f}(\mathfrak{b}_o)+\mathfrak{J}^{\lambda_o}_{\mathfrak{b}_o-}\mathrm{f}(\mathfrak{a}_o)\right) \\
&\quad \leq \frac{2\lambda_o}{(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}} \int_{\mathfrak{a}_o}^{\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}} \left(\mathrm{f}\left(\mathfrak{a}_o\right)+\mathrm{f}(\mathfrak{b}_o)-\mathrm{f}(\mathfrak{b}_o-\varkappa)-\mathrm{f}(\varkappa-\mathfrak{a}_o)\right) \\
&\qquad \times \left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1}+(\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right) d\varkappa \\
&\quad = 2\left(\mathrm{f}(\mathfrak{a}_o)+\mathrm{f}(\mathfrak{b}_o)\right)-\frac{\lambda_o}{(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}} \\
&\qquad \times \int_{\mathfrak{a}_o}^{\mathfrak{b}_o} \left(\mathrm{f}(\mathfrak{b}_o-\varkappa)+\mathrm{f}(\varkappa-\mathfrak{a}_o)\right)\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1}+(\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right) d\varkappa.
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{\Gamma(1+\lambda_o)}{(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}} \left(\mathfrak{J}^{\lambda_o}_{\mathfrak{a}_o+}\mathrm{f}(\mathfrak{b}_o)+\mathfrak{J}^{\lambda_o}_{\mathfrak{b}_o-}\mathrm{f}(\mathfrak{a}_o)\right) \\
&\quad \leq 2\left(\mathrm{f}(\mathfrak{a}_o)+\mathrm{f}(\mathfrak{b}_o)\right)-\frac{\lambda_o}{(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}} \\
&\qquad \times \int_{\mathfrak{a}_o}^{\mathfrak{b}_o} \left(\mathrm{f}(\mathfrak{b}_o-\varkappa)+\mathrm{f}(\varkappa-\mathfrak{a}_o)\right)\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1}+(\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right) d\varkappa. \square
\end{aligned}$$

Theorem 4.5. *Let* f *exhibits a P-superquadraticity on* $[\mathfrak{a}_o,\mathfrak{b}_o]$ *then the subsequent inequality holds*

$$\begin{aligned}
&\mathrm{f}\left(\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}\right)+\frac{\lambda_o}{(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}} \int_{\mathfrak{a}_o}^{\mathfrak{b}_o} \mathrm{f}\left(\left|\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}-\varkappa\right|\right) \\
&\qquad \times \left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1}+(\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right) d\varkappa \\
&\quad \leq \frac{\Gamma(1+\lambda_o)}{(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}} \left(\mathfrak{J}^{\lambda_o}_{\mathfrak{a}_o+}\mathrm{f}(\mathfrak{b}_o)+\mathfrak{J}^{\lambda_o}_{\mathfrak{b}_o-}\mathrm{f}(\mathfrak{a}_o)\right) \\
&\quad \leq 2\left(\mathrm{f}(\mathfrak{a}_o)+\mathrm{f}(\mathfrak{b}_o)\right)-\frac{\lambda_o}{(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}} \\
&\qquad \times \int_{\mathfrak{a}_o}^{\mathfrak{b}_o} \left(\mathrm{f}(\mathfrak{b}_o-\varkappa)+\mathrm{f}(\varkappa-\mathfrak{a}_o)\right)\left((\mathfrak{b}_o-\varkappa)^{\lambda_o-1}+(\varkappa-\mathfrak{a}_o)^{\lambda_o-1}\right) d\varkappa.
\end{aligned}$$

Proof. We have seen that Theorem 4.2 attains

$$\mathrm{f}\left(\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}\right)+\frac{\lambda_o}{(\mathfrak{b}_o-\mathfrak{a}_o)^{\lambda_o}} \int_{\mathfrak{a}_o}^{\mathfrak{b}_o} \mathrm{f}\left(\left|\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}-\varkappa\right|\right) \tag{4.13}$$

$$\times \left((\mathfrak{b}_o - \varkappa)^{\lambda_o - 1} + (\varkappa - \mathfrak{a}_o)^{\lambda_o - 1} \right) d\varkappa$$
$$\leq \frac{\Gamma(1+\lambda_o)}{(\mathfrak{b}_o - \mathfrak{a}_o)^{\lambda_o}} \left(\mathfrak{J}^{\lambda_o}_{\mathfrak{a}_o +} \mathrm{f}(\mathfrak{b}_o) + \mathfrak{J}^{\lambda_o}_{\mathfrak{b}_o -} \mathrm{f}(\mathfrak{a}_o) \right).$$

From Theorem 4.4 we have

(4.14)

$$\frac{\Gamma(1+\lambda_o)}{(\mathfrak{b}_o - \mathfrak{a}_o)^{\lambda_o}} \left(\mathfrak{J}^{\lambda_o}_{\mathfrak{a}_o +} \mathrm{f}(\mathfrak{b}_o) + \mathfrak{J}^{\lambda_o}_{\mathfrak{b}_o -} \mathrm{f}(\mathfrak{a}_o) \right)$$
$$\leq 2 \left(\mathrm{f}(\mathfrak{a}_o) + \mathrm{f}(\mathfrak{b}_o) \right) - \frac{\lambda_o}{(\mathfrak{b}_o - \mathfrak{a}_o)^{\lambda_o}}$$
$$\times \int_{\mathfrak{a}_o}^{\mathfrak{b}_o} \left(\mathrm{f}(\mathfrak{b}_o - \varkappa) + \mathrm{f}(\varkappa - \mathfrak{a}_o) \right) \left((\mathfrak{b}_o - \varkappa)^{\lambda_o - 1} + (\varkappa - \mathfrak{a}_o)^{\lambda_o - 1} \right) d\varkappa.$$

Combining (4.13) and (4.14) we get what we want. □

Remark 4.6. If we pick $\lambda_o = 1$, in Theorem 4.5, we approach to (2.16).

Consider the subsequent example for the authenticity of Theorem 4.5.

Example 4.7. Since the function $\varkappa^{\mathfrak{p}}$ is a P-superquadratic on $I = [\mathfrak{a}_o, \mathfrak{b}_o]$, $\forall \varkappa \geq 2$ and $\mathfrak{p} \geq 2$ such that $|I| = \mathfrak{b}_o - \mathfrak{a}_o = 1$. The following figure describes the validity of Theorem 4.5.

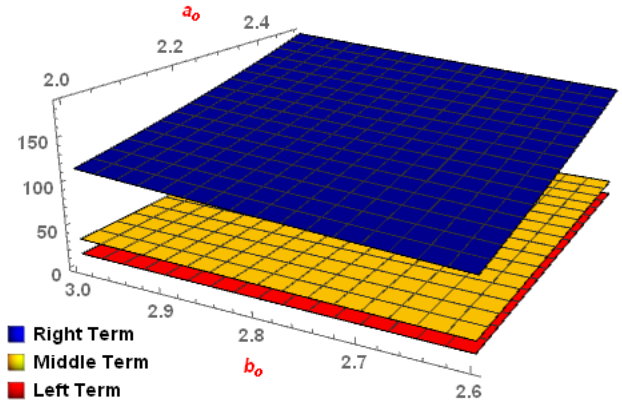


FIGURE 9. Graphical illustration that confirm the authenticity of Theorem 4.5 for $\mathfrak{a}_o \in [2, 2.5]$, $\mathfrak{b}_o \in [2.6, 3]$ $\alpha_o = 0.5$.

5. Applications

This section illustrates the applications of the obtained results in terms of special means, special functions and moments of random variables by which we get new recurrence relations.

5.1. **Special Means.** Let $\mathfrak{a}_o < \mathfrak{b}_o$ and $\mathfrak{a}_o, \mathfrak{b}_o \in \mathrm{R}$, then Arithmetic, $\mathfrak{p}$-logarithmic and Geometric means which are denoted by $\mathrm{A}(\mathfrak{a}_o, \mathfrak{b}_o)$, $\mathrm{L}_{\mathfrak{p}}(\mathfrak{a}_o, \mathfrak{b}_o)$ and $\mathrm{G}(\mathfrak{a}_o, \mathfrak{b}_o)$ respectively are defined as

$$\mathrm{A}(\mathfrak{a}_o, \mathfrak{b}_o) = \frac{\mathfrak{a}_o + \mathfrak{b}_o}{2}.$$

$$\mathrm{L}_{\mathfrak{p}}(\mathfrak{a}_o, \mathfrak{b}_o) = \left[\frac{\Gamma(1+\mathfrak{p})}{\Gamma(1+(\mathfrak{p}+1))} \frac{{\mathfrak{b}_o}^{(\mathfrak{p}+1)} - {\mathfrak{a}_o}^{(\mathfrak{p}+1)}}{(\mathfrak{b}_o - \mathfrak{a}_o)}\right]^{\frac{1}{\mathfrak{p}}},$$

where $\mathfrak{p} \in \mathrm{Z} \backslash \{-1, 0\}$, $\mathfrak{a}_o, \mathfrak{b}_o \in \mathrm{R}$, $\mathfrak{a}_o \neq \mathfrak{b}_o$.

$$\mathrm{G}(\mathfrak{a}_o, \mathfrak{b}_o) = \sqrt{\mathfrak{a}_o \mathfrak{b}_o}.$$

Proposition 5.1. *For $\mathfrak{a}_o, \mathfrak{b}_o \in [1, \infty)$, $0 < \mathfrak{a}_o < \mathfrak{b}_o$ and $\mathfrak{p} \geq 2$ we have*

$$(\mathrm{A}(\mathfrak{a}_o, \mathfrak{b}_o))^{\mathfrak{p}} \leq 2\mathrm{A}({\mathfrak{a}_o}^{\mathfrak{p}}, {\mathfrak{b}_o}^{\mathfrak{p}}) - 2\left(\mathrm{A}(-\mathfrak{a}_o, \mathfrak{b}_o)\right)^{\mathfrak{p}}. \tag{5.1}$$

Proof. The required result is obtained by utilising the function $\mathrm{f}(\varkappa) = \varkappa^{\mathfrak{p}}$, in Theorem 2.12 for n=2, and taking $\lambda_{o1} = \lambda_{o2} = \frac{1}{2}$. □

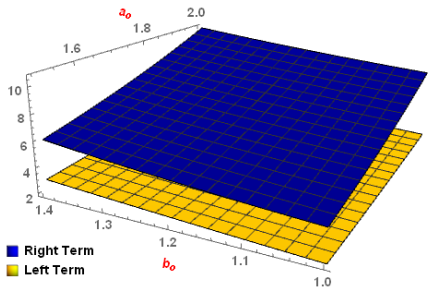
The subsequent example confirms the veracity of Proposition 5.1.

Example 5.2. We attain the subsequent graphs given by Figure 10 for various values of $\mathfrak{a}_o, \mathfrak{b}_o$ and $\mathfrak{p}$ via Proposition (5.1).

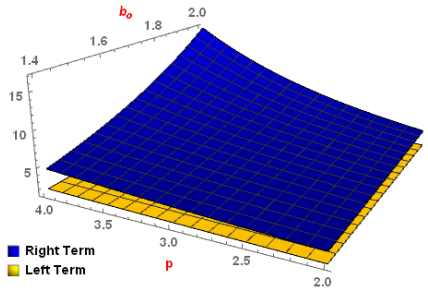
Proposition 5.3. *For $\mathfrak{a}_o, \mathfrak{b}_o \in [1, \infty)$ and $0 < \mathfrak{a}_o < \mathfrak{b}_o$ we attain*

$$\begin{aligned}(\mathfrak{p}+1)\left(\mathrm{A}(\mathfrak{a}_o, \mathfrak{b}_o)\right)^{\mathfrak{p}} + 2\left(\mathrm{A}(-\mathfrak{a}_o, \mathfrak{b}_o)\right)^{\mathfrak{p}} \\ \leq 2(\mathfrak{p}+1)\left(\mathrm{L}_{\mathfrak{p}}(\mathfrak{a}_o, \mathfrak{b}_o)\right)^{\mathfrak{p}} \\ \leq 4(\mathfrak{p}+1)\mathrm{A}\left({\mathfrak{a}_o}^{\mathfrak{p}}, {\mathfrak{b}_o}^{\mathfrak{p}}\right) - 2^{\mathfrak{p}+1}\left(\mathrm{A}(-\mathfrak{a}_o, \mathfrak{b}_o)\right)^{\mathfrak{p}}.\end{aligned} \tag{5.2}$$

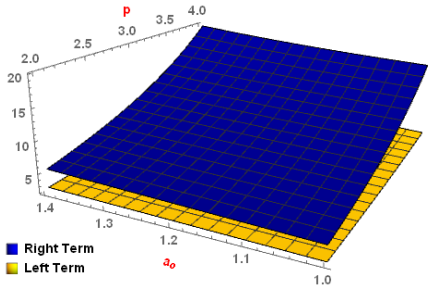
Proof. The required result is determined by utilising the function $\mathrm{f}(\varkappa) = \varkappa^{\mathfrak{p}}$ in Theorem 2.16. □



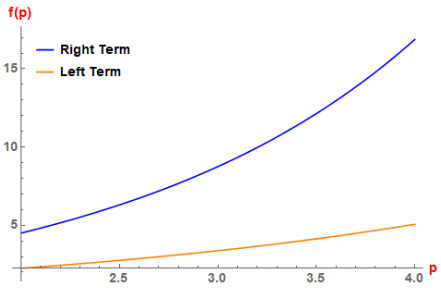
(A) $\mathfrak{p}$ is fixed while $\mathfrak{a}_o$ and $\mathfrak{b}_o$ vary such that $\mathfrak{p} = 3$, $\mathfrak{a}_o \in [1, 1.4]$ and $\mathfrak{b}_o \in [1.5, 2]$.



(B) $\mathfrak{a}_o$ is fixed while $\mathfrak{b}_o$ and $\mathfrak{p}$ vary such that $\mathfrak{a}_o = 1$, $\mathfrak{b}_o \in [1.4, 2]$ and $\mathfrak{p} \in [2, 4]$.



(C) $\mathfrak{b}_o$ is fixed while $\mathfrak{a}_o$ and $\mathfrak{p}$ vary such that $\mathfrak{b}_o = 2$, $\mathfrak{a}_o \in [1.4, 2]$ and $\mathfrak{p} \in [2, 4]$.



(D) $\mathfrak{a}_o$ and $\mathfrak{b}_o$ are fixed while $\mathfrak{p}$ varies such that $\mathfrak{a}_o = 1$, $\mathfrak{b}_o = 2$ and $\mathfrak{p} \in [2, 4]$.

FIGURE 10. Graphical depiction of Proposition 5.1.

The subsequent example confirms the veracity of Proposition 5.3.

Example 5.4. The subsequent graphs given by Figure 11 for various values of $\mathfrak{a}_o, \mathfrak{b}_o$ and $\mathfrak{p}$ via Proposition 5.3.

5.2. **Type-1 Modified Bessel Function.** Let's look at the modified Bessel function of type-1 i.e., $\mathfrak{J}_{\mathfrak{p}}$ [34].

$$\mathfrak{J}_{\mathfrak{p}}(\varkappa) = \sum_{\mathfrak{n}=0}^{\infty} \frac{\left(\frac{\varkappa}{2}\right)^{\mathfrak{p}+2\mathfrak{n}}}{\mathfrak{n}!\Gamma(\mathfrak{p}+\mathfrak{n}+1)}, \quad \text{where } \varkappa \in \Re.$$

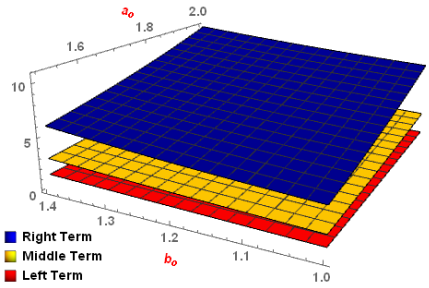
We graphically showed that $\mathfrak{J}_{\mathfrak{p}}(\varkappa)$ is P-superquadratic on $[\mathfrak{a}_o, \mathfrak{b}_o]$, $\forall \mathfrak{p} \geq 2$ and $\varkappa \geq 1$.

Let us define a new function $\mathfrak{f}_{\mathfrak{p}} : [0, \infty[\to [0, \infty[$ for $\mathfrak{p} \geq 2$ such that

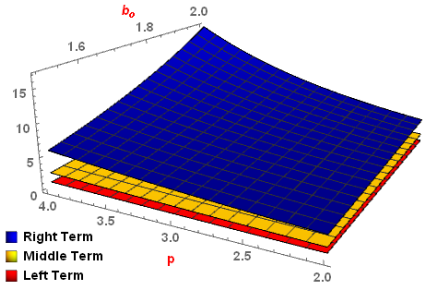
$$\mathfrak{f}_{\mathfrak{p}}(\varkappa) = \varkappa^{\mathfrak{p}} \mathfrak{J}_{\mathfrak{p}}(\varkappa).$$

$$\mathfrak{f}'_{\mathfrak{p}}(\varkappa) = \varkappa^{\mathfrak{p}} \mathfrak{J}_{\mathfrak{p}-1}(\varkappa).$$

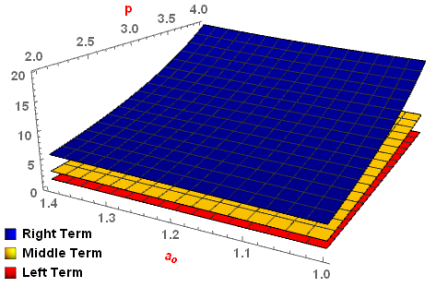
Through Definition 2.1, it can be seen that the functions $\mathfrak{f}_{\mathfrak{p}}(\varkappa)$ and $\mathfrak{f}_{\mathfrak{p}}^{'}(\varkappa)$ are P-superquadratic for $\forall \quad \mathfrak{p} \geq 2$ and $\mathfrak{p} \geq 3$ respectively.



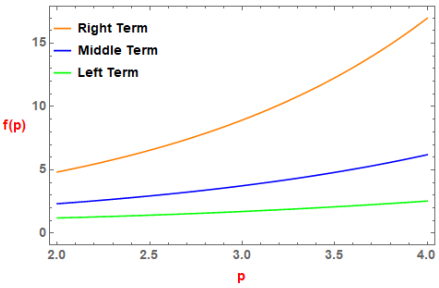
(A) $\mathfrak{p}$ is fixed while $\mathfrak{a}_o$ and $\mathfrak{b}_o$ vary such that $\mathfrak{p} = 3$, $\mathfrak{a}_o \in [1, 1.4]$ and $\mathfrak{b}_o \in [1.5, 2]$.



(B) $\mathfrak{a}_o$ is fixed while $\mathfrak{b}_o$ and $\mathfrak{p}$ vary such that $\mathfrak{a}_o = 1$, $\mathfrak{b}_o \in [1.5, 2]$ and $\mathfrak{p} \in [2, 4]$.



(C) $\mathfrak{b}_o$ is fixed while $\mathfrak{a}_o$ and $\mathfrak{p}$ vary such that $\mathfrak{b}_o = 2$, $\mathfrak{a}_o \in [1, 1.4]$ and $\mathfrak{p} \in [2, 4]$.



(D) $\mathfrak{a}_o$ and $\mathfrak{b}_o$ are fixed while $\mathfrak{p}$ varies such that $\mathfrak{a}_o = 1$, $\mathfrak{b}_o = 2$ and $\mathfrak{p} \in [2, 4]$.

FIGURE 11. Graphical depiction of Proposition 5.3.

Proposition 5.5. *The inequality*

(5.3)

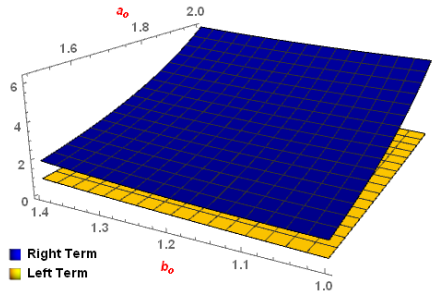
$$\left(\frac{\mathfrak{a}_o + \mathfrak{b}_o}{2}\right)^{\mathfrak{p}} \mathfrak{J}_{\mathfrak{p}-1}\left(\frac{\mathfrak{a}_o + \mathfrak{b}_o}{2}\right) \leq (\mathfrak{a}_o)^{\mathfrak{p}} \mathfrak{J}_{\mathfrak{p}-1}(\mathfrak{a}_o) + (\mathfrak{b}_o)^{\mathfrak{p}} \mathfrak{J}_{\mathfrak{p}-1}(\mathfrak{b}_o) - 2\left(\frac{\mathfrak{b}_o - \mathfrak{a}_o}{2}\right)^{\mathfrak{p}} \mathfrak{J}_{\mathfrak{p}-1}\left(\frac{\mathfrak{b}_o - \mathfrak{a}_o}{2}\right),$$

holds $\forall\ \mathfrak{p} \geq 3$ *and* $\mathfrak{a}_o, \mathfrak{b}_o \in [1, \infty)$.

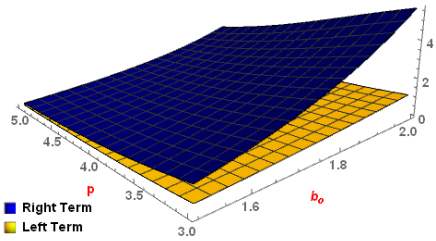
Proof. The required result is developed by setting $\mathfrak{f}(\varkappa) = \mathfrak{f}_{\mathfrak{p}}^{'}(\varkappa) = \varkappa^{\mathfrak{p}} \mathfrak{J}_{\mathfrak{p}-1}(\varkappa)$ in Theorem 2.12 for n=2, and taking $\lambda_{o1} = \lambda_{o2} = \frac{1}{2}$. □

The subsequent example confirms the veracity of Proposition 5.5.

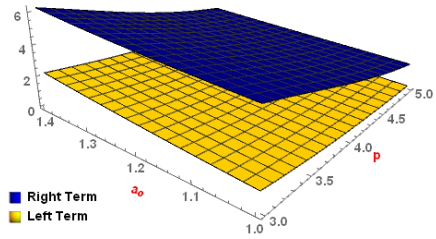
Example 5.6. The graphs given by Figure 12 for various values of $\mathfrak{a}_o, \mathfrak{b}_o$ and $\mathfrak{p}$ confirms the veracity of Proposition 5.5.



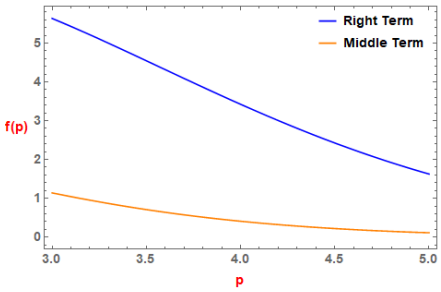
(A) $\mathfrak{p}$ is fixed while $\mathfrak{a}_o$ and $\mathfrak{b}_o$ vary such that $\mathfrak{p} = 3$, $\mathfrak{a}_o \in [1, 1.4]$ and $\mathfrak{b}_o \in [1.5, 2]$.



(B) $\mathfrak{a}_o$ is fixed while $\mathfrak{b}_o$ and $\mathfrak{p}$ vary such that $\mathfrak{a}_o = 1$, $\mathfrak{b}_o \in [1.5, 2]$ and $\mathfrak{p} \in [3, 5]$.



(C) $\mathfrak{b}_o$ is fixed while $\mathfrak{a}_o$ and $\mathfrak{p}$ vary such that $\mathfrak{b}_o = 2$, $\mathfrak{a}_o \in [1, 1.4]$ and $\mathfrak{p} \in [3, 5]$.



(D) $\mathfrak{a}_o$ and $\mathfrak{b}_o$ are fixed while $\mathfrak{p}$ varies such that $\mathfrak{a}_o = 1$, $\mathfrak{b}_o = 2$ and $\mathfrak{p} \in [3, 5]$.

FIGURE 12. Graphical depiction of Proposition 5.5.

Proposition 5.7. *The inequality*

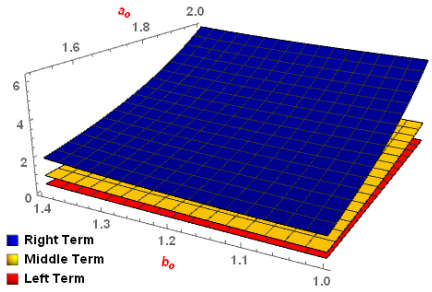
$$\begin{aligned}(5.4)\quad &\frac{1}{2}\left(\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}\right)^{\mathfrak{p}}\mathfrak{J}_{\mathfrak{p}-1}\left(\frac{\mathfrak{a}_o+\mathfrak{b}_o}{2}\right)+\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{2}\right)^{(\mathfrak{p}-1)}\mathfrak{J}_{\mathfrak{p}}\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{2}\right)\\ &\leq \frac{1}{\mathfrak{b}_o-\mathfrak{a}_o}\left[\mathfrak{b}_o{}^{\mathfrak{p}}\mathfrak{J}_{\mathfrak{p}}(\mathfrak{b}_o)-\mathfrak{a}_o{}^{\mathfrak{p}}\mathfrak{J}_{\mathfrak{p}}(\mathfrak{a}_o)\right]\\ &\leq \mathfrak{b}_o{}^{\mathfrak{p}}\mathfrak{J}_{\mathfrak{p}-1}(\mathfrak{b}_o)+\mathfrak{a}_o{}^{\mathfrak{p}}\mathfrak{J}_{\mathfrak{p}-1}(\mathfrak{a}_o)-2(\mathfrak{b}_o-\mathfrak{a}_o)^{\mathfrak{p}-1}\mathfrak{J}_{\mathfrak{p}}(\mathfrak{b}_o-\mathfrak{a}_o),\end{aligned}$$

holds $\forall\ \mathfrak{p} \geq 3$ *and* $\mathfrak{a}_o, \mathfrak{b}_o \in [1, \infty)$.

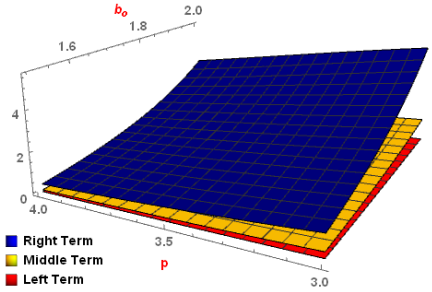
Proof. The required result is established by setting the function $\mathrm{f}(\varkappa) = \mathrm{f}_{\mathfrak{p}}^{'}(\varkappa) = \varkappa^{\mathfrak{p}}\mathfrak{J}_{\mathfrak{p}-1}(\varkappa)$ in Theorem 2.16. □

The subsequent example confirms the veracity of Proposition 5.7.

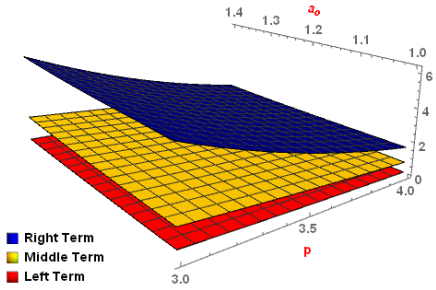
Example 5.8. The graphs given by Figure 13 for various values of $\mathfrak{a}_o, \mathfrak{b}_o$ and $\mathfrak{p}$ of Proposition 5.7.



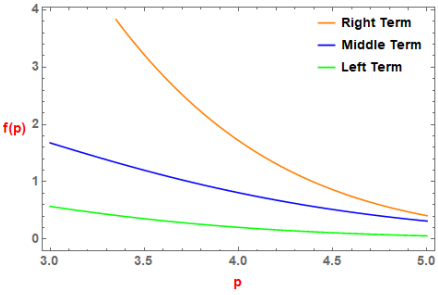
(A) $\mathfrak{p}$ is fixed while $\mathfrak{a}_o$ and $\mathfrak{b}_o$ vary such that $\mathfrak{p} = 3$, $\mathfrak{a}_o \in [1, 1.4]$ and $\mathfrak{b}_o \in [1.5, 2]$.



(B) $\mathfrak{a}_o$ is fixed while $\mathfrak{b}_o$ and $\mathfrak{p}$ vary such that $\mathfrak{a}_o = 1$, $\mathfrak{b}_o \in [1.5, 2]$ and $\mathfrak{p} \in [3, 4]$.



(C) $\mathfrak{b}_o$ is fixed while $\mathfrak{a}_o$ and $\mathfrak{p}$ vary such that $\mathfrak{b}_o = 2$, $\mathfrak{a}_o \in [1, 1.4]$ and $\mathfrak{p} \in [3, 4]$.



(D) $\mathfrak{a}_o$ and $\mathfrak{b}_o$ are fixed while $\mathfrak{p}$ varies such that $\mathfrak{a}_o = 1$, $\mathfrak{b}_o = 2$ and $\mathfrak{p} \in [3, 5]$.

FIGURE 13. Graphical illustration of Proposition 5.7.

Hence the aforementioned graph confirms the validity of Proposition 5.7.

5.3. **Moment of Random Variables.** The probability distribution of a random variable may be represented in terms of distribution and density functions which are telling information about what may occur for the values to be different. However these functions alone are not very effective in drawing out differences between different distributions especially when comparing two distributions that look fairly similar. It is often helpful to consider observations that determine each distribution in particular conditions, such as the mean, variance, skewness, and kurtosis, attempting to be able to better spot them. Moments provide a more subtle way of discriminating across distributions by capturing important facets of their spread and form. Knowing the probability

function of a random variable also enables us to infer whether its moments exist and up to what extent they are well defined. In recent years, specific bounding and estimation of those moments using mathematical inequalities applied which facilitate distribution comparison and analysis better have been noted by some scholars. For more details on these estimates, see [11, 25].

Consider $\mathrm{M}_{\mathfrak{r}}(\mathrm{x})$ as the r-th moment of a random variable χ in probability theory, where $\varkappa$ represents an arbitrary reference point of the random variable χ. For any $\mathfrak{r} \geq 1$, the r-th moment $\mathrm{M}_{\mathfrak{r}}(\mathrm{x})$, is expressed as:

$$\mathrm{M}_{\mathfrak{r}}(\varkappa) = \int_{\mathfrak{a}_o}^{\mathfrak{b}_o} (\mathrm{y} - \varkappa)^{\mathfrak{r}} \mathfrak{g}(\mathrm{y}) d\mathrm{y}, \quad \mathfrak{r} = 1, 2, \ldots. \tag{5.5}$$

The function $\mathfrak{f}$ is taken to be the probability function in (5.5) which is in this instance the probability density function (pdf). In particular it is assumed that the pdf is uniform on the interval $[\mathfrak{a}_o, \mathfrak{b}_o]$. This indicates that there is an equal chance for the random variable to take any value within this range. When there is no preference for any specific result within the given range, the uniform distribution is frequently employed.

Keep in view (5.5) we derive the subsequent proposition.

Proposition 5.9. *Let* $\mathfrak{g} : \mathfrak{I} \subseteq \mathrm{R} \to \mathrm{R}_+$, *is a uniform* pdf *on* $\mathfrak{I} = [\mathfrak{r}_o, \mathfrak{s}_o]$, *for a random variable* χ, *then* $\forall \quad \mathfrak{r} \geq 1$, *we attain*

(5.6)

$$\begin{aligned}
&\frac{1}{2}\mathrm{M}_{\mathfrak{r}}\left(\frac{\mathfrak{a}_o + \mathfrak{b}_o}{2}\right) + \frac{1}{\mathfrak{b}_o - \mathfrak{a}_o}\int_{\mathfrak{a}_o}^{\mathfrak{b}_o} \mathrm{M}_{\mathfrak{r}}\left(\left|\frac{\mathfrak{a}_o + \mathfrak{b}_o}{2} - \varkappa\right|\right) d\varkappa \\
&\quad \geq \frac{1}{\mathfrak{b}_o - \mathfrak{a}_o}\int_{\mathfrak{a}_o}^{\mathfrak{b}_o} \mathrm{M}_{\mathfrak{r}}(\varkappa) d\varkappa \\
&\quad \geq (\mathrm{M}_{\mathfrak{r}}(\mathfrak{a}_o) + \mathrm{M}_{\mathfrak{r}}(\mathfrak{b}_o)) - \frac{1}{\mathfrak{b}_o - \mathfrak{a}_o}\int_{\mathfrak{a}_o}^{\mathfrak{b}_o} (\mathrm{M}_{\mathfrak{r}}(\mathfrak{b}_o - \varkappa) + \mathrm{M}_{\mathfrak{r}}(\varkappa - \mathfrak{a}_o))\, d\varkappa.
\end{aligned}$$

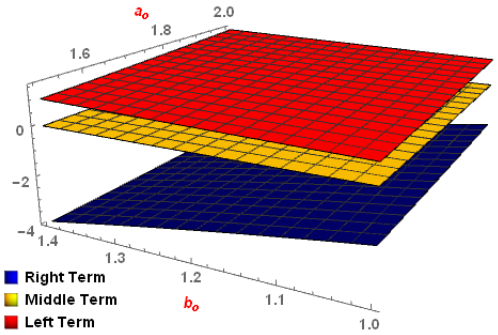
Proof. The function $\mathrm{M}_{\mathfrak{r}}(\varkappa)$, in which $\mathfrak{g}$, is taken as uniform *pdf* on $\mathfrak{I} \subseteq \mathrm{R}$, is P-subquadratic. So using this function in the Theorem 2.16 reverses its inequalities as a result we get (5.6). □

Example 5.10. The graphs given by Figure 14 are derived on the bases of values of Right, Left and Middle terms and verifies the authenticity of (5.6) of Proposition 5.9 taking various ranges of the variables of $\mathfrak{a}_o$ and $\mathfrak{b}_o$ in the interval $[1, 2]$ and for $\mathfrak{r}$=2.

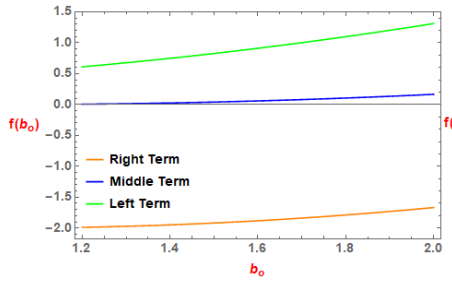
$$\text{Left Term} = \frac{(\mathfrak{b}_o - \mathfrak{a}_o)^2}{6} + \frac{7\mathfrak{b}_o^2 + 10\mathfrak{a}_o\mathfrak{b}_o + 7\mathfrak{a}_o^2}{48}$$

$$\text{Middle Term} = \frac{(\mathfrak{b}_o - \mathfrak{a}_o)^2}{6}$$

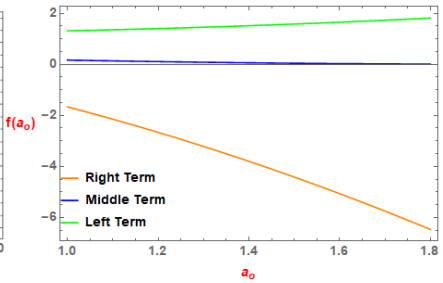
$$\text{Right Term} = \frac{2(\mathfrak{b}_o - \mathfrak{a}_o)^2}{3} - \frac{\mathfrak{b}_o^2 - 2\mathfrak{a}_o\mathfrak{b}_o + 7\mathfrak{a}_o^2}{3}.$$



(A) $\mathfrak{a}_o$ and $\mathfrak{b}_o$ vary such that $\mathfrak{a}_o \in [1, 1.4]$ and $\mathfrak{b}_o \in [1.5, 2]$.



(B) $\mathfrak{a}_o$ is fixed while $\mathfrak{b}_o$ varies such that $\mathfrak{a}_o = 1, \mathfrak{b}_o \in [1.2, 2]$.



(C) $\mathfrak{b}_o$ is fixed while $\mathfrak{a}_o$ varies such that $\mathfrak{b}_o = 2, \mathfrak{a}_o \in [1, 1.8]$.

FIGURE 14. Graphical depiction of the Proposition 5.9 for $\mathfrak{r} = 2$.

6. Conclusion

The primary novelty of this paper lies in the introduction of the concept of P-superquadratic functions, which extends the classical notion of superquadraticity. This new class of functions offers unique properties and inequalities that have not been previously explored in depth. By establishing key inequalities such as Jensen's, Mercer Jensen's, and H.H's inequalities specifically for P-superquadratic functions, the paper provides a fresh perspective on these well-known results, demonstrating their applicability in a new context. The derivation of fractional versions of H.H-type inequalities, which expands the utility of these inequalities into the realm of fractional calculus. A comparative analysis of inequalities for P-superquadratic and P-convex functions, highlighting the relationships and distinctions between these two classes. These

results not only deepen the understanding of P-superquadratic functions but also provide valuable tools for further theoretical exploration and practical applications. Applying the methods developed in this study to derive additional inequalities for P-superquadratic functions, potentially leading to new theoretical insights. Exploring the application of the P-superquadratic function concept in various mathematical frameworks, such as fractal calculus, fractional calculus, multiplicative calculus, interval calculus, quantum calculus, and fuzzy mathematics as well as finding the further fractional versions via other fractional operators such that proportional Caputo-hybrid fractional integrals, Katugampola fractional integrals, generalized R.L fractional integrals, ψ-R.L fractional integrals, k-R.L fractional integrals, Atangana-Baleanu fractional integrals and conformable fractional integrals. This interdisciplinary approach could yield innovative results and broaden the understanding of these functions in diverse contexts. Investigating the potential applications of the established inequalities in fields such as differential equations and optimization, where enhanced mathematical tools are often required for effective problem-solving.

Acknowledgment. The author wish to thank $\cdots$

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