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Analysis On Simpson's Type Inequalities Through Generalized Convexity with Applications

Arslan Munir[1*] and Artion Kashuri[2]

Abstract. Fractional operators and integral inequalities have become a focal point of research due to their applications in mathematics, physics, engineering, and applied sciences. This paper introduces a new identity for the Caputo-Fabrizio fractional integral operators. Employing the Peano kernel method, we derive Simpson's type inequalities for (s, m)-convex functions through twice-differentiable functions, accompanied by graphical illustrations to analyze their behavior. Several new corollaries are established, with insightful remarks enhancing their interpretation. Additionally, applications to special means, q-digamma functions, modified Bessel function, Simpson's formula, matrix inequality, and midpoint formula are explored, underscoring the utility and adaptability of these results across various mathematical and applied domains.

1. Introduction

Convex functions and integral inequalities remain an active and fertile area of research, with ongoing efforts to deepen our understanding and uncover new applications across various disciplines. Convex functions have played a crucial role in investigating various types of inequalities, including the Hermite-Hadamard type, Simpson's type, Hermite-Hadamard-Mercer type, Ostrowski type, and many others. Mathematicians have been actively establishing new refinements of the Hermite-Hadamard type inequality for different classes of convex functions and

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mappings. Some notable examples include refinements for harmonically convex functions [26], quasi-convex functions [1], convex functions [30], m-convex functions [19], s-convex functions of Raina type [6], and Riemann-Liouville operators [23]. Additionally, operators like k-Riemann-Liouville [22], Caputo-Fabrizio [14, 21], generalized Atangana-Baleanu operator [3] have been employed to establish refined versions of the Hermite-Hadamard inequality and other related inequalities.

In addition fractional calculus has indeed garnered increasing interest in recent years, owing to its efficacy in tackling a diverse array of real-world problems spanning various physical and engineering domains. The exploration of fractional-order integral and derivative functions across both real and complex domains, along with their applications, has emerged as a focal point within the realm of fractional calculus. References such as [4, 7, 8, 11, 17] likely provide valuable insights and applications of fractional calculus, contributing to the growing body of knowledge in this field. Fractional integral operators play a crucial role in this regard, serving as powerful tools for demonstrating well-known integral inequalities. As mathematicians delve deeper into fractional calculus, they continue to explore innovative theories and methodologies, paving the way for novel applications and advancements in mathematics and its applications. However, in all of this years, Thomas Simpson established fundamental methods for numerical integration and estimate of definite integrals that are now known as Simpson's law. (1710-1761). But J. Kepler utilized an identical approximation over a century before, which is because it is often referred to as Kepler's law. Estimates based just on a three-step quadratic kernel are often referred to as Newton-type results as Simpson's method utilizes the three-point Newton-Cotes quadrature rule. Let $\mathcal{F}:[\Phi,\Omega]\to\mathbb{R}$ be a continuous function on a finite interval $[\Phi,\Omega]$ where $\Phi,\Omega\in\mathbb{R}$ and $\Phi<\Omega$. Then the following Simpson's quadrature formula holds:

(1) Simpson's quadrature formula (Simpson's 1/3):

$$\int_{\Phi}^{\Omega}\mathcal{F}(z)\,dz\approx\frac{\Omega-\Phi}{6}\left[\mathcal{F}(\Phi)+4\mathcal{F}\left(\frac{\Phi+\Omega}{2}\right)+\mathcal{F}(\Omega)\right].$$

(2) Simpson's second formula or Newton-Cotes quadrature formula (Simpson's 3/8):

$$\int_{\Phi}^{\Omega}\mathcal{F}(z)\,dz\approx\frac{\Omega-\Phi}{8}\left[\mathcal{F}(\Phi)+3\mathcal{F}\left(\frac{2\Phi+\Omega}{2}\right)+3\mathcal{F}\left(\frac{\Phi+2\Omega}{2}\right)+\mathcal{F}(\Omega)\right].$$

Theorem 1.1 ([10])**.** *Let $\mathcal{F}:[\Phi,\Omega]\to\mathbb{R}$ be a four times continuously differentiable function on (Φ,Ω) and $\left\|\mathcal{F}^{(4)}\right\|_{\infty}:=\sup_{z\in(\Phi,\Omega)}\left|\mathcal{F}^{(4)}\right|<\infty$, then following inequality holds:*

$$\left|\frac{F(\Phi)+F(\Omega)}{6}+\frac{2}{3}F\left(\frac{\Phi+\Omega}{2}\right)-\frac{1}{\Omega-\Phi}\int_{\Phi}^{\Omega}F(z)\,dz\right|$$
$$\leq\frac{1}{2880}\left\|F^{(4)}\right\|_{\infty}(\Omega-\Phi)^{4}.$$

The exploration of Simpson's type inequality in various mappings has indeed garnered attention from several authors in recent years. Several authors focusing on the results of Simpson's and Newton's type inequalities to derive convex mappings highlights the utility of convexity theory in solving a broad spectrum of mathematical problems. In particular, Dragomir *et al.* [10] introduced the most recent Simpson's inequalities and their applications in quadrature formulas. Additionally, Alomari *et al.* [2] established some of Simpson's type inequalities for s-convex functions. Sarikaya *et al.* [25] then identified the significance of the dependence of the variance of the Simpson's type inequality on convexity. For harmonic convex, and p-harmonic convex maps the authors established Newton's type inequality in [5]. A novel generalized, convex Newton-type inequality for functions with the local fractional derivative was described by Iftikhar *et al.* [13].

Theorem 1.2 ([16])**.** *Let $F:[\Phi,\Omega]\to\mathbb{R}$ be a differentiable function whose derivative is continuous on (Φ,Ω) and $F'\in L_1[\Phi,\Omega]$ (the set of all Lebesgue integrable functions on a finite interval $[\Phi,\Omega]$), then the following inequality holds:*

$$\left|\left[\frac{F(\Phi)+F(\Omega)}{6}+\frac{2}{3}F\left(\frac{\Phi+\Omega}{2}\right)\right]-\frac{1}{\Omega-\Phi}\int_{\Phi}^{\Omega}F(z)\,dz\right|\leq\frac{\Omega-\Phi}{3}\left\|F'\right\|_{1},$$

where $\|F'\|_1:=\int_{\Phi}^{\Omega}|F'(z)|\,dz$.

Additionally, [10] provided the Simpson's type inequality that is given below.

Theorem 1.3. *Let $F:[\Phi,\Omega]\to\mathbb{R}$ be an absolutely continuous function on (Φ,Ω) whose derivative belongs to $L_p[\Phi,\Omega]$, where $\frac{1}{p}+\frac{1}{q}=1$ and $q>1$, then the following inequality holds:*

$$\left|\frac{F(\Phi)+F(\Omega)}{6}+\frac{2}{3}F\left(\frac{\Phi+\Omega}{2}\right)-\frac{1}{\Omega-\Phi}\int_{\Phi}^{\Omega}F(z)\,dz\right|$$
$$\leq\frac{1}{6}\left(\frac{1+2^{q+1}}{3(q+1)}\right)^{\frac{1}{q}}(\Omega-\Phi)^{\frac{1}{q}}\left\|F'\right\|_{p}.$$

Definition 1.4 ([15])**.** A function $F:I\subseteq\mathbb{R}\to\mathbb{R}$ is called convex on I, if

$$F(\Upsilon\Phi+(1-\Upsilon)\Omega)\leq\Upsilon F(\Phi)+(1-\Upsilon)F(\Omega),$$

for all $\Phi, \Omega \in I$ and $\Upsilon \in [0,1]$.

Hudzik in [12], introduced and investigated the concept of s-convexity.

Definition 1.5. A function $\digamma : I \subset [0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense on I, if

$$\digamma\left(\Upsilon\Phi + (1-\Upsilon)\Omega\right) \leq \Upsilon^s \digamma(\Phi) + (1-\Upsilon)^s \digamma(\Omega),$$

holds for all $\Phi, \Omega \in I$ and $\Upsilon \in [0,1]$, for some fixed $s \in (0,1]$.

Definition 1.6 ([20])**.** A function $\digamma : I \subset [0,\infty) \to \mathbb{R}$ is said to be (s,m)-convex in the second sense on I, if

$$\digamma\left(\Upsilon\Phi + m(1-\Upsilon)\Omega\right) \leq \Upsilon^s \digamma(\Phi) + m(1-\Upsilon)^s \digamma(\Omega),$$

holds for all $\Phi, \Omega \in I$ and $\Upsilon \in [0,1]$, for some fixed $s \in (0,1]$, and $m \in (0,1]$.

Definition 1.7 ([18])**.** Suppose that $\digamma \in L_1[\Phi,\Omega]$. The left and right-sided Riemann-Liouville fractional integrals of order $\rho > 0$ are defined by:

$$I^{\rho}_{\Phi^+}\digamma(\Upsilon) = \frac{1}{\Gamma(\rho)}\int_{\Phi}^{z}(z-\Upsilon)^{\rho-1}\digamma(\Upsilon)\,d\Upsilon, \quad z > \Phi,$$

$$I^{\rho}_{\Omega^-}\digamma(\Upsilon) = \frac{1}{\Gamma(\rho)}\int_{z}^{\Omega}(\Upsilon-z)^{\rho-1}\digamma(\Upsilon)\,d\Upsilon, \quad z < \Omega.$$

where $\Gamma(\rho)$ is the gamma function and $I^{0}_{\Phi^+}\digamma(\Upsilon) = I^{0}_{\Omega^-}\digamma(\Upsilon) = \digamma(\Upsilon)$.

Definition 1.8 ([9])**.** Let $\digamma \in H^1(\Phi,\Omega)$ (Sobolev space), $0 < \Phi < \Omega$, for all $\rho \in [0,1]$, where $\beta(\rho) > 0$ is a normalizer function satisfying $\beta(0) = \beta(1) = 1$, then the left and right Caputo-Fabrizio fractional integrals are defined by:

$$\left({}^{CF}_{\Phi}I^{\rho}\digamma\right)(z) = \frac{1-\rho}{\beta(\rho)}\digamma(z) + \frac{\rho}{\beta(\rho)}\int_{\Phi}^{z}\digamma(\Upsilon)\,d\Upsilon, \quad z > \Phi,$$

$$\left({}^{CF}I^{\rho}_{\Omega}\digamma\right)(z) = \frac{1-\rho}{\beta(\rho)}\digamma(z) + \frac{\rho}{\beta(\rho)}\int_{z}^{\Omega}\digamma(\Upsilon)\,d\Upsilon, \quad z < \Omega.$$

In this article, the primary objective is to establish a new integral identity utilizing the Caputo-Fabrizio fractional integral operators. This identity will serve as a foundational tool for generalizing the novel Simpson's type inequality through (s,m)-convex functions. Furthermore, some applications will be explores in different directions.

2. Main Results

In this section, we use Peano kernel to present the Simpson's type integral identity for twice differential functions, which is required to prove our main results.

Lemma 2.1. *Let $[m\Phi,\Omega] \subset I \subset [0,+\infty)$ and $F : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I^o (the interior set of I), where $\Phi,\Omega \in I$ with $\Phi < \Omega$ and $m \in (0,1]$. If $F'' \in L_1[m\Phi,\Omega]$ for all $k \in (m\Phi,\Omega)$, then the following equality holds:*

$$\frac{\beta(\rho)}{\rho(\Omega-m\Phi)}\left[\left({}^{CF}_{m\Phi}I^{\rho}F\right)(k)+\left({}^{CF}I^{\rho}_{\Omega}F\right)(k)\right]-\frac{2(1-\rho)}{\rho(\Omega-m\Phi)}F(k)$$
$$-\frac{1}{6}\left[F(m\Phi)+4F\left(\frac{m\Phi+\Omega}{2}\right)+F(\Omega)\right]$$
$$=(\Omega-m\Phi)^2\int_0^1 K(\Upsilon)\,F''(\Upsilon\Omega+m(1-\Upsilon)\Phi)\,d\Upsilon,$$

where

$$K(\Upsilon):=\begin{cases}\frac{1}{6}\,\Upsilon(3\Upsilon-1), & \Upsilon\in\left[0,\frac{1}{2}\right],\\ \frac{1}{6}\,(\Upsilon-1)(3\Upsilon-2), & \Upsilon\in\left(\frac{1}{2},1\right].\end{cases}$$

Proof. Let us consider

$$I=(\Omega-m\Phi)^2\int_0^1 K(\Upsilon)\,F''(\Upsilon\Omega+m(1-\Upsilon)\Phi)\,d\Upsilon$$
$$=\int_0^{\frac{1}{2}}\frac{1}{6}\,\Upsilon(3\Upsilon-1)\,F''(\Upsilon\Omega+m(1-\Upsilon)\Phi)\,d\Upsilon$$
$$+\int_{\frac{1}{2}}^1\frac{1}{6}\,(\Upsilon-1)(3\Upsilon-2)\,F''(\Upsilon\Omega+m(1-\Upsilon)\Phi)\,d\Upsilon$$
$$=(\Omega-m\Phi)^2(I_1+I_2).$$

Integration by parts, we have

$$I_1=\int_0^{\frac{1}{2}}\frac{1}{6}\,\Upsilon(3\Upsilon-1)F''(\Upsilon\Omega+m(1-\Upsilon)\Phi)\,d\Upsilon \tag{2.1}$$
$$=\frac{1}{6}\int_0^{\frac{1}{2}}(3\Upsilon^2-\Upsilon)F''(\Upsilon\Omega+m(1-\Upsilon)\Phi)\,d\Upsilon$$
$$=\frac{1}{6}\left[\left.\frac{(3\Upsilon^2-\Upsilon)F'(\Upsilon\Omega+m(1-\Upsilon)\Phi)}{\Omega-m\Phi}\right|_0^{\frac{1}{2}}\right.$$

$$-\frac{1}{\Omega - m\Phi}\int_0^{\frac{1}{2}}(6\Upsilon - 1)F'(\Upsilon\Omega + m(1-\Upsilon)\Phi)\,d\Upsilon\Bigg]$$

$$= \frac{1}{24(\Omega - m\Phi)}F'\left(\frac{m\Phi+\Omega}{2}\right)$$

$$- \frac{1}{6(\Omega - m\Phi)}\int_0^{\frac{1}{2}}(6\Upsilon - 1)F'(\Upsilon\Omega + m(1-\Upsilon)\Phi)\,d\Upsilon$$

$$= \frac{1}{24(\Omega - m\Phi)}F'\left(\frac{m\Phi+\Omega}{2}\right) - \frac{1}{3(\Omega - m\Phi)^2}F\left(\frac{m\Phi+\Omega}{2}\right)$$

$$- \frac{1}{6(\Omega - m\Phi)^2}F(m\Phi) + \frac{1}{(\Omega - m\Phi)^3}\int_{m\Phi}^{\frac{m\Phi+\Omega}{2}}F(u)\,du.$$

Similarly, we get

$$I_2 = \int_{\frac{1}{2}}^{1}\frac{1}{6}(\Upsilon - 1)(3\Upsilon - 2)F''(\Upsilon\Omega + m(1-\Upsilon)\Phi)\,d\Upsilon \tag{2.2}$$

$$= \frac{1}{6}\left[\frac{(3\Upsilon^2 - 5\Upsilon + 2)F'(\Upsilon\Omega + m(1-\Upsilon)\Phi)}{\Omega - m\Phi}\Bigg|_{\frac{1}{2}}^{1}\right.$$

$$\left.-\frac{1}{\Omega - m\Phi}\int_{\frac{1}{2}}^{1}(6\Upsilon - 5)F'(\Upsilon\Omega + m(1-\Upsilon)\Phi)\,d\Upsilon\right]$$

$$= \frac{-1}{24(\Omega - m\Phi)}F'\left(\frac{m\Phi+\Omega}{2}\right) - \frac{1}{3(\Omega - m\Phi)^2}F\left(\frac{m\Phi+\Omega}{2}\right)$$

$$- \frac{1}{6(\Omega - m\Phi)^2}F(\Omega) + \frac{1}{(\Omega - m\Phi)^3}\int_{\frac{m\Phi+\Omega}{2}}^{\Omega}F(u)\,du.$$

Adding the equalities (2.1) and (2.2) and multiplying $(\Omega - m\Phi)^2$, we obtain

$$(\Omega - m\Phi)^2(I_1 + I_2) \tag{2.3}$$

$$= \frac{\Omega - m\Phi}{24}F'\left(\frac{m\Phi+\Omega}{2}\right) - \frac{1}{3}F\left(\frac{m\Phi+\Omega}{2}\right) - \frac{1}{6}F(m\Phi)$$

$$+ \frac{1}{\Omega - m\Phi}\int_{m\Phi}^{\frac{m\Phi+\Omega}{2}}F(u)\,du - \frac{\Omega - m\Phi}{24}F'\left(\frac{m\Phi+\Omega}{2}\right)$$

$$- \frac{1}{3}F\left(\frac{m\Phi+\Omega}{2}\right) - \frac{1}{6}F(\Omega) + \frac{1}{\Omega - m\Phi}\int_{\frac{m\Phi+\Omega}{2}}^{\Omega}F(u)\,du$$

$$= -\frac{1}{6}\left[F(m\Phi) + 4F\left(\frac{m\Phi+\Omega}{2}\right) + F(\Omega)\right]$$

$$+\frac{1}{\Omega-m\Phi}\int_{m\Phi}^{\Omega}F(u)\,du.$$

Adding both sides $\frac{2(1-\rho)}{\rho(\Omega-\Phi)}F(k)$ with the equality (2.3), we obtain

$$\begin{aligned}
&(\Omega-m\Phi)^2(I_1+I_2)+\frac{2(1-\rho)}{\rho(\Omega-\Phi)}F(k)\\
&=-\frac{1}{6}F(m\Phi)-\frac{2}{3}F\left(\frac{m\Phi+\Omega}{2}\right)-\frac{1}{6}F(\Omega)\\
&\quad+\frac{1}{\Omega-m\Phi}\int_{m\Phi}^{\Omega}F(u)\,du+\frac{2(1-\rho)}{\rho(\Omega-\Phi)}F(k)\\
&=-\frac{1}{6}F(m\Phi)-\frac{2}{3}F\left(\frac{m\Phi+\Omega}{2}\right)-\frac{1}{6}F(\Omega)\\
&\quad+\frac{\beta(\rho)}{\rho(\Omega-m\Phi)}\left[\frac{\rho}{\beta(\rho)}\int_{m\Phi}^{k}F(u)\,du+\frac{1-\rho}{\beta(\rho)}F(k)\right.\\
&\quad\left.+\frac{\rho}{\beta(\rho)}\int_{k}^{\Omega}F(u)\,du+\frac{1-\rho}{\beta(\rho)}F(k)\right]\\
&=-\frac{1}{6}F(m\Phi)-\frac{2}{3}F\left(\frac{m\Phi+\Omega}{2}\right)-\frac{1}{6}F(\Omega)\\
&\quad+\frac{\beta(\rho)}{\rho(\Omega-m\Phi)}\left[\left({}_{m\Phi}^{CF}I^{\rho}F\right)(k)+\left({}^{CF}I_{\Omega}^{\rho}F\right)(k)\right].
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&(\Omega-m\Phi)^2\int_0^1 K(\Upsilon)F''(\Upsilon\Omega+m(1-\Upsilon)\Phi)\,d\Upsilon\\
&=\frac{\beta(\rho)}{\rho(\Omega-m\Phi)}\left[\left({}_{m\Phi}^{CF}I^{\rho}F\right)(k)+\left({}^{CF}I_{\Omega}^{\rho}F\right)(k)\right]-\frac{2(1-\rho)}{\rho(\Omega-m\Phi)}F(k)\\
&\quad-\frac{1}{6}\left[F(m\Phi)+4F\left(\frac{m\Phi+\Omega}{2}\right)+F(\Omega)\right].
\end{aligned}$$

The proof of Lemma 2.1 is completed. □

Theorem 2.2. *Suppose that all the assumptions of Lemma 2.1 are satisfied. If $|F''|$ is (s,m)-convex function on $[\Phi,\Omega]$, for some fixed $s\in(0,1]$ and $m\in(0,1]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{\beta(\rho)}{\rho(\Omega-m\Phi)}\left[\left({}_{m\Phi}^{CF}I^{\rho}F\right)(k)+\left({}^{CF}I_{\Omega}^{\rho}F\right)(k)\right]-\frac{2(1-\rho)}{\rho(\Omega-m\Phi)}F(k)\right.\\
&\quad\left.-\frac{1}{6}\left[F(m\Phi)+4F\left(\frac{m\Phi+\Omega}{2}\right)+F(\Omega)\right]\right|
\end{aligned}$$

$$\leq \frac{\Omega - m\Phi}{6}\left[Z_3\left(m\left|F''(\Phi)\right| + \left|F''(\Omega)\right|\right)\right].$$

where

$$\begin{aligned}
Z_3 :=& \int_0^{\frac{1}{2}} |\Upsilon(3\Upsilon-1)|\,\Upsilon^s d\Upsilon + \int_{\frac{1}{2}}^1 |(\Upsilon-1)(3\Upsilon-2)|\,\Upsilon^s d\Upsilon \\
&+ m\int_0^{\frac{1}{2}} |\Upsilon(3\Upsilon-1)|\,(1-\Upsilon)^s d\Upsilon \\
&+ m\int_{\frac{1}{2}}^1 |(\Upsilon-1)(3\Upsilon-2)|\,(1-\Upsilon)^s d\Upsilon \\
=& \Bigg[\Bigg\{\frac{1}{(s+3)3^{s+2}} - \frac{1}{2^{s+2}(s+3)} - \frac{1}{2^{s+2}(s+2)} + \frac{1}{3^{s+2}(s+2)} \\
&+ \frac{3}{2^{s+3}(s+3)} - \frac{1}{3^{s+2}(s+3)} - 2\left(\frac{2^{s+1}}{3^{s+1}(s+1)} - \frac{1}{2^{s+1}(s+1)}\right) \\
&+ 5\left(\frac{2^{s+2}}{3^{s+2}(s+2)} - \frac{1}{2^{s+2}(s+2)}\right) - 3\left(\frac{2^{s+3}}{3^{s+3}(s+3)} - \frac{1}{2^{s+3}(s+3)}\right)\Bigg\} \\
&+ m\Bigg\{\frac{1}{3(s+3)} + \frac{2^{s+3}}{3^{s+2}(s+3)} + 5\left(\frac{1}{s+1} - \frac{2^{s+2}}{3^{s+2}(s+1)}\right) \\
&- 2\left(\frac{1}{s+1} - \frac{2^{s+1}}{3^{s+1}(s+1)}\right) + \frac{3\times 2^{s+3}}{3^{s+3}(s+3)} - \frac{3}{2^{s+3}(s+3)} \\
&- 5\left(\frac{2^{s+2}}{3^{s+2}(s+2)} - \frac{1}{2^{s+2}(s+2)}\right) + \left(\frac{2^{s+1}}{3^{s+1}(s+1)} - \frac{1}{2^{s+1}(s+1)}\right) \\
&- \frac{1}{3^{s+2}(s+3)(s+2)}\left(\frac{2^{s+2}(s+2)+1}{2^{s+3}} - \frac{3^{s+2}(s+2)}{2^{s+2}}\right. \\
&\left. + \frac{1}{(s+3)3^{s+3}} - \frac{1}{3^{s+2}(s+2)}\right)\Bigg\}\Bigg].
\end{aligned}$$

Proof. By using the Lemma 2.1, since $|F''|$ is (s,m)-convex, we have

$$\begin{aligned}
&\Bigg|\frac{\beta(\rho)}{\rho(\Omega - m\Phi)}\left[\left({}^{CF}_{m\Phi}I^{\rho}F\right)(k) + \left({}^{CF}I^{\rho}_{\Omega}F\right)(k)\right] - \frac{2(1-\rho)}{\rho(\Omega - m\Phi)}F(k) \\
&\quad - \frac{1}{6}\left[F(m\Phi) + 4F\left(\frac{m\Phi+\Omega}{2}\right) + F(\Omega)\right]\Bigg| \\
&\leq (\Omega - m\Phi)^2\int_0^1 |K(\Upsilon)|\left|F''(\Upsilon\Omega + m(1-\Upsilon)\Phi)\right| d\Upsilon
\end{aligned}$$

$$\leq \frac{(\Omega - m\Phi)^2}{6}\left[\int_0^{1/2} \left|\Upsilon(3\Upsilon-1)\right| \left(\Upsilon^s \left|F''(\Omega)\right| + m(1-\Upsilon)^s \left|F''(\Phi)\right|\right) d\Upsilon\right.$$
$$\left. + \int_{1/2}^1 \left|(\Upsilon-1)(3\Upsilon-2)\right| \left(\Upsilon^s \left|F''(\Omega)\right| + m(1-\Upsilon)^s \left|F''(\Phi)\right|\right) d\Upsilon\right]$$
$$= \frac{\Omega - m\Phi}{6}\left[Z_3\left(m\left|F''(\Phi)\right| + \left|F''(\Omega)\right|\right)\right].$$

This completes the proof. □

Corollary 2.3. *If we choose $m = s = 1$ in Theorem 2.2, then we have*

$$\left|\frac{\beta(\rho)}{\rho(\Omega-\Phi)}\left[\left({}^{CF}_{\Phi}I^{\rho}F\right)(k) + \left({}^{CF}I^{\rho}_{\Omega}F\right)(k)\right] - \frac{2(1-\rho)}{\rho(\Omega-\Phi)}F(k)\right.$$
$$\left. - \frac{1}{6}\left[F(\Phi) + 4F\left(\frac{\Phi+\Omega}{2}\right) + F(\Omega)\right]\right|$$
$$\leq \frac{(\Omega-\Phi)^2}{162}\left[\left|F''(\Phi)\right| + \left|F''(\Omega)\right|\right].$$

Corollary 2.4. *If we choose $F(\Phi) = F\left(\frac{\Phi+\Omega}{2}\right) = F(\Omega)$ in Corollary 2.3, then we get*

$$\left|F\left(\frac{\Phi+\Omega}{2}\right) - \frac{\beta(\rho)}{\rho(\Omega-\Phi)}\left[\left({}^{CF}_{\Phi}I^{\rho}F\right)(k) + \left({}^{CF}I^{\rho}_{\Omega}F\right)(k)\right] - \frac{2(1-\rho)}{\rho(\Omega-\Phi)}F(k)\right|$$
$$\leq \frac{(\Omega-\Phi)^2}{162}\left[\left|F''(\Phi)\right| + \left|F''(\Omega)\right|\right].$$

Remark 2.5. If we choose $\rho = 1$ and $\beta(0) = \beta(1) = 1$ in Corollary 2.3, then we get

$$\left|\frac{1}{6}\left[F(\Phi) + 4F\left(\frac{\Phi+\Omega}{2}\right) + F(\Omega)\right] - \frac{1}{\Omega-\Phi}\int_\Phi^\Omega F(z)\,dz\right|$$
$$\leq \frac{(\Omega-\Phi)^2}{162}\left[\left|F''(\Phi)\right| + \left|F''(\Omega)\right|\right].$$

Remark 2.5 was proved by Sarikaya *et al.* in [24, Corollary 2.3].

Theorem 2.6. *Assume that all the assumptions of Lemma 2.1 are satisfied. If $|F''|^q$ is (s,m)-convex function on $[\Phi, \Omega]$ for some fixed $s \in (0,1]$, and $m \in (0,1]$ with $q \geq 1$, then the following inequality holds:*

$$\left|\frac{\beta(\rho)}{\rho(\Omega-m\Phi)}\left[\left({}^{CF}_{m\Phi}I^{\rho}F\right)(k) + \left({}^{CF}I^{\rho}_{\Omega}F\right)(k)\right] - \frac{2(1-\rho)}{\rho(\Omega-m\Phi)}F(k)\right.$$

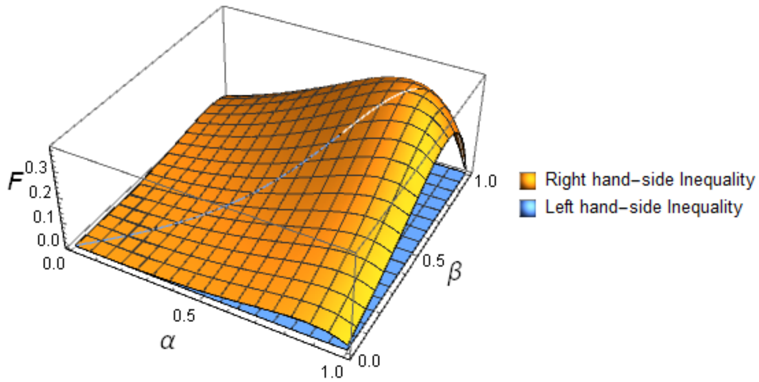

FIGURE 1. Graphical representation of the error bound for Corollary 2.4, let $F(x) = x^2, \Phi, \Omega \in [0,1]$ and $\rho = 1$ and $\beta(0) = \beta(1) = 1$, where the left side of the inequality is depicted in Sky blue and the right side in yellow.

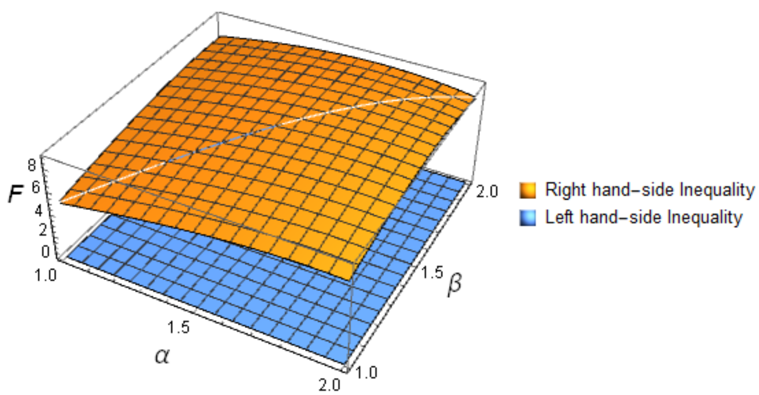

FIGURE 2. Graphical representation of the error bound for Corollary 2.3, let $F(x) = x^2, \Phi, \Omega \in [1,2]$ and $\rho = 1$ and $\beta(0) = \beta(1) = 1$, where the left side of the inequality is depicted in Sky blue and the right side in yellow.

$$- \frac{1}{6}\left[F(m\Phi) + 4F\left(\frac{m\Phi+\Omega}{2}\right) + F(\Omega)\right]\Bigg|$$

$$\leq (\Omega - m\Phi)^2 \left(\frac{1}{162}\right)^{1-\frac{1}{q}} (Z_5 + Z_6)\left[m\left|F''(\Phi)\right|^q + \left|F''(\Omega)\right|^q\right]^{\frac{1}{q}}.$$

where

$$Z_5 := \int_0^{\frac{1}{2}} |\Upsilon(3\Upsilon - 1)|\, \Upsilon^s d\Upsilon + m\int_0^{\frac{1}{2}} |\Upsilon(3\Upsilon-1)|\,(1-\Upsilon)^s d\Upsilon$$

$$= \Bigg[\Bigg\{\frac{3^{-s-2}-2^{-s-2}+3^{-s-2}}{s+2}+\frac{2^{-s-2}+3\cdot 2^{-s-3}-3^{-s-2}}{s+3}\Bigg\} + m\Bigg\{\frac{2^{s+3}\cdot 3^{-s-2}-3\cdot 2^{-s-3}}{s+3}+\frac{5\cdot 2^{-s-2}-5\cdot 2^{s+2}}{s+2} + \frac{2^{s+2}\cdot 3^{-s-1}-2^{-s-1}}{s+1}\Bigg\}\Bigg],$$

and

$$Z_6 := \int_{\frac{1}{2}}^{1}|(\Upsilon-1)(3\Upsilon-2)|\,\Upsilon^s d\Upsilon + m\int_{\frac{1}{2}}^{1}|(\Upsilon-1)(3\Upsilon-2)|\,(1-\Upsilon)^s d\Upsilon$$

$$= \Bigg[\Bigg\{\frac{2^{s+3}\cdot 3^{-s-2}-3\cdot 2^{-s-3}}{s+3}+\frac{5\cdot 2^{-s-2}-5\cdot 2^{s+2}}{s+2} + \frac{2^{s+2}\cdot 3^{-s-1}-2^{-s-1}}{s+1}\Bigg\} + m\Bigg\{\frac{3^{-s-2}-2^{-s-2}+3^{-s-2}}{s+2}+\frac{2^{-s-2}+3\cdot 2^{-s-3}-3^{-s-2}}{s+3}\Bigg\}\Bigg].$$

Proof. By using the Lemma 2.1, with the help of power-mean inequality and (s,m)-convexity of $|F''|^q$, we have

$$\Bigg|\frac{\beta(\rho)}{\rho(\Omega-m\Phi)}\left[\left({}^{CF}_{m\Phi}I^{\rho}F\right)(k)+\left({}^{CF}I^{\rho}_{\Omega}F\right)(k)\right]-\frac{2(1-\rho)}{\rho(\Omega-m\Phi)}F(k) -\frac{1}{6}\left[F(m\Phi)+4F\left(\frac{m\Phi+\Omega}{2}\right)+F(\Omega)\right]\Bigg|$$

$$= (\Omega-m\Phi)^2\int_0^1\big|K(\Upsilon)\big|\big|F''(\Upsilon\Omega+m(1-\Upsilon)\Phi)\big|\,d\Upsilon$$

$$\le (\Omega-m\Phi)^2\left[\int_0^{\frac{1}{2}}\Big|\tfrac{1}{6}\Upsilon(3\Upsilon-1)\Big|\big|F''(\Upsilon\Omega+m(1-\Upsilon)\Phi)\big|\,d\Upsilon + \int_{\frac{1}{2}}^{1}\Big|\tfrac{1}{6}(\Upsilon-1)(3\Upsilon-2)\Big|\big|F''(\Upsilon\Omega+m(1-\Upsilon)\Phi)\big|\,d\Upsilon\right]$$

$$\le (\Omega-m\Phi)^2\Bigg[\left(\int_0^{\frac{1}{2}}\Big|\tfrac{1}{6}\Upsilon(3\Upsilon-1)\Big|d\Upsilon\right)^{1-\frac{1}{q}}$$

$$\times \left(\int_0^{\frac{1}{2}} \left| \tfrac{1}{6} \Upsilon (3\Upsilon - 1) \right| \left(\Upsilon^s |F''(\Omega)|^q + m(1-\Upsilon)^s |F''(\Phi)|^q \right) d\Upsilon \right)^{\frac{1}{q}}$$

$$+ \left(\int_{\frac{1}{2}}^1 \left| \tfrac{1}{6} (\Upsilon - 1)(3\Upsilon - 2) \right| d\Upsilon \right)^{1-\frac{1}{q}}$$

$$\times \left(\int_{\frac{1}{2}}^1 \left| \tfrac{1}{6} (\Upsilon - 1)(3\Upsilon - 2) \right| \left(\Upsilon^s |F''(\Omega)|^q + m(1-\Upsilon)^s |F''(\Phi)|^q \right) d\Upsilon \right)^{\frac{1}{q}} \Bigg]$$

$$= (\Omega - m\Phi)^2 \left(\tfrac{1}{162}\right)^{1-\frac{1}{q}} (Z_5 + Z_6) \left[m|F''(\Phi)|^q + |F''(\Omega)|^q \right]^{\frac{1}{q}}.$$

This completes the proof. □

Corollary 2.7. *If we choose $m = 1$ in Theorem 2.6, then we have*

$$\Bigg| \frac{\beta(\rho)}{\rho(\Omega - \Phi)} \left[{}^{CF}_{\Phi} I^{\rho} F(k) + {}^{CF} I^{\rho}_{\Omega} F(k) \right] - \frac{2(1-\rho)}{\rho(\Omega - \Phi)} F(k) - \frac{1}{6} \left[F(\Phi) + 4F\left(\frac{\Phi + \Omega}{2}\right) + F(\Omega) \right] \Bigg|$$

$$\leq (\Omega - \Phi)^2 \left(\frac{1}{162} \right)^{1-\frac{1}{q}} (Z_5 + Z_6) \left[\left|F''(\Phi)\right|^q + \left|F''(\Omega)\right|^q \right]^{\frac{1}{q}}.$$

Corollary 2.8. *If we choose $F(\Phi) = F\left(\frac{\Phi+\Omega}{2}\right) = F(\Omega)$ in Corollary 2.7, then we get*

$$\left| F\left(\frac{\Phi + \Omega}{2}\right) - \frac{\beta(\rho)}{\rho(\Omega - \Phi)} \left[{}^{CF}_{\Phi} I^{\rho} F(k) + {}^{CF} I^{\rho}_{\Omega} F(k) \right] - \frac{2(1-\rho)}{\rho(\Omega - \Phi)} F(k) \right|$$

$$\leq (\Omega - \Phi)^2 \left(\frac{1}{162} \right)^{1-\frac{1}{q}} (Z_5 + Z_6) \left[\left|F''(\Phi)\right|^q + \left|F''(\Omega)\right|^q \right]^{\frac{1}{q}}.$$

Remark 2.9. If we choose $q = 1$ in Theorem 2.6, then Theorem 2.6 reduces to Theorem 2.2.

Corollary 2.10. *If we choose $s = m = 1$ in Theorem 2.6, then we obtain*

$$\Bigg| \frac{\beta(\rho)}{\rho(\Omega - \Phi)} \left[{}^{CF}_{\Phi} I^{\rho} F(k) + {}^{CF} I^{\rho}_{\Omega} F(k) \right] - \frac{2(1-\rho)}{\rho(\Omega - \Phi)} F(k) - \frac{1}{6} \left[F(\Phi) + 4F\left(\frac{\Phi + \Omega}{2}\right) + F(\Omega) \right] \Bigg|$$

$$\leq \frac{(\Omega-\Phi)^2}{162}\left[\frac{59\,|F''(\Phi)|^q+133\,|F''(\Omega)|^q}{3\cdot 2^6}+\frac{133\,|F''(\Phi)|^q+59\,|F''(\Omega)|^q}{3\cdot 2^6}\right]^{\frac{1}{q}}.$$

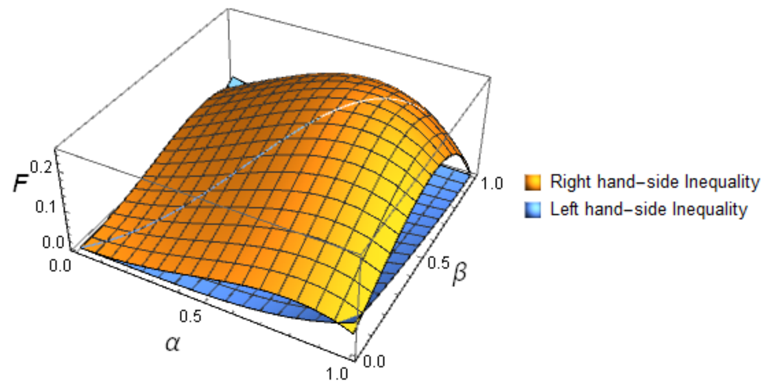

FIGURE 3. Graphical representation of the error bound for Corollary 2.7, let $F(x)=x^3, \Phi,\Omega\in[0,1]$ and $\rho=1$ and $\beta(0)=\beta(1)=1$, where the left side of the inequality is depicted in Sky blue and the right side in yellow.

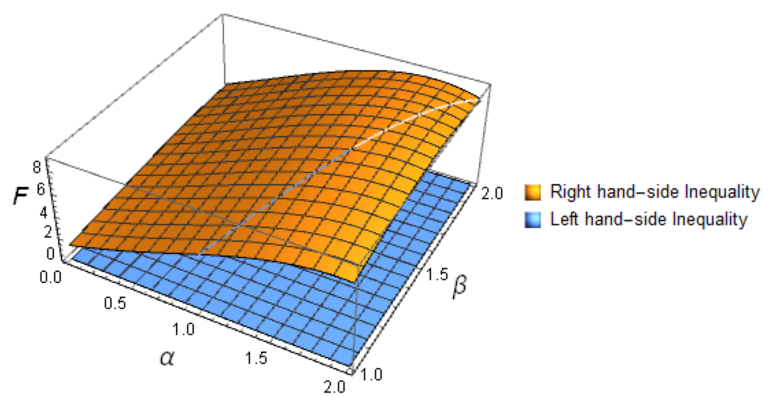

FIGURE 4. Graphical representation of the error bound for Corollary 2.8, let $F(x)=x^3, \Phi,\Omega\in[1,2]$ and $\rho=1$ and $\beta(0)=\beta(1)=1$, where the left side of the inequality is depicted in sky blue and the right side in yellow.

Remark 2.11. If we choose $\rho=1$ and $\beta(0)=\beta(1)=1$ in Theorem 2.2, then we get

$$\left|\frac{1}{6}\left[F(\Phi)+4F\left(\frac{\Phi+\Omega}{2}\right)+F(\Omega)\right]-\frac{1}{\Omega-\Phi}\int_{\Phi}^{\Omega}F(z)\,dz\right|$$

$$\leq \frac{(\Omega-\Phi)^2}{162}\left[\frac{59\left|F''(\Phi)\right|^q+133\left|F''(\Omega)\right|^q}{3\cdot 2^6}+\frac{133\left|F''(\Phi)\right|^q+59\left|F''(\Omega)\right|^q}{3\cdot 2^6}\right]^{\frac{1}{q}}.$$

Remark 2.11 was proved by Sarikaya *et al.* in [24, Remark 2.7].

Theorem 2.12. *Under the assumptions of Lemma 2.1, if $|F''|^q$ is concave function on $[\Phi,\Omega]$ and $q\geq 1$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{\beta(\rho)}{\rho(\Omega-m\Phi)}\left[{}^{CF}_{m\Phi}I^{\rho}F(k)+{}^{CF}I^{\rho}_{\Omega}F(k)\right]-\frac{2(1-\rho)}{\rho(\Omega-m\Phi)}F(k)\right.\\&\left.\quad-\frac{1}{6}\left[F(m\Phi)+4F\left(\frac{m\Phi+\Omega}{2}\right)+F(\Omega)\right]\right|\\&\quad\leq\frac{(\Omega-m\Phi)^2}{162}\left\{\left|F''\left(\frac{133m\Phi+59\Omega}{192}\right)\right|+\left|F''\left(\frac{59m\Phi+133\Omega}{192}\right)\right|\right\}.\end{aligned}$$

Proof. By using the Lemma 2.1, we have

$$\begin{aligned}&\left|\frac{\beta(\rho)}{\rho(\Omega-m\Phi)}\left[{}^{CF}_{m\Phi}I^{\rho}F(k)+{}^{CF}I^{\rho}_{\Omega}F(k)\right]-\frac{2(1-\rho)}{\rho(\Omega-m\Phi)}F(k)\right.\\&\left.\quad-\frac{1}{6}\left[F(m\Phi)+4F\left(\frac{m\Phi+\Omega}{2}\right)+F(\Omega)\right]\right|\\&\leq(\Omega-m\Phi)^2\int_0^1|K(\Upsilon)|\left|F''(\Upsilon\Omega+m(1-\Upsilon)\Phi)\right|d\Upsilon\\&=(\Omega-m\Phi)^2\left\{\int_0^{\frac{1}{2}}\left|\frac{1}{6}\Upsilon(3\Upsilon-1)\right|\left|F''(\Upsilon\Omega+m(1-\Upsilon)\Phi)\right|d\Upsilon\right.\\&\quad\left.+\int_{\frac{1}{2}}^1\left|\frac{1}{6}(\Upsilon-1)(3\Upsilon-2)\right|\left|F''(\Upsilon\Omega+m(1-\Upsilon)\Phi)\right|d\Upsilon\right\}.\end{aligned}\tag{2.4}$$

By Jensen integral inequality, we obtain

$$\begin{aligned}&\int_0^{\frac{1}{2}}\left|\frac{1}{6}\Upsilon(3\Upsilon-1)\right|\left|F''(\Upsilon\Omega+m(1-\Upsilon)\Phi)\right|d\Upsilon\\&\leq\left(\int_0^{\frac{1}{2}}\left|\frac{1}{6}\Upsilon(3\Upsilon-1)\right|d\Upsilon\right)\\&\quad\times\left|F''\left(\frac{\int_0^{\frac{1}{2}}\left|\frac{1}{6}\Upsilon(3\Upsilon-1)\right|(\Upsilon\Omega+m(1-\Upsilon)\Phi)\,d\Upsilon}{\int_0^{\frac{1}{2}}\left|\frac{1}{6}\Upsilon(3\Upsilon-1)\right|d\Upsilon}\right)\right|\\&=\frac{1}{162}\left|F''\left(\frac{133m\Phi+59\Omega}{192}\right)\right|.\end{aligned}\tag{2.5}$$

and

$$\begin{aligned}(2.6)\quad &\int_{\frac{1}{2}}^{1}\left|\frac{1}{6}(\Upsilon-1)(3\Upsilon-2)\right|\left|F''(\Upsilon\Omega+m(1-\Upsilon)\Phi)\right|d\Upsilon\\ &\leq\left(\int_{\frac{1}{2}}^{1}\left|\frac{1}{6}(\Upsilon-1)(3\Upsilon-2)\right|d\Upsilon\right)\\ &\quad\times\left|F''\left(\frac{\int_{\frac{1}{2}}^{1}\left|\frac{1}{6}(\Upsilon-1)(3\Upsilon-2)\right|(\Upsilon\Omega+m(1-\Upsilon)\Phi)\,d\Upsilon}{\int_{\frac{1}{2}}^{1}\left|\frac{1}{6}(\Upsilon-1)(3\Upsilon-2)\right|d\Upsilon}\right)\right|\\ &=\frac{1}{162}\left|F''\left(\frac{59m\Phi+133\Omega}{192}\right)\right|.\end{aligned}$$

Using the inequalities (2.5) and (2.6) in (2.4), we have

$$\begin{aligned}&\left|\frac{\beta(\rho)}{\rho(\Omega-m\Phi)}\left[{}^{CF}_{m\Phi}I^{\rho}F(k)+{}^{CF}I^{\rho}_{\Omega}F(k)\right]-\frac{2(1-\rho)}{\rho(\Omega-m\Phi)}F(k)\right.\\ &\qquad\left.-\frac{1}{6}\left[F(m\Phi)+4F\left(\frac{m\Phi+\Omega}{2}\right)+F(\Omega)\right]\right|\\ &\quad\leq\frac{(\Omega-m\Phi)^2}{162}\left[\left|F''\left(\frac{133m\Phi+59\Omega}{192}\right)\right|+\left|F''\left(\frac{59m\Phi+133\Omega}{192}\right)\right|\right].\end{aligned}$$

This completes the proof. □

Corollary 2.13. *If we choose $m=1$ in Theorem 2.12, then we get*

$$\begin{aligned}&\left|\frac{\beta(\rho)}{\rho(\Omega-\Phi)}\left[{}^{CF}_{\Phi}I^{\rho}F(k)+{}^{CF}I^{\rho}_{\Omega}F(k)\right]-\frac{2(1-\rho)}{\rho(\Omega-m\Phi)}F(k)\right.\\ &\qquad\left.-\frac{1}{6}\left[F(\Phi)+4F\left(\frac{\Phi+\Omega}{2}\right)+F(\Omega)\right]\right|\\ &\quad\leq\frac{(\Omega-\Phi)^2}{162}\left[\left|F''\left(\frac{133\Phi+59\Omega}{192}\right)\right|+\left|F''\left(\frac{59\Phi+133\Omega}{192}\right)\right|\right].\end{aligned}$$

3. Applications

3.1. **Special Means:** First, we recall the following special means for different positive real numbers Φ and Ω where $\Phi<\Omega$:

(a) The Arithmetic mean:

$$A(\Phi,\Omega):=\frac{\Phi+\Omega}{2},$$

(b) The Harmonic mean:

$$H(\Phi,\Omega) := \frac{2}{\frac{1}{\Phi}+\frac{1}{\Omega}},$$

(c) The Logarithmic mean:

$$L(\Phi,\Omega) := \begin{cases} \Phi, & \text{if } \Phi{=}\Omega, \\ \frac{\Omega-\Phi}{\ln\Omega-\ln\Phi}, & \text{if } \Phi, \neq \Omega,\ \Phi,\Omega>0. \end{cases}$$

(d) The Generalized Logarithmic mean:

$$L_r(\Phi,\Omega) := \left[\frac{\Omega^{r+1}-\Phi^{r+1}}{(r+1)(\Omega-\Phi)}\right]^{\frac{1}{r}}, \quad r\in\mathbb{Z}\setminus\{-1,0\}.$$

Using above special means and our main results, we get the following inequalities.

Proposition 3.1. *Let* $\Phi,\Omega\in\mathbb{R}$, $0<\Phi<\Omega$, $n\in\mathbb{N}$ *and* $n\geq 2$. *Then, we have*

$$\left|\frac{1}{6}A(\Phi^n,\Omega^n)+\frac{2}{3}A^n(\Phi,\Omega)-L_n^n(\Phi,\Omega)\right| \leq \frac{n(n-1)(\Omega-\Phi)^2}{162}\left[\Phi^{n-2}+\Omega^{n-2}\right].$$

Proof. The assertion follows from Corollary 2.3 for $\digamma(z)=z^n$, $\rho=1$ and $\beta(0)=\beta(1)=1$. □

Proposition 3.2. *Let* $\Phi,\Omega\in\mathbb{R}$, $0<\Phi<\Omega$ *and* $q\geq 1$. *Then, we have*

$$\left|\frac{1}{3}H^{-1}(\Phi,\Omega)+\frac{2}{3}A(\Phi,\Omega)-L^{-1}(\Phi,\Omega)\right| \leq \frac{(\Omega-\Phi)^2}{162}\left[\left(\frac{59}{192}\left(\frac{2}{\Phi^3}\right)^q+\frac{133}{192}\left(\frac{2}{\Omega^3}\right)^q\right)^{\frac{1}{q}} + \left(\frac{133}{192}\left(\frac{2}{\Phi^3}\right)^q+\frac{59}{192}\left(\frac{2}{\Omega^3}\right)^q\right)^{\frac{1}{q}}\right].$$

Proof. The assertion follows from Corollary 2.10 for $\digamma(z)=\frac{1}{z}$, $\rho=1$ and $\beta(0)=\beta(1)=1$. □

3.2. **q-Digamma Function:** Let $0<q<1$, the q-digamma(psi) function φ_q, is the q-analogue of the digamma function q defined as [28]:

$$\varphi_q = -\ln(1-q)+\ln q\sum_{k=0}^{\infty}\frac{q^{k+z}}{1-q^{k+z}}$$

$$= -\ln(1-q) + \ln q \sum_{k=1}^{\infty} \frac{q^{kz}}{1-q^k}.$$

For $q > 1$ and $z > 0$, the q-digamma function φ_q is defined as:

$$\varphi_q = -\ln(q-1) + \ln q \left[z - \frac{1}{2} - \sum_{k=0}^{\infty} \frac{q^{-(k+z)}}{1-q^{-(k+z)}} \right]$$
$$= -\ln(q-1) + \ln q \left[z - \frac{1}{2} - \sum_{k=1}^{\infty} \frac{q^{-kz}}{1-q^{-k}} \right].$$

Proposition 3.3. *Let* $\Phi, \Omega \in \mathbb{R}$, $0 < \Phi < \Omega$ *and* $0 < q < 1$. *Then, we have*

$$\left| \frac{1}{6} \left[\varphi_q'(\Phi) + 4\varphi_q' \left(\frac{\Phi+\Omega}{2} \right) + \varphi_q'(\Omega) \right] - \frac{\varphi_q(\Omega) - \varphi_q(\Phi)}{\Omega - \Phi} \right| \le \frac{(\Omega-\Phi)^2}{162} \left[\left| \varphi_q'''(\Phi) \right| + \left| \varphi_q'''(\Omega) \right| \right]. \tag{3.1}$$

Proof. By utilizing the definition of the q-digamma function φ_q it becomes apparent that the q-trigamma function $z \to \varphi_q'(z)$ is completely monotonic on the interval $(0, \infty)$. This property ensures that the function. This property ensures that the function $\varphi_q'''(z)$ is again completely monotonic on the interval $(0, \infty)$ for each $q \in (0,1)$ and as a result, it is convex (refer to [29], p.167). Furthermore, applying Remark 2.5 allows us to deduce that the inequality (3.1) holds true for $q \in (0,1)$. □

Presently, another application of inequality (3.1), we can give the accompanying inequalities for the q-trigamma and q-polygamma functions and the simple of Harmonic numbers $H_{n\psi}$ characterized by

$$H_{nq} = \sum_{k=1}^{n} \frac{q^k}{1-q^k}, \quad n \in \mathbb{N}.$$

So, from inequality (3.1) and using the equation

$$\varphi_q(n+1) = \varphi_q(1) - \log(q) H_{nq},$$

we obtain the following result.

Corollary 3.4. *Let* $n \in \mathbb{N}$, $0 < q < 1$, *then we have*

$$\left| \frac{1}{6} \left[\varphi_q'(1) + 4\varphi_q' \left(\frac{n}{2} + 1 \right) + \varphi_q'(n+1) \right] + \frac{\log(q) H_{nq}}{n} \right| \le \frac{n^2}{162} \left[\left| \varphi_q'''(1) \right| + \left| \varphi_q'''(n+1) \right| \right].$$

3.3. **Modified Bessel Function:** Let $\sigma > -1$. The second kind of modified Bessel function K_σ [28] is defined as:

$$K_\sigma(z) = \frac{\pi}{2} \frac{I_{-\sigma}(z) - I_\sigma(z)}{\sin \sigma\pi}.$$

Let the function $\Psi_\sigma(z) : \mathbb{R} \to [1, \infty)$ defined by

$$\Psi_\sigma(z) = 2^\sigma \Gamma(\sigma + 1) z^{-\sigma} K_\sigma(z).$$

Proposition 3.5. *Let $\sigma > -1$ and $0 < \Phi < \Omega$, then we have*

$$\left| \frac{1}{6} \left[\Psi_\sigma(\Phi) + 4\Psi_\sigma\left(\frac{\Phi + \Omega}{2}\right) + \Psi_\sigma(\Omega) \right] - \frac{1}{\Omega - \Phi} \int_\Phi^\Omega \Psi_\sigma(z)\, dz \right| \leq \frac{(\Omega - \Phi)^2}{162} \left[\left|\Psi''_{\sigma+1}(\Phi)\right| + \left|\Psi''_{\sigma+1}(\Omega)\right| \right].$$

Proof. The assertion can be obtained immediately by using the Corollary 2.3 to $F(z) = \Psi_\sigma(z)$ and $\rho = 1$, $\beta(0) = \beta(1) = 1$. □

Proposition 3.6. *Let $\sigma > -1$ and $0 < \Phi < \Omega$, then we have*

$$\left| \Psi_\sigma\left(\frac{\Phi + \Omega}{2}\right) - \frac{1}{\Omega - \Phi} \int_\Phi^\Omega \Psi_\sigma(z)\, dz \right| \leq \frac{(\Omega - \Phi)^2}{162} \left[\left|\Psi''_{\sigma+1}(\Phi)\right| + \left|\Psi''_{\sigma+1}(\Omega)\right| \right].$$

Proof. The assertion can be obtained immediately by Corollary 2.4 to $F(z) = \Psi_\sigma(z)$ and $\rho = 1$, $\beta(0) = \beta(1) = 1$. □

3.4. **Simpson's Formula:** Let d is the partition of the interval $[\Phi, \Omega]$, $d : \Phi = z_0 < z_1 < z_2 < \cdots < z_{n-1} < z_n = \Omega$, then the Simpson's formula is given by:

$$S(F, d) := \sum_{i=0}^{n-1} \frac{F(z_i) + 4F(z_i + h_i) + F(z_{i+1})}{6} (z_{i+1} - z_i),$$

where

$$\int_\Phi^\Omega F(z)\, dz := S(F, d) + E_s(F, d)$$

and $E_s(F, d)$ is the approximation error.

Proposition 3.7. *Under the assumptions of Lemma 2.1, for every division d of $[\Phi, \Omega]$ the following inequality holds:*

$$|E_s(F, d)| \leq \frac{1}{162} \sum_{i=0}^{n-1} (z_{i+1} - z_i)^2 \left[\left|F''(z_i)\right| + \left|F''(z_{i+1})\right| \right].$$

Proof. Applying the Corollary 2.3 on the subinterval $[z_i, z_{i+1}]$, of the division d and taking $\rho = 1$, $\beta(0) = \beta(1) = 1$, we have

$$\left|\frac{z_{i+1}-z_i}{6}\left[F(z_i) + 4F\left(\frac{z_i+z_{i+1}}{2}\right) + F(z_{i+1})\right] - \int_{z_i}^{z_{i+1}} F(z)\,dz\right|$$
$$\leq \frac{(z_{i+1}-z_i)^2}{162}\left[\left|F''(z_i)\right| + \left|F''(z_{i+1})\right|\right].$$

Summing over i from 0 to $n-1$ the above inequality and applying the triangle inequality, we have

$$\left|S(F,d) - \int_{\Phi}^{\Omega} F(z)\,dz\right| \leq \frac{1}{162}\sum_{i=0}^{n-1}(z_{i+1}-z_i)^2\left[\left|F''(z_i)\right| + \left|F''(z_{i+1})\right|\right].$$

This completes the proof. □

Proposition 3.8. *Under the assumptions of Lemma 2.1, for every division d of $[\Phi, \Omega]$ the following inequality holds:*

$$|E_S(F,d)|$$
$$\leq \frac{1}{162}\sum_{i=0}^{n-1}(z_{i+1}-z_i)^2\left[\left|F''\left(\frac{133z_{i+1}+59z_i}{192}\right)\right| + \left|F''\left(\frac{59z_{i+1}+133z_i}{192}\right)\right|\right].$$

Proof. The proof is similar to the Proposition 3.7, using the Theorem 2.12 choosing $\rho = m = 1$, $\beta(0) = \beta(1) = 1$. □

Proposition 3.9. *Let P is the partition of the interval $[\Phi, \Omega]$, $P : \Phi = z_0 < z_1 < z_2 < \cdots < z_{n-1} < z_n = \Omega$, then the Midpoint's formula is defined by: $M(F,P) := \sum_{i=0}^{n-1} F\left(\frac{z_i+z_{i+1}}{2}\right)(z_{i+1}-z_i)$, where*

$$\int_{\Phi}^{\Omega} F(z)\,dz := M(F,P) + E(F,P)$$

and $E(F,P)$ is the approximation error.

Proposition 3.10. *Under the assumptions of Lemma 2.1, for every division P of $[\Phi, \Omega]$ the following inequality holds:*

$$|E(F,P)| \leq \frac{1}{162}\sum_{i=1}^{n-1}(z_{i+1}-z_i)^3\left[\left|F''(z_i)\right| + \left|F''(z_{i+1})\right|\right].$$

Proof. By using the Corollary 2.3 on the subinterval $[z_i, z_{i+1}]$, $i = 0, 1, 2, \ldots, n-1$ of division P choosing $\rho = 1$, $\beta(0) = \beta(1) = 1$, we have

$$\left|F\left(\frac{z_i+z_{i+1}}{2}\right)(z_{i+1}-z_i) - \int_{z_i}^{z_{i+1}} F(z)\,dz\right|$$

$$\leq \frac{(z_{i+1}-z_i)^3}{162}\left[\left|F''(z_i)\right|+\left|F''(z_{i+1})\right|\right].$$

Summing over i from 0 to $n-1$ the above inequality and applying the triangle inequality, we have

$$\left|M\left(F,P\right)-\int_{z_i}^{z_{i+1}}F\left(z\right)dz\right|\leq\frac{1}{162}\sum_{i=1}^{n-1}(z_{i+1}-z_i)^3\left[\left|F''(z_i)\right|+\left|F''(z_{i+1})\right|\right].$$

This completes the proof. □

3.5. **Matrix Inequality:** Example: In [27] Sababheh proved that the function $\psi\left(\theta\right)=\left\|A^{\theta}XB^{1-\theta}+A^{1-\theta}XB^{\theta}\right\|$ for $A,B\in M_n^+$ and $X\in M_n$ is convex for all $\theta\in[0,1]$. Then by using the Corollary 2.4 for $\Phi,\Omega,k\in[0,1]$ with $\rho>0$ and $m=1$, we have

$$\begin{aligned}&\Bigg|\left\|A^{\frac{\Phi+\Omega}{2}}XB^{1-\frac{\Phi+\Omega}{2}}+A^{1-\frac{\Phi+\Omega}{2}}XB^{\frac{\Phi+\Omega}{2}}\right\|\\&\quad+\frac{\beta(\rho)}{\rho(\Omega-\Phi)}\left[{}^{CF}_{\Phi}I^{\rho}\left\|A^{k}XB^{1-k}+A^{1-k}XB^{k}\right\|\right.\\&\quad\left.+{}^{CF}I^{\rho}_{\Omega}\left\|A^{k}XB^{1-k}+A^{1-k}XB^{k}\right\|\right]\\&\quad-\frac{2(1-\rho)}{\rho(\Omega-\Phi)}\left\|A^{k}XB^{1-k}+A^{1-k}XB^{k}\right\|\Bigg|\\&\leq\frac{(\Omega-\Phi)^2}{162}\left[\left\|A^{\Phi}XB^{1-\Phi}+A^{1-\Phi}XB^{\Phi}\right\|\right.\\&\quad\left.+\left\|A^{\Omega}XB^{1-\Omega}+A^{1-\Omega}XB^{\Omega}\right\|\right].\end{aligned}$$

4. Conclusions

The recent developments in the field of inequalities have indeed been focused on discovering new bounds and generalized versions of well-known inequalities, often by utilizing various fractional integral operators. In this article, we established the new identity for Caputo-Fabrizio fractional integral operators. Employing this new identity some generalizations of Simpson's type inequality for (s,m)-convex functions are obtained. Moreover, we also included some applications to special means, q-digamma functions, modified Bessel function, Simpson's and midpont's formula, and matrix inequality. In the future, this work can be extended by using different convexity classes with modified Caputo-Fabrizio and modified AB-fractional operators.

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