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Exploring Fractional q-Kinetic Equations via Generalized q-Mittag-Leffler Type Functions: Applications and Analysis

Mulugeta Dawud Ali[1], Dayalal Suthar[2] and Sunil Dutt Purohit[3*]

Abstract. In this study, the q-calculus is employed to introduce a novel generalization of the Mittag-Leffler function. In the following, we investigate a novel q-exponential function with five parameters, resulting in the generalized q-Mittag-Leffler function. Some q-integral representations and fractional q-derivative (Caputo and Hilfer) for this q-Mittag-Leffler type function are derived. Moreover, we obtain the solutions to the q-fractional kinetic equations includes this function by applying q-Laplace and q-Sumudu transforms defined using fractional q-calculus operators of the Riemann-Liouville (R-L) type. A few particular scenarios to illustrate the use of our primary finding. Further, we state some significant and special cases of our main results. Finally, we present the obtained solutions in the form of numerical graphs using MATLAB 16.

1. Introduction

The Mittag-Leffler function (MLF) plays a pivotal role in contemporary research on fractional calculus because of its critical significance in differential equations of both fractional and integral order. Its adaptability and applicability have led to a recent surge in attention among scholars and researchers. Many mathematicians view the MLF as a fundamental tool in fractional calculus because of its special characteristics and broad applicability in physics and mathematics. Several academic studies have thoroughly investigated and recorded the various aspects of this function, including its generalizations, unique features, and variety of uses. Interested readers are urged to peruse the wealth of literature

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on the MLF for a deeper understanding, as it offers thorough insights into both its theoretical foundations and practical applications. Shyamsunder and Gangwar [33] used incomplete $\mathcal{N}$-functions to analyze the effect of fractional orders on reaction rates, resulting in analytical solutions for fractional kinetic equations. Shyamsunder [32] expanded on this method for sustainable energy systems, highlighting the importance of such functions in simulating real-world dynamics. Ata and Kıymaz [9] demonstrated the extensive applicability of fractional partial differential equations with special functions and general kernels, complementing previous developments. Nishant et al. [20] investigated the interplay of the incomplete Aleph-function with Srivastava polynomials, expanding the analytical toolkit for fractional models and contributing to a better understanding of their structural features. Öztürk et al. [21] suggested a fractional-order SAQ (Susceptible-Addicted-Quit) alcohol model and examined its stability qualities. Their research also included an application to real-world data from Turkey, demonstrating the efficacy of fractional calculus in capturing the intricate behavior of alcohol addiction dynamics.

Within the field of q-theory, several researchers are investigating the q-analogue of the generalized MLF as an extension of the MLF. Among them are Rajković et al. [24], Purohit and Kalla [22], Garg et al. [12], Sharma and Jain [26]. Nadeem et al. [19], Bairwa et al. [10] and, Ali and Suthar [6–8] presented an innovative approach to extending the q-analogue of the MLF and examined its characteristics.

q-calculus refers to the extensions of ordinary calculus that include q-analogues. Current uses of q-calculus operators include q-transform analysis, which is useful in signal processing and communications; solving q-difference equations that occur in combinatorics and physics; solving q-integral equations that are pertinent to mathematical physics and probability theory; and investigating connections with ordinary fractional calculus, which sheds light on continuous systems and phenomena with non-integer dimensions. These uses demonstrate the q-calculus's adaptability and significance to a wide range of fields in contemporary scientific inquiry.

The q-exponential functions [13] can be written by the power series

$$e_q(x) = \frac{1}{(x,q)_\infty} = \sum_{\omega=0}^{\infty} \frac{x^\omega}{(q;q)_\omega}, \quad (|x|<1),$$

and

$$E_q(x) = (-x,q)_\infty = \sum_{\omega=0}^{\infty} \frac{q^{\omega(\omega-1)/2}}{(q;q)_\omega} x^\omega. \tag{1.1}$$

In 2007, Rajkovic et al. [24] introduced the small q-MLF and the big q-MLF under the condition of $q, x, c, \vartheta, \xi \in \mathbb{C}; \Re(\vartheta) > 0, \Re(\xi) > 0, |q| < 1$, as:

$$e_{q;\vartheta,\xi}(x;c) = \sum_{\omega=0}^{\infty} \frac{x^{\vartheta\omega+\xi-1}(c/x;q)_{\vartheta\omega+\xi-1}}{(q;q)_{\vartheta\omega+\xi-1}}, \quad (|c| < |x|),$$

and

$$E_{q;\vartheta,\xi}(x;c) = \sum_{\omega=0}^{\infty} \frac{q^{\binom{\vartheta\omega+\xi-1}{2}}}{(-c;q)_{\vartheta\omega+\xi-1}} \frac{x^{\vartheta\omega+\xi-1}(c/x;q)_{\vartheta\omega+\xi-1}}{(q;q)_{\vartheta\omega+\xi-1}}.$$

The new form of the q-analogue of the MLF is introduced by Mansour [18] as:

$$E_{\vartheta,\xi}\left(x;q\right) = \sum_{\omega=0}^{\infty} \frac{x^{\omega}}{\Gamma_q(\vartheta\omega+\xi)}, \quad \left(\vartheta > 0, \xi \in \mathbb{C}, |x| < (1-q)^{-\vartheta}\right). \tag{1.2}$$

In 2011, Purohit and Kalla [22] introduced another small q-MLF and big q-MLF with the condition of $|x| < (1-q)^{-\vartheta}$ defined as:

$$e^{\gamma}_{\vartheta,\xi}(x;q) = \sum_{\omega=0}^{\infty} \frac{x^{\omega}(q^{\gamma};q)_{\omega}}{\Gamma_q(\vartheta\omega+\xi)(q;q)_{\omega}}, \tag{1.3}$$

and

$$E^{\gamma}_{\vartheta,\xi}(x;q) = \sum_{\omega=0}^{\infty} \frac{x^{\omega}q^{\omega(\omega-1)/2}(q^{\gamma};q)_{\omega}}{\Gamma_q(\vartheta\omega+\xi)(q;q)_{\omega}}, \tag{1.4}$$

respectively.

Recently, the q-analogue of MLF was introduced by Sharma and Jain [26] in 2016, and its definition is as follows:

$$E^{\gamma,\delta}_{\vartheta,\xi}\left(x;q\right) = \sum_{\omega=0}^{\infty} \frac{\left(q^{\gamma};q\right)_{\omega}}{\left(q^{\delta};q\right)_{\omega}} \frac{x^{\omega}}{\Gamma_q\left(\vartheta\omega+\xi\right)}, \tag{1.5}$$

where $\vartheta, \xi, \gamma, \delta \in \mathbb{C}, \Re(\gamma) > 0, \Re(\xi) > 0, \Re(\vartheta) > 0, \Re(\delta) > 0$ and $|q| < 1$.

2. q-Notations

In the theory of q-calculus (see [13]), there is a q-analogue of the standard Pochhammer symbol, or the q-shifted factorial. The following product yields the q-Pochhammer symbol, which is indicated as $(a;q)_n$:

$$(a;q)_n = \prod_{k=0}^{n-1}\left(1-aq^k\right) = (1-a)(1-aq)\left(1-aq^2\right)\cdots\left(1-aq^{n-1}\right),$$

Here, a is a complex or real number, and $|q| < 1$.

One can extend the q-Pochhammer symbol to an infinite product:

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

The given infinite product simultaneously serves as a formal power series in q and as an analytic function of q for $|q|<1$. Euler's function is the particular circumstances where $n=\infty$. It is important in number theory, modular form theory, and combinatorics.

One way to express the finite product is in terms of the infinite product:

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \quad n \in \mathbb{C}.$$

The q-Pochhammer symbol is subject to various q-series identities, including infinite series expansions and the q-binomial theorem.

Equivalently, in terms of q-gamma function it is defined as:

$$(a;q)_n = \frac{\Gamma_q(a+n)}{\Gamma_q(a)}(1-q)^n,$$

where $\Gamma_q(.)$ is the q-analogue of the gamma function [13] provided by

$$\Gamma_q(\delta) = \frac{(q;q)_\infty}{(q^\delta;q)_\infty}(1-q)^{1-\delta} = \frac{(q;q)_{\delta-1}}{(1-q)^{\delta-1}}, \quad (\delta \neq 0,-1,-2,\ldots).$$

As stated by [18], the q-analogue of the beta function is

$$\begin{aligned} \mathfrak{B}_q(r,s) &= \int_0^1 z^{r-1}(zq;q)_{y-1}d_qz \\ &= \frac{\Gamma_q(r)\,\Gamma_q(s)}{\Gamma_q(r+s)}, \quad \Re(s)>0, \Re(r)>0. \end{aligned} \tag{2.1}$$

Additionally, q-derivative of a function $f(t)$ defined on a subset of $\mathbb{C}$ is provided by [13] respectively

$$\mathfrak{D}_q f(t) = \frac{f(t)-f(tq)}{t(1-q)}, \quad (t\neq 0,\, q\neq 1), \tag{2.2}$$

$$\mathfrak{D}_q\left\{(t-\varepsilon)_q^\mu\right\} = [\mu]_q (t-\varepsilon)_q^{\mu-1}, \tag{2.3}$$

$$\mathfrak{D}_q\left\{(\eta-t)_q^\mu\right\} = -[\mu]_q(\eta-qt)_q^{\mu-1}, \tag{2.4}$$

and

$$\int_0^t f(t)d(t;q) = t(1-q)\sum_{k=0}^{\infty} q^k f(tq^k).$$

The fractional q-integral operator is defined by

$$\left[\mathfrak{I}_{q,a+}^{\delta}(t-\varepsilon)_q^\mu\right](z) = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\mu+\delta+1)}(t-\varepsilon)_q^{\mu+\delta}, \quad \delta\in(-1,\infty). \tag{2.5}$$

3. q-analogue of the Generalized MLF

Recently, Ali and Suthar [8] introduced the q-GMLF by using the following relation to generalize the definition of the q-analogue of the MLF found in (1.5).

$$\frac{(q^\gamma;q)_\omega}{(q^c;q)_\omega} = \frac{\mathfrak{B}_q(\gamma+\omega, c-\gamma)}{\mathfrak{B}_q(\gamma, c-\gamma)}.$$

Definition 3.1. For $\vartheta,\xi,\gamma,c,\delta \in \mathbb{C}, \Re(\vartheta) > 0, \Re(\xi) > 0, \Re(\gamma) > 0, \Re(c) > 0, \Re(\delta) > 0$ and $|q| < 1$, the q-GMLF is defined as:

$$\mathfrak{E}^{(\gamma,c);\delta}_{\vartheta,\xi}(z;q) = \sum_{\omega=0}^{\infty} \frac{\mathfrak{B}_q(\gamma+\omega, c-\gamma)}{\mathfrak{B}_q(\gamma, c-\gamma)} \frac{(q^c;q)_\omega}{(q^\delta;q)_\omega} \frac{z^\omega}{\Gamma_q(\vartheta\omega+\xi)}, \tag{3.1}$$

where $\mathfrak{B}(.)$ is a q-beta function defined in (2.1).

The specific instances of the q-GMLF with additional unique functions are now listed as follows:

- Letting $\delta = 1$ in (3.1), we have

$$\begin{aligned}\mathfrak{E}^{(\gamma,c);1}_{\vartheta,\xi}(z;q) &= \sum_{\omega=0}^{\infty} \frac{\mathfrak{B}_q(\gamma+\omega, c-\gamma)}{\mathfrak{B}_q(\gamma, c-\gamma)} \frac{(q^c;q)_\omega}{(q;q)_\omega} \frac{z^\omega}{\Gamma_q(\vartheta\omega+\xi)} \\ &= E^{(\gamma,c)}_{\vartheta,\xi}(z;q),\end{aligned} \tag{3.2}$$

  which is given by Nadeem et al. [19].
- If $c = 1$ in (3.1), we get

$$\mathfrak{E}^{(\gamma,1);c}_{\vartheta,\xi}(z;q) = \sum_{\omega=0}^{\infty} \frac{(q^\gamma;q)_\omega}{(q^\delta;q)_\omega} \frac{z^\omega}{\Gamma_q(\vartheta\omega+\xi)} = E^{\gamma,\delta}_{\vartheta,\xi}(z;q), \tag{3.3}$$

  which is defined in (1.5).
- Letting $\gamma = 1$ and $\delta = 1$ in (3.1), we have

$$\begin{aligned}\mathfrak{E}^{(1,c);1}_{\vartheta,\xi}(z;q) &= \sum_{\omega=0}^{\infty} \frac{(q;q)_\omega}{(q;q)_\omega} \frac{z^\omega}{\Gamma_q(\vartheta\omega+\xi)} \\ &= E_{\vartheta,\xi}(z;q),\end{aligned} \tag{3.4}$$

  which is given in (1.2).
- Letting $\delta = 1$ and $c = 1$ as parameters in (3.1), we get

$$\begin{aligned}\mathfrak{E}^{(\gamma,1);1}_{\vartheta,\xi}(z;q) &= \sum_{\omega=0}^{\infty} \frac{(q^\gamma;q)_\omega}{(q;q)_\omega} \frac{z^\omega}{\Gamma_q(\vartheta\omega+\xi)} \\ &= e^{\gamma}_{\vartheta,\xi}(z;q),\end{aligned} \tag{3.5}$$

  which is given by Equation (1.3).

- Letting $\vartheta = \xi = \gamma = \delta = 1$ as parameters in (3.1), we get

$$\mathfrak{E}_{1,1}^{(1,c);1}(z;q) = \sum_{\omega=0}^{\infty} \frac{(q^c;q)_\omega}{(q;q)_\omega} z^\omega = \frac{(q^c z;q)_\infty}{(q;q)_\infty} = {}_1\phi_0(q^c;-;q,z), \tag{3.6}$$

where the function ${}_1\phi_0(q^c;-;q,z) = (1-z)^{-c}$ can be considered as a q-binomial function.

4. Main Results

Within this section, our objective is to thoroughly evaluate the fractional q-integral representations and fractional q-derivatives associated with the recently introduced five-parameter q-Mittag-Leffler functions. Additionally, we will investigate their practical application in solving q-fractional kinetic equations that are constructed utilizing the q-Riemann-Liouville (q-R-L) fractional integral operator. q-analogue of Mittag-Leffler function. The main theorems are as under:

4.1. **Integral Representations.** In this part, we derive and illustrate numerous theorems based on q-integral representations of the q-GMLF. These theorems offer insights and analytical results into the properties and applications of the q-GMLF in a variety of mathematical situations.

Theorem 4.1. *For $z, \vartheta, \xi, \delta, \gamma, c \in \mathbb{C}, \Re(\delta) > 0, \Re(\xi) > \Re(\vartheta) > 0, \Re(\vartheta) > 0, \Re(c) > 0,$ and $z \neq 0$ then*

$$\mathfrak{E}_{\vartheta,\xi}^{(\gamma,c);\delta}(z;q) = \frac{z^{\vartheta-\xi}}{(1-q^{1/n})} \int_0^\infty e_q(-t^n/z^n)\, t^{\xi-\vartheta-1} \times \sum_{\omega=0}^{\infty} \frac{\mathfrak{B}_q(\gamma+\omega, c-\gamma)}{\mathfrak{B}_q(\gamma, c-\gamma)} \frac{(q^c;q)_\omega}{(q^\delta;q)_\omega} \frac{q^{\sigma(\sigma-1)/2} t^\omega}{\Gamma_q(\vartheta\omega+\xi)(q;q)_{\sigma-1}} d_q t, \tag{4.1}$$

where $\sigma = (\xi - \vartheta + \omega)/n$ and n is any non-zero positive number.

Proof. To show the outcomes of Equation (4.1), let the RHS of Equation (4.1) by R.

$$R = \frac{z^{\vartheta-\xi}}{(1-q^{1/n})} \int_0^\infty e_q(-t^n/z^n)\, t^{\xi-\vartheta-1} \times \sum_{\omega=0}^{\infty} \frac{\mathfrak{B}_q(\gamma+\omega, c-\gamma)}{\mathfrak{B}_q(\gamma, c-\gamma)} \frac{(q^c;q)_\omega}{(q^\delta;q)_\omega} \frac{q^{\sigma(\sigma-1)/2} t^\omega}{\Gamma_q(\vartheta\omega+\xi)(q;q)_{\sigma-1}} d_q t.$$

Substituting $t^n/z^n = \Omega$, into (2.2), we have

$$d_q t = \frac{(1-q^{1/n})}{(1-q)} z\,\Omega^{1/n-1} d_q\Omega.$$

Again, we can write

$$R = \frac{1}{(1-q)} \int_0^\infty e_q(-\Omega)\,\Omega^{((\xi-\vartheta)/n)-1} \times \sum_{\omega=0}^{\infty} \frac{\mathfrak{B}_q(\gamma+\omega, c-\gamma)}{\mathfrak{B}_q(\gamma, c-\gamma)} \frac{(q^c;q)_\omega}{(q^\delta;q)_\omega} \frac{q^{\sigma(\sigma-1)/2}\left(z\Omega^{1/n}\right)^\omega}{\Gamma_q(\vartheta\omega+\xi)(q;q)_{\sigma-1}} d_q\Omega.$$

By Interchanging the order of summation and integration, subject to the condition specified in Equation (4.1), we get

$$\begin{aligned} R &= \frac{1}{(1-q)} \sum_{\omega=0}^{\infty} \frac{\mathfrak{B}_q(\gamma+\omega, c-\gamma)}{\mathfrak{B}_q(\gamma, c-\gamma)} \frac{(q^c;q)_\omega}{(q^\delta;q)_\omega} \frac{q^{\sigma(\sigma-1)/2}\,(z)^\omega}{\Gamma_q(\vartheta\omega+\xi)(q;q)_{\sigma-1}} \\ &\quad \times \int_0^\infty e_q(-\Omega)\,\Omega^{\sigma-1} d_q\Omega \\ &= \frac{1}{(1-q)} \sum_{\omega=0}^{\infty} \frac{\mathfrak{B}_q(\gamma+\omega, c-\gamma)}{\mathfrak{B}_q(\gamma, c-\gamma)} \frac{(q^c;q)_\omega}{(q^\delta;q)_\omega} \frac{q^{\sigma(\sigma-1)/2}\,(z)^\omega}{\Gamma_q(\vartheta\omega+\xi)(q;q)_{\sigma-1}} \\ &\quad \times L_q\left\{\Omega^{\sigma-1};1\right\}, \end{aligned} \tag{4.2}$$

here $L_q\{f(\Omega);s\}$ indicates the q-Laplace transform of $f(\Omega)$, which is introduced by Hann [14] and given as

$$L_q\{f(\Omega);s\} = \frac{1}{(1-q)} \int_0^\infty e_q(-s\Omega)\, f(\Omega) d_q\Omega.$$

Using Abdi's result [1], namely

$$L_q\left\{\Omega^{\sigma-1};s\right\} = \frac{q^{-\sigma(\sigma-1)/2}(q;q)_{\sigma-1}}{s^\sigma}, \quad \Re(\delta) > 0.$$

We have

$$L_q\left\{\Omega^{\sigma-1};1\right\} = q^{-\sigma(\sigma-1)/2}(q;q)_{\sigma-1}.$$

As a result, (4.2) corresponds to the LHS of (4.1). Thus, it completes in support of (4.1). □

Theorem 4.2. *For* $z, \vartheta, \delta, \xi, \gamma, c \in \mathbb{C}, \Re(c) > 0, \Re(\vartheta) > 0, \Re(\delta) > 0$ *and* $\Re(\xi) > \Re(\vartheta) > 0$, *then*

$$\mathfrak{E}^{(\gamma,c);\delta}_{\vartheta,\xi}(z;q) = \frac{(1-q)}{(1-q^\vartheta)\Gamma_q(\xi-\vartheta)} \int_0^1 \left(qt^{1/\vartheta};q\right)_{\xi-\vartheta-1} \mathfrak{E}^{(\gamma,c);\delta}_{\vartheta,\vartheta}(zt;q) d_q t. \tag{4.3}$$

Proof. Using the definition in Equation (3.1) for the RHS of Equation (4.3), designated as (L), we get

$$L = \frac{(1-q)}{(1-q^{\vartheta})\Gamma_q(\xi-\vartheta)} \int_0^1 \left(qt^{1/\vartheta};q\right)_{\xi-\vartheta-1} \times \sum_{\omega=0}^{\infty} \frac{\mathfrak{B}_q(\gamma+\omega,c-\gamma)}{\mathfrak{B}_q(\gamma,c-\gamma)} \frac{(q^c;q)_\omega}{(q^\delta;q)_\omega} \frac{(zt)^\omega}{\Gamma_q(\vartheta\omega+\vartheta)} d_qt$$
$$= \frac{(1-q)}{(1-q^{\vartheta})\Gamma_q(\xi-\vartheta)} \sum_{\omega=0}^{\infty} \frac{\mathfrak{B}_q(\gamma+\omega,c-\gamma)}{\mathfrak{B}_q(\gamma,c-\gamma)} \frac{(q^c;q)_\omega}{(q^\delta;q)_\omega} \frac{(z)^\omega}{\Gamma_q(\vartheta\omega+\vartheta)} \times \int_0^1 t^\omega \left(qt^{1/\vartheta};q\right)_{\xi-\vartheta-1} d_qt.$$

Let $t = y^\vartheta$ then $d_qt = \left((1-q^\vartheta)/(1-q)\right) y^{\vartheta-1} d_qy$ and using the Equation (2.1), we have

$$L = \frac{(1-q)}{(1-q^{\vartheta})\Gamma_q(\xi-\vartheta)} \times \sum_{\omega=0}^{\infty} \frac{\mathfrak{B}_q(\gamma+\omega,c-\gamma)}{\mathfrak{B}_q(\gamma,c-\gamma)} \frac{(q^c;q)_\omega}{(q^\delta;q)_\omega} \frac{(1-q^\vartheta)\Gamma_q(\vartheta\omega+\vartheta)\Gamma_q(\xi-\vartheta)(z)^\omega}{\Gamma_q(\vartheta\omega+\xi)\Gamma_q(\vartheta\omega+\vartheta)(1-q)}.$$

After simplifying the given expression, we arrive at the LHS of Equation (4.3). □

Theorem 4.3. *For* $\vartheta,\delta,\xi,\gamma,c,\in\mathbb{C}, \Re(c)>0, \Re(\vartheta)>0, \Re(\delta)>0$ *and* $\Re(\xi)>\Re(\vartheta)>0$, *then*

(4.4)
$$\mathfrak{E}_{\vartheta,\xi}^{(\gamma,c);\delta}(z;q) = \frac{1}{\Gamma_q(\vartheta)} \int_0^1 t^{\vartheta-1} (qt;q)_{\xi-\vartheta-1}\ \mathfrak{E}_{\vartheta,\xi-\vartheta}^{(\gamma,c);\delta}(z(1-tq^{\xi-\vartheta})^{(\vartheta)};q) d_qt.$$

Proof. Applying the definition (3.1), we obtain

$$\mathfrak{E}_{\vartheta,\xi}^{(\gamma,c);\delta}(z;q) = \frac{1}{\Gamma_q(\vartheta)} \int_0^1 t^{\vartheta-1} (qt;q)_{\xi-\vartheta-1} \sum_{\omega=0}^{\infty} \frac{\mathfrak{B}_q(\gamma+\omega,c-\gamma)}{\mathfrak{B}_q(\gamma,c-\gamma)} \frac{(q^c;q)_\omega}{(q^\delta;q)_\omega} \times \frac{(z)^\omega}{\Gamma_q(\vartheta\omega+\xi-\vartheta)} (tq^{\xi-\vartheta};q)_{(\omega\vartheta)} d_qt.$$

By applying the q-identity and switching the order of summation and integration referenced in ([24], P. 234, 1.17), the above equation simplifies to:

(4.5)
$$\mathfrak{E}_{\vartheta,\xi}^{(\gamma,c);\delta}(z;q) = \frac{1}{\Gamma_q(\vartheta)} \sum_{\omega=0}^{\infty} \frac{\mathfrak{B}_q(\gamma+\omega,c-\gamma)}{\mathfrak{B}_q(\gamma,c-\gamma)} \frac{(q^c;q)_\omega}{(q^\delta;q)_\omega} \frac{(z)^\omega}{\Gamma_q(\vartheta\omega+\xi-\vartheta)}$$

$$\times \int_0^1 t^{\vartheta-1} (tq;q)_{\omega\vartheta+\xi-\vartheta-1}\, d_q t.$$

From (2.1) and (3.1), Equation (4.5) becomes to the LHS of (4.4). □

4.2. Fractional q-derivative of $\mathfrak{E}^{(\gamma,c);\delta}_{\vartheta,\xi}(z;q)$.

Theorem 4.4. *(Caputo fractional q-derivative) Let* $\vartheta,\delta,\xi,\gamma,c,\in\mathbb{C},\Re(c)>0,\Re(\vartheta)>0,\Re(\delta)>0$ *and* $\Re(\xi)>\Re(\vartheta)>0,$ *then for* $\mu\in\mathbb{N}$

(4.6)

$$\left[{}_c\mathfrak{D}^{\mu}_{q,\,a+}\left\{\mathfrak{E}^{(\gamma,c);\delta}_{\vartheta,\xi}((t-qa)^{\mu}_q,\,q)\right\}\right](z) = (z-qa)^{-\mu}_q\sum_{\omega=0}^{\infty}\frac{\mathfrak{B}_q(\gamma+\omega,c-\gamma)}{\mathfrak{B}_q(\gamma,c-\gamma)}\frac{(q^c;q)_\omega}{(q^\delta;q)_\omega}\frac{\Gamma_q(1+\mu\omega)(z-qa)^{\mu\omega}_q}{\Gamma_q(\vartheta\omega+\xi)\Gamma_q(1+\mu\omega-\mu)}.$$

Proof. The Caputo fractional q-derivative is provided by [25] as

(4.7)

$$\left[{}_c\mathfrak{D}^{\mu}_{q,a+}f\right](z) = \left[\mathfrak{I}^{1-\mu}_{q,a+}\,\mathfrak{D}_q f\right](z) = \frac{1}{\Gamma_q(1-\mu)}\int_a^z (z-qt)^{-\mu}_q f(t)d_q t.$$

Now, by using (4.7) and (3.1), we have

$$\begin{aligned}
&\left[{}_c\mathfrak{D}^{\mu}_{q,\,a+}\left\{\mathfrak{E}^{(\gamma,c);\delta}_{\vartheta,\xi}((t-qa)^{\mu}_q,\,q)\right\}\right](z)\\
&= \left[{}_c\mathfrak{D}^{\mu}_{q,\,a+}\sum_{\omega=0}^{\infty}\frac{\mathfrak{B}_q(\gamma+\omega,c-\gamma)}{\mathfrak{B}_q(\gamma,c-\gamma)}\frac{(q^c;q)_\omega}{(q^\delta;q)_\omega}\frac{(t-qa)^{\mu\,\omega}_q}{\Gamma_q(\vartheta\omega+\xi)}\right](z)\\
&= \sum_{\omega=0}^{\infty}\frac{\mathfrak{B}_q(\gamma+\omega,c-\gamma)}{\mathfrak{B}_q(\gamma,c-\gamma)}\frac{(q^c;q)_\omega}{(q^\delta;q)_\omega}\frac{1}{\Gamma_q(\vartheta\omega+\xi)}\left[{}_c\mathfrak{D}^{\mu}_{q,\,a+}\,(t-qa)^{\mu\,\omega}_q\right](z)\\
&= \sum_{\omega=0}^{\infty}\frac{\mathfrak{B}_q(\gamma+\omega,c-\gamma)}{\mathfrak{B}_q(\gamma,c-\gamma)}\frac{(q^c;q)_\omega}{(q^\delta;q)_\omega}\frac{1}{\Gamma_q(\vartheta\omega+\xi)}\left[\mathfrak{I}^{1-\mu}_{q,a+}\,\mathfrak{D}_q\,(t-qa)^{\mu\,\omega}_q\right](z).
\end{aligned}$$

Using (2.3) and (2.4), we get

$$\begin{aligned}
&\left[{}_c\mathfrak{D}^{\mu}_{q,\,a+}\left\{\mathfrak{E}^{(\gamma,c);\delta}_{\vartheta,\xi}((t-qa)^{\mu}_q,\,q)\right\}\right](z)\\
&= \sum_{\omega=0}^{\infty}\frac{\mathfrak{B}_q(\gamma+\omega,c-\gamma)}{\mathfrak{B}_q(\gamma,c-\gamma)}\frac{(q^c;q)_\omega}{(q^\delta;q)_\omega}\frac{1}{\Gamma_q(\vartheta\omega+\xi)}\left[\mathfrak{I}^{1-\mu}_{q,a+}\,[\mu\omega]_q\,(t-qa)^{\mu\,\omega-1}_q\right](z)\\
&= \sum_{\omega=0}^{\infty}\frac{\mathfrak{B}_q(\gamma+\omega,c-\gamma)}{\mathfrak{B}_q(\gamma,c-\gamma)}\frac{(q^c;q)_\omega}{(q^\delta;q)_\omega}\frac{1}{\Gamma_q(\vartheta\omega+\xi)}\,[\mu\omega]_q\left[\mathfrak{I}^{1-\mu}_{q,a+}\,(t-qa)^{\mu\,\omega-1}_q\right](z).
\end{aligned}$$

Using (2.5), we obtain

$$\left[{}_c\mathfrak{D}^{\mu}_{q,\,a+}\left\{\mathfrak{E}^{(\gamma,c);\delta}_{\vartheta,\xi}((t-qa)^{\mu}_{q},\,q)\right\}\right](z)$$
$$=\sum_{\omega=0}^{\infty}\frac{\mathfrak{B}_q(\gamma+\omega,c-\gamma)}{\mathfrak{B}_q(\gamma,c-\gamma)}\frac{(q^c;q)_\omega}{(q^\delta;q)_\omega}\frac{\Gamma_q(\mu\omega)}{\Gamma_q(\vartheta\omega+\xi)\Gamma_q(1+\mu\omega-\mu)}[\mu\omega]_q$$
$$\times\left[(t-qa)^{\mu\omega-\mu}_{q}\right]$$
$$=\sum_{\omega=0}^{\infty}\frac{\mathfrak{B}_q(\gamma+\omega,c-\gamma)}{\mathfrak{B}_q(\gamma,c-\gamma)}\frac{(q^c;q)_\omega}{(q^\delta;q)_\omega}\frac{\Gamma_q(1+\mu\omega)(t-qa)^{\mu\omega-\mu}_{q}}{\Gamma_q(\vartheta\omega+\xi)\Gamma_q(1+\mu\omega-\mu)}$$

This implies that

(4.8)

$$\left[{}_c\mathfrak{D}^{\mu}_{q,\,a+}\left\{\mathfrak{E}^{(\gamma,c);\delta}_{\vartheta,\xi}((t-qa)^{\mu}_{q},\,q)\right\}\right](z)$$
$$=(z-qa)^{-\mu}_{q}\sum_{\omega=0}^{\infty}\frac{\mathfrak{B}_q(\gamma+\omega,c-\gamma)}{\mathfrak{B}_q(\gamma,c-\gamma)}\frac{(q^c;q)_\omega}{(q^\delta;q)_\omega}\frac{\Gamma_q(1+\mu\omega)(z-qa)^{\mu\omega}_{q}}{\Gamma_q(\vartheta\omega+\xi)\Gamma_q(1+\mu\omega-\mu)}.$$

The intended outcome is (4.6). □

Theorem 4.5. *(Hilfer fractional q-derivative) If* $\vartheta,\delta,\xi,\gamma,c,\in\mathbb{C},\Re(c)>0,\Re(\vartheta)>0,\Re(\delta)>0$ *and* $\Re(\xi)>\Re(\vartheta)>0$, *then for* $\mu\in\mathbb{N}$

(4.9)

$$\left[{}_c\mathfrak{D}^{\mu,\tau}_{q,a+}\left\{\mathfrak{E}^{(\gamma,c);\delta}_{\vartheta,\xi}((t-qa)^{\mu}_{q},\,q)\right\}\right](z)$$
$$=(z-qa)^{-\mu}_{q}\sum_{\omega=0}^{\infty}\frac{\mathfrak{B}_q(\gamma+\omega,c-\gamma)}{\mathfrak{B}_q(\gamma,c-\gamma)}\frac{(q^c;q)_\omega}{(q^\delta;q)_\omega}\frac{\Gamma_q(1+\mu\omega)(z-qa)^{\mu\omega}_{q}}{\Gamma_q(\vartheta\omega+\xi)\Gamma_q(1+\mu\omega-\mu)}.$$

Proof. The Hilfer fractional q-derivative [16, 17] of $0<\mu<1$ and $0\le\tau\le 1$ is defined as

$$\left[{}_c\mathfrak{D}^{\mu,\tau}_{q,a+}f\right](z)=\left[\mathfrak{I}^{(1-\mu)\tau}_{q,a+}\mathfrak{D}_q\left(\mathfrak{I}^{(1-\mu)(1-\tau)}_{q,a+}\right)\right](z).\tag{4.10}$$

Using (4.10) and (3.1), we have

(4.11)

$$\left[\mathfrak{D}^{\mu,\tau}_{q,a+}\left\{\mathfrak{E}^{(\gamma,c);\delta}_{\vartheta,\xi}((t-qa)^{\mu}_{q},\,q)\right\}\right](z)$$
$$=\mathfrak{D}^{\mu,\tau}_{q,a+}\left[\sum_{\omega=0}^{\infty}\frac{\mathfrak{B}_q(\gamma+\omega,c-\gamma)}{\mathfrak{B}_q(\gamma,c-\gamma)}\frac{(q^c;q)_\omega}{(q^\delta;q)_\omega}\frac{(t-qa)^{\mu\,\omega}_{q}}{\Gamma_q(\vartheta\omega+\xi)}\right](z)$$
$$=\sum_{\omega=0}^{\infty}\frac{\mathfrak{B}_q(\gamma+\omega,c-\gamma)}{\mathfrak{B}_q(\gamma,c-\gamma)}\frac{(q^c;q)_\omega}{(q^\delta;q)_\omega}\frac{1}{\Gamma_q(\vartheta\omega+\xi)}\left[\mathfrak{D}^{\mu,\tau}_{q,a+}(t-qa)^{\mu\omega}_{q}\right](z)$$

$$= \sum_{\omega=0}^{\infty} \frac{\mathfrak{B}_q(\gamma+\omega, c-\gamma)}{\mathfrak{B}_q(\gamma, c-\gamma)} \frac{(q^c;q)_\omega}{(q^\delta;q)_\omega} \frac{1}{\Gamma_q(\vartheta\omega+\xi)} \times \left[\mathfrak{I}_{q,a+}^{(1-\mu)\tau}\, \mathfrak{D}_q \left(\mathfrak{I}_{q,a+}^{(1-\mu)(1-\tau)}\right)(t-qa)_q^{\mu\omega}\right](z).$$

Using (2.5), we get

$$\left[\mathfrak{D}_{q,a+}^{\mu,\tau}\left\{\mathfrak{E}_{\vartheta,\xi}^{(\gamma,c);\delta}((t-qa)_q^{\mu},\, q)\right\}\right](z) = \sum_{\omega=0}^{\infty} \frac{\mathfrak{B}_q(\gamma+\omega, c-\gamma)}{\mathfrak{B}_q(\gamma, c-\gamma)} \frac{(q^c;q)_\omega}{(q^\delta;q)_\omega} \frac{1}{\Gamma_q(\vartheta\omega+\xi)} \times \left[\frac{\left(\mathfrak{I}_{q,a+}^{(1-\mu)\tau}\, \mathfrak{D}_q\,(t-qa)_q^{1-\mu-\tau+\mu\tau+\mu\omega}\right)(z)}{\left(\frac{\Gamma_q(2-\mu-\tau+\mu\tau+\mu\omega)}{\Gamma_q(1+\mu\omega)}\right)}\right].$$

Using (2.3) and (2.4), we get

$$\left[\mathfrak{D}_{q,a+}^{\mu,\tau}\left\{\mathfrak{E}_{\vartheta,\xi}^{(\gamma,c);\delta}((t-qa)_q^{\mu},\, q)\right\}\right](z) = \sum_{\omega=0}^{\infty} \frac{\mathfrak{B}_q(\gamma+\omega, c-\gamma)}{\mathfrak{B}_q(\gamma, c-\gamma)} \frac{(q^c;q)_\omega}{(q^\delta;q)_\omega} \frac{1}{\Gamma_q(\vartheta\omega+\xi)} \times \left[\frac{\left(\mathfrak{I}_{q,a+}^{(1-\mu)\tau}\, \mathfrak{D}_q\,(t-qa)_q^{-\mu-\tau+\mu\tau+\mu\omega}\right)(z)}{\left(\frac{\Gamma_q(2-\mu-\tau+\mu\tau+\mu\omega)}{(1-\mu-\tau+\mu\tau+\mu\omega)_q\,\Gamma_q(1+\mu\omega)}\right)}\right].$$

Again, using (2.5), we get

$$\left[\mathfrak{D}_{q,a+}^{\mu,\tau}\left\{\mathfrak{E}_{\vartheta,\xi}^{(\gamma,c);\delta}((t-qa)_q^{\mu},\, q)\right\}\right](z) = \sum_{\omega=0}^{\infty} \frac{\mathfrak{B}_q(\gamma+\omega, c-\gamma)}{\mathfrak{B}_q(\gamma, c-\gamma)} \frac{(q^c;q)_\omega}{(q^\delta;q)_\omega} \frac{1}{\Gamma_q(\vartheta\omega+\xi)} \times \left[\frac{\left(\frac{\Gamma_q(1-\mu-\tau+\mu\tau+\mu\omega)}{\Gamma_q(1-\mu-\tau+\mu\tau+\mu\omega+\tau-\mu\tau)}\right)(z-qa)_q^{\tau-\mu\tau-\mu-\tau+\mu\tau+\mu\omega}}{\left(\frac{\Gamma_q(1-\mu-\tau+\mu\tau+\mu\omega)}{\Gamma_q(1+\mu\omega)}\right)}\right]$$

$$= \sum_{\omega=0}^{\infty} \frac{\mathfrak{B}_q(\gamma+\omega, c-\gamma)}{\mathfrak{B}_q(\gamma, c-\gamma)} \frac{(q^c;q)_\omega}{(q^\delta;q)_\omega} \frac{\Gamma_q\,(1+\mu\omega)\,(z-qa)_q^{\mu\omega-\mu}}{\Gamma_q(\vartheta\omega+\xi)\Gamma_q\,(1+\mu\omega-\mu)}.$$

This implies that

$$\left[\mathfrak{D}_{q,a+}^{\mu,\tau}\left\{\mathfrak{E}_{\vartheta,\xi}^{(\gamma,c);\delta}((t-qa)_q^{\mu},\, q)\right\}\right](z) = (z-qa)_q^{-\mu} \sum_{\omega=0}^{\infty} \frac{\mathfrak{B}_q(\gamma+\omega, c-\gamma)}{\mathfrak{B}_q(\gamma, c-\gamma)} \frac{(q^c;q)_\omega}{(q^\delta;q)_\omega} \frac{\Gamma_q\,(1+\mu\omega)\,(z-qa)_q^{\mu\omega}}{\Gamma_q(\vartheta\omega+\xi)\Gamma_q\,(1+\mu\omega-\mu)}.$$

This is the required result (4.9). □

4.3. **Generalized Fractional Kinetic Equation.** A key idea in mathematical analysis is the generalized fractional kinetic equation, which expands the classical theory of differential equations to include fractional calculus. Fractional kinetic equations, in contrast to conventional equations employing integer-order derivatives, explain systems that show memory effects and non-local rates of change. These formulas are useful in many domains, including physics, biology, and economics, where power-law tendencies and anomalous diffusion are observed. In addition to enhancing our theoretical knowledge of complex systems, the study of generalized fractional kinetic equations offers strong tools for more flexible and accurate modeling and prediction of real-world dynamics. Furthermore [27, 28], studying these equations with the q-calculus using fractional q-calculus operators of the Riemann-Liouville (R-L) type may lead to innovative research and applications.

The mathematical definitions of these three numbers are as follows assuming that $\mathfrak{N}(t)$ is an arbitrary time-dependent reaction, p is the production rate on $\mathfrak{N}$, and d is the destruction rate.

$$\frac{d\mathfrak{N}}{dt} = -d + p.$$

The following differential fractional equation, presented by Haubold and Mathai [15], is for $\mathfrak{N}(t)$, p, and d:

$$\frac{d\mathfrak{N}}{dt} = -d(\mathfrak{N}_t) + p(\mathfrak{N}_t), \tag{4.12}$$

where, $\mathfrak{N}_t(t^*) = \mathfrak{N}(t - t^*)$ for $t^* > 0$.

Furthermore, it was found by Haubold and Mathai [15], if inhomognities or spatial shifts in quantities $\mathfrak{N}(t)$ are disregarded, Equation (4.12) is now,

$$\frac{d\mathfrak{N}_i}{dt} = -c_i \mathfrak{N}_i(t). \tag{4.13}$$

Regarding the initial circumstances $(t = 0) = \mathfrak{N}_0\, \mathfrak{N}_i$. The typical kinetic equation is denoted by the constant $c_i > 0$, where $t = 0$ represents the numerical density of species i at the moment. The solution to Equations (4.13) can be obtained by: $\mathfrak{N}_i(t) = \mathfrak{N}_0 e^{-c_i t}$. Otherwise, by removing the index i and integrating the usual kinetic Equation (4.13), we obtain:

$$\mathfrak{N}(t) - \mathfrak{N}_0 = c_0 \mathfrak{D}_t^{-1} \mathfrak{N}(t). \tag{4.14}$$

The commonly used operator for the integral is ${}_0\mathfrak{D}_t^{-1}$.

The fractional extension of Equations (4.14) is given by Haubold and Mathai [15]:

$$\mathfrak{N}(t) - \mathfrak{N}_0 = c^v {}_0\mathfrak{D}_t^{-v} \mathfrak{N}(t). \tag{4.15}$$

The standard R-L fractional integral operator ${}_0\mathfrak{D}_t^{-v}$ can be represented as follows:

$$ {}_0\mathfrak{D}_t^{-v} f(t) = \frac{1}{\Gamma(v)} \int_0^t (t-u)^{v-1} f(u)\, du, \quad \Re(v) > 0. $$

The following is obtained by solving the fractional kinetic Equation (4.15)

$$ \mathfrak{N}(t) = \mathfrak{N}_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(vk+1)} (ct)^{vk}. $$

4.4. **Solution of Fractional q-Kinetic Equations by using q-Laplace Transform.** At this point, we use the q-Laplace transform method to calculate the fractional q-kinetic equation (q-FKE), which is represented by Equation (3.1) and involves a q-generalized Mittag-Leffler function (q-GMLF). The behavior and solutions of the q-FKE can be systematically analyzed with this methodology.

Hahn [14] (also see [2]) introduced the q-analogues of the classical Laplace transforms that are widely used. He presented these analogies with the following expression for the q-integral:

$$ {}_q\mathfrak{L}_\kappa \{f(h)\} = \frac{1}{1-q} \int_0^{\kappa^{-1}} E_q^{-q\kappa h} f(h) d_q h, $$

where the q-exponential, defined in (1.1), is denoted by E_q^h.

In addition, the q-Laplace convolution theorem was given by Garg and Chanchlani [11]. It is expressed as follows:

$$ {}_q\mathfrak{L}_\kappa \{f *_q g\}(h) = {}_q\mathfrak{L}_\kappa \{f(h)\}\, {}_q\mathfrak{L}_\kappa \{g(h)\}, $$

The q-convolution of two analytic functions, $f(h)$ and $g(h)$, is denoted as $f *_q g$.

$$ f(h) *_q g(h) = \frac{1}{1-q} \int_0^h f(u) g[h-qu]\, d_q h, $$

where the function $g(h) = \sum_{n=0}^{\infty} a_n h^n$,

$$ g[h-qu] = \sum_{n=0}^{\infty} a_n (h-qu)\left(h-q^2u\right)\ldots(h-q^n u). $$

Theorem 4.6. *Let $\mathfrak{A} > 0$, $\vartheta > 0, 0 < |q| < 1$, then an associated q-FKE*

$$ \mathfrak{N}_q(t) - \mathfrak{N}_0\, t^{\xi-1} \left\{ \mathfrak{E}_{\vartheta,\xi}^{(\gamma,c);\delta}(-\mathfrak{A}t^\vartheta, q) \right\} = -\mathfrak{A}\, \mathfrak{I}_q^\vartheta \mathfrak{N}_q(t), \tag{4.16} $$

provides a subsequent result.

(4.17)

$$ \mathfrak{N}_q(t) = \mathfrak{N}_0\, t^{\xi-1} \sum_{\omega=0}^{\infty} \frac{\mathfrak{B}_q(\gamma+\omega, c-\gamma)}{\mathfrak{B}_q(\gamma, c-\gamma)} \frac{(q^c;q)_\omega}{(q^\delta;q)_\omega} (-\mathfrak{A}t^\vartheta)^\omega\, E_{\vartheta,\vartheta\omega+\xi}\left(-\mathfrak{A}t^\vartheta; q\right). $$

Proof. Use the q-Laplace transform on the two sides of the Equation (4.16), the equation now looks like this:

(4.18)
$$ {}_q\mathfrak{L}_\kappa\left\{\mathfrak{N}_q(t)\right\} - {}_q\mathfrak{L}_\kappa\left\{\mathfrak{N}_0\, t^{\xi-1}\{\mathfrak{E}_{\vartheta,\xi}^{(\gamma,c);\delta}(-\mathfrak{A}t^{\vartheta},\, q)\}\right\} = {}_q\mathfrak{L}_\kappa\left\{-\mathfrak{A}\,\mathfrak{I}_q^{\vartheta}\mathfrak{N}_q(t)\right\}. $$

Now, using Equation (3.1) in Equation (4.18), we get

(4.19)
$$ \begin{aligned} &{}_q\mathfrak{L}_\kappa\left\{\mathfrak{N}_q(t)\right\} + {}_q\mathfrak{L}_\kappa\left\{\mathfrak{A}\,\mathfrak{I}_q^{\vartheta}\mathfrak{N}_q(t)\right\} \\ &\quad= {}_q\mathfrak{L}_\kappa\left\{\mathfrak{N}_0\, t^{\xi-1}\sum_{\omega=0}^{\infty}\frac{\mathfrak{B}_q(\gamma+\omega, c-\gamma)}{\mathfrak{B}_q(\gamma, c-\gamma)}\frac{(q^c;q)_\omega}{(q^\delta;q)_\omega}\frac{\left(-\mathfrak{A}t^{\vartheta}\right)^{\omega}}{\Gamma_q(\vartheta\omega+\xi)}\right\} \\ &\quad= \mathfrak{N}_0\sum_{\omega=0}^{\infty}\frac{\mathfrak{B}_q(\gamma+\omega, c-\gamma)}{\mathfrak{B}_q(\gamma, c-\gamma)}\frac{(q^c;q)_\omega}{(q^\delta;q)_\omega}\frac{\left(-\mathfrak{A}\right)^{\omega}}{\Gamma_q(\vartheta\omega+\xi)}\,{}_q\mathfrak{L}_\kappa\left\{t^{\xi+\vartheta\omega-1}\right\}. \end{aligned} $$

As shown in [11], the q-Laplace transform of the fractional q-integral operator of the R-L is as follows:

$$ {}_q\mathfrak{L}_\kappa\left\{\mathfrak{I}_q^{\vartheta}f(t)\right\} = \left\{\tfrac{1-q}{\kappa}\right\}^{\vartheta}{}_q\mathfrak{L}_\kappa f(t), \quad \Re(\vartheta) > 0, \tag{4.20} $$

wherein [4] provides the R-L fractional q-integral operator as

$$ \mathfrak{I}_q^{\vartheta}f(t) = \frac{1}{\Gamma_q(\vartheta)}\int_0^t (t-zq)_q^{\vartheta-1}f(z)d_qz, \quad \Re(\vartheta) > 0. \tag{4.21} $$

Using Equation (4.20), the power function of q-Laplace transform

$$ {}_q\mathfrak{L}_\kappa\left\{t^{\vartheta}\right\} = \frac{\Gamma_q(\vartheta+1)(1-q)^{\vartheta}}{\kappa^{\vartheta+1}}, \quad \Re(\vartheta) > 0, $$

and ${}_q\mathfrak{L}_\kappa\left\{\mathfrak{N}_q(t)\right\} = \mathfrak{N}_q^*(\kappa)$ in Equation (4.19) , we get

$$ \begin{aligned} \mathfrak{N}_q^*(\kappa)\left(1+\frac{\mathfrak{A}(1-q)^{\vartheta}}{\kappa^{\vartheta}}\right) &= \mathfrak{N}_0\sum_{\omega=0}^{\infty}\frac{\mathfrak{B}_q(\gamma+\omega, c-\gamma)}{\mathfrak{B}_q(\gamma, c-\gamma)}\frac{(q^c;q)_\omega}{(q^\delta;q)_\omega} \\ &\quad\times\frac{(-\mathfrak{A})^{\omega}(1-q)^{\xi+\vartheta\omega-1}\Gamma_q(\vartheta\omega+\xi)}{\Gamma_q(\vartheta\omega+\xi)\;\kappa^{\vartheta\omega+\xi}}. \end{aligned} $$

The following expression is obtained by rearranging the terms and then applying the geometric series expansion formula:

$$ \begin{aligned} \mathfrak{N}_q^*(\kappa) &= \mathfrak{N}_0\sum_{\omega=0}^{\infty}\frac{\mathfrak{B}_q(\gamma+\omega, c-\gamma)}{\mathfrak{B}_q(\gamma, c-\gamma)}\frac{(q^c;q)_\omega}{(q^\delta;q)_\omega}\frac{(-\mathfrak{A})^{\omega}(1-q)^{\xi+\vartheta\omega-1}}{\kappa^{\vartheta\omega+\xi}} \\ &\quad\times\sum_{r=0}^{\infty}\frac{(-1)^r(-\mathfrak{A})^r(1-q)^{\vartheta r}}{\kappa^{\vartheta r}}. \end{aligned} \tag{4.22} $$

On both sides of (4.22), we have applied the inverse q-Laplace transform, then

$$
\begin{aligned}
{}_q\mathfrak{L}_\kappa^{-1}\left\{\mathfrak{N}_q^*(\kappa)\right\} = \mathfrak{N}_0 \sum_{\omega=0}^{\infty} \frac{\mathfrak{B}_q(\gamma+\omega, c-\gamma)}{\mathfrak{B}_q(\gamma, c-\gamma)} \frac{(q^c;q)_\omega}{(q^\delta;q)_\omega} (-\mathfrak{A})^\omega (1-q)^{\xi+\vartheta\omega-1} \\
\times \sum_{r=0}^{\infty} (-1)^r (-\mathfrak{A})^r (1-q)^{\vartheta r} {}_q\mathfrak{L}_\kappa^{-1}\left\{\kappa^{-(\vartheta r+\vartheta\omega+\xi)}\right\}.
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{N}_q(t) &= \mathfrak{N}_0 \sum_{\omega=0}^{\infty} \frac{\mathfrak{B}_q(\gamma+\omega, c-\gamma)}{\mathfrak{B}_q(\gamma, c-\gamma)} \frac{(q^c;q)_\omega}{(q^\delta;q)_\omega} (-\mathfrak{A})^\omega (1-q)^{\xi+\vartheta\omega-1} \\
&\quad \times \sum_{r=0}^{\infty} \frac{(-\mathfrak{A})^r (1-q)^{\vartheta r} t^{(\vartheta r+\vartheta\omega+\xi)}}{(1-q)^{\vartheta r+\vartheta\omega+\xi-1}\,\Gamma_q(\vartheta r+\vartheta\omega+\xi)} \\
&= \mathfrak{N}_0\, t^{\xi-1} \sum_{\omega=0}^{\infty} \frac{\mathfrak{B}_q(\gamma+\omega, c-\gamma)}{\mathfrak{B}_q(\gamma, c-\gamma)} \frac{(q^c;q)_\omega}{(q^\delta;q)_\omega} (-\mathfrak{A}t^\vartheta)^\omega \\
&\quad \times \sum_{r=0}^{\infty} \frac{\left(-\mathfrak{A}t^\vartheta\right)^r}{\Gamma_q(\vartheta r+\vartheta\omega+\xi)}.
\end{aligned} \tag{4.23}
$$

By analyzing the result, we arrive at the intended consequence from (4.23) in the context of (3.4), specifically as follows:

$$
\mathfrak{N}_q(t) = \mathfrak{N}_0 t^{\xi-1} \sum_{\omega=0}^{\infty} \frac{\mathfrak{B}_q(\gamma+\omega, c-\gamma)}{\mathfrak{B}_q(\gamma, c-\gamma)} \frac{(q^c;q)_\omega}{(q^\delta;q)_\omega} (-\mathfrak{A}t^\vartheta)^\omega E_{\vartheta,\vartheta\omega+\xi}\left(-\mathfrak{A}t^\vartheta; q\right). \quad \square
$$

4.5. **An Alternative Method Using the Sumudu Transform for Generalized Fractional Kinetic Equations.** The use of the q-Sumudu transform in the solution of a generalized fractional q-kinetic equation involving the q-generalized Mittag-Leffler function (q-GMLF), as given by Equation (3.1), is examined in this section.

The q-Sumudu transform was introduced by Albayrak et al. [5] as a tool for solving differential and integral equations in the time domain, which can be used in a variety of practical physics and engineering domains. Because of its special qualities, the q-Sumudu transform is especially useful for addressing kinetic equation issues in scientific and technical settings. It is q-Laplace transforms, expanding the set of analytical instruments for dealing with complicated systems and phenomena.

The following expression is the q-Sumudu transform, which was introduced by Albayrak et al. [5] (also described in [23]):

$$
S_q\left\{f(z);\sigma\right\} = \frac{1}{(1-q)\,\sigma} \int_0^\sigma E_q\left(\frac{q}{\sigma}t\right) d_q z, \quad \sigma \in (-\tau_1, \tau_2), \tag{4.24}
$$

$$= (q;q)_\infty \sum_{k=0}^{\infty} \frac{q^k f\left(\sigma q^k\right)}{(q;q)_k}.$$

across the set of functions

$$A = \left\{ f(z) \, \exists M, \tau_1 \tau_2 > 0, \; \left| f(z) < M E_q\left(|t|/\tau_j\right), \; z \in (-1)^j \times [0,\infty) \right| \right\}.$$

The q-Sumudu transforms of the following special functions that are utilized in the sequel:

$$S_q\left\{z^{h-1};\sigma\right\} = \sigma^{h-1}(1-q)^{h-1}\Gamma_q(h), \quad (\Re(h) > 0). \tag{4.25}$$

$$S_q\left\{\frac{(1-q)}{z} E_{h,0}\left(-\mathfrak{A} z^h; q\right); \sigma\right\} = \frac{1}{\sigma\left(1 + \mathfrak{A}\sigma^h (1-q)^h\right)}, \quad (\Re(h) > 0).$$

The following is a q-Sumudu convolution Theorem given by Albayrak et al. [5]:

$$S_q\{(f *_q g)(z);\sigma\} = \sigma \, S_q\{f(z);\sigma\} \, S_q\{g(z);\sigma\},$$

where the two analytic functions $f(t)$ and $g(t)$ provided by are q-convolutions represented by $f *_q g$.

$$(f *_q g)(z) = \frac{z}{1-q} \int_0^1 f(z\,x)\, g\left[z(1-qx)\right] d_q x,$$

where $g[z-x] = \sum_{k=0}^{\infty} a_n (z-x)_q^n$ and $g(z) = \sum_{k=0}^{\infty} a_n z^n$.

Discussion: The solution to the generalized q-FKE (4.16) is given by (4.17), assuming $0 < |q| < 1, \mathfrak{A} > 0$, $\vartheta > 0$,. By applying the q-Sumudu on the two ways of Equation (4.16), the result is

(4.26)

$$S_q\{\mathfrak{N}_q(t);\sigma\} - S_q\left\{\mathfrak{N}_0\, t^{\xi-1}\{\mathfrak{E}_{\vartheta,\xi}^{(\gamma,c);\delta}(-\mathfrak{A}t^\vartheta, q)\};\, \sigma\right\} = S_q\left\{-\mathfrak{A}\, \mathfrak{I}_q^\vartheta \mathfrak{N}_q(t);\, \sigma\right\}.$$

The q-Sumudu transform of the R-L fractional q-integral operator was defined as follows by Purohit and Ucar [23]:

$$S_q\left\{\mathfrak{I}_q^\vartheta f(t); s\right\} = s^\vartheta (1-q)^\vartheta S_q\{f(t);s\} \tag{4.27}$$

where Al-Salam [4] established the R-L fractional q-integral operator, which Agarwal [3] independently introduced:

$$\mathfrak{I}_q^\vartheta f(t) = \frac{t^{\vartheta-1}}{\Gamma_q(\vartheta)} \int_0^t (qz/t;q)_{\vartheta-1}\, d_q x, \;\; \Re\ (\vartheta) > 0\ , \quad \vartheta \notin \{-1,-2,\ldots\}.$$

This gives us the following when we use Equations (3.1) and (4.27) in Equation (4.26):

$$S_q\{\mathfrak{N}_q(t);\sigma\} + \mathfrak{A}\,\sigma^\vartheta (1-q)^\vartheta S_q\{\mathfrak{N}_q(t);\sigma\}$$

$$= S_q \left\{ \mathfrak{N}_0 t^{\xi-1} \sum_{\omega=0}^{\infty} \frac{\mathfrak{B}_q(\gamma+\omega, c-\gamma)}{\mathfrak{B}_q(\gamma, c-\gamma)} \frac{(q^c;q)_\omega}{(q^\delta;q)_\omega} \frac{\left(-\mathfrak{A} t^\vartheta\right)^\omega}{\Gamma_q\left(\vartheta\omega+\xi\right)} \right\}.$$

Substituting $S_q\left\{\mathfrak{N}_q\left(t\right);\sigma\right\} = \mathfrak{N}\left(\sigma\right)$, we have

(4.28)

$$\mathfrak{N}(\sigma) + \mathfrak{A}\,\sigma^\vartheta\left(1-q\right)^\vartheta \mathfrak{N}(\sigma) = \mathfrak{N}_0 \sum_{\omega=0}^{\infty} \frac{\mathfrak{B}_q(\gamma+\omega, c-\gamma)}{\mathfrak{B}_q(\gamma, c-\gamma)} \frac{(q^c;q)_\omega}{(q^\delta;q)_\omega} \frac{(-\mathfrak{A})^\omega}{\Gamma_q\left(\vartheta\omega+\xi\right)} \times S_q\left\{t^{\xi+\vartheta\omega-1}\right\}.$$

Using Equation (4.25), we get

$$\mathfrak{N}(\sigma)\left\{1+\mathfrak{A}\,\sigma^\vartheta\left(1-q\right)^\vartheta\right\} = \mathfrak{N}_0 \sum_{\omega=0}^{\infty} \frac{\mathfrak{B}_q(\gamma+\omega, c-\gamma)}{\mathfrak{B}_q(\gamma, c-\gamma)} \frac{(q^c;q)_\omega}{(q^\delta;q)_\omega} \frac{(-\mathfrak{A})^\omega}{\Gamma_q\left(\vartheta\omega+\xi\right)} \times \sigma^{\xi+\vartheta\omega-1}\left(1-q\right)^{\xi+\vartheta\omega-1}\Gamma_q\left(\xi+\vartheta\omega\right).$$

Following term rearrangement, we apply a geometric series expansion formula to get

$$\mathfrak{N}(\sigma) = \mathfrak{N}_0 \sum_{\omega=0}^{\infty} \frac{\mathfrak{B}_q(\gamma+\omega, c-\gamma)}{\mathfrak{B}_q(\gamma, c-\gamma)} \frac{(q^c;q)_\omega}{(q^\delta;q)_\omega} (-\mathfrak{A})^\omega\ \sigma^{\xi+\vartheta\omega-1}\left(1-q\right)^{\xi+\vartheta\omega-1} \times \sum_{r=0}^{\infty}(-1)^r(-\mathfrak{A}\sigma^\vartheta)^r(1-q)^{\vartheta r}.$$

By applying both sides of (4.27) to the inverse q-Sumudu transform, we obtain

$$\mathfrak{N}_q\left(t\right) = \mathfrak{N}_0\, t^{\xi-1} \sum_{\omega=0}^{\infty} \frac{\mathfrak{B}_q(\gamma+\omega, c-\gamma)}{\mathfrak{B}_q(\gamma, c-\gamma)} \frac{(q^c;q)_\omega}{(q^\delta;q)_\omega} (-\mathfrak{A}t^\vartheta)^\omega \sum_{r=0}^{\infty} \frac{\left(-\mathfrak{A}t^\vartheta\right)^r}{\Gamma_q(\vartheta r+\vartheta\omega+\xi)}.$$

The required result (4.17) may be readily obtained by interpreting the result (4.28) in light of (3.4).

5. Graphical Interpretation

In this part, we illustrate the graphical interpretation of the key findings using varying parameter values in (4.17). Figure 1 is the graphical solutions of Equation (4.17) by using the parameters $\mathfrak{N}_0 = 0.086625$, $\vartheta = 0.6$, $\xi = 0.5$, $\delta = 0.8$, $\gamma = 0.75$, $c = 0.25$ and $q = 0.25, 0.5, 0.75, 0.85$. Also Figure 2 is the graphical solutions of Equation (4.17) by using the parameters $\mathfrak{N}_0 = 0.086625$, $q = 0.5$, $\xi = 0.5$, $\delta = 0.8$, $\gamma = 0.75$, $c = 0.25$ and $\vartheta = 0.25, 0.5, 0.75, 0.85$. Similarly varying the values of ξ and δ, we get a graphical solution of Equation (4.17) described in Figure 3.

Figure 1 through Figure 3 present the graphical solutions of Equation (4.17), illustrating the time-dependent behavior of $N(t)$ under variations in the parameters ϑ, ξ and δ. Figure 1 explores the influence of ϑ (ranging from 0.25 to 1) on the decay profile, revealing a rapid decline in for higher values. Figure 2 demonstrates the effect of ξ (similarly varied) over an extended timescale, showing analogous decay dynamics but with adjusted temporal resolution. Figure 3 mirrors the analysis for δ, confirming consistent decay trends across parameters. Collectively, these plots validate the analytical solution of (4.17) and highlight the sensitivity of the system's relaxation rate to parameter choices. The uniformity in decay behavior suggests a universal first-order kinetic structure, while subtle differences in timescales may reflect context-dependent roles of ϑ, ξ and δ in physical applications.

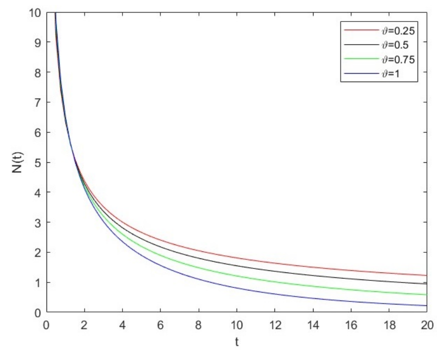

FIGURE 1. Plot of Solutions of (4.17) with $\mathcal{N}_0 = 0.086625$, $q = 0.5$, $\xi = 0.5$, $\delta = 0.8$, $\gamma = 0.75$, $c = 0.25$ and $\vartheta = 0.25, 0.5, 0.75, 0.85$.

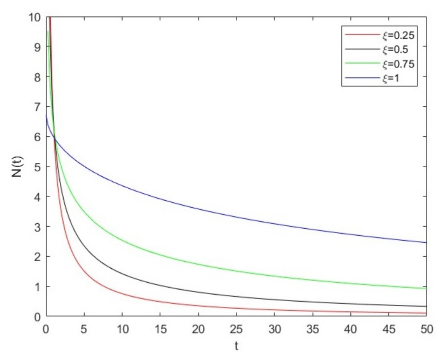

FIGURE 2. Plot of Solutions of (4.17) with $\mathcal{N}_0 = 0.086625$, $q = 0.5$, $\vartheta = 0.5$, $\delta = 0.8$, $\gamma = 0.75$, $c = 0.25$ and $\xi = 0.25, 0.5, 0.75, 0.85$.

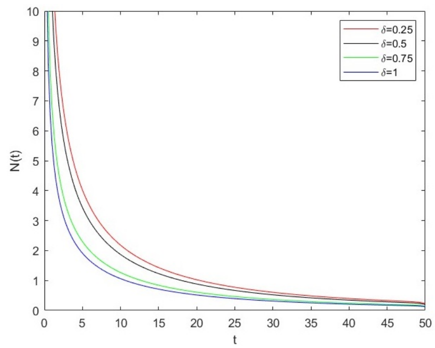

FIGURE 3. Plot of Solutions of (4.17) with $\mathcal{N}_0 = 0.086625$, $q = 0.5$, $\xi = 0.5$, $\vartheta = 0.5$, $\gamma = 0.75$, $c = 0.25$ and $\delta = 0.25, 0.5, 0.75, 0.85$.

The graph demonstrates that $N(t)$ undergoes a decaying process over time, and the parameters ϑ, ξ and δ plays a crucial role in controlling the speed of this decay. Higher values of the parameters lead to a faster decrease in $N(t)$, while lower values of the parameters result in a slower decrease.

6. Concluding Observations

We briefly discuss some ramifications of the results that came before. For example, using Theorem 4.1 and setting n=1 results in the following:

Corollary 6.1. *For* $\vartheta,\xi,\gamma,c,\delta \in \mathbb{C}, \Re(\vartheta) > 0, \Re(\delta) > 0$ *and* $\Re(\xi) > \Re(\vartheta) > 0, \Re(c) > 0$, *then*

$$\mathfrak{E}^{(\gamma,c);\delta}_{\vartheta,\xi}(z;q) = \frac{z^{\vartheta-\xi}}{(1-q)} \int_0^\infty e_q(-t/z)\, t^{\xi-\vartheta-1} \sum_{\omega=0}^{\infty} \frac{\mathfrak{B}_q(\gamma+\omega, c-\gamma)}{\mathfrak{B}_q(\gamma, c-\gamma)} \frac{(q^c;q)_\omega}{(q^\delta;q)_\omega} \times \frac{q^{\xi-\vartheta+\omega(\xi-\vartheta+\omega-1)/2} t^\omega}{\Gamma_q(\vartheta\omega+\xi)(q;q)_{\xi-\vartheta+\omega-1}} d_q t.$$

Corollary 6.1 and Theorem 4.2-4.3 provide the integral representation for the MLF in the following results for $q \to 1^-$.

Corollary 6.2. *If* $\vartheta,\xi,\gamma,c,\delta \in \mathbb{C}, \Re(\vartheta) > 0, \Re(\delta) > 0$ *and* $\Re(\xi) > \Re(\vartheta) > 0, \Re(c) > 0$, *then*

$$\mathfrak{E}^{(\gamma,c);\delta}_{\vartheta,\xi}(z) = z^{\vartheta-\xi} \int_0^\infty exp(-t/z)\, t^{\xi-\vartheta-1} \sum_{\omega=0}^{\infty} \frac{\mathfrak{B}(\gamma+\omega, c-\gamma)}{\mathfrak{B}(\gamma, c-\gamma)} \frac{(c)_\omega}{(\delta)_\omega} \times \frac{t^\omega}{\Gamma(\vartheta\omega+\xi)\Gamma(\xi-\vartheta+\omega)} dt.$$

Corollary 6.3. *For* $z, \vartheta, \xi, \gamma, c, \delta \in \mathbb{C}, \Re(\vartheta) > 0, \Re(\delta) > 0$ *and* $\Re(\xi) > \Re(\vartheta) > 0, \Re(c) > 0$, *then*

$$\mathfrak{E}_{\vartheta,\xi}^{(\gamma,c);\delta}(z) = \frac{1}{\vartheta\,\Gamma(\xi-\vartheta)} \int_0^1 \left(1 - t^{1/\vartheta}\right)^{\xi-\vartheta-1} \mathfrak{E}_{\vartheta,\vartheta}^{(\gamma,c);\delta}(zt)dt.$$

Corollary 6.4. *For* $\vartheta,\xi, \gamma, c, \delta \in \mathbb{C}, \Re(\vartheta) > 0, \Re(\delta) > 0$ *and* $\Re(\xi) > \Re(\vartheta) > 0, \Re(c) > 0$, *then*

$$\mathfrak{E}_{\vartheta,\xi}^{(\gamma,c);\delta}(z) = \frac{1}{\Gamma(\vartheta)} \int_0^1 t^{\vartheta-1} \left(1 - t\right)^{\xi-\vartheta-1} \; \mathfrak{E}_{\vartheta,\xi-\vartheta}^{(\gamma,c);\delta}(z(1-t)^{(\vartheta)}dt.$$

Further, by using the relation (3.5), we observe that the Corollaries 6.2-6.4 have the special case of the known results due to Purohit and Kalla [[22], p.23, eq. 5.3-5.5].

Moreover, we can deal with additional fractional q-kinetic equations in addition to regular q-kinetic equations by taking into account the variables indicated in the main Theorem.

The following results, which correspond to a q-analog of findings reported by Garg and Chanchlani [[11], p.39, eq. 30], are obtained using relation (3.5) from Theorem 4.6.

Corollary 6.5. *Let* $\omega > 0$, $\vartheta > 0, 0 < |q| < 1$, *then the subsequent FqKE*

$$\mathfrak{N}_q\left(t\right) - \mathfrak{N}_0\, t^{\xi-1} E_{\vartheta,\xi}^{\gamma}(-\mathfrak{A}t^{\vartheta}, q) = -\mathfrak{A}\, \mathfrak{I}_q^{\vartheta} \mathfrak{N}_q\left(t\right), \tag{6.1}$$

has the subsequent result

$$\mathfrak{N}_q\left(t\right) = \mathfrak{N}_0\, t^{\xi-1} \sum_{\omega=0}^{\infty} \frac{(q^{\gamma};q)_{\omega}}{\omega!} (-\mathfrak{A}t^{\vartheta})^{\omega}\, E_{\vartheta,\vartheta\omega+\xi}\left(-\mathfrak{A}t^{\vartheta};q\right). \tag{6.2}$$

If we take $q \to 1$ in our Theorem 4.6 with $\delta = 1$ and $c = 1$, furthermore, we claim that the results reported in this work are comprehensive enough to provide solutions to a large class of known or unknown fractional kinetic equations with special functions like those studied by Saxena et al. [29]. Moreover, our findings also tackle the issues examined by Saxena et al. ([30], [31]) as well as Haubold and Mathai [15]. Finally, we emphasize the q-Sumudu transform approach as a very efficient and useful substitute for solving fractional q-differential equations.

In conclusion, this paper makes a significant contribution by demonstrating that q-exponential functions, q-Mittag-Leffler functions (q-MLF), and q-hypergeometric functions can all be effectively extended. This is achieved through the novel application of q-Generalized Mittag-Leffler function (q-GMLF) properties and their associated integrals, as thoroughly explored within this article. We anticipate these findings will have important implications for solving various fractional q-integral and q-difference equations.

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