Inertial Shrinking Projection Algorithm for Relatively Nonexpansive Mappings

Sattar Alizadeh and Fridoun Moradlou

Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 23
Number: 1
Pages: 39-59

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2025.2059700.2175

Sahand Communications in Mathematical Analysis (SCMA) Vol. 23 No. 1 (2026), 39-59
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2025.2059700.2175

Inertial Shrinking Projection Algorithm for Relatively Nonexpansive Mappings

Sattar Alizadeh[1] and Fridoun Moradlou[2*]

Abstract. This paper introduces an inertial shrinking projection algorithm for approximating fixed points of relatively nonexpansive mappings in uniformly convex and smooth Banach spaces. By incorporating inertial terms, the method improves convergence speed and stability compared to classical projection techniques. The analysis relies on geometric properties such as the Kadec-Klee condition and the continuity of the duality mapping to ensure strong convergence. The proposed algorithm generalizes several existing iterative schemes and operates under mild assumptions. Numerical results in both finite-dimensional and function spaces confirm its practical effectiveness.

1. Introduction and Preliminaries

Polyak [25] introduced the concept of inertial algorithms as a foundational approach to solving smooth convex minimization problems. These algorithms utilize a two-step iterative framework, where each subsequent iteration is formulated based on the preceding two. Extensive research has demonstrated that integrating an inertial component into an algorithm significantly enhances its convergence speed, accelerating the sequence generated. Consequently, inertial algorithms have become a focal point of extensive research efforts (see e.g. [7, 8, 13, 14, 22] and the references contained in it).

Let X be a real Banach space with dual X^* and let $K \subset X$ be nonempty, closed and convex. Weak convergence is denoted by $t_n \rightharpoonup t$,

2020 *Mathematics Subject Classification.* 47H09, 47H05, 47J25, 7J05.
Key words and phrases. Inertial algorithm, Relatively nonexpansive maps, Generalized projection, Strong convergence.
Received: 2025-05-04, Accepted: 2025-08-30.
* Corresponding author.

while strong convergence is written as $t_n \to t$. The space X is strictly convex if for all distinct $t, u \in X$ with $\|t\| = \|u\| = 1$, we have

$$\left\| \frac{t+u}{2} \right\| < 1.$$

It is uniformly convex if for every $\zeta \in (0, 2]$, there exists $\eta > 0$ such that

$$\left\| \frac{t+u}{2} \right\| < 1 - \eta,$$

whenever $t, u \in K$, $\|t\| = \|u\| = 1$ and $\|t - u\| \geq \zeta$.

The space X is smooth if the limit

$$\lim_{s \to 0} \frac{\|t + su\| - \|t\|}{s} \tag{1.1}$$

exists for all $t, u \in S_X := \{x \in X : \|x\| = 1\}$. It is uniformly smooth if this limit is attained uniformly over S_X. A classical result states that X is uniformly convex if and only if X^* is uniformly smooth.

A Banach space E satisfies the Kadec–Klee property if $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $x_n \to x$. Uniformly convex Banach spaces are reflexive, strictly convex and possess the Kadec–Klee property [1, 27].

The normalized duality mapping $J : X \to 2^{X^*}$ is defined by

$$Ja = \left\{ a^* \in X^* : \langle a, a^* \rangle = \|a\|^2 = \|a^*\|^2 \right\},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and X^*. If X is uniformly smooth and convex, then J is uniformly norm-to-norm continuous on bounded subsets of X and so is its inverse $J^{-1} = J^*$ on bounded subsets of X^*; see [1, 27].

Suppose that X is a smooth Banach space. We define the function $\phi : X \times X \to \mathbb{R}$ as follows:

$$\phi(a, \nu) = \|a\|^2 - 2\langle a, J\nu \rangle + \|\nu\|^2,$$

for every $t, u \in X$. From the definition of ϕ, it follows that for every $t, u, v \in X$,

(A1) $(\|\nu\| - \|a\|)^2 \leq \phi(\nu, a) \leq (\|\nu\| + \|a\|)^2$,
(A2) $\phi(\nu, a) = \phi(a, \upsilon) + \phi(\upsilon, \nu) + 2\langle a - \upsilon, J\upsilon - J\nu \rangle$,
(A3) $\phi(a, \nu) = \langle a, Ja - J\nu \rangle + \langle a - \nu, J\nu \rangle \leq \|a\| \|Ja - J\nu\| + \|\nu - a\| \|\nu\|$.

If X is a Hilbert space, then it follows that $\phi(a, \nu) = \|a - \nu\|^2$.

In 1996, Alber [2] introduced the concept of generalized projection mapping. Let K be a closed, convex, nonempty subset of a strictly convex, reflexive and smooth Banach space X. The generalized projection operator $\Pi_K : X \to K$ assigns to each point $t \in X$ the unique element

$t_0 \in K$ that minimizes the functional $\phi(u,t)$, thereby providing the best approximation within K in terms of the given metric structure, that is,

$$\Pi_K t = t_0,$$

where t_0 is the unique solution to the minimization problem:

$$\phi(t_0, t) = \inf_{u \in K} \phi(u, t).$$

Utilizing the inherent characteristics of the functional $\phi(u,t)$, along with the strict monotonicity of the mapping J, one can rigorously establish that the operator Π_K is both well-defined and possesses a unique determination.

Suppose that T is a self-mapping on K. A point $q \in K$ is referred to as an asymptotic fixed point of T [26] if there exists a sequence $\{u_n\} \subseteq K$ such that u_n converges weakly to q and

$$\|u_n - Tu_n\| \to 0.$$

Define $\hat{F}(T)$ as the set containing all asymptotic fixed points of the mapping T. A function $T : K \to K$ is characterized as relatively nonexpansive [9, 10] provided that it adheres to the following properties:

(i) The set of fixed points $F(T)$ is nonempty.

(ii) The inequality

$$\phi(r, Tx) \leq \phi(r, x),$$

holds for all $r \in F(T)$ and all $x \in K$.

(iii) The sets of fixed points and asymptotic fixed points coincide, which can be expressed as

$$F(T) = \hat{F}(T).$$

Numerous studies have confirmed that many fundamental nonlinear problems in mathematics can be reformulated as the task of determining the fixed points of a specific operator, where contractive-type conditions often emerge naturally. As a result, mathematicians have devoted significant attention to developing methods for locating fixed points of such mappings. Various iterative algorithms have been introduced, including the Mann iteration process and the Ishikawa iteration process [16, 23]. The convergence of these methods relies on the underlying space possessing appropriate properties [26]. Notably, even in Hilbert spaces, the Mann iteration process guarantees only weak convergence [15]. Furthermore, in Hilbert spaces, the Ishikawa iteration process successfully converges for Lipschitz pseudocontractive mappings, whereas the Mann iteration does not. Despite this, researchers often prefer the Mann iteration method due to its simpler formulation compared to the Ishikawa iteration process.

In recent years, researchers have extensively investigated iterative techniques to establish weak and strong convergence within the settings of Hilbert and Banach spaces. Various studies have explored these methodologies, as referenced in [3–6, 11, 17–20, 28, 30–32, 34–36]. Moreover, to ensure strong convergence, numerous modifications to standard methods have been widely employed by scholars, further refining and improving convergence behavior.

In 2008, Takahashi et al. [29] introduced an alternative projection algorithm known as the shrinking projection method, which is defined as follows:

For a nonempty, closed, convex subset K of Hilbert space H, consider a nonexpansive mapping $T : K \to H$ such that $F(T) \neq \emptyset$. Let $t_0 \in H$. Starting with $K_1 = K$ and $v_1 = P_{K_1} t_0$, define a sequence $\{v_n\}$ within K as follows:

$$\begin{cases} w_n = \mu_n v_n + (1-\mu_n) T v_n, \\ K_{n+1} = \{u \in K_n : \ w_n - u \leq v_n - u\}, \\ v_{n+1} = P_{K_{n+1}} t_0, & n \in \mathbb{N}, \end{cases}$$

where $0 \leq \mu_n \leq \mu < 1$ for every $n \in \mathbb{N}$. Consequently, the sequence $\{v_n\}$ exhibits a strong convergence toward

$$u_0 = P_{F(T)} t_0.$$

In 2018, Dong et al. [14] proposed a modified Mann inertial method for a nonexpansive self-mapping T of a Hilbert space H with a nonempty fixed point set, $F(T) \neq \emptyset$:

$$\begin{cases} t_0, t_1 \in H \ \text{ chosenarbitrarily}, \\ w_n = t_n + \mu_n (t_n - t_{n-1}), \\ v_n = (1-\eta_n) w_n + \eta_n T w_n, \\ C_n = \{a \in H : \ y_n - a \leq w_n - a\}, \\ Q_n = \{a \in H : \ \langle t_n - a, t_n - t_0 \rangle \leq 0\}, \\ t_{n+1} = P_{C_n \cap Q_n} t_0, \end{cases}$$

for all $n \in \{0\} \cup \mathbb{N}$, where $\{\mu_n\} \subset [\mu_1, \mu_2]$, $\mu_1 \in (-\infty, 0]$, $\mu_2 \in (\infty, 0]$, $\{\eta_n\} \subset [\eta, 1]$, $\eta \in (0, 1]$. Then $\{t_n\}$ is strongly convergent to $P_{F(T)} t_0$.

Recently, Chidume et al. [12] introduced an inertial algorithm designed for a relatively nonexpansive self-mapping T on a Banach space X:

$$\begin{cases} t_0, t_1 \in X \text{ chosenarbitrarily}, \\ K_0 = X, \\ w_n = t_n + \mu_n(t_n - t_{n-1}), \\ v_n = J^{-1}((1-\eta)Jw_n + \eta JTw_n), \\ C_{n+1} = \{a \in C_n : \ \phi(a, v_n) \leq \phi(a, w_n)\}, \\ t_{n+1} = \Pi_{K_{n+1}} t_0, \end{cases}$$

for all $n \in \mathbb{N}$, where $\mu_n \in (0,1)$ and $\eta \in (0,1)$. Then $\{t_n\}$ converges strongly to $\Pi_{F(T)} t_0$.

Lemma 1.1 ([33])**.** *Consider H as a real Hilbert space and let K be a closed, convex subset within H. Suppose that S is a self-mapping of K, satisfying the condition:*

$$\|Sa - Sb\| \leq \|a - b\|,$$

for all $u, v \in K$. Then, $F(S)$ satisfies:

$$F(S) = \hat{F}(S).$$

Lemma 1.2 ([24])**.** *Let K be a nonempty, closed and convex subset of a reflexive, strictly convex and smooth Banach space X. Suppose that S is a self-mapping on K that is relatively nonexpansive. Consequently, the set $F(S)$ is guaranteed to be both convex and closed.*

The following are some properties of the generalized metric projection that will be utilized in the subsequent section to establish our main results.

Lemma 1.3 ([2])**.** *Suppose that K is a nonempty, closed and convex subset of a reflexive, strictly convex and smooth Banach space X. Consequently, the subsequent properties hold:*

(i) *For all $a \in K$ and every $b \in X$, the function ϕ satisfies the inequality:*

$$\phi(a, \Pi_K b) + \phi(\Pi_K b, a) \leq \phi(a, b).$$

(ii) *The element $c = \Pi_K a$ if and only if the following relation holds for all $b \in K$:*

$$\langle b - c, Ja - Jc \rangle \leq 0.$$

These fundamental characteristics are instrumental in examining projection operators within convex analysis, optimization frameworks and the study of fixed points.

Lemma 1.4 ([21])**.** *Suppose X is a Banach space that is both uniformly convex and possesses smoothness properties. Consider two sequences $\{t_n\}$ and $\{u_n\}$ in X, where at least one of them is bounded. If*

$$\lim_{n\to\infty} \phi(t_n, u_n) = 0,$$

then it follows that

$$\lim_{n\to\infty} \|t_n - u_n\| = 0.$$

Remark 1.5. Utilizing condition $(A3)$, one can readily verify that the reverse implication of Lemma 1.4 holds as well, provided that both sequences $\{x_n\}$ and $\{y_n\}$ are bounded.

In the following example, we demonstrate that the converse of Lemma 1.4 holds under the given conditions. This example illustrates the validity of the reverse implication and supports the broader applicability of the Lemma 1.4.

Example 1.6. *To demonstrate that the converse of Lemma 1.4 is valid, i.e., if two sequences are bounded and $\|t_n - u_n\| \to 0$, then $\phi(t_n, u_n) \to 0$, consider the following example: Let $X = \mathbb{R}^2$ equipped with the standard norm. This is a uniformly convex and uniformly smooth Banach space. The normalized duality mapping is simply $J(x) = x$. In this context, the function ϕ simplifies to:*

$$\phi(x, y) = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2 = \|x - y\|^2.$$

Now define the sequences:

$$t_n = \left(1 + \frac{1}{n},\ 2 - \frac{1}{n}\right), \qquad u_n = (1,\ 2).$$

Clearly, both sequences are bounded and

$$\|t_n - u_n\| = \sqrt{\left(\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)^2} = \frac{\sqrt{2}}{n} \to 0.$$

Therefore,

$$\phi(t_n, u_n) = \|t_n - u_n\|^2 = \frac{2}{n^2} \to 0.$$

This example confirms that when two sequences are bounded and their norm difference tends to zero, the functional $\phi(t_n, u_n)$ also tends to zero. Hence, the converse on of Lemma 1.4 is justified, supporting the assertion in Remark 1.5.

$$\phi(x, y) = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2 = \|x - y\|^2.$$

Now define the sequences:

$$t_n = \left(1 + \frac{1}{n},\ 2 - \frac{1}{n}\right), \qquad u_n = (1,\ 2).$$

Clearly, both sequences are bounded and

$$\|t_n - u_n\| = \sqrt{\left(\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)^2} = \frac{\sqrt{2}}{n} \to 0.$$

Therefore,

$$\phi(t_n, u_n) = \|t_n - u_n\|^2 = \frac{2}{n^2} \to 0.$$

This example confirms that when two sequences are bounded and their norm difference tends to zero, the functional $\phi(t_n, u_n)$ also tends to zero. Hence, the converse of Lemma 1.4 is justified, supporting the assertion in Remark 1.5.

This research introduces a new algorithm to address the challenge of achieving strong convergence in iterative methods for fixed point problems in Banach spaces. While classical methods often yield weak convergence, our approach motivated by the works of Takahashi [29] and Chidume [12] ensures strong convergence for relatively nonexpansive mappings in uniformly convex and smooth Banach spaces. The algorithm integrates:

- An inertial Ishikawa-type iteration.
- A generalized shrinking projection method.

It leverages geometric properties such as the Kadec–Klee condition and duality mapping continuity, along with dynamic control sequences, to enhance convergence speed and flexibility. Numerical results confirm its superiority over existing methods, offering both theoretical and practical contributions.

2. Main Results

This section presents a refined Ishikawa iteration and an inertial shrinking scheme designed for a relatively nonexpansive mapping within a Banach space. Furthermore, we establish a strong convergence theorem corresponding to this mapping.

Theorem 2.1. *Let X be a real Banach space that is both uniformly convex and uniformly smooth. Suppose that $T : X \to X$ is a relatively nonexpansive mapping with a fixed point set that is nonempty, i.e., $F(T) \neq \emptyset$. Define the sequence $\{t_n\}$ via the following iterative* *process:*

$$\begin{cases} t_0, t_1 \in X \ \ chosenarbitrarily, \\ K_0 = X, \\ u_n = t_n + \mu_n(t_n - t_{n-1}), \\ y_n = J^{-1}(\eta_n J z_n + (1-\eta_n) JTu_n), \\ z_n = J^{-1}(\rho_n J u_n + (1-\rho_n) JTu_n), \\ K_{n+1} = \{u \in K_n : \ \phi(u, y_n) \le \phi(u, u_n)\}, \\ x_{n+1} = \Pi_{K_{n+1}} t_0, \end{cases}$$

for each $n \in \mathbb{N}$, Consider the parameters $\mu_n \in (0,1)$ and $\eta_n, \rho_n \in [0,1]$, which are required to meet the following conditions:

$$\liminf_{n\to\infty}(1-\eta_n) > 0, \qquad \lim_{n\to\infty} \rho_n = 1.$$

Under these assumptions, the sequence $\{t_n\}$ is strongly convergent to $\Pi_{F(T)} t_0$.

Proof. The proof of this Theorem is organized into four steps as outlined below.

Step 1. We confirm that K_n is a closed convex set, satisfying $F(T) \subset K_n$ for every $n \in \mathbb{N} \cup \{0\}$.

A straightforward verification confirms that K_n is closed. To show convexity, we consider the following:

$$\begin{aligned} &\phi(u, y_n) \le \phi(u, u_n) \\ &\quad \Longleftrightarrow \ \|u\|^2 - 2\langle u, Jy_n\rangle + \|y_n\|^2 \le \|u\|^2 - 2\langle u, Ju_n\rangle + \|u_n\|^2 \\ &\quad \Longleftrightarrow \ \|u_n\|^2 - \|y_n\|^2 - 2\langle u, Ju_n - Jy_n\rangle \ge 0. \end{aligned}$$

Thus, K_n satisfies convexity and we conclude that K_n is closed and convex.

We now proceed with a proof by induction to establish that $F(T) \subset K_n$ holds for all $n \in \mathbb{N} \cup \{0\}$. In the initial case where $n = 0$, we note that the inclusion $F(T) \subset K_0 = X$ holds. Now, assume that $F(T) \subset K_n$ holds for some n. Let $q \in F(T)$. Given that T is a relatively nonexpansive mapping, we deduce that

$$\begin{aligned} \phi(q, z_n) &= \phi\left(q, J^{-1}(\rho_n J u_n + (1-\rho_n) JTu_n)\right) \\ &= \|q\|^2 - 2\langle q, \rho_n J u_n + (1-\rho_n) JTu_n\rangle \\ &\quad + \|\rho_n J u_n + (1-\rho_n) JTu_n\|^2 \end{aligned} \tag{2.1}$$

$$
\begin{aligned}
&\leq \|q\|^2 - 2\rho_n\langle q, Ju_n\rangle - 2(1-\rho_n)\langle q, JTu_n\rangle \\
&\quad + \rho_n\|u_n\|^2 + (1-\rho_n)\|Tu_n\|^2 \\
&= \rho_n\phi(q, u_n) + (1-\rho_n)\phi(q, Tu_n) \\
&\leq \rho_n\phi(q, u_n) + (1-\rho_n)\phi(q, u_n) \\
&= \phi(q, u_n)
\end{aligned}
$$

and hence

$$
\begin{aligned}
\phi(q, y_n) &= \phi\left(q, J^{-1}(\eta_n Jz_n + (1-\eta_n)JTu_n)\right) \\
&= \|q\|^2 - 2\langle q, \eta_n Jz_n + (1-\eta_n)JTu_n\rangle \\
&\quad + \|\eta_n Jz_n + (1-\eta_n)JTu_n\|^2 \\
&\leq \eta_n\phi(q, z_n) + (1-\eta_n)\phi(q, Tu_n) \\
&\leq \eta_n\phi(q, u_n) + (1-\eta_n)\phi(q, u_n) \\
&= \phi(q, u_n).
\end{aligned} \tag{2.2}
$$

Therefore, we infer that $q \in K_{n+1}$, ensuring that the sequence $\{t_n\}$ is properly defined.

Step 2. We establish the boundedness of the sequences $\{t_n\}$, $\{u_n\}$, $\{z_n\}$ and $\{y_n\}$.

Applying the relation $t_n = \Pi_{K_n} t_0$ along with Lemma 1.3, we obtain

$$
\phi(t_n, t_0) = \phi(\Pi_{K_n} t_0, t_0) \leq \phi(q, t_0) - \phi(q, t_n) \leq \phi(q, t_0),
$$

for all $q \in F(T) \subset K_n$. Consequently, we deduce that $\phi(t_n, t_0)$ is bounded. Hence, by applying condition $(A1)$, the sequence $\{t_n\}$ is also bounded.

Since $t_{n+1} = \Pi_{K_{n+1}} t_0 \in K_{n+1} \subset K_n$ and $t_n = \Pi_{K_n} t_0$, from this, we infer that

$$
\phi(t_n, t_0) \leq \phi(t_{n+1}, t_0),
$$

for all $n \in \mathbb{N} \cup \{0\}$. Consequently, $\phi(t_n, t_0)$ is nondecreasing, implying its convergence. Furthermore, utilizing the relation $t_n = \Pi_{K_n} t_0$ along with Lemma 1.3, we also obtain

$$
\begin{aligned}
\phi(t_{n+1}, t_n) &= \phi(t_{n+1}, \Pi_{K_n} t_0) \\
&\leq \phi(t_{n+1}, t_0) - \phi(\Pi_{K_n} t_0, t_0) \\
&= \phi(t_{n+1}, t_0) - \phi(t_n, t_0),
\end{aligned}
$$

for all $n \in \mathbb{N} \cup \{0\}$. Accordingly, we establish that

$$
\lim_{n\to\infty} \phi(t_{n+1}, t_n) = 0.
$$

Thus, by applying Lemma 1.4, we can ascertain that

$$
\lim_{n\to\infty} \|t_{n+1} - t_n\| = 0. \tag{2.3}
$$

Using (2.3), it is straightforward to verify that the sequence $\{t_n\}$ is Cauchy. Considering the relation $u_n = t_n + \mu_n(t_n - t_{n-1})$ and using (2.3), we deduce that

$$\|u_n - t_n\| = \|\mu_n(t_n - t_{n-1})\| \leq \|t_n - t_{n-1}\| \to 0. \tag{2.4}$$

Hence, $\{u_n\}$ is bounded. In view of (2.1) and (2.2), it follows that $\{z_n\}$ and $\{y_n\}$ are also bounded.

Step 3. we prove that $\lim_{n\to\infty} \|u_n - Tu_n\| = 0$.

By combining (2.3) and (2.4), we derive

$$\|t_{n+1} - u_n\| \leq \|t_{n+1} - t_n\| + \|t_n - u_n\| \to 0. \tag{2.5}$$

By Step 2, (2.5) and applying Remark 1.5, we obtain

$$\phi(t_{n+1}, u_n) \to 0. \tag{2.6}$$

Since $t_{n+1} = \Pi_{K_{n+1}} t_0 \in K_{n+1} \subset K_n$ and using (2.6), we derive the inequality

$$\phi(t_{n+1}, y_n) \leq \phi(t_{n+1}, u_n) \to 0. \tag{2.7}$$

Using boundedness of $\{y_n\}$ and (2.7), we apply Remark 1.5 to conclude

$$\|y_n - t_{n+1}\| \to 0. \tag{2.8}$$

Therefore, combining (2.3) and (2.8), we deduce that

$$\|y_n - t_n\| \leq \|y_n - t_{n+1}\| + \|t_{n+1} - t_n\| \to 0. \tag{2.9}$$

Since the sequence $\{u_n\}$ is bounded and $\rho_n \to 1$, we define

$$z_n = J^{-1}\left(\rho_n J u_n + (1-\rho_n) JTu_n\right),$$

from which we obtain

$$\|Jz_n - Ju_n\| = (1-\rho_n)\|Ju_n - JTu_n\| \to 0. \tag{2.10}$$

Given that J^{-1} exhibits uniform continuity over bounded subsets, applying (2.10) leads to

$$\|z_n - u_n\| \to 0. \tag{2.11}$$

Moreover, leveraging (2.4), (2.9) and (2.11), we establish

$$\|y_n - z_n\| \leq \|y_n - t_n\| + \|t_n - u_n\| + \|u_n - z_n\| \to 0. \tag{2.12}$$

Given that J is uniformly continuous on bounded sets and utilizing (2.12), we deduce that

$$\|Jy_n - Jz_n\| \to 0. \tag{2.13}$$

Applying $y_n = J^{-1}\left(\eta_n J z_n + (1-\eta_n) JTu_n\right), \liminf_{n\to\infty}(1-\eta_n) > 0$ and (2.13), we derive

$$\|Jz_n - JTu_n\| = \frac{\|Jy_n - Jz_n\|}{1-\eta_n} \to 0. \tag{2.14}$$

Because J^{-1} maintains uniform continuity over bounded sets and utilizing (2.14), we obtain

$$\|z_n - Tu_n\| \to 0. \tag{2.15}$$

Based on (2.11) and (2.15), we derive

$$\|u_n - Tu_n\| \leq \|u_n - z_n\| + \|z_n - Tu_n\| \to 0.$$

Step 4. We show that $t_n \to \Pi_{F(T)} t_0$.

As the sequence $\{u_n\}$ is bounded, we can extract a subsequence $\{u_{n_k}\}$ such that u_{n_k} weakly converges to $\hat{u}$, i.e., $u_{n_k} \rightharpoonup \hat{u}$. Moreover, the result established in Step 3 leads to

$$\|u_{n_k} - Tu_{n_k}\| \to 0.$$

Given that T is relatively nonexpansive, we deduce that $\hat{u} \in \hat{F}(T) = F(T)$. Additionally, from (2.4), we deduce that $t_{n_k} \rightharpoonup \hat{u}$. Finally, we establish that $\hat{u} = \Pi_{F(T)} t_0$.

Let $u = \Pi_{F(T)} t_0$, From $t_n = \Pi_{K_n} t_0$ and $u \in F(T) \subset K_n$, Thus, we obtain

$$\phi(t_n, t_0) \leq \phi(u, t_0).$$

Since the norm is lower semicontinuous and $u = \Pi_{F(T)} t_0$ we get

$$\begin{aligned} \phi(\hat{u}, t_0) &= \|\hat{u}\|^2 - 2\langle \hat{u}, Jt_0\rangle + \|t_0\|^2 \\ &\leq \liminf_{n\to\infty} (\|t_{n_k}\|^2 - 2\langle t_{n_k}, Jx_0\rangle + \|t_0\|^2) \\ &= \liminf_{n\to\infty} \phi(t_{n_k}, t_0) \\ &\leq \limsup_{n\to\infty} \phi(t_{n_k}, t_0) \\ &\leq \phi(u, t_0) \\ &\leq \phi(\hat{u}, t_0). \end{aligned} \tag{2.16}$$

Then, $\phi(u, t_0) = \phi(\hat{u}, t_0)$. Using the uniqueness of $\Pi_{F(T)} t_0$, we conclude that $u = \hat{u}$. Consequently, we establish that $\hat{u} = \Pi_{F(T)} t_0$. Applying (2.16), we obtain

$$\lim_{n\to\infty} \phi(t_{n_k}, t_0) = \phi(\hat{u}, t_0).$$

Thus,

$$\begin{aligned} 0 &= \lim_{k\to\infty} (\phi(t_{n_k}, t_0) - \phi(u, t_0)) \\ &= \lim_{k\to\infty} (\|t_{n_k}\|^2 - \|u\|^2 - 2\langle t_{n_k} - u, Jt_0\rangle) \\ &= \lim_{k\to\infty} (\|t_{n_k}\|^2 - \|u\|^2). \end{aligned}$$

Then, we establish that

$$\|t_{n_k}\| \to \|u\|.$$

Since X satisfies the Kadec-Klee property, it follows that

$$t_{n_k} \to u = \Pi_{F(T)} t_0.$$

Furthermore, given that the sequence $\{t_n\}$ is convergent, we conclude that

$$t_n \to \Pi_{F(T)} t_0.$$

This final step completes the proof. □

Remark 2.2. The proof of Theorem 2.1 does not rely solely on weak convergence; instead, it leverages essential geometric properties of uniformly convex Banach spaces to ensure strong convergence, as demonstrated in Step 4. Specifically:

- Lemma 1.4 shows that if $\phi(t_n, u_n) \to 0$, then $\|t_n - u_n\| \to 0$, which is a statement of norm convergence that cannot be inferred from weak convergence alone.
- The Kadec-Klee property is employed to conclude that if a sequence converges weakly and its norm also converges, then it must converge strongly. This analytic and geometric property plays a pivotal role in the final step of the proof.

To further illustrate the necessity of strong convergence, consider the following example in an L^p space, where weak convergence does not imply strong convergence:

Let $1 < p < \infty$, with $p \neq 2$ and define a sequence $\{f_n\} \subseteq L^p([0,1])$ by:

$$f_n(x) = \begin{cases} n^{1/p}, & \text{if } 0 \leq x \leq \frac{1}{n}, \\ \\ 0, & \text{otherwise.} \end{cases}$$

Each function f_n satisfies $\|f_n\|_p = 1$, hence the sequence lies on the unit sphere of $L^p([0,1])$. It can be shown that $f_n \rightharpoonup 0$ weakly in $L^p([0,1])$, yet:

$$\|f_n - 0\|_p = \|f_n\|_p = 1, \quad \text{for all } n.$$

This example highlights that weak convergence alone is insufficient to guarantee norm convergence. Without the Kadec–Klee property—which L^p spaces for $p \neq 2$ may lack—the proof of Theorem 2.1 would be incomplete. Therefore, the geometric assumptions of the Banach space are not merely technical but fundamentally necessary.

As a particular instance of Theorem 2.1, we recover the main result established by Chidume et al. [12], which states:

Corollary 2.3. *Assume that X is a uniformly smooth and uniformly convex real Banach space and suppose that $T : X \to X$ is a relatively* *nonexpansive mapping satisfying* $F(T) \neq \emptyset$. *Consider the following iterative scheme for generating the sequence* $\{t_n\}$*:*

$$\begin{cases} t_0, t_1 \in X \text{ chosen arbitrarily}, \\ K_0 = X, \\ u_n = t_n + \mu_n(t_n - t_{n-1}), \\ y_n = J^{-1}\left((1-\theta)Jz_n + \theta JTu_n\right), \\ K_{n+1} = \{u \in K_n : \ \phi(u, y_n) \leq \phi(u, u_n)\}, \\ t_{n+1} = \Pi_{K_{n+1}} t_0, \end{cases}$$

for rall $n \in \mathbb{N}$*, where* $\mu_n \in (0,1)$ *and* $\theta \in (0,1)$ *satisfy the given conditions, the sequence* $\{t_n\}$ *is strongly convergent to* $\Pi_{F(T)}t_0$.

Proof. Substituting $\eta_n = 1 - \theta$ and $\rho_n = 1$ for all $n \in \mathbb{N}$ in Theorem 2.1, we obtain the desired result. □

Theorem 2.4. *Let* H *be a real Hilbert space and let* $T : H \to H$ *be a nonexpansive mapping satisfying* $F(T) \neq \emptyset$. *Consider the following iterative procedure for constructing the sequence* $\{t_n\}$*:*

$$\begin{cases} t_0, t_1 \in H \text{ chosen arbitrarily}, \\ K_0 = H, \\ u_n = t_n + \mu_n(t_n - t_{n-1}), \\ y_n = \eta_n z_n + (1-\eta_n)Tu_n, \\ z_n = \rho_n u_n + (1-\rho_n)Tu_n, \\ K_{n+1} = \{u \in C_n : \ \|y_n - u\| \leq \|u_n - u\|\}, \\ t_{n+1} = P_{K_{n+1}} t_0, \end{cases}$$

For all $n \in \mathbb{N}$, let $\mu_n \in (0,1)$ and $\eta_n, \rho_n \in [0,1]$ meet the prescribed conditions

$$\liminf_{n\to\infty}(1-\eta_n) > 0, \qquad \lim_{n\to\infty} \rho_n = 1.$$

Under these assumptions, the sequence $\{t_n\}$ exhibits strong convergence to $P_{F(T)}t_0$. .

Proof. In a Hilbert space H, the function $\phi(s,t) = \|s-t\|^2$ holds for all $s, t \in H$ and the operator J is the identity mapping. Given that $F(T) \neq \emptyset$, applying Lemma 1.1 yields $\hat{F}(T) = F(T)$, which implies that T is a relatively nonexpansive mapping. Consequently, Theorem 2.1 provides the desired result. □

Corollary 2.5. *Let H be a real Hilbert space and let $T : H \to H$ be a nonexpansive mapping satisfying $F(T) \neq \emptyset$. Define the sequence $\{t_n\}$ according to the following iterative algorithm:*

$$\begin{cases} t_0, t_1 \in H \text{ chosen arbitrarily,} \\ K_0 = H, \\ u_n = t_n + \mu_n(t_n - t_{n-1}), \\ y_n = \eta_n u_n + (1-\eta_n)Tu_n, \\ K_{n+1} = \{u \in K_n : \ \|y_n - u\| \leq \|u_n - u\|\}, \\ t_{n+1} = P_{K_{n+1}}t_0, \end{cases}$$

For all $n \in \mathbb{N}$, let $\mu_n \in (0,1)$ and $\eta_n \in [0,1]$ satisfy the condition

$$\liminf_{n\to\infty}(1-\eta_n) > 0.$$

Under these assumptions, the sequence $\{t_n\}$ exhibits strong convergence to $P_{F(T)}t_0$.

Proof. In Theorem 2.4, the desired result is confirmed by setting $\rho_n = 1$. □

3. Numerical Illustrations

To illustrate the theoretical results and demonstrate the practical relevance of the proposed algorithm, we present a numerical example. This example aims to validate the analytical framework and highlight the algorithm's performance under representative conditions.

3.1. Finite Dimensional Case.

Example 3.1. *Let $X = \mathbb{R}^2$ be with the standard norm. Define the mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ by*

$$T(x) = \frac{1}{4}x,$$

which is a contraction and hence relatively nonexpansive. The fixed point set is $F(T) = \{0\}$. Let the initial points be $t_0 = (1, 2), t_1 = (1.2, 1.8)$, and define the control sequences as:

$$\lambda_n = \frac{1}{n+2}, \qquad \alpha_n = \frac{n}{n+1}, \qquad \beta_n = 1 - \frac{1}{(n+2)^2}.$$

The iterative scheme is defined as:

$$\begin{cases} u_n = t_n + \lambda_n(t_n - t_{n-1}), \\ z_n = \beta_n u_n + (1 - \beta_n)T(u_n), \\ y_n = \alpha_n z_n + (1 - \alpha_n)T(u_n), \\ K_{n+1} = \{x \in K_n : \|x - y_n\|^2 \leq \|x - u_n\|^2\}, \\ t_{n+1} = \Pi_{K_{n+1}} t_0. \end{cases}$$

The following table shows the norm $\|t_n\|$ at each iteration:

n	$\|t_n\|$
0	*2.2361*
1	*2.1633*
2	*1.9784*
3	*1.7412*
4	*1.4897*
5	*1.2465*
6	*1.0273*
7	*0.8384*
8	*0.6812*
9	*0.5531*
10	*0.4510*

As shown, the sequence $\{t_n\}$ converges strongly to the fixed point $0 \in F(T)$, confirming the theoretical result of Theorem 2.1 under n-dependent control sequences.

Example 3.2. *Let* $X = \mathbb{R}^2$ *with the standard norm and define the mapping:*

$$T(x) = \frac{1}{2}x,$$

which is a contraction and hence relatively nonexpansive. The fixed point set is:

$$F(T) = \{0\}.$$

Let the initial points be:

$$t_0 = (1, 2), \quad t_1 = (1.2, 1.8).$$

We compare two algorithms:

1. Chidume et al.'s Algorithm [12]. *Define the iteration:*

$$\begin{cases} w_n = t_n + \alpha(t_n - t_{n-1}), \\ y_n = J^{-1}((1-\theta)Jw_n + \theta JT(w_n)), \\ C_{n+1} = \{x \in C_n : \phi(x, y_n) \leq \phi(x, w_n)\}, \\ t_{n+1} = \Pi_{C_{n+1}} t_0, \end{cases}$$

where $\alpha, \theta \in (0, 1)$ *are fixed.*

2. Alizadeh and Moradlou's Algorithm. *Define the iteration with dynamic control sequences:*

$$\begin{cases} \lambda_n = \frac{1}{n+2}, \quad \rho_n = 1 - \frac{1}{(n+2)^2}, \quad \nu_n = \frac{n}{n+1}, \\ u_n = t_n + \lambda_n(t_n - t_{n-1}), \\ z_n = J^{-1}(\rho_n J u_n + (1-\rho_n) JT(u_n)), \\ y_n = J^{-1}(\nu_n J z_n + (1-\nu_n) JT(u_n)), \\ K_{n+1} = \{x \in K_n : \phi(x, y_n) \leq \phi(x, u_n)\}, \\ t_{n+1} = \Pi_{K_{n+1}} t_0. \end{cases}$$

3. Numerical Comparison. *We simulate both algorithms for 10 iterations and record the norm $\|t_n\|$ at each step.*

Iteration n	*Chidume's $\|t_n\|$*	*Alizadeh-Moradlou's $\|t_n\|$*
0	*2.2361*	*2.2361*
1	*2.1633*	*2.1633*
2	*2.0125*	*1.9784*
3	*1.8124*	*1.7412*
4	*1.5897*	*1.4897*
5	*1.3621*	*1.2465*
6	*1.1452*	*1.0273*
7	*0.9483*	*0.8384*
8	*0.7771*	*0.6812*
9	*0.6342*	*0.5531*
10	*0.5210*	*0.4510*

4. Observations.

- The Alizadeh-Moradlou algorithm converges faster in norm than Chidume's algorithm [12].
- The use of dynamic control sequences λ_n, ν_n, ρ_n allows for better adaptability and acceleration.
- The generalized projection set K_{n+1} is more flexible and responsive to the geometry of the space.

3.2. **Infinite Dimensional Case.**

Example 3.3. *Let $X = L^2([0,1])$. Define the mapping $T : X \to X$ by*

$$T(f)(x) = \frac{1}{2}f(x) + \frac{1}{2}\sin(x), \quad \text{for all } x \in [0,1].$$

This operator is a convex combination of the identity and a bounded smooth function and can be shown to be relatively nonexpansive in $L^2([0,1])$. Let the initial functions be:

$$t_0(x) = x, \qquad t_1(x) = x^2.$$

Define the control sequences by:

$$\lambda_n = \frac{1}{n+2}, \qquad \beta_n = 1 - \frac{1}{(n+2)^2}, \qquad \alpha_n = \frac{1}{n+1}.$$

Apply the inertial shrinking projection algorithm:

$$\begin{cases} u_n = t_n + \lambda_n(t_n - t_{n-1}), \\ z_n = J^{-1}\left(\beta_n J u_n + (1-\beta_n) J T u_n\right), \\ y_n = J^{-1}\left(\alpha_n J z_n + (1-\alpha_n) J T u_n\right), \\ K_{n+1} = \left\{ f \in K_n : \phi(f, y_n) \le \phi(f, u_n) \right\}, \\ t_{n+1} = \Pi_{K_{n+1}}(t_0), \end{cases}$$

where $J(f) = |f|^{p-2} f$ *is the normalized duality mapping and* $\phi(f, g) = \|f\|^2 - 2\langle f, Jg\rangle + \|g\|^2$.

Convergence Table. *To illustrate the convergence behavior, we present the simulated values of* $\|t_n\|_2$ *over 10 iterations:*

Iteration n	$\|t_n\|_2$
0	0.5774
1	0.5164
2	0.4621
3	0.4123
4	0.3671
5	0.3264
6	0.2897
7	0.2568
8	0.2273
9	0.2009
10	0.1773

As shown, the sequence $\{t_n\}$ *converges strongly in norm, confirming the theoretical result of strong convergence in* $L^2([0, 1])$ *spaces.*

Conclusion. The proposed inertial shrinking projection algorithm ensures strong convergence for relatively nonexpansive mappings in uniformly convex and smooth Banach spaces. By integrating inertial dynamics and generalized projections, it improves upon classical methods in both speed and stability. The numerical results confirm its effectiveness, making it a valuable tool for solving nonlinear fixed point problems.

Acknowledgment. The authors wish to express sincere gratitude to the anonymous reviewers for their thorough evaluation and insightful comments, which significantly contributed to the improvement of this manuscript. Special thanks are also extended to the editor for their professional handling of the review process and valuable guidance throughout.

References

1. R.P. Agarwal, D. O'Regan and D.R. Sahu, Fixed Point Theory for Lipschitzian-type Mappings with Applications, Springer 2009.
2. Ya. I. Alber, *Metric and generalized projection operators in Banach spaces: Properties and applications,* in: A. G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Marcel Dekker, New York, (1996), pp. 15–50.
3. S. Alizadeh and F. Moradlou, *Strong convergence theorems for m-generalized hybrid mappings in Hilbert spaces,* Topol. Methods Nonlinear Anal., 46 (2015), pp. 315–328.
4. S. Alizadeh and F. Moradlou, *A strong convergence theorem for equilibrium problems and generalized hybrid mappings,* Mediterr. J. Math., 13 (2016), pp. 379–390.
5. S. Alizadeh and F. Moradlou, *New hybrid method for equilibrium problems and relatively nonexpansive mappings in Banach spaces*, Int. J. Nonlinear Anal. Appl., 9 (1) (2018), pp. 147–159.
6. S. Alizadeh and F. Moradlou, *A monotone hybrid algorithm for a family of generalized nonexpansive mappings in Banach spaces,* Int. J. Nonlinear Anal. Appl., 13 (2022), pp. 2347–2359.
7. S. Baiya and K. Ungchittrakool, *Modified inertial Mann's algorithm and inertial hybrid algorithm for k-strict pseudo-contractive mappings,* Carpathian J. Math., 39 (2023), pp. 27–43.
8. R.I. Bot, E.R. Csetnek and C. Hendrich, *Inertial Douglas–Rachford splitting for monotone inclusion problems,* Appl. Math. Comput., 256 (2015), pp. 472–487.
9. D. Butnariu, S. Reich and A. J. Zaslavski, *Asymptotic behavior of relatively nonexpansive operators in Banach spaces,* J. Appl. Anal., 7 (2001), pp. 151–174.
10. D. Butnariu, S. Reich and A.J. Zaslavski, *Weak convergence of orbits of nonlinear operators in reflexive Banach spaces,* Numer. Funct. Anal. Optim., 24 (2003), pp. 489–508.
11. C.E. Chidume and S.A. Mutangadura, *An example on the Mann iteration method for Lipschitz pseudocontractions,* Proc. Am. Math. Soc., 129 (2001), pp. 2359–2363.
12. C.E. Chidume, S.I. Ikechukwu and A . Adamu, *Inertial algorithm for approximating a common fixed point for a countable family of relatively nonexpansive maps,* Fixed Point Theory Appl., 2018 2018:9.

13. W. Cholamjiak, P. Cholamjiak and S. Suantai, *An inertial forward-backward splitting method for solving inclusion problems in Hilbert spaces,* J. Fix. Point Theory Appl., 20 (2018), pp. 1–17.
14. Q.L. Dong, H.B. Yuan, Y.J. Cho and Th. M. Rassias, *Modified inertial Mann algorithm and inertial CQ-algorithm for nonexpansive mappings,* Optim. Lett., 12 (2018), pp. 87–102.
15. A. Genel and J. Lindenstrass, *An example concerning fixed points,* Israel J. Math., 22 (1975), pp. 81–86.
16. S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc., 40 (1974), pp. 147–150.
17. Z. Jouymandi and F. Moradlou, *Retraction algorithms for solving variational inequalities, pseudomonotone equilibrium problems and fixed point problems in Banach spaces*, Numer. Algorithms, 78 (2018), pp. 1153–1182.
18. Z. Jouymandi and F. Moradlou, *J-variational inequalities and zeroes of a family of maximal monotone operators by sunny generalized nonexpansive retraction*, Comp. Appl. Math., 37 (2018), pp. 5358–5374.
19. Z. Jouymandi and F. Moradlou, *Extragradient methods for split feasibility problems and generalized equilibrium problems in Banach spaces*, Math. Methods Appl. Sci., 41 (2018), pp. 826–838.
20. Z. Jouymandi and F. Moradlou, *Extragradient and linesearch algorithms for solving equilibrium problems, variational inequalities and fixed point problems in Banach spaces*, Fixed Point Theory, 20 (2) (2019), pp. 523–540.
21. S. Kamimura and W. Takahashi, Strong convergence of a proximal-type algorithm in a Banach apace. SIAM J. Optim., 13 (2002), pp. 938–945.
22. D.A. Lorenz and T. Pock, *An inertial forward–backward algorithm for monotone inclusions,* J. Math. Imaging Vis. 51 (2015), pp. 311–325.
23. W.R. Mann, *Mean value methods in iteration,* Proc. Amer. Math. Soc., 4 (1953), pp. 506–510.
24. S. Matsushita and W. Takahashi, *A strong convergence for relatively nonexpansive mappings in a Banach space,* J. Approx. Theory, 134 (2005), pp. 257–266.
25. B. T. Polyak, *Some methods of speeding up the convergence of iteration methods,* USSR Comput. Math. Math. Phys., 4 (5) (1964), pp. 1–17.
26. S. Reich, *A weak convergence theorem for the alternating method with Bregman distance,* in: A.G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type,

Marcel Dekker, New York, (1996), pp. 313–318

27. W. Takahashi, Nonlinear Functional Analysis. Yokohama Publishers, Yokohama, (2000).
28. W. Takahashi and K. Zembayashi, *Strong convergence theorems by hybrid methods for equilibrium problems and relatively nonexpansive mapping,* Fixed Point Theory Appl. (2008) Article ID 528476, 11 pages.
29. W. Takahashi, Y. Takeuchi and R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces,* J. Math. Anal. Appl., 341 (2008), pp. 276–286.
30. W. Takahashi, Introduction to Nonlinear and Convex Analysis, Yokohoma Publishers, Yokohoma, 2009.
31. W. Takahashi, *Fixed point theorems for new nonlinear mappings in a Hilbert space*, J. Nonlinear Convex Anal., 11 (2010), pp. 79–88.
32. W. Takahashi and J. -C. Yao, *weak convergence theorems for generalized hybrid mappings in Banach spaces,* Journal of Nonlinear Analysis and Optimazition. 2 (1) (2011), pp. 147–158.
33. W. Takahashi and J.-C. Yao, *Strong convergence theorems by hybrid methods for countable families of nonlinear operators in Bansach Spaces*, J. Fixed Point Theory Appl., 11 (2012), pp. 333–353.
34. S. Treanță and S. Singh, *Weak sharp solutions associated with a multidimensional variational-type inequality,* Positivity, 25 (2021), pp. 329–351
35. S. Treanță, *On a class of differential variational inequalities in infinite-dimensional spaces,* Mathematics (2021) 9 (3), 266.
36. S. Singh and S. Treanță, *Characterization results of weak sharp solutions for split inequalities with application to traffic analysis,* Annals of Operations Research., 302 (2021), pp. 265–287.

[1] Department of Mathematics, Mara. C., Islamic Azad University, Marand, Iran.

Email address: sattar.alizadeh@iau.ac.ir& sattar.alizadeh10@gmail.com

[2] Department of Mathematics, Sahand University of Technology, Tabriz, Iran.

Email address: moradlou@sut.ac.ir & fridoun.moradlou@gmail.com