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On Generalized Normed Spaces

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Abstract. In this paper, we introduce a generalized normed space, which we refer to as a G-normed space. We define the concepts of a G-continuous and a G-bounded linear operator on this space and explore some related properties. Given the central role of the Banach-Steinhaus Theorem in the theory of normed spaces, particularly in the study of bounded linear operators, we conclude by formulating and proving a version of the Banach-Steinhaus Theorem for G-normed spaces.

1. Introduction

The concept of a norm is fundamental in mathematics, with different types used depending on the requirements of various areas. The norm was first introduced by Stefan Banach [3] and was used to solve several problems related to Lebesgue spaces [7]. As we know, norms have been extensively used in the study of function spaces (see, e.g. [1, 6, 10, 12]) and new types of norms are regularly introduced, such as Orlicz's, Luxemburg's and Amemiya's norms. For more details, see e.g. [2, 8, 13].

The Banach-Steinhaus Theorem, also known as the Uniform Boundedness Principle, plays an important role in the theory of normed spaces, particularly in the context of bounded linear operators (see [9, 11]). An important aspect of the theorem is its contribution to understanding the boundedness of a family of operators. The theorem guarantees that if each operator in a collection is pointwise bounded, then there is a uniform bound for all operators in the collection [4, 5]. This is an important result for the study of operator theory in Banach spaces.

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In this paper, we formulate the Banach-Steinhaus Theorem in a new type of generalized normed space, which we call a G-normed space. Firstly, we construct a G-normed space and prove key properties of the space. Additionally, we also introduce the concept of G-continuous and G-bounded linear operators on the space and explore related properties. Finally, we prove the Banach-Steinhaus Theorem in space.

2. G-Normed Spaces and Some Basic Properties

We begin this section by defining a G-norm. The definition is motivated by the concept of a norm.

Definition 2.1. Let X be a linear space over the scalar field $\mathbb{R}$ (or $\mathbb{C}$). A function $\|\cdot\|_G : X \to \mathbb{R}$ is called a G-norm provided for all x and y in X,

(G1) $\|x\|_G = 0$ if and only if $x = \theta$;
(G2) for any scalar $\alpha \neq 0$, there are scalars $\beta, \gamma > 0$ such that
$$\gamma\|x\|_G \leq \|\alpha x\|_G \leq \beta\|x\|_G;$$
(G3) $\|x + y\|_G \leq \|x\|_G + \|y\|_G$.

A linear space X equipped with a G-norm $\|\cdot\|_G$, denoted by $(X, \|\cdot\|_G)$, is called a G-normed space. The G-normed space $(X, \|\cdot\|_G)$ is often written simply by X, as long as it does not cause any confusion. It is clear that the G-norm $\|\cdot\|_G$ is non-negative. Hereinafter, it is also trivial that every norm is a G-norm; however, the converse is not always true.

Example 2.2. Let $\phi : \mathbb{R} \to \mathbb{R}$ be defined by $\phi(x) = x^2$, for every $x \in \mathbb{R}$. Let ω denote the collection of all sequences in $\mathbb{R}$. It can be verified that the set
$$X_\phi = \left\{ x = (x_k) \in \omega : \sum_{k=1}^{\infty} \phi(x_k) < \infty \right\},$$
is a linear space. The function $\|\cdot\|_G : X_\phi \to \mathbb{R}$ defined by
$$\|x\|_G = \inf \left\{ \varepsilon > 0 : \sum_{k=1}^{\infty} \phi\left(\frac{x_k}{\varepsilon}\right) \leq \varepsilon \right\},$$
is a G-norm, but not a norm on X_ϕ.

Proof. Let $x = (x_k) \in X_\phi$ be given. If $\sum_{k=1}^{\infty} \phi(x_k) \in [0, 1]$, then we can choose $\varepsilon = 1$ such that $\sum_{k=1}^{\infty} \phi\left(\frac{x_k}{\varepsilon}\right) = \sum_{k=1}^{\infty} \phi(x_k) \leq 1 = \varepsilon$. If $\sum_{k=1}^{\infty} \phi(x_k) > 1$, then there exists $\varepsilon = \sum_{k=1}^{\infty} \phi(x_k) > 0$ such that

$$\sum_{k=1}^{\infty} \phi\left(\frac{x_k}{\varepsilon}\right) = \frac{1}{\varepsilon^2} \sum_{k=1}^{\infty} \phi(x_k) = \frac{1}{\varepsilon^2} \cdot \varepsilon = \frac{1}{\varepsilon} < \varepsilon.$$

Thus, $\left\{\varepsilon > 0 : \sum_{k=1}^{\infty} \phi\left(\frac{x_k}{\varepsilon}\right) \leq \varepsilon\right\} \neq \emptyset$. Further, we need to show that $\|\cdot\|_G$ is a G-norm on X_ϕ.

(G1) It is obvious that for $x = \theta$, then $\|x\|_G = 0$. Conversely, if $\|x\|_G = 0$, then for every $0 < \varepsilon < 1$, there exists $\varepsilon_0 > 0$ such that $\varepsilon_0 < \varepsilon$ and $\sum_{k=1}^{\infty} \phi\left(\frac{x_k}{\varepsilon_0}\right) \leq \varepsilon_0$. Notice that

$$\sum_{k=1}^{\infty} \phi(x_k) \leq \sum_{k=1}^{\infty} \phi\left(\frac{x_k}{\varepsilon_0}\right) \leq \varepsilon_0 < \varepsilon.$$

Since $0 < \varepsilon < 1$ is arbitrary, we conclude that

$$\sum_{k=1}^{\infty} \phi(x_k) = 0 \quad \Leftrightarrow \quad \sum_{k=1}^{\infty} (x_k)^2 = 0.$$

Therefore, it follows that $x_k = 0$, for each $k \in \mathbb{N}$, i.e. $x = \theta$.

(G2) Let $\alpha \neq 0$, $x \in X_\phi$ and $\mu > 0$. Then there exists $\varepsilon_0 > 0$ such that $\sum_{k=1}^{\infty} \phi\left(\frac{x_k}{\varepsilon_0}\right) \leq \varepsilon_0 < \|x\|_G + \mu$. Let $\beta = \max\{\alpha^2, 1\}$, then

$$\sum_{k=1}^{\infty} \phi\left(\frac{\alpha x_k}{\beta \varepsilon_0}\right) = \sum_{k=1}^{\infty} \alpha^2 \phi\left(\frac{x_k}{\beta \varepsilon_0}\right) \leq \alpha^2 \sum_{k=1}^{\infty} \phi\left(\frac{x_k}{\varepsilon_0}\right) \leq \beta \varepsilon_0.$$

This implies,

$$\|\alpha x\|_G \leq \beta \varepsilon_0 < \beta(\|x\|_G + \mu),$$

so

$$\|\alpha x\|_G \leq \beta \|x\|_G. \tag{2.1}$$

Moreover, since

$$\|\alpha x\|_G = \inf\left\{\delta > 0 : \sum_{k=1}^{\infty} \phi\left(\frac{\alpha x_k}{\delta}\right) \leq \delta\right\},$$

then there is $\delta_0 > 0$ such that $\sum_{k=1}^{\infty} \phi\left(\frac{\alpha x_k}{\delta_0}\right) \leq \delta_0 < \|\alpha x\|_G + \mu$.
Let $\gamma = \min\{1, \alpha^2\}$, then

$$\sum_{k=1}^{\infty} \phi\left(\frac{x_k}{\frac{\delta_0}{\gamma}}\right) \leq \sum_{k=1}^{\infty} \phi\left(\frac{x_k}{\delta_0}\right) \leq \frac{\delta_0}{\gamma}.$$

This implies,

$$\|x\|_G \leq \frac{\delta_0}{\gamma} < \frac{\|\alpha x\|_G + \mu}{\gamma},$$

so

$$\gamma \|x\|_G \leq \|\alpha x\|_G. \tag{2.2}$$

Based on inequalities (2.1) and (2.2), we conclude that there exist $\beta, \gamma > 0$ such that

$$\gamma \|x\|_G \leq \|\alpha x\|_G \leq \beta \|x\|_G.$$

(G3) Let $x, y \in X_\phi$ and $\mu > 0$. Then there are $\varepsilon_0, \delta_0 > 0$ such that

$$\sum_{k=1}^{\infty} \phi\left(\frac{x_k}{\varepsilon_0}\right) \leq \varepsilon_0 < \|x\|_G + \frac{\mu}{2}$$

and

$$\sum_{k=1}^{\infty} \phi\left(\frac{y_k}{\delta_0}\right) \leq \delta_0 < \|y\|_G + \frac{\mu}{2}.$$

As ϕ is convex, then we get

$$\begin{aligned}
\sum_{k=1}^{\infty} \phi\left(\frac{x_k + y_k}{\varepsilon_0 + \delta_0}\right) &= \sum_{k=1}^{\infty} \phi\left(\frac{\varepsilon_0}{\varepsilon_0 + \delta_0}\frac{x_k}{\varepsilon_0} + \frac{\delta_0}{\varepsilon_0 + \delta_0}\frac{y_k}{\delta_0}\right) \\
&\leq \sum_{k=1}^{\infty} \left(\frac{\varepsilon_0}{\varepsilon_0 + \delta_0}\phi\left(\frac{x_k}{\varepsilon_0}\right) + \frac{\delta_0}{\varepsilon_0 + \delta_0}\phi\left(\frac{y_k}{\delta_0}\right)\right) \\
&= \frac{\varepsilon_0}{\varepsilon_0 + \delta_0}\sum_{k=1}^{\infty} \phi\left(\frac{x_k}{\varepsilon_0}\right) + \frac{\delta_0}{\varepsilon_0 + \delta_0}\sum_{k=1}^{\infty} \phi\left(\frac{y_k}{\delta_0}\right) \\
&\leq \frac{\varepsilon_0}{\varepsilon_0 + \delta_0}\varepsilon_0 + \frac{\delta_0}{\varepsilon_0 + \delta_0}\delta_0 \\
&< \varepsilon_0 + \delta_0.
\end{aligned}$$

This implies that

$$\|x + y\|_G \leq \varepsilon_0 + \delta_0 < \|x\|_G + \|y\|_G + \mu.$$

Hence,

$$\|x+y\|_G \le \|x\|_G + \|y\|_G.$$

Thus, $\|\cdot\|_G$ is a G-norm on X_ϕ. Furthermore, if we take $\alpha = 2$ and $x = (1, 0, 0, \dots) \in X_\phi$, then we get

$$\|2x\|_G = \inf\left\{\varepsilon > 0 : \sum_{k=1}^{\infty} \phi\left(\frac{2x_k}{\varepsilon}\right) \le \varepsilon\right\} = 4^{\frac{1}{3}}.$$

On the other hand,

$$\|x\|_G = \inf\left\{\varepsilon > 0 : \sum_{k=1}^{\infty} \phi\left(\frac{x_k}{\varepsilon}\right) \le \varepsilon\right\} = 1.$$

So, we obtain that $\|2x\|_G \ne 2\|x\|_G$. Thus, $\|\cdot\|_G$ is not a norm on X_ϕ. □

Example 2.3. A function $\|\cdot\|_G^* : \ell_1 \to \mathbb{R}$ defined by

$$\|x\|_G^* = \left(\sum_{k=1}^{\infty} |x_k|\right)^{\frac{1}{2}}$$

is a G-norm, but not a norm.

Proof. It is clear that $\|\cdot\|_G^*$ satisfies the conditions ($G1$) and ($G3$) in Definition 2.1. It now remains to verify ($G2$). For any $x \in \ell_1$ and any scalars $\alpha \ne 0$, we get

$$\|\alpha x\|_G^* = \left(\sum_{k=1}^{\infty} |\alpha x_k|\right)^{\frac{1}{2}} = |\alpha|^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} |x_k|\right)^{\frac{1}{2}} = |\alpha|^{\frac{1}{2}} \|x\|_G^*. \tag{2.3}$$

So, by taking $\beta = \gamma = |\alpha|^{\frac{1}{2}} > 0$, then we get

$$\gamma\|x\|_G^* \le \|\alpha x\|_G^* \le \beta\|x\|_G^*.$$

It means that $\|\cdot\|_G^*$ is a G-norm on ℓ_1. Moreover, following (2.3) we deduce that $\|\cdot\|_G^*$ is not a norm on ℓ_1. □

Analogous to defining the concept related to sequences in normed spaces, we define the same concepts in G-normed spaces.

Definition 2.4. Let $(X, \|\cdot\|_G)$ be a G-normed space.
A sequence (x_n) in X is said to be G-convergent if there exists an element $x \in X$ such that

$$\|x_n - x\|_G \to 0, \quad \text{as } n \to \infty.$$

In this case, x is called a G-limit of (x_n) and we write

$$x = G\text{-}\lim x_n.$$

The sequence (x_n) is called a G-Cauchy sequence if

$$\|x_m - x_n\|_G \to 0, \quad \text{as } m, n \to \infty. \tag{2.4}$$

The sequence (x_n) is said to be G-bounded if there exists a real number $C > 0$ such that $\|x_n\|_G \leq C$ for every $n \in \mathbb{N}$.

The G-normed space $(X, \|\cdot\|_G)$ is said to be G-complete if every G-Cauchy sequence in $(X, \|\cdot\|_G)$ G-converges in $(X, \|\cdot\|_G)$. A complete G-normed space is then called a G-Banach space.

Notice that a G-norm does not satisfy the absolute homogeneity property. However, following the condition $(G2)$ in Definition 2.1, it can be verified that a sequence (x_n) in a G-normed space X is G-convergent to some $x \in X$ if and only if $\|x - x_n\|_G \to 0$ as $n \to \infty$. Consequently, it is obvious that every G-convergent sequence in the space X has a unique G-limit. By considering the condition $(G2)$ in Definition 2.1, the expression $\|x_m - x_n\|_G$ in (2.4) does not depend on the order of the indices m and n. It can be also verified that every G-convergent sequence in the space X is a G-Cauchy sequence and every G-Cauchy sequence in the space X is G-bounded.

The following theorem presents some properties that also hold in any G-normed space.

Theorem 2.5. *Let $(X, \|\cdot\|_G)$ be a G-normed space.*

(a) *If (x_n) and (y_n) are sequences in X that G-converge to $x \in X$ and $y \in X$, respectively, then $(x_n + y_n)$ G-converges to $x + y$.*
(b) *For any elements $x, y \in X$ there is a real number $\beta > 0$ such that $|\|x\|_G - \|y\|_G| \leq \beta \|x - y\|_G$.*
(c) *If (x_n) G-converges to $x \in X$, then the sequence of real numbers $(\|x_n\|_G)$ converges to $\|x\|_G$.*

Proof. The proof is routine, so the details are omitted. □

We also observe the following theorem.

Theorem 2.6. *Let (x_n) be a G-Cauchy sequence in the G-normed space $(X, \|\cdot\|_G)$ and let $x \in X$ have the property that every subsequence of (x_n) G-converges to x. Then the sequence (x_n) G-converges to x.*

Proof. The proof is standard, so the details are omitted. □

3. A Linear Operator on a G-Normed Space

In this section, we introduce some types of operators on a G-normed space. First, we define the notion of a G-continuous operator on a G-normed space as a generalization of the concept of a continuous operator on a normed space.

Definition 3.1. Let $\left(X, \|\cdot\|_{G_X}\right)$ and $\left(Y, \|\cdot\|_{G_Y}\right)$ be G-normed spaces.

(i) The operator $T : X \to Y$ is said to be G-continuous at $c \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|T(x) - T(c)\|_{G_Y} < \varepsilon,$$

for all $x \in X$ satisfying $\|x-c\|_{G_X} < \delta$. The operator $T : X \to Y$ is said to be G-continuous if it is G-continuous at each $c \in X$.

(ii) The operator $T : X \to Y$ is said to be sequentially G-continuous at $c \in X$, if for every sequence (x_n) in X which G-converges to c, the sequence $(T(x_n))$ G-converges to $T(c)$.

The relationship between G-continuity and sequentially G-continuity is given in the following theorem.

Theorem 3.2. *Let* $\left(X, \|\cdot\|_{G_X}\right)$ *and* $\left(Y, \|\cdot\|_{G_Y}\right)$ *be* G*-normed spaces and* $T : X \to Y$ *be an operator. The operator* T *is* G*-continuous if and only if it is sequentially* G*-continuous.*

Proof. Since T is G-continuous, then for any $c \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for every $x \in X$ with $\|x - c\|_{G_X} < \delta$,

$$\|T(x) - T(c)\|_{G_Y} < \varepsilon.$$

Let (x_n) be any sequence which G-converges to c, then there exists a natural number $n(\delta)$ such that for all $n \geq n(\delta)$, we have $\|x_n - c\|_{G_X} < \delta$. This implies

$$\|T(x_n) - T(c)\|_{G_Y} < \varepsilon,$$

for all $n \geq n(\delta)$. Therefore, the assertion follows.

Conversely, suppose that T is not G-continuous, say at $c \in X$. This means that there exists $\varepsilon > 0$ such that for all $n \in \mathbb{N}$, there exists $x_n \in X$ such that $\|x_n - c\|_{G_X} < \dfrac{1}{n}$ and

$$\|T(x_n) - T(c)\|_{G_Y} \geq \varepsilon.$$

Hence, we get a sequence (x_n) which G-converges to c, but the sequence $(T(x_n))$ does not G-converge to $T(c)$. This contradicts the hypothesis; thus, T is G-continuous. □

Let's recall the definition of a linear operator on a linear space. Let X and Y be linear spaces over $\mathbb{R}$. An operator T from X to Y is said to be linear if the domain of T is a linear subspace $D(T) \subset X$ and

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y),$$

for every $x, y \in D(T)$ and $\alpha, \beta \in \mathbb{R}$. We now extend the concept of a bounded linear operator on a normed space by introducing the notion of a G-bounded linear operator on a G-normed space.

Definition 3.3. Let $\left(X, \|\cdot\|_{G_X}\right)$ and $\left(Y, \|\cdot\|_{G_Y}\right)$ be G-normed spaces. A linear operator $T : D(T) \subset X \to Y$ is said to be G-bounded if there is a constant $K > 0$ such that

$$\|T(x)\|_{G_Y} \leq K\|x\|_{G_X},$$

for every $x \in D(T)$.

Now, we will observe the connection between boundedness and continuity for a linear operator on a G-normed space.

Theorem 3.4. *Let $\left(X, \|\cdot\|_{G_X}\right)$ and $\left(Y, \|\cdot\|_{G_Y}\right)$ be G-normed spaces and $T : D(T) \subset X \to Y$ be a linear operator. If T is G-bounded, then T is G-continuous.*

Proof. Given $c \in D(T)$ and (x_n) is arbitrary sequence in $D(T)$ that G-converges to c. Since T is a G-bounded linear operator, then there is a constant $K > 0$ such that

$$\|T(x_n) - T(c)\|_{G_Y} = \|T(x_n - c)\|_{G_Y} \leq K\|x_n - c\|_{G_X}.$$

Since the sequence (x_n) is G-converges to c, then for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$ then $\|x_n - c\|_{G_X} < \dfrac{\varepsilon}{K}$. Consequently,

$$\|T(x_n) - T(c)\|_{G_Y} \leq K\|x_n - c\|_{G_X} < K\frac{\varepsilon}{K} = \varepsilon.$$

Thus, $(T(x_n))$ is G-converges to $T(c)$. In other words, T is G-continuous on $D(T)$. □

By Theorem 3.4, we conclude that every G-bounded linear operator on the G-normed space is G-continuous. However, the converse remains an open problem.

Let $\left(X, \|\cdot\|_{G_X}\right)$ and $\left(Y, \|\cdot\|_{G_Y}\right)$ be real G-normed spaces. A collection of all G-bounded linear operators from $\left(X, \|\cdot\|_{G_X}\right)$ to $\left(Y, \|\cdot\|_{G_Y}\right)$ will be denoted by $L_{B_G}(X, Y)$. We observe that $L_{B_G}(X, Y)$ is a linear space over addition and scalar multiplication operations as given by:

(i) $(T_1 + T_2)(x) = T_1(x) + T_2(x)$
(ii) $(\alpha T_1)(x) = \alpha T_1(x)$

for all $T_1, T_2 \in L_{B_G}(X, Y), \alpha \in \mathbb{R}$, and $x \in X$.

Theorem 3.5. *Let $\left(X, \|\cdot\|_{G_X}\right)$ and $\left(Y, \|\cdot\|_{G_Y}\right)$ be G-normed spaces. The set $L_{B_G}(X, Y)$ is a linear space.*

Proof. Given $S, T \in L_{B_G}(X, Y)$ and $\alpha \in \mathbb{R}$. It is clear that $\alpha S + T$ is a linear operator. Moreover, we can choose $K_1, K_2 > 0$ such that for all $x \in X$,

$$\|S(x)\|_{G_Y} \le K_1 \|x\|_{G_X} \quad \text{and} \quad \|T(x)\|_{G_Y} \le K_2 \|x\|_{G_X}.$$

So, for every $x \in X$ we obtain

$$\begin{aligned} \|(\alpha S + T)(x)\|_{G_Y} &= \|\alpha S(x) + T(x)\|_{G_Y} \\ &\le \|\alpha S(x)\|_{G_Y} + \|T(x)\|_{G_Y} \\ &\le \beta \|S(x)\|_{G_Y} + \|T(x)\|_{G_Y}, \text{for some } \beta > 0 \\ &\le \beta K_1 \|x\|_{G_X} + K_2 \|x\|_{G_X} \\ &= K \|x\|_{G_X}, \quad \text{where } K = \beta K_1 + K_2 > 0. \end{aligned}$$

Hence, $\alpha S + T$ is G-bounded. Then, the assertion follows. □

It can be verified that the set $\left\{ \frac{\|T(x)\|_{G_Y}}{\|x\|_{G_X}} : x \in X, \|x\|_{G_X} \neq 0 \right\}$ is bounded for any $T \in L_{B_G}(X, Y)$. Therefore, we can define a function $\|\cdot\|_{L_{B_G}} : L_{B_G}(X, Y) \to \mathbb{R}$ by

$$\|T\|_{L_{B_G}} = \sup \left\{ \frac{\|T(x)\|_{G_Y}}{\|x\|_{G_X}} : x \in X, \|x\|_{G_X} \neq 0 \right\},$$

for every $T \in L_{B_G}(X, Y)$. Further, as given in the following theorem, we prove that the function $\|\cdot\|_{L_{B_G}}$ is a G-norm on $L_{B_G}(X, Y)$.

Theorem 3.6. *Let $\left(X, \|\cdot\|_{G_X}\right)$ and $\left(Y, \|\cdot\|_{G_Y}\right)$ be G-normed spaces. The space $L_{B_G}(X, Y)$ is a G-normed space with the G-norm*

$$\|T\|_{L_{B_G}} = \sup \left\{ \frac{\|T(x)\|_{G_Y}}{\|x\|_{G_X}} : x \in X, \|x\|_{G_X} \neq 0 \right\},$$

for every $T \in L_{B_G}(X, Y)$.

Proof. It is easy to verify that $\|\cdot\|_{L_{B_G}}$ satisfies the conditions $(G1)$ and $(G3)$ in Definition 2.1. It remains to show that $\|\cdot\|_{L_{B_G}}$ satisfies the axiom $(G2)$ in the definition. For any scalars $\alpha \neq 0$ and $T \in L_{B_G}(X, Y)$, we get

$$\begin{aligned} \|\alpha T\|_{L_{B_G}} &= \sup \left\{ \frac{\|\alpha T(x)\|_{G_Y}}{\|x\|_{G_X}} : x \in X, \|x\|_{G_X} \neq 0 \right\} \\ &\le \sup \left\{ \frac{\beta \|T(x)\|_{G_Y}}{\|x\|_{G_X}} : x \in X, \|x\|_{G_X} \neq 0 \right\} \\ &= \beta \sup \left\{ \frac{\|T(x)\|_{G_Y}}{\|x\|_{G_X}} : x \in X, \|x\|_{G_X} \neq 0 \right\} \end{aligned}$$

$$= \beta\|T\|_{L_{B_G}}$$

and

$$\begin{aligned}\|\alpha T\|_{L_{B_G}} &= \sup\left\{\frac{\|\alpha T(x)\|_{G_Y}}{\|x\|_{G_X}} : x \in X, \|x\|_{G_X} \neq 0\right\} \\ &\geq \sup\left\{\frac{\gamma\|T(x)\|_{G_Y}}{\|x\|_{G_X}} : x \in X, \|x\|_{G_X} \neq 0\right\} \\ &= \gamma \sup\left\{\frac{\|T(x)\|_{G_Y}}{\|x\|_{G_X}} : x \in X, \|x\|_{G_X} \neq 0\right\} \\ &= \gamma\|T\|_{L_{B_G}},\end{aligned}$$

for some $\beta, \gamma > 0$. Therefore, for any scalars $\alpha \neq 0$, there exist $\beta, \gamma > 0$ such that

$$\gamma\|T\|_{L_{B_G}} \leq \|\alpha T\|_{L_{B_G}} \leq \beta\|T\|_{L_{B_G}},$$

for every $T \in L_{B_G}(X, Y)$. Hence, we conclude that $\left(L_{B_G}(X, Y), \|\cdot\|_{L_{B_G}}\right)$ is a G-normed space. □

Remark 3.7. As in the case of the normed space, it can be proved that the space $L_{B_G}(X, Y)$ is bounded and for any $T \in L_{B_G}(X, Y)$, it holds that

$$\|T(x)\|_{G_Y} \leq \|T\|_{L_{B_G}}\|x\|_{G_X},$$

for each $x \in X$.

Now we are in a position to prove the G-completeness of the G-normed space $\left(L_{B_G}(X, Y), \|\cdot\|_{L_{B_G}}\right)$.

Theorem 3.8. *Let* $\left(X, \|\cdot\|_{G_X}\right)$ *be a* G*-normed space and* $(Y, \|\cdot\|_{G_Y})$ *be a* G*-Banach space. Then* $\left(L_{B_G}(X, Y), \|\cdot\|_{L_{B_G}}\right)$ *is a* G*-Banach space.*

Proof. By Theorem 3.6, the space $\left(L_{B_G}(X, Y), \|\cdot\|_{L_{B_G}}\right)$ is a G-normed space. Then, we will show that $L_{B_G}(X, Y)$ is G-complete. Let (T_n) be an arbitrary G-Cauchy sequence in $L_{B_G}(X, Y)$. Because, (T_n) is a G-Cauchy sequence in $L_{B_G}(X, Y)$, then $\|T_m - T_n\|_{L_{B_G}} \to 0$, as $m, n \to \infty$. Consequently, by using Remark 3.7, we get

$$\|T_m(x) - T_n(x)\|_{G_Y} \leq \|T_m - T_n\|_{L_{B_G}}\|x\|_{G_X} \to 0, \text{ as } m, n \to \infty, \tag{3.1}$$

for all $x \in X$. So, for every $x \in X$, $(T_n(x))$ is a G-Cauchy sequence in Y. As $\left(Y, \|\cdot\|_{G_Y}\right)$ is G-complete, then for any $x \in X$, the sequence $(T_n(x))$ G-converges in Y. It means, for any $x \in X$, there is $y \in Y$ such that the sequence $(T_n(x))$ G-converges to y. For each $x \in X$ we define

$$T(x) = y = \lim_{n\to\infty} T_n(x).$$

It is easy to verify that $T \in L_{B_G}(X, Y)$. So, the remaining part that we have to prove is that the sequence (T_n) G-converges to T.

Let $\varepsilon > 0$ be given. Since (T_n) is a G-Cauchy sequence, then there is $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$ we have $\|T_m - T_n\|_{L_{B_G}} < \varepsilon$. By inequality (3.1), we have

$$\|T_m(x) - T_n(x)\|_{G_Y} < \varepsilon \|x\|_{G_X},$$

for every $m, n \geq n_0$. For $\|x\|_{G_X} \neq 0$, we get

$$\frac{\|T_m(x) - T_n(x)\|_{G_Y}}{\|x\|_{G_X}} < \varepsilon,$$

for every $m, n \geq n_0$. Moreover, as $m \to \infty$, it follows that

$$\frac{\|T(x) - T_n(x)\|_{G_Y}}{\|x\|_{G_X}} \leq \varepsilon,$$

for every $n \geq n_0$. Therefore,

$$\sup \left\{ \frac{\|T(x) - T_n(x)\|_{G_Y}}{\|x\|_{G_X}} : x \in X, \|x\|_{G_X} \neq 0 \right\} \leq \varepsilon.$$

So, $\|T - T_n\|_{L_{B_G}} \leq \varepsilon$, for every $n \geq n_0$. In other words, the sequence (T_n) G-converges to T. Thus, the space $\left(L_{B_G}(X, Y), \|\cdot\|_{L_{B_G}}\right)$ is a G-Banach space. □

We need the following definition for proving the main theorem.

Definition 3.9. Let $\left(X, \|\cdot\|_{G_X}\right)$ be a G-normed space. A G-norm $\|\cdot\|_{G_X}$ is said to satisfy the (S)-condition if for any scalar $\alpha > 0$, we have

$$\|\alpha x\|_{G_X} < \sqrt{2\alpha} \|x\|_{G_X},$$

for every non-zero vector $x \in X$.

Example 3.10. Let us consider Example 2.3. The function $\|\cdot\|_G^*$ defined by

$$\|x\|_G^* = \left(\sum_{k=1}^{\infty} |x_k| \right)^{\frac{1}{2}}, \quad \text{for any } x \in \ell_1,$$

is a G-norm, but not a norm on ℓ_1. Moreover, for any scalar $\alpha > 0$, we have

$$\|\alpha x\|_G^* = \left(\sum_{k=1}^{\infty} |\alpha x_k| \right)^{\frac{1}{2}} = \sqrt{\alpha} \left(\sum_{k=1}^{\infty} |x_k| \right)^{\frac{1}{2}} = \sqrt{\alpha} \|x\|_G^* < \sqrt{2\alpha} \|x\|_G^*,$$

for any non-zero vector $x \in \ell_1$. So, the G-norm $\|\cdot\|_G^*$ satisfies the (S)-condition.

Example 3.11. Let $(X, \|\cdot\|)$ be a normed space. It is easy to verify that every norm is G-norm. However,

$$\|3x\| = |3|\|x\| \nless \sqrt{6}\|x\|,$$

for every non-zero vector $x \in X$. Thus, the norm $\|\cdot\|$ does not satisfy the (S)-condition.

Let $(X, \|\cdot\|_G)$ be a G-normed space. For any $x \in X$ and $r > 0$, we define

$$B(x, r) = \{y \in X : \|y - x\|_G < r\},$$

which is called an open ball with center x and radius r. Since $\|y-x\|_G \neq \|x-y\|_G$, the condition $\|y-x\|_G < r$ for the open ball $B(x, r)$ should not be replaced by $\|x-y\|_G < r$. Some notions in a G-normed space, such as interior points, limit points and closure points, are defined analogously to those in normed spaces. A set $F \subseteq X$ is said to be open in X if every element in F is an interior point. A set $F \subseteq X$ is said to be closed in X if F is equal to its closure in X.

The upcoming theorem will outline some properties that are also true in any G-normed space.

Theorem 3.12. *Let $(X, \|\cdot\|_G)$ be a G-normed space and $F \subseteq X$.*

(i) *The set F is closed in X if and only if the complement of F is open in X.*
(ii) *The set F is closed in X if and only if F contains all its limit points.*
(iii) *If $x \in X$ is a limit point of F, then there exists a sequence (x_k) in F such that (x_k) G-converges to x.*

Proof. As the proof is standard, we omit the details. □

Theorem 3.13. *Let $(X, \|\cdot\|_G)$ be a G-Banach space where $X \neq \emptyset$. If*

$$X = \bigcup_{k=1}^{\infty} A_k \text{ where } A_k \text{ is closed subsets,}$$

then there exists at least one $k_0 \in \mathbb{N}$ such that A_{k_0} contains a non-empty open set.

Proof. Assume that none of the A_k's contains a non-empty open set. This means that none of the A_k's is equal to X. Since $A_1 \neq X$, then $A_1^c = X - A_1$ is a non-empty open set in X. Take $x_1 \in A_1^c$, then we can choose $0 < r_1 < \dfrac{1}{2}$ such that $B_1 = B(x_1, r_1) \subseteq A_1^c$. By the hypothesis, A_2 does not contain the open ball $B\left(x_1, \dfrac{r_1}{2}\right)$. As a result,

$A_2^c \cap B\left(x_1, \frac{r_1}{2}\right)$ is a non-empty open set, so we can choose an open ball inside it, namely

$$B_2 = B(x_2, r_2) \subseteq A_2^c \cap B\left(x_1, \frac{r_1}{2}\right),$$

with $0 < r_2 < \frac{r_1}{2} < \frac{1}{2^2}$. If we continue the process inductively, we will obtain a sequence $(B(x_k, r_k))$ of open balls with $0 < r_k < \frac{1}{2^k}$ such that $B_k \cap A_k = \emptyset$ and $B_{k+1} \subseteq B\left(x_k, \frac{r_k}{2}\right)$, $k = 1, 2, \ldots$. Since $r_k < \frac{1}{2^k}$, then (x_k) is a G-Cauchy sequence in X. Because X is G-complete, then the sequence (x_k) G-converges to some $x \in X$. For any $m, n \in \mathbb{N}$ with $m > n \geq 1$, we obtain

$$\|x - x_n\|_G \leq \|x - x_m\|_G + \|x_m - x_n\|_G < \|x - x_m\|_G + \frac{r_n}{2}.$$

As $m \to \infty$, we get $\|x - x_n\|_G \leq \frac{r_n}{2} < r_n$. This means that $x \in B(x_n, r_n)$, for every $n \geq 1$. Since $B(x_n, r_n) \cap A_n = \emptyset$, then $x \notin A_n$, for every $n \geq 1$. This implies that $x \notin X$. This contradicts the hypothesis. Hence, there exists at least one $k_0 \in \mathbb{N}$ such that A_{k_0} contains a non-empty open set. □

We are now ready to prove the Banach-Steinhaus theorem for G-normed spaces.

Theorem 3.14. *Let* $\left(X, \|\cdot\|_{G_X}\right)$ *and* $\left(Y, \|\cdot\|_{G_Y}\right)$ *be* G*-Banach spaces that satisfy the* (S)*-condition. If for every* $x \in X$*, there exists a scalar* $c_x > 0$ *such that*

$$\|T(x)\|_{G_Y} \leq c_x, \text{ for every } T \in L_{B_G}(X, Y),$$

then there exists a scalar $c > 0$ *such that* $\|T\|_{L_{B_G}} \leq c$*, for every* $T \in L_{B_G}(X, Y)$.

Proof. For every $k \in \mathbb{N}$, we define

$$A_k = \left\{x \in X : \|T(x)\|_{G_Y} \leq k, \forall\, T \in L_{B_G}(X, Y)\right\}.$$

It will be shown that A_k is closed. Take any limit point x of A_k, so there exists a sequence (x_j) in A_k such that (x_j) G-converges to x. Since for each $j \in \mathbb{N}$, $x_j \in A_k$, then we have $\|T(x_j)\|_{G_Y} \leq k$, for every $T \in L_{B_G}(X, Y)$. Because the operator T is G-bounded, then T is G-continuous and thus the sequence $(T(x_j))$ G-converges to $T(x)$. Furthermore, we obtain

$$\|T(x)\|_{G_Y} = \left\|\lim_{j \to \infty} T(x_j)\right\|_{G_Y} = \lim_{j \to \infty} \|T(x_j)\|_{G_Y} \leq \lim_{j \to \infty} k = k.$$

Therefore, $\|T(x)\|_{G_Y} \le k$, which implies $x \in A_k$. Thus, A_k is closed.

Furthermore, it will be shown that $X = \bigcup_{k=1}^{\infty} A_k$. From the definition of A_k, we know that for each $k \in \mathbb{N}$, $A_k \subseteq X$. Therefore, $\bigcup_{k=1}^{\infty} A_k \subseteq X$. Now, take any $x \in X$. By assumption, there exists a scalar $c_x > 0$ such that $\|T(x)\|_{G_Y} \le c_x$. By the Archimedean property, there exists $k \in \mathbb{N}$ such that $c_x < k$. Thus, there exists $k \in \mathbb{N}$ such that $\|T(x)\|_{G_Y} \le k$. Therefore, $x \in A_k \subseteq \bigcup_{k=1}^{\infty} A_k$. This shows that, $X \subseteq \bigcup_{k=1}^{\infty} A_k$. Hence, $X = \bigcup_{k=1}^{\infty} A_k$.

Since $(X, \|\cdot\|_G)$ is a G-Banach space, then by Theorem 3.13, there exists $k_0 \in \mathbb{N}$ such that A_{k_0} contains an open ball, i.e.

$$B(x_0, r) \subseteq A_{k_0} \text{ for some } x_0 \in X \text{ and for some } r > 0. \tag{3.2}$$

Now, take any $x \in X$ with $x \neq \theta$. Define

$$z = x_0 + \alpha x, \quad \text{where } \alpha = \frac{r^2}{2\|x\|_{G_X}^2}. \tag{3.3}$$

We obtain

$$\|z - x_0\|_{G_X} = \|\alpha x\|_{G_X} < \sqrt{2\alpha}\|x\|_{G_X} = r.$$

Therefore, $z \in B(x_0, r)$. From the definition of A_{k_0} and the formula (3.2), we have $\|T(z)\|_{G_Y} \le k_0$. Because of $x_0 \in B(x_0, r)$, we also have $\|T(x_0)\|_{G_Y} \le k_0$. From the formula (3.3), we have $x = \frac{1}{\alpha}(z - x_0)$. As a result, for every $x \in X$, we have

$$\begin{aligned}
\|T(x)\|_{G_Y} &= \left\| T\left(\frac{1}{\alpha}(z - x_0)\right) \right\|_{G_Y} \\
&= \left\| \frac{1}{\alpha} T(z - x_0) \right\|_{G_Y} \\
&< \sqrt{\frac{2}{\alpha}} \, \|T(z - x_0)\|_{G_Y} \\
&\le \frac{2\|x\|_{G_X}}{r} \left(\|T(z)\|_{G_Y} + \beta \|T(x_0)\|_{G_Y} \right), \quad \text{for some } \beta > 0 \\
&\le \frac{2k_0}{r}(1 + \beta)\|x\|_{G_X}, \quad \text{for some } \beta > 0.
\end{aligned}$$

Thus, for every $T \in L_{B_G}(X, Y)$, we obtain

$$\|T\|_{B_G} = \sup\left\{ \frac{\|T(x)\|_{G_Y}}{\|x\|_{G_X}} : x \in X, \|x\|_{G_X} \neq 0 \right\} \le \frac{2k_0}{r}(1 + \beta).$$

Hence, there is a scalar $c = \dfrac{2k_0}{r}(1+\beta) > 0$ such that $\|T\|_{L_{B_G}} \leq c$. $\square$

4. Concluding Remarks

In this paper, we have successfully constructed a new type of generalized normed space, which we refer to as a G-normed space. Furthermore, we proved that if every operator in a collection of G-bounded linear operators from $(X, \|\cdot\|_{G_X})$ to $(Y, \|\cdot\|_{G_Y})$ is pointwise bounded, then there is a uniform bound for all operators in the collection. Although we have successfully shown that every G-bounded linear operator is G-continuous, the converse remains an open problem—specifically, whether every G-continuous linear operator is necessarily G-bounded on a G-normed space. In the future, we plan to formulate sufficient conditions under which every G-continuous linear operator is necessarily G-bounded on a G-normed space. Additionally, we will develop related theories for these spaces.

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