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## Fundamental System and Boundary Structure of Topological Krasner Hypermodules

Azam Zare<sup>1</sup> and Bijan Davvaz<sup>2\*</sup>

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ABSTRACT. In this article, we first define hyperstructures known as Krasner hypermodules. Then, the concept of topological Krasner hypermodules is explored, examining their fundamental properties and the notion of continuous mappings that exist between such topological hyperstructures. Next, the concept of Hausdorff topology is introduced and its relation to Krasner hypermodules is examined. The relationship between locally compact Krasner hypermodules and the role of open neighborhoods in their topological structure is then analyzed. Several theorems are presented and proven to clarify these relationships. By applying relative topology to subhypermodules, their associated properties are analyzed. In other words, the aim is to use specific topologies to identify the various substructural features of this type of hypermodule. Additionally, the quotient topology induced by the  $\theta^*$ -relation on the Krasner hypermodule is investigated to understand how this relation affects the topological structure of the hypermodule. Finally, it is shown that the topological Krasner hypermodule induced by  $\tau_\theta$ , the finest and strongest topology on it, ultimately forms a module.

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### 1. FOREWORD

In algebraic hyperstructure theory, a hyperoperation is a generalized operation defined as  $+ : H \times H \rightarrow \mathcal{P}^*(H)$ , where  $\mathcal{P}^*(H)$  represents the collection of all non-empty subsets of  $H$ . Unlike standard operations, a hyperoperation locates a non-empty subset of  $H$  for both members of  $H$ . For any subsets  $A, D \in \mathcal{P}^*(H)$  and an element  $r \in H$ , we define  $A + D = \bigcup_{a \in A, d \in D} (a + d)$  and  $p + A$  is symbol of  $\{p\} + A$  and  $A + p$  is  $A + \{p\}$ .

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A structure  $(H, +)$  is named a semihypergroup if the hyperoperation is associative, i.e., for all  $p, q, t \in H$ , we have,  $(p + q) + t = p + (q + t)$ . If for any member  $p$  in semihypergroup  $H$ ,  $p + H = H + p = H$ , this means that it is a hypergroup. This means adding any element to the whole set does not change the set. A non-empty subset  $\mathcal{J} \subseteq H$  is a subhypergroup if for any  $k \in \mathcal{J}$ , it holds that  $k + \mathcal{J} = \mathcal{J} + k = \mathcal{J}$ .

**Definition 1.1** ([5, 11]). A hyperstructure that has the following conditions is named a Krasner hyperring:

- (1)  $(H, +)$  is a canonical hypergroup, i.e., “+” is a hyperoperation on  $H$  so that,
  - (a) for each  $\mathbf{a}, \mathbf{b}$  and  $\mathbf{c}$  in  $H$ ,  $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$ ,
  - (b) for all  $\mathbf{a}, \mathbf{b} \in H$ ,  $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ ,
  - (c) there is  $0 \in H$  so that,  $0 + \mathbf{a} = \mathbf{a} + 0$  for every  $\mathbf{a} \in H$ ,
  - (d) for all  $\mathbf{a} \in H$  is only a  $\mathbf{a}' \in H$  so that,  $0 \in \mathbf{a} + \mathbf{a}'$ ,
  - (e) when  $\mathbf{a} \in \mathbf{b} + \mathbf{c}$ , then  $\mathbf{b} \in \mathbf{a} - \mathbf{c}$  and  $\mathbf{c} \in -\mathbf{b} + \mathbf{a}$ , for any  $\mathbf{a}, \mathbf{b}, \mathbf{c} \in H$ .
- (2)  $(H, \cdot)$  is a semigroup with zero as a two-way absorbing element, i.e.,  $0 \cdot \mathbf{a} = \mathbf{a} \cdot 0 = 0$  for any  $\mathbf{a} \in H$ .
- (3) The multiplication operation is denoted by “ $\cdot$ ” distributes over the hyperoperation represented by “+”.

A non-empty subset  $\mathcal{J}$  of the hyperring  $H$  is said to be a subhyperring of  $H$  if  $(\mathcal{J}, +, \cdot)$  is itself a hyperring. The subhyperring  $\mathcal{J}$  is a hyperideal of  $H$  if  $h \cdot k \in \mathcal{J}$  and  $k \cdot h \in \mathcal{J}$  for all  $h \in H$  and  $k \in \mathcal{J}$ . The subhyperring  $\mathcal{J}$  is said to be normal hyperideal in  $H$  if and only if  $h + \mathcal{J} - h \subseteq \mathcal{J}$  for all  $h \in H$ .

**Definition 1.2.** Let  $\mathcal{Q}$  represent a hyperideal within a Krasner hyperring denoted as  $(\mathcal{H}, +, \cdot)$ . The quotient  $\mathcal{H}/\mathcal{Q} = \{\mathbf{b} + \mathcal{Q} : \mathbf{b} \in \mathcal{H}\}$  is classified as a Krasner hyperring, specifically termed the quotient Krasner hyperring formed by  $\mathcal{H}$  and  $\mathcal{Q}$ . The operations for this hyperring are defined in the following manner: for any  $\mathbf{b}, \mathbf{s} \in \mathcal{H}$ ,

$$\begin{aligned} (\mathbf{b} + \mathcal{Q}) \oplus (\mathbf{s} + \mathcal{Q}) &= \{\mathbf{t} + \mathcal{Q} : \mathbf{t} \in \mathbf{b} + \mathbf{s}\}, \\ (\mathbf{b} + \mathcal{Q}) \odot (\mathbf{s} + \mathcal{Q}) &= (\mathbf{b} \cdot \mathbf{s}) + \mathcal{Q}. \end{aligned}$$

**Lemma 1.3** ([13]). Let  $q : X_1 \rightarrow X_2$  be a map from topological space  $X_1$  to  $X_2$ . The given statements are equivalent:

- (1)  $q : X_1 \rightarrow X_2$  is continuous,
- (2) for each open subset  $S$  of  $X_2$ ,  $q^{-1}(S)$  is open in  $X_1$ ,
- (3) for every  $t \in X_1$  and each open subset  $S$  of  $X_2$  containing  $q(t)$ , there is an open subset  $W$  of  $X_1$  containing  $t$  so that  $q(W) \subseteq S$ .

**Remark 1.4.** Consider  $H$  as a set so that  $\tau$  is a topology on it and define

$$S_V = \{S \in \mathcal{P}^*(H) \mid S \subseteq V, S \in \tau\}.$$

We put a topology on power set  $H$  which is produced by  $\mathcal{B} = \{S_V \mid V \in \tau\}$  [10].

By Lemma 1.3, hyperoperation “+” from  $H \times H$  to  $\mathcal{P}^*(H)$  is continuous, if for every basis element  $S_V$  that  $V \in \tau$ , the set

$$\{W_1 \times W_2 : W_1 + W_2 \in S_V\}$$

is open [15].

**Definition 1.5** ([17]). Suppose that  $H$  is a Krasner hyperring with topology  $\tau$  on it. Thus  $H$  is named a topological Krasner hyperring, indicated by  $(H, +, \cdot, \tau)$ , if according to the product topology on  $H \times H$  and the topology  $\tau^*$  on  $\mathcal{P}^*(H)$ ,

- (1)  $(\mathbf{r}_1, \mathbf{r}_2) \mapsto \mathbf{r}_1 + \mathbf{r}_2$  from  $H \times H$  to  $\mathcal{P}^*(H)$ ,
- (2)  $\mathbf{r} \mapsto -\mathbf{r}$  from  $H$  to  $H$ ,
- (3)  $(\mathbf{r}_1, \mathbf{r}_2) \mapsto \mathbf{r}_1 \cdot \mathbf{r}_2$  from  $H \times H$  to  $H$ ,

are continuous. Having these conditions, this topology on a topological Krasner hyperring  $H$  is called a Krasner hyperring topology.

**Example 1.6.** Consider the Krasner hyperring  $(H, +, \cdot)$ , where  $H = \{0, 1, 2\}$ , the hyperoperation “+” and the binary operation “.” defined as follows :

+	0	1	2
0	{0}	{1}	{2}
1	{1}	{1}	H
2	{2}	H	{2}

.	0	1	2
0	0	0	0
1	0	1	2
2	0	2	2

Let  $H$  be topological with  $\tau = \{\emptyset, H, \{0\}, \{0, 1\}, \{0, 2\}\}$ . Then  $(H, +, \cdot, \tau)$  is a topological Krasner hyperring.

**Definition 1.7** ([17]). A non-empty subset  $C$  of a Krasner hyperring  $H$  is a complete part, if  $C \cap \sum_{j=1}^n \mathbf{r}_j \neq \emptyset$ , this implies that  $\sum_{j=1}^n \mathbf{r}_j \subseteq C$  for any natural number  $n$  and for all  $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n$  of  $H$ .

## 2. TOPOLOGICAL KRASNER HYPERMODULES

We begin with the following definition.

**Definition 2.1.** Suppose that  $(H, +, \cdot)$  is a Krasner hyperring. The canonical hypergroup  $(M, +)$  along with the map  $\cdot : H \times M \rightarrow M$  is named a Krasner hypermodule over  $H$  if for each  $h, h_1, h_2 \in H$  and  $\mathbf{m}, \mathbf{m}_1, \mathbf{m}_2 \in M$  the following conditions are true:

- (1)  $h.(\mathbf{m}_1 + \mathbf{m}_2) = h.\mathbf{m}_1 + h.\mathbf{m}_2$ ,
- (2)  $(h_1 + h_2).\mathbf{m} = h_1.\mathbf{m} + h_2.\mathbf{m}$ ,

- (3)  $(h_1 \cdot h_2) \cdot \mathbf{m} = h_1 \cdot (h_2 \cdot \mathbf{m})$ ,  
(4)  $0_{\mathbf{H}} \cdot \mathbf{m} = 0_{\mathbf{M}}$ .

**Example 2.2.** Consider the set  $\mathbf{R} = \{0, 1, 2\}$ , so that “+” is a hyperoperation and “ $\cdot$ ” is an operation that are stated below:

+	0	1	2
0	{0}	{1}	{2}
1	{1}	{1}	R
2	{2}	R	{2}

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	2

So  $\mathbf{H}$  is a Krasner  $\mathbf{H}$ -hypermodule.

Recently, applications of approximation spaces and rough sets have been extensively explored in medical and data-driven contexts. For instance, novel neighborhood-based rough set approaches have been applied to the diagnosis of Covid-19 variants [7], while tritopological and primal approximation spaces have proven effective in medical data reduction and structural analysis [8, 14]. Moreover, the topological study of covering soft rough sets [4] and predictive models for lung cancer [6] highlight the interplay between algebraic structure and real-world problems. These works suggest potential application domains for topological Krasner hypermodules.

**Definition 2.3.** If  $(M_1, +)$  and  $(M_2, +')$  are two hypermodules over a Krasner hyperring  $\mathbf{H}$ , then a function  $\xi : M_1 \rightarrow M_2$  is named an inclusion homomorphism from  $M_1$  into  $M_2$ , if for every  $r \in \mathbf{H}$ ,  $b, z \in M_1$ ,  $\xi(b+z) \subseteq \xi(b) +' \xi(z)$  and  $\xi(r \cdot z) \subseteq r \cdot \xi(z)$  and if  $\xi(b+z) = \xi(b) +' \xi(z)$  and  $\xi(r \cdot z) = r \cdot \xi(z)$  then  $\xi$  is named a homomorphism or good homomorphism.

Assume that  $\mathbf{M}$  is a Krasner hypermodule over  $\mathbf{H}$  and  $\mathcal{N}$  is a non-empty subset of  $\mathbf{M}$ . Hence,  $\mathcal{N}$  is named a subhypermodule of  $\mathbf{M}$  if  $\mathcal{N}$  is itself a Krasner hypermodule over  $\mathbf{H}$ . On the other hand,  $(\mathcal{N}, +)$  is canonical subhypergroup of  $(\mathbf{M}, +)$  and for all  $h \in \mathbf{H}$  and  $n \in \mathcal{N}$ ,  $h \cdot n \in \mathcal{N}$ . Assume that  $\emptyset \neq I$  is a subset of a Krasner hyperring  $\mathbf{H}$ . It is classified as a left hyperideal if and only if for any elements  $\mathbf{u}, \mathbf{v} \in I$  and every  $\mathbf{w} \in \mathbf{H}$ , the set inclusion  $\mathbf{u} - \mathbf{v} \subseteq I$  is satisfied and the product  $\mathbf{w} \cdot \mathbf{u}$  belongs to  $I$ . In a similar manner, consider a non-empty subset  $\mathcal{N}$  of an  $\mathbf{H}$ -hypermodule  $\mathbf{M}$ . This subset is recognized as a subhypermodule precisely when, for any  $\mathbf{u}, \mathbf{v} \in \mathcal{N}$ , the inclusion  $\mathbf{u} - \mathbf{v} \subseteq \mathcal{N}$  holds and for each  $\mathbf{w} \in \mathbf{H}$  and  $\mathbf{u} \in \mathcal{N}$ , the element  $\mathbf{w} \cdot \mathbf{u}$  is contained in  $\mathcal{N}$ . Moreover, a subhypermodule  $\mathcal{N}$  of an  $\mathbf{H}$ -hypermodule  $\mathbf{M}$  is termed normal if and only if, for every  $m \in \mathbf{M}$ , the condition  $m + \mathcal{N} - m \subseteq \mathcal{N}$  is satisfied.

**Definition 2.4.** Suppose that  $\mathbf{H}$  is a topological Krasner hyperring and  $\mathbf{M}$  a Krasner  $\mathbf{H}$ -hypermodule that furnished with topology  $\tau$ , then  $\mathbf{M}$  is said a topological Krasner  $\mathbf{H}$ -hypermodule if:

- (1)  $(m_1, m_2) \rightarrow m_1 + m_2$  from  $M \times M$  to  $\mathcal{P}^*(M)$ ,
- (2)  $m \rightarrow -m$  from  $M$  to  $M$ ,
- (3)  $(h, m) \rightarrow h.m$  from  $H \times M$  to  $M$ ,

are continuous where  $M$  is specified topology  $\tau$  a topological space and  $M \times M$  the Cartesian product topology influenced by  $\tau$  and  $H \times M$  the Cartesian product topology determined by the topology of  $H$  and  $\tau$ . Having these conditions, topology  $\tau$  on a Krasner hypermodule is a Krasner hypermodule topology.

**Example 2.5.** Consider the Krasner hyperring  $(H, +, \cdot)$ , where  $H = \{0, 1, 2\}$ , the hyperoperation “+” and the binary operation “ $\cdot$ ” are defined as follows:

+	0	1	2
0	{0}	{1}	{2}
1	{1}	{0, 2}	{1}
2	{2}	{1}	{0}

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	0

Let  $H$  be endowed with the topology  $\tau = \{\emptyset, \{1\}, \{0, 2\}, H\}$ . Then  $(H, +, \cdot, \tau)$  is a topological Krasner hyperring.

Let  $M = \{a, b, c, d, e, f, g, h, k\}$  be a set with the hyperoperation as follows:

$\oplus$	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>h</i>	<i>k</i>
<i>a</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>h</i>	<i>k</i>
<i>b</i>	<i>b</i>	{ <i>a, c</i> }	<i>b</i>	<i>e</i>	{ <i>d, f</i> }	<i>e</i>	<i>h</i>	{ <i>g, k</i> }	<i>h</i>
<i>c</i>	<i>c</i>	<i>b</i>	<i>a</i>	<i>f</i>	<i>e</i>	<i>d</i>	<i>k</i>	<i>h</i>	<i>g</i>
<i>d</i>	<i>d</i>	<i>e</i>	<i>f</i>	{ <i>a, g</i> }	{ <i>b, h</i> }	{ <i>c, k</i> }	<i>d</i>	<i>e</i>	<i>f</i>
<i>e</i>	<i>e</i>	{ <i>d, f</i> }	<i>e</i>	{ <i>b, h</i> }	{ <i>a, c, g, k</i> }	{ <i>b, h</i> }	<i>e</i>	{ <i>d, f</i> }	<i>e</i>
<i>f</i>	<i>f</i>	<i>e</i>	<i>d</i>	{ <i>k, c</i> }	{ <i>b, h</i> }	{ <i>a, g</i> }	<i>f</i>	<i>e</i>	<i>d</i>
<i>g</i>	<i>g</i>	<i>h</i>	<i>k</i>	<i>d</i>	<i>e</i>	<i>f</i>	<i>a</i>	<i>b</i>	<i>c</i>
<i>h</i>	<i>h</i>	{ <i>g, k</i> }	<i>h</i>	<i>e</i>	{ <i>d, f</i> }	<i>e</i>	<i>b</i>	{ <i>a, c</i> }	<i>b</i>
<i>k</i>	<i>k</i>	<i>h</i>	<i>g</i>	<i>f</i>	<i>e</i>	<i>d</i>	<i>c</i>	<i>b</i>	<i>a</i>

Then  $(M, \oplus)$  is a canonical hypergroup. Now, we define the external product from  $H \times M \rightarrow M$  as follows:

$\otimes$	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>h</i>	<i>k</i>
0	<i>a</i>								
1	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>h</i>	<i>k</i>
2	<i>a</i>	<i>c</i>	<i>a</i>	<i>g</i>	<i>k</i>	<i>g</i>	<i>a</i>	<i>c</i>	<i>a</i>

Let  $M$  be topological with  $\tau' = \{\emptyset, \{d, e, f\}, \{a, b, c, g, h, k\}, M\}$ . Therefore  $(M, \oplus, \otimes)$  is a topological  $H$ -hypermodule.

The theory of topological Krasner hypermodules provides a comprehensive framework for the study of algebraic structures endowed with

topological properties. These structures facilitate the investigation of fundamental concepts such as commutativity, groups generated by subsets, boundedness and connected subsets, with potential applications in functional analysis, fuzzy systems, and generalized algebraic systems. Future research directions include extending these concepts to broader classes of topological spaces, examining interactions with other algebraic structures such as groups and hyperrings and developing homological and categorical frameworks for topological Krasner hypermodules. Such studies are expected to enhance the understanding of the interplay between algebra and topology, thereby opening new avenues in both theoretical and applied mathematics.

Here we examine the proof of the topological nature of the hypermodule. So first we recall the Lemma related to the definition of continuity:

**Lemma 2.6.** *Let  $(\mathcal{G}, +, \tau)$  be a topological hypergroup. Then, the hyperoperation  $+ : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{P}^*(\mathcal{G})$  is continuous if and only if for all  $x, y \in \mathcal{G}$  and  $U \in \tau$  so that  $x + y \subseteq U$  then there are  $V, W \in \tau$  so that  $x \in V$  and  $y \in W$  and  $V + W \subseteq U$ .*

Now we examine the three conditions for the definition of a topological hypermodule for the above example which are:

- (1)  $(\mathbf{m}_1, \mathbf{m}_2) \rightarrow \mathbf{m}_1 \oplus \mathbf{m}_2$  from  $M \times M$  to  $\mathcal{P}^*(M)$ ,
- (2)  $\mathbf{m} \rightarrow \mathbf{m}'$  from  $M$  to  $M$  ( $\mathbf{m}'$  is the inverse of the hyperoperation  $\oplus$ ),
- (3)  $(\mathbf{h}, \mathbf{m}) \rightarrow \mathbf{h} \otimes \mathbf{m}$  from  $H \times M$  to  $M$ ,

are continuous where  $M$  is specified topology  $\tau'$  a topological space and  $M \times M$  the Cartesian product topology influenced by  $\tau'$  and  $H \times M$  the Cartesian product topology determined by the topology of  $H$  and  $\tau'$ . In the example we have:

$$\tau = \{\emptyset, \{1\}, \{0, 2\}, H\}, \quad \tau' = \{\emptyset, \{d, e, f\}, \{a, b, c, g, h, k\}, M\},$$

then put:

$$U = \{d, e, f\}, \quad V = \{a, b, c, g, h, k\}, \quad S = \{1\}, \quad W = \{0, 2\},$$

then according to the definition the hyperoperation  $\oplus$ :

$$U \oplus U = V, \quad U \oplus V = U, \quad V \oplus V = V.$$

Then for every  $x \in U = \{d, e, f\}$  there are  $y \in U, z \in V$  so that  $x = y \oplus z \subseteq U \oplus V = U$  and for every  $x \in V = \{a, b, c, g, h, k\}$ , we have  $x = y \oplus z$  so that  $y \oplus z \subseteq U \oplus U = V$  or  $y \oplus z \subseteq V \oplus V = V$ . So condition (1) holds.

Also for every  $\mathbf{m} \in M$ , we have  $a \in \mathbf{m} \oplus \mathbf{m}$  so that  $a$  is zero member of  $M$ , then put  $\mathbf{m}' = \mathbf{m}$ . Then map  $\mathbf{m} \rightarrow \mathbf{m}'$  from  $M$  to  $M$  is equivalent to map  $\mathbf{m} \rightarrow \mathbf{m}$  from  $M$  to  $M$  is continuous. So condition (2) holds.

We have according to the definition operation  $\otimes$ :

$$S \otimes U = U, \quad S \otimes V = V, \quad W \otimes U = \{a, c, g, k\} \subseteq V, \quad W \otimes V = \{a, c\} \subseteq V.$$

Thus

$$\forall x \in U = \{d, e, f\}, \quad x = 1 \otimes x \in S \otimes U = U$$

and for every  $x \in V = \{a, b, c, g, h, k\}$  the following situations occur:  $a = 0 \otimes m$ , for every  $m \in M$  that  $m \in U$  or  $m \in V$ , on the other hand  $W \otimes U, W \otimes V \subseteq V$ .

$$\begin{aligned} a &= 1 \otimes a \in S \otimes V = V, \\ a &= 2 \otimes a = 2 \otimes c = 2 \otimes g = 2 \otimes k \in W \otimes V \subseteq V, \\ b &= 1 \otimes b \in S \otimes V = V, \\ c &= 1 \otimes c \in S \otimes V = V, \\ c &= 2 \otimes b = 2 \otimes h \in W \otimes V \subseteq V, \\ g &= 1 \otimes g \in S \otimes V = V, \\ g &= 2 \otimes d = 2 \otimes f \in W \otimes U \subseteq V, \\ h &= 1 \otimes h \in S \otimes V = V, \\ k &= 1 \otimes k \in S \otimes V = V, \\ k &= 2 \otimes e \in W \otimes U \subseteq V. \end{aligned}$$

So condition (3) holds.

**Theorem 2.7.** *Let  $M$  be a topological  $H$ -hypermodule.*

- (1) *For every  $a \in M$ ,  $r \rightarrow r.a$  is continuous from  $H$  to  $M$  and for each  $b \in H$ ,  $x \rightarrow b.x$  is continuous from  $M$  to  $M$ . Also if  $b$  is invertible,  $x \rightarrow b.x$  is a homeomorphism.*
- (2) *Assume that  $g$  is a function from a topological space  $\mathbf{T}$  to  $M$  that is continuous at  $\mathbf{t} \in \mathbf{T}$ , this implies that for every  $\alpha \in H$ ,  $\alpha g$  is continuous at  $\mathbf{t}$ .*

*Proof.* (1) Consider  $g$  as  $g(h, m) = h.m$  for all  $(h, m) \in H \times M$ , from  $H \times M$  to  $M$ . Also  $f_b$  is a function from  $M$  to  $H \times M$  defined by  $f_b(x) = (b, x)$ . It is obvious that  $f_b$  is continuous for every topology on  $H$  and the Cartesian product topology it defines on  $H \times M$ . Finally the function  $g \circ f_b : x \rightarrow b.x$  is continuous. Also  $g \circ f_b^{-1}$  is similarly continuous. Thus  $x \rightarrow b.x$  is a homeomorphism.

- (2) Assume that  $h \times g$  is the function from  $\mathbf{T}$  to  $H \times M$  defined by  $(h \times g)(\mathbf{t}) = (h(\mathbf{t}), g(\mathbf{t})) = (\alpha, g(\mathbf{t}))$ . As  $g$  and  $h$  are continuous at  $\mathbf{t}$ , so is  $h \times g$ , then  $\alpha g$  is simply the composite of that function with multiplication.  $\square$

**Lemma 2.8.** *Consider  $(\mathcal{G}, +)$  as a topological hypergroup and  $\mathfrak{a} \in \mathcal{G}$ . The functions  $\mathfrak{g} \rightarrow -\mathfrak{g}$  from  $\mathcal{G}$  to  $\mathcal{G}$  and  $\mathfrak{g} \rightarrow \mathfrak{g} + \mathfrak{a}$  and  $\mathfrak{g} \rightarrow \mathfrak{a} + \mathfrak{g}$  from  $\mathcal{G}$  to  $\mathcal{P}^*(\mathcal{G})$  are continuous. Also, for any open subset  $\mathcal{S}$  of  $\mathcal{G}$ ,  $-\mathcal{S}$  is open and if  $\mathcal{S}$  is a complete part of  $\mathcal{G}$ , then  $\mathfrak{a} + \mathcal{S}$  is open.*

*Proof.* To prove the this Lemma, we have taken ideas from sources [9] and [18, Theorem 2.7].  $\square$

**Remark 2.9.** Consider  $M$  as a Krasner  $H$ -hypermodule, then according to the Definition 2.1 and the properties of the distribution (1) and (2) in this definition, the function  $r \rightarrow r.a$  from  $H$  to  $M$  for every  $a \in M$  and the function  $x \rightarrow b.x$  is from  $M$  to  $M$  for each  $b \in H$  are homomorphism on the underlying additive hypergroups (in Krasner hyperring  $H$  and Krasner hypermodule  $M$ ,  $(H, +)$  and  $(M, +)$  are commutative). Clearly, if  $g : H \times M \rightarrow M$  be function scalar multiplication of an  $H$ -hypermodule  $M$ , then  $g(h, 0) = 0 = g(0, h)$ ,  $g(-h, m) = -g(h, m) = g(h, -m)$ ,  $g(-h, -m) = g(h, m)$ .

**Theorem 2.10.** *Let  $H$  be a topological Krasner hyperring,  $M$  is an  $H$ -hypermodule such that the open subsets of  $M$  are complete parts. If a topology  $\tau$  on  $M$  satisfies (1) and (2) of Definition 2.4, then  $\tau$  satisfies (3) of Definition 2.4, if and only if:*

- (1)  $(r, m) \rightarrow r.m$  from  $H \times M$  to  $M$  is continuous at  $(0, 0)$ ,
- (2) for all  $x \in M$ ,  $r \rightarrow r.x$  from  $H$  to  $M$  is continuous at zero,
- (3) for all  $r \in H$ ,  $y \rightarrow r.y$  from  $M$  to  $M$  is continuous at zero.

*Proof.* ( $\Rightarrow$ ): Considering the form of the theorem, we see that conditions (1) and (2) for defining topological Krasner hypermodule (Definition 2.4) are part of the assumptions of this theorem. Then, if condition (3) of the definition also holds,  $M$  is a topological  $H$ -hypermodule. Thus  $(r, m) \rightarrow r.m$  is continuous at  $(0, 0)$  and by Theorem 2.7 for every  $x \in M$ ,  $r \rightarrow r.x$  is continuous from  $H$  to  $M$  and for each  $r \in H$ ,  $y \rightarrow r.y$  is continuous from  $M$  to  $M$  and therefore, proofs (2) and (3) are obtained.

( $\Leftarrow$ ): Now, assuming that parts (1), (2) and (3) of the theorem hold, we must prove that  $(h, m) \rightarrow h.m$  from  $H \times M$  to  $M$  is continuous. To do this, review the Lemma 2.6. Let  $Y$  be an open set containing  $r_1.m_1$  in  $M$ . Then by Lemma 2.8, we have a neighborhood  $S$  of zero in  $M$  so that  $r_1.m_1 + S = Y$  and we have a neighborhood  $V$  of zero in  $M$  so that  $V + V + V \subseteq S$ . By part (1),  $(r, m) \rightarrow r.m$  from  $H \times M$  to  $M$  is continuous at  $(0, 0)$ . So for every neighborhood  $V$  of zero there is a neighborhood  $V_1$  of zero in  $H$  and a neighborhood  $V_2$  of zero in  $M$  such that  $V_1 \times V_2 \subseteq V$ , that is  $v.v' \in V$  for each  $(v, v') \in V_1 \times V_2$ . By part (2), for all  $x \in M$ ,  $r \rightarrow r.x$  from  $H$  to  $M$  is continuous at zero. So for every neighborhood  $V$  of zero there is a neighborhood  $U_1$  of zero in  $H$  such that  $U_1.x \subseteq V$  for every  $x \in M$ , that is  $u_1.m_1 \in V$

for every  $u_1 \in U_1$ . By part (3), for all  $r \in H$ ,  $y \rightarrow r.y$  from  $M$  to  $M$  is continuous at zero. So for every neighborhood  $V$  of zero there is a neighborhood  $U_2$  of zero in  $M$  such that  $r.U_2 \subseteq V$  for every  $r \in H$ , that is  $r_1.u_2 \in V$  for every  $u_2 \in U_2$ . Consider  $S_1 = r_1 + (U_1 \cap V_1)$  and  $S_2 = m_1 + (U_2 \cap V_2)$ . Then by Lemma 2.8,  $S_1$  and  $S_2$  are neighborhoods  $r_1$  and  $m_1$ , respectively. Suppose that  $t_1 \in S_1$  and  $t_2 \in S_2$ . Then  $t_1 - r_1 \subseteq U_1 \cap V_1$  and  $t_2 - m_1 \subseteq U_2 \cap V_2$ . Therefore, by Remark 2.9, we have  $g(t_1, t_2) = g(r_1, m_1) + g(t_1 - r_1, m_1) + g(t_1 - r_1, t_2 - m_1) + g(r_1, t_2 - m_1) \subseteq g(r_1, m_1) + V + V + V \subseteq g(r_1, m_1) + S = r_1.m_1 + S = Y$ . Thus,  $t_1, t_2 \in Y$ . That is, it was proved that there are neighborhoods  $S_1$  of  $r_1$  and  $S_2$  of  $m_1$  so that  $S_1 \times S_2 \subseteq Y$ .  $\square$

**Theorem 2.11.** *The Cartesian product of a family of topological H-hypermodules is a topological H-hypermodule.*

*Proof.* See the proof of [18, Theorem 1.6].  $\square$

**Corollary 2.12.** *Assume that  $(\tau_j)_{j \in I}$  is a family of H-hypermodule topologies on a H-hypermodule  $M$ , then  $\sup \{\tau_j | j \in I\}$  is an H-hypermodule topology.*

**Definition 2.13.** In a topological space  $\mathcal{X}$ , a fundamental system  $\mathcal{F}$  for  $\mathbf{p} \in \mathcal{X}$  is a non-empty collection of open sets that all contain the point  $\mathbf{p}$  and for any open set  $W$  that contains  $\mathbf{p}$ , there is at least one set  $V \in \mathcal{F}$  so that  $V \subseteq W$ . In other words, for every open set  $W$  containing  $\mathbf{p}$ , you can always find a set from the collection  $\mathcal{F}$  that is contained within  $W$  and still contains  $\mathbf{p}$ .

**Theorem 2.14.** *Let  $M$  be a Krasner H-hypermodule. If  $\mathcal{W}$  is a fundamental system of neighborhood of zero for a Krasner H-hypermodule topology on  $M$ , Then*

- (1) *for all  $Q \in \mathcal{W}$  there is  $P \in \mathcal{W}$  so that  $P + P \subseteq Q$ ,*
- (2) *for all  $Q \in \mathcal{W}$  there is  $P \in \mathcal{W}$  so that  $P \subseteq -Q$ ,*
- (3) *for all  $Q \in \mathcal{W}$  there is a neighborhood  $O$  of zero in  $H$  and  $P \in \mathcal{W}$  so that  $O.P \subseteq Q$ ,*
- (4) *for all  $Q \in \mathcal{W}$  and every  $m \in M$  there is a neighborhood  $O$  of zero in  $H$  so that  $O.m \subseteq Q$ ,*
- (5) *for all  $Q \in \mathcal{W}$  and every  $r \in H$  there is  $P \in \mathcal{W}$  so that  $r.P \subseteq Q$ .*

*Also, if  $\mathcal{W}$  is a filter base on  $M$  (A set  $\mathcal{W}$  of subsets of  $M$  is a filter base if and only if  $\mathcal{W} \neq \emptyset$ ,  $\emptyset \notin \mathcal{W}$  and the intersection of two members of  $\mathcal{W}$  contains a member of  $\mathcal{W}$ ) satisfying (1)-(5), then there is a unique Krasner H-hypermodule topology on  $M$  for which  $\mathcal{W}$  is a fundamental system of neighborhood of zero.*

*Proof.* The proofs of (1) and (2) is similar to [18, Theorem 3.1] and (3)-(5) follows from Theorem 2.10. Conversely, conditions (3)-(5) restate Theorem 2.10, thus  $M$  is a topological Krasner hypermodule over  $H$ . Therefore the theorem follows from conditions (1) and (2) by [18, Corollary 3.2].  $\square$

**Theorem 2.15.** *Let  $M$  be a Krasner  $H$ -hypermodule and  $I$  a hyperideal of  $H$ . If  $H$  is furnished with Krasner hyperring topology for which  $(I^k)_{k \geq 1}$  is a fundamental system of neighborhood of zero, then  $M$ , furnished with additive hypergroup topology for which  $(I^k M)_{k \geq 1}$  is a fundamental system of neighborhood of zero, is a topological Krasner hypermodule over  $H$ .*

*Proof.* Clearly  $(I^k M)_{k \geq 1}$  satisfies (3)-(5) of Theorem 2.14 if  $H$  is furnished with Krasner hyperring topology for which  $(I^k)_{k \geq 1}$  is a fundamental system of neighborhood of zero.  $\square$

### 3. BOUNDED AND LOCALLY BOUNDED KRASNER HYPERMODULES

Bounded and locally bounded Krasner hypermodules constitute a central topic, which we introduce here. Also we define locally compact Krasner hypermodules and check its relationship with locally boundedness. We remind you that compactness itself refers to a property of a space where, for any open cover of the space (a collection of open sets whose union covers the entire space), there is a finite subcover (a finite subset of the open sets that still cover the entire space).

**Definition 3.1.** Assume that  $T$  is a topological space. The space  $T$  is Hausdorff if and only if for any two different points  $\mathfrak{p}$  and  $\mathfrak{q}$  in  $T$ , there are two open neighborhoods  $P$  and  $Q$  so that  $\mathfrak{p} \in P$ ,  $\mathfrak{q} \in Q$ ,  $P \cap Q = \emptyset$  (the neighborhoods  $P$  and  $Q$  are disjoint).

**Definition 3.2** ([13]). A topological space  $\mathcal{X}$  is named to be locally compact if, for every point in the space, there is a compact neighborhood around that point. This means that around each point  $\mathfrak{p} \in \mathcal{X}$ , you can find a neighborhood that is compact.

**Definition 3.3.** A locally compact Krasner hypermodule is a Krasner hypermodule that is also a locally compact topological space. In other words, it has the structure of both a Krasner hypermodule (with its hyperoperation and multiplication) and a locally compact space.

**Definition 3.4.** Consider  $M$  as a topological Krasner  $H$ -hypermodule. A subset  $B$  of  $M$  is bounded if for any neighborhood  $\mathbf{U}$  of zero in  $M$  there is a neighborhood  $\mathbf{V}$  of zero in  $H$  so that  $\mathbf{V} \cdot B \subseteq \mathbf{U}$ .

**Definition 3.5.** A topological Krasner hypermodule  $M$  and its topology are called bounded if  $M$  is a bounded set and are locally bounded if there is a bounded neighborhood of zero.

Every subset consisting of one element of a topological hypermodule is bounded by (2) of Theorem 2.14. More generally:

**Theorem 3.6.** *If  $\mathcal{K}$  is a compact subset in topological Krasner H-hypermodule  $M$ , then  $\mathcal{K}$  is bounded.*

*Proof.* Let  $M$  be a topological Krasner hypermodule and consider a compact subset  $\mathcal{K}$  within  $M$  and an arbitrary neighborhood  $\mathcal{N}$  of zero in  $M$ . For all  $x$  member of  $M$  the mapping  $(b, x) \rightarrow b.x$  from  $H \times M$  to  $M$  is continuous, which implies continuity at  $(0, a)$ . Consequently, there exists an open set  $W_a$  containing  $a$  and an open set  $U_a$  containing zero so that  $U_a W_a \subseteq \mathcal{N}$ . Furthermore, the collection  $\{W_a : a \in \mathcal{K}\}$  forms an open cover of  $\mathcal{K}$ , so by compactness, there is a finite subset  $\mathcal{S} \subseteq \mathcal{K}$  so that  $\mathcal{K} \subseteq \bigcup_{a \in \mathcal{S}} W_a$ . Defining  $U = \bigcap_{a \in \mathcal{S}} U_a$ , we obtain an open neighborhood of zero in  $M$  that satisfies  $U\mathcal{K} \subseteq \mathcal{N}$ . Similarly, we deduce that  $\mathcal{K}U \subseteq \mathcal{N}$ , leading to the conclusion that  $\mathcal{K}$  is bounded.  $\square$

**Corollary 3.7.** *A (locally) compact Krasner hypermodule is (locally) bounded.*

**Lemma 3.8.** *Assume that  $A$  and  $B$  are subsets of a topological Krasner hyperring  $H$  and topological Krasner H-hypermodule  $M$ , respectively. Then,  $\overline{A.B} \subseteq \overline{A.B}$ .*

*Proof.* Given the third condition of the definition of topological Krasner hypermodule, Definition 2.4, its proof is straightforward.  $\square$

**Theorem 3.9.** *Suppose that  $B_1$  and  $B_2$  are bounded subsets of a topological Krasner H-hypermodule  $M$ , then so are  $\overline{B_1}$ ,  $B_1 \cup B_2$  and  $B_1 + B_2$ .*

*Proof.* Assume that  $U$  is a closed neighborhood of 0 in  $M$ . Given that  $B_1$  is bounded, using the definition of boundedness there is a neighborhood  $V$  of 0 in  $H$  so that  $V.B_1 \subseteq U$  and as scalar multiplication is continuous and according to the previous lemma, then  $V.\overline{B_1} \subseteq \overline{V.B_1} \subseteq \overline{U} = U$ . Thus  $\overline{B_1}$  is bounded. Consider  $W$  as a neighborhood of 0 so that  $W+W \subseteq U$  and  $V_1, V_2$  be a neighborhood of 0 in  $H$  in which  $V_1.B_1 \subseteq W$  and  $V_2.B_2 \subseteq W$ . This implies that  $(V_1 \cap V_2)(B_1 \cup B_2) \subseteq V_1(B_1 \cup B_2) \subseteq V_1.B_1$  and  $(V_1 \cap V_2)(B_1 \cup B_2) \subseteq V_2(B_1 \cup B_2) \subseteq V_2.B_2$ . Then,  $(V_1 \cap V_2)(B_1 \cup B_2) \subseteq V_1.B_1 \cup V_2.B_2 \subseteq W \subseteq W+W \subseteq U$ . Now, According to the first condition of the definition of Krasner hypermodule, Definition 2.1, we have  $(V_1 \cap V_2)(B_1 + B_2) \subseteq (V_1 \cap V_2).B_1 + (V_1 \cap V_2).B_2 \subseteq V_1.B_1 + V_2.B_2 \subseteq W+W \subseteq U$ .  $\square$

**Theorem 3.10.** *Suppose that  $\mathfrak{g}$  from a topological H-hypermodule  $M$  to a topological H-hypermodule  $\mathcal{N}$  is a continuous homomorphism and  $\mathcal{B}$  is a bounded subset of  $M$ , then  $\mathfrak{g}(\mathcal{B})$  is bounded subset of  $\mathcal{N}$ .*

*Proof.* Suppose that  $U$  is an open set containing zero in  $\mathcal{N}$ . Thus  $\mathfrak{g}^{-1}(U)$  is an open set containing zero in  $M$ , so there is a neighborhood  $I$  of zero in  $H$  so that  $I\mathcal{B} \subseteq \mathfrak{g}^{-1}(U)$ . Therefore  $I\mathfrak{g}(\mathcal{B}) = \mathfrak{g}(I\mathcal{B}) \subseteq \mathfrak{g}(\mathfrak{g}^{-1}(U)) \subseteq U$ .  $\square$

**Theorem 3.11.** *Consider  $\mathcal{N}$  as a subhypermodule of a topological  $H$ -hypermodule  $M$  and  $B \subseteq \mathcal{N}$ , then  $\mathcal{B}$  is bounded subset of  $M$  if and only if it is a subset of  $\mathcal{N}$  that is bounded.*

*Proof.* A subset  $\mathcal{B}$  of  $M$  is bounded if for each neighborhood  $U$  of zero in  $M$  there is a neighborhood  $V$  of zero in  $H$  so that  $V\mathcal{B}$  is a subset of  $U$ . And according to the assumption this leads to that  $V\mathcal{B} \subseteq U \cap \mathcal{N}$  and conversely. Therefore the proof is obvious.  $\square$

**Theorem 3.12.** *Let  $pr_j$  be a canonical projection from Cartesian product  $(M_j)_{j \in I}$  to  $M_j$ , where  $M_j$  is a topological Krasner  $H$ -hypermodule. Every subset  $\mathcal{B}$  of Cartesian product  $(M_j)_{j \in I}$  is bounded if and only if  $pr_j(\mathcal{B})$  is bounded of  $M_j$  for all  $j \in I$ .*

*Proof.* Consider  $\mathcal{B}$  as a bounded set. Then, by before theorem, the result is obtained. Now we check the other side. Let  $\mathcal{K}$  be the Cartesian product of  $(\mathcal{K}_j)_{j \in I}$ , that  $\mathcal{K}_j$  is a neighborhood of zero in  $M_j$  for all  $j \in I$  and for a subset  $S$  of  $I$  that is finite,  $\mathcal{K}_j = M_j$  for every  $j \in I \setminus S$ . By assumption,  $pr_j(\mathcal{B})$  is bounded, then for every  $j \in S$  there is an open set  $V_j$  containing zero in  $H$  so that  $V_j.pr_j(\mathcal{B}) \subseteq \mathcal{K}_j$ . This shows that  $(\bigcap_{j \in S} V_j) \cdot \mathcal{B} \subseteq \mathcal{K}$ .  $\square$

**Theorem 3.13.** (1) *Any subhypermodule of a (locally) bounded topological Krasner hypermodule  $\mathcal{H}$  is (locally) bounded.*  
 (2) *Consider  $M$  as a (locally) bounded topological Krasner hypermodule and  $\mathcal{I}$  is a subhypermodule of  $M$ , then  $M/\mathcal{I}$  is (locally) bounded.*  
 (3) *Assume that  $(M_j)_{j \in I}$  is a family of topological Krasner hypermodules and  $M$  is their Cartesian product, then  $M$  is bounded if and only if every  $M_j$  is bounded and  $M$  is locally bounded if and only if every  $M_j$  is locally bounded and for almost all, except a finite few  $j \in I$ ,  $M_j$  is bounded.*

*Proof.* The proofs of (1) and (2) are obtained from Theorem 3.10 and (3) from Theorem 3.12.  $\square$

The condition given in the after theorem is the original definition of a bounded set in real topological vector spaces.

**Theorem 3.14.** *A necessary condition for a subset  $B$  of a topological  $H$ -hypermodule  $M$  to be bounded is that for every sequence  $(a_n)_{n \geq 1}$  of elements of  $B$  and each sequence  $(r_n)_{n \geq 1}$  of scalars, if  $\lim_{n \rightarrow \infty} r_n = 0$ ,*

then  $\lim_{n \rightarrow \infty} r_n \cdot a_n = 0$ . If this condition exists and  $H$  is metrizable,  $B$  is bounded.

*Proof.* Consider  $\mathbf{U}$  as an open set containing zero in  $M$ , we have an open set  $\mathbf{W}$  containing zero in  $H$  so that  $\mathbf{W} \cdot B \subseteq \mathbf{U}$ . Let  $r_n \in \mathbf{W}$  for all  $n \geq m$ , then  $r_n \cdot a_n \in \mathbf{U}$  for all  $n \geq m$ . Conversely, suppose that  $H$  is metrizable and  $(\mathbf{W}_n)_{n \geq 1}$  is a fundamental decreasing sequence of neighborhood of zero in  $H$ . Suppose that  $B$  is not bounded. This implies that there is a neighborhood  $\mathbf{U}$  of zero in  $M$  so that for any  $n \geq 1$  there are  $r_n \in \mathbf{W}_n$  and  $a_n \in B$  so that  $r_n \cdot a_n \notin \mathbf{U}$ . Therefore  $\lim_{n \rightarrow \infty} r_n = 0$ , but  $(r_n \cdot a_n)_{n \geq 1}$  does not converge to zero.  $\square$

**Theorem 3.15.** *Assume that  $H$  is a topological Krasner hyperring with identity, containing a subset  $A$  of invertible elements so that  $0$  member of the closure of  $A$ . If  $U$  is a bounded neighborhood of zero in a unitary topological Krasner  $H$ -hypermodule  $M$ , then the collection  $\{a \cdot U : a \in A\}$  constitutes a fundamental system of neighborhoods of zero.*

*Proof.* Let  $r \in H^\times$  ( $H^\times$  shows the multiplicative group of its invertible elements), so  $x \rightarrow r \cdot x$  is homeomorphism from  $M$  to  $M$ , thus  $r \cdot U$  is a neighborhood of zero. Since  $U$  is bounded, If  $V$  is each neighborhood of zero in  $M$ , there is a neighborhood  $S$  of zero in  $H$  so that  $S \cdot U \subseteq V$  and there is  $a \in A \cap S$ , thus  $a \cdot U \subseteq V$ . That is, the Definition 2.13 is valid for each neighborhood  $V$  of zero.  $\square$

**Theorem 3.16.** *Suppose that  $H$  is topological Krasner hyperring with identity and if zero is adherent to  $H^\times$ , the only Hausdorff bounded unitary Krasner  $H$ -hypermodule is the zero hypermodule.*

*Proof.* Let  $M$  be a Hausdorff bounded unitary Krasner  $H$ -hypermodule, thus  $\{r \cdot M : r \in H^\times\}$  has the conditions of the previous theorem. Also  $r \cdot M = M$  for every  $r \in H^\times$ . It follow that  $M = (0)$ . (Since  $M$  is Hausdorff, this means that if  $0 \neq u \in M$ , there are open neighborhoods  $P$  and  $V$  so that  $0 \in V$ ,  $u \in P$ ,  $P \cap V = \emptyset$ . On the other hand, there is  $r \in H^\times$ , so that  $r \cdot M = M \subseteq V$ , which is a contradiction).  $\square$

#### 4. THE QUOTIENT HYPERMODULES AND $\theta^*$ -RELATION IN TOPOLOGICAL KRASNER HYPERMODULES

In the section we study topological quotient Krasner hypermodules and define a fundamental relation on Krasner hypermodules and show that by defining a finest topology we will have a module.

**Definition 4.1** ([5]). Consider Krasner hyperring  $(H, +, \cdot)$  and hyperideal  $I$  of it. Hence, we define the following relation:

$$\mathbf{a} \equiv \mathbf{b}(\text{mod } I) \leftrightarrow \mathbf{a} \in \mathbf{b} + I$$

for every  $\mathbf{a}, \mathbf{b} \in \mathbf{H}$ . The fact that it is an equivalence relation and the set of all classes is  $\mathbf{H}/\mathbf{I} = \{\mathbf{a} + \mathbf{I} : \mathbf{a} \in \mathbf{H}\}$ .

**Theorem 4.2.** *Let  $\mathbf{I}$  be a hyperideal of a Krasner hyperring  $(\mathbf{H}, +, \cdot)$ . Then,  $(\mathbf{H}/\mathbf{I}, \oplus, \odot)$  is Krasner hyperring, called quotient Krasner hyperring of  $\mathbf{H}$  by  $\mathbf{I}$ , where  $(\mathbf{b} + \mathbf{I}) \oplus (\mathbf{s} + \mathbf{I}) = \{\mathbf{t} + \mathbf{I} : \mathbf{t} \in \mathbf{b} + \mathbf{s}\}$  and  $(\mathbf{b} + \mathbf{I}) \odot (\mathbf{s} + \mathbf{I}) = (\mathbf{b} \cdot \mathbf{s}) + \mathbf{I}$  for  $\mathbf{b}, \mathbf{s} \in \mathbf{H}$  [5].*

**Remark 4.3** ([16]). Suppose that  $\mathbf{I}$  is a hyperideal of a krasner hyper-ring  $\mathbf{H}$  that is normal, so  $\mathbf{b} \in \mathbf{s} + \mathbf{I}$  if and only if  $\mathbf{s} - \mathbf{b} \cap \mathbf{I} \neq \emptyset$ .

**Definition 4.4** ([5]). Let  $\mathbf{H}$  represent a Krasner hyperring and let  $\mathbf{I}$  be a normal hyperideal. In this context, we introduce the following relation:

$$\mathbf{r} \equiv \mathbf{s}(\text{mod } \mathbf{I}) \leftrightarrow \mathbf{r} - \mathbf{s} \cap \mathbf{I} \neq \emptyset.$$

This relation is denoted by  $\mathbf{rI}^*\mathbf{s}$ . The relation  $\mathbf{I}^*$  is an equivalence relation.

Assume that  $\mathbf{I}$  is a normal hyperideal of  $\mathbf{H}$ , hence  $\mathbf{I} + x = \mathbf{I}^*(x)$  for all  $x \in \mathbf{H}$ . Consider the set  $[\mathbf{H} : \mathbf{I}^*] = \{\mathbf{I}^*(x) | x \in \mathbf{H}\}$ , which is defined on it the hyperoperation  $\oplus$  as

$$\mathbf{I}^*(x_1) \oplus \mathbf{I}^*(x_2) = \{\mathbf{I}^*(z) | z \in \mathbf{I}^*(x_1) + \mathbf{I}^*(x_2)\},$$

and the multiplication  $\odot$  as

$$\mathbf{I}^*(x_1) \odot \mathbf{I}^*(x_2) = \mathbf{I}^*(x_1 \cdot x_2).$$

If  $\mathbf{I}$  is a normal hyperideal of  $\mathbf{H}$ , then  $(\mathbf{I} + x_1) + (\mathbf{I} + x_2) = \mathbf{I} + x_1 + x_2$  for every  $x_1, x_2 \in \mathbf{H}$  and  $\mathbf{I} + x_1 = \mathbf{I} + x_2$  for all  $x_2 \in \mathbf{I} + x_1$  [5]. So we show:

$$\begin{aligned} \mathbf{I}^*(x_1) \oplus \mathbf{I}^*(x_2) &= \{\mathbf{I}^*(z) | z \in \mathbf{I}^*(x_1) + \mathbf{I}^*(x_2)\} \\ &= \{\mathbf{I} + z | z \in \mathbf{I} + x_1 + \mathbf{I} + x_2 = \mathbf{I} + x_1 + x_2\} \\ &= \{\mathbf{I} + z | z \in \mathbf{I} + h, h \in x_1 + x_2\} \\ &= \{\mathbf{I} + h | h \in x_1 + x_2\} = \mathbf{I}^*(h), \quad \forall h \in x_1 + x_2. \end{aligned}$$

Thus  $[\mathbf{H} : \mathbf{I}^*]$  is a krasner ring.

**Remark 4.5.** Consider  $\mathbf{I}$  as a hyperideal of a topological Krasner hyperring  $(\mathbf{H}, +, \cdot, \tau)$  and  $\phi : \mathbf{H} \rightarrow \mathbf{H}/\mathbf{I}$ , so that  $\phi(\mathbf{h}) = \mathbf{h} + \mathbf{I}$  for each  $\mathbf{h} \in \mathbf{H}$ . We topologize the set  $\mathbf{H}/\mathbf{I}$  by announcing the map  $\phi$  to be a quotient, i.e., a subset  $\mathcal{B}$  of  $\mathbf{H}/\mathbf{I}$  is open in  $\mathbf{H}/\mathbf{I}$  if and only if  $\phi^{-1}(\mathcal{B})$  is open in  $\mathbf{H}$ .

**Lemma 4.6.** *Let  $\mathcal{B}$  be a topological Krasner hyperring(hypermodule) and  $\mathcal{D}$  be a normal hyperideal(subhypermodule) of  $\mathcal{B}$ . If each open subset of  $\mathcal{B}$  is a complete part, natural mapping  $\mathbf{b} \mapsto \mathbf{b} + \mathcal{D}$  of  $\mathcal{B}$  onto  $\mathcal{B}/\mathcal{D}$  is open.*

*Proof.* To prove this lemma, we use [9, Lemma 4.4]. □

**Theorem 4.7.** *Let  $\mathcal{H}$  be a topological Krasner hyperring and every open subset of  $\mathcal{C}$  is a complete part. Then,  $(\mathcal{H}/\mathcal{C}, \oplus, \odot)$  is a topological Krasner hyperring.*

*Proof.* We prove that the hyperoperation  $\oplus$  and operation  $\odot$  and the map  $\mathbf{h} + \mathcal{C} \rightarrow -(\mathbf{h} + \mathcal{C})$  are continuous. Consider  $\mathbf{r} + \mathcal{C}, \mathbf{s} + \mathcal{C} \in \mathcal{H}/\mathcal{C}$  and  $\mathcal{J}$  is an open subset of  $\mathcal{H}/\mathcal{C}$  so that  $\mathbf{r} + \mathcal{C} \oplus \mathbf{s} + \mathcal{C} \subseteq \mathcal{J}$ . According to Lemma 4.6, we consider natural mapping  $\rho : \mathcal{H} \rightarrow \mathcal{H}/\mathcal{C}$ . Then,  $\mathbf{r} + \mathbf{s} \subseteq \rho^{-1}(\mathcal{J})$ . Since  $\rho^{-1}(\mathcal{J})$  is open in  $\mathcal{H}$ , there is open subset  $\mathcal{T}$  and  $\mathcal{Q}$  of  $\mathcal{H}$  including  $\mathbf{r}$  and  $\mathbf{s}$ , respectively, so that  $\mathcal{T} + \mathcal{Q} \subseteq \rho^{-1}(\mathcal{J})$ . Thus by Lemma 4.6 implies that  $\rho(\mathcal{T})$  and  $\rho(\mathcal{Q})$  are open in  $\mathcal{H}/\mathcal{C}$  including  $\mathbf{r} + \mathcal{C}$  and  $\mathbf{s} + \mathcal{C}$ , respectively, so that  $\rho(\mathcal{T}) \oplus \rho(\mathcal{Q}) \subseteq \mathcal{J}$ . Thus, the hyperoperation  $\oplus$  is continuous. It is proved similarly, the operation  $\odot$  is continuous.

To complete the proof, we consider  $\mathcal{J}$  as an open subset of  $\mathcal{H}/\mathcal{C}$  so that  $-(\mathbf{h} + \mathcal{C}) = -\mathbf{h} + \mathcal{C} \in \mathcal{J}$ . Then,  $-\mathbf{h} \in \rho^{-1}(\mathcal{J})$ . Thus, there is an open subset  $S$  in  $\mathcal{H}$  so that  $-\mathbf{h} \in -S \subseteq \rho^{-1}(\mathcal{J})$ , so  $\rho(-\mathbf{h}) = -\mathbf{h} + \mathcal{C} \in \rho(-S) \subseteq \mathcal{J}$  and  $\rho(-S)$  is open in  $\mathcal{H}/\mathcal{C}$ .  $\square$

**Corollary 4.8.** *If  $\mathbf{H}$  is a topological Krasner hyperring and  $\mathbf{I}$  a normal hyperideal of  $\mathbf{H}$ , then  $(\mathbf{H}/\mathbf{I}, \oplus, \odot)$  is a topological Krasner ring.*

**Definition 4.9** ([16]). Assume that  $\mathcal{K}$  is a subhypermodule of a Krasner hypermodule  $\mathbf{M}$ , then we define

$$m \equiv m' \pmod{\mathcal{K}} \leftrightarrow m \in m' + \mathcal{K}$$

for all  $m, m' \in \mathbf{M}$ . This relation is denoted by  $m\mathcal{K}^*m'$ . The fact that the relation is an equivalence relation.

**Theorem 4.10.** *Let  $\mathbf{M}$  be a Krasner  $\mathbf{H}$ -hypermodule and  $\mathbf{I}$  be a hyperideal of  $\mathbf{H}$  and  $\mathcal{K}$  be a subhypermodule of  $\mathbf{M}$ . Then, for all  $m, m_1, m_2 \in \mathbf{M}$ ,  $r \in \mathbf{H}$ ,  $[\mathbf{M} : \mathcal{K}^*]$  is a  $[\mathbf{H} : \mathbf{I}^*]$ -hypermodule so that*

$$\begin{aligned} \mathcal{K}^*(m_1) \oplus \mathcal{K}^*(m_2) &= \{\mathcal{K}^*(x) \mid x \in \mathcal{K}^*(m_1) + \mathcal{K}^*(m_2)\}, \\ \mathbf{I}^*(r) \odot \mathcal{K}^*(m) &= \mathcal{K}^*(r.m) \end{aligned}$$

and  $[\mathbf{M} : \mathcal{K}^*]$  is an  $\mathbf{H}$ -hypermodule so that

$$\begin{aligned} \mathcal{K}^*(m_1) \oplus \mathcal{K}^*(m_2) &= \{\mathcal{K}^*(x) \mid x \in \mathcal{K}^*(m_1) + \mathcal{K}^*(m_2)\}, \\ r \odot \mathcal{K}^*(m) &= \mathcal{K}^*(r.m), \end{aligned}$$

*Proof.* The theorem is easily proved by examining the definition of Krasner hypermodule.  $\square$

**Remark 4.11.** Consider  $\mathcal{K}$  as a subhypermodule of a topological Krasner hypermodule  $\mathbf{M}$  so that it is normal and  $\phi : \mathbf{M} \rightarrow \mathbf{M}/\mathcal{K}$ , defined by  $\phi(\mathbf{r}) = \mathbf{r} + \mathcal{K}$  for  $\mathbf{r} \in \mathbf{M}$ . Considering  $\phi$  as a quotient mapping, the set

$M/\mathcal{K}$  is a topological space, i.e., a subset  $\mathcal{Z}$  of  $M/\mathcal{K}$  is open in  $M/\mathcal{K}$  if and only if  $\phi^{-1}(\mathcal{Z})$  is open in  $M$ .

**Theorem 4.12.** *Let  $M$  be a topological Krasner  $H$ -hypermodule and  $I$  be a hyperideal of  $H$  and  $\mathcal{K}$  be a subhypermodule of  $M$ . This implies that,  $[M : \mathcal{K}^*]$  is a topological Krasner  $[H : I^*]$ -hypermodule.*

*Proof.* We show that the hyperoperation  $\oplus$  and operation  $\odot$  and the map  $m + \mathcal{K} \rightarrow -(m + \mathcal{K})$  are continuous. To check continuity, we need to recall Lemma 2.6, which will be used in the proof. We consider natural mappings  $\rho$  from  $M$  onto  $M/\mathcal{K}$  and  $\rho'$  from  $H$  onto  $H/I$ . Consider  $\mathbf{r} + \mathcal{K}, \mathbf{s} + \mathcal{K} \in M/\mathcal{K}$  and  $\mathcal{J}$  which is a member of the topology defined on  $M/\mathcal{K}$  so that  $\mathbf{r} + \mathcal{K} \oplus \mathbf{s} + \mathcal{K} \subseteq \mathcal{J}$ . Then,  $\mathbf{r} + \mathbf{s} \subseteq \rho^{-1}(\mathcal{J}) = \{t | \rho(t) = t + \mathcal{K} \in \mathcal{J}\}$ . Since  $\rho^{-1}(\mathcal{J})$  is open in  $M$ , by Lemma 2.6, there is open subset  $\mathcal{T}$  and  $\mathcal{Q}$  of  $M$  including  $\mathbf{r}$  and  $\mathbf{s}$ , respectively, so that  $\mathcal{T} + \mathcal{Q} \subseteq \rho^{-1}(\mathcal{J})$ . We claim that  $\rho(\mathcal{T})$  and  $\rho(\mathcal{Q})$  are open subsets in  $M/\mathcal{K}$  including  $\mathbf{r} + \mathcal{K}$  and  $\mathbf{s} + \mathcal{K}$ , respectively, so that  $\rho(\mathcal{T}) \oplus \rho(\mathcal{Q}) \subseteq \mathcal{J}$ , which shows, the hyperoperation  $\oplus$  is continuous. The validity of this claim follows from the definition of  $\oplus$  in the previous theorem, since we have:

$$\begin{aligned} \rho(\mathcal{T}) \oplus \rho(\mathcal{Q}) &= \mathcal{T} + \mathcal{K} \oplus \mathcal{Q} + \mathcal{K} \\ &= \bigcup_{x \in \mathcal{T}, y \in \mathcal{Q}} x + \mathcal{K} \oplus y + \mathcal{K} \\ &= \bigcup_{x \in \mathcal{T}, y \in \mathcal{Q}} \{a + \mathcal{K} | a \in (x + \mathcal{K}) + (y + \mathcal{K})\}. \end{aligned}$$

So if  $z + \mathcal{K} \in \rho(\mathcal{T}) \oplus \rho(\mathcal{Q})$ , we have  $z + \mathcal{K} \in (x + \mathcal{K}) + (y + \mathcal{K}) = (x + y) + \mathcal{K} \subseteq \mathcal{J}$  because,  $x + y \subseteq \mathcal{T} + \mathcal{Q} \subseteq \rho^{-1}(\mathcal{J})$  and so  $\rho(x + y) = (x + y) + \mathcal{K} \subseteq \mathcal{J}$ . Consider  $m + \mathcal{K} \in M/\mathcal{K}$ ,  $r + I \in H/I$  and  $\mathcal{W}$  is an open subset of  $M/\mathcal{K}$  so that  $r + I \odot m + \mathcal{K} \subseteq \mathcal{W}$ , so  $r.m + \mathcal{K} \subseteq \mathcal{W}$ . Then,  $r.m \subseteq \rho^{-1}(\mathcal{W})$ . Since  $\rho^{-1}(\mathcal{W})$  is open in  $M$ , by Lemma 2.6, there is open subset  $\mathcal{Q}$  of  $M$  and  $\mathcal{T}$  of  $H$  including  $m$  and  $r$ , respectively, so that  $\mathcal{T}.\mathcal{Q} \subseteq \rho^{-1}(\mathcal{W})$ . This yields that,  $\rho(\mathcal{Q})$  is open in  $M/\mathcal{K}$  including  $m + \mathcal{K}$  and  $\rho'(\mathcal{T})$  is open in  $H/I$  including  $r + I$ , so that  $\rho'(\mathcal{T}) \odot \rho(\mathcal{Q}) \subseteq \mathcal{W}$ . Thus, the operation  $\odot$  is continuous.

Next, we want to prove the map  $m + \mathcal{K} \rightarrow -(m + \mathcal{K})$  from  $M/\mathcal{K}$  to  $M/\mathcal{K}$  is continuous, so we assume  $m + \mathcal{K} \in M/\mathcal{K}$  and  $\mathcal{J}$  an open subset of  $M/\mathcal{K}$  so that  $-(m + \mathcal{K}) = -m + \mathcal{K} \in \mathcal{J}$ . Then,  $-m \in \rho^{-1}(\mathcal{J})$ . Since  $\rho^{-1}(\mathcal{J})$  is open subset in  $M$  including  $-m$ , according to the second condition of the Definition 2.4, there exists open subset  $S$  of  $M$  including  $m$  so that  $-S \subseteq \rho^{-1}(\mathcal{J})$  and so  $\rho(-S) \subseteq \mathcal{J}$  such that  $\rho(S)$  is open subset in  $M/\mathcal{K}$  including  $m + \mathcal{K}$ .  $\square$

**Theorem 4.13.** *Let  $M$  be a Krasner  $H$ -hypermodule. If  $\mathcal{K}$  is a normal subhypermodule of  $M$  and  $I$  is a normal hyperideal of  $H$ , then  $[M : \mathcal{K}^*]$  is a  $[H : I^*]$ -module.*

*Proof.* For proof, see the similar theorem in [16]. □

**Theorem 4.14.** *Let  $M$  be a topological Krasner  $H$ -hypermodule and every open subset of  $H$  and  $M$  are complete part. If  $I$  be a normal hyperideal of  $H$  and  $\mathcal{K}$  be a normal subhypermodule of  $M$ , then  $[M : \mathcal{K}^*]$  is a topological  $[H : I^*]$ -module.*

*Proof.* For proof, we use Theorems 4.12, 4.13. □

**Definition 4.15** ([3]). Assume that  $H$  is a Krasner hyperring and  $M$  is a hypermodule over  $H$ . We define the relation  $\theta: a\theta b \iff \exists n \in \mathbb{N}, \exists(m_1, \dots, m_n) \in M^n, \exists(k_1, \dots, k_n) \in \mathbb{N}^n, \exists\sigma \in \mathbb{S}_n$  and  $\exists(x_{i1}, \dots, a_{ik_i}) \in H^{k_i}, \exists\sigma_{ij} \in \mathbb{S}_{k_{ij}}, \exists\sigma_i \in \mathbb{S}_{n_i}, (i = 1, \dots, n)$  so that  $a \in \sum_{i=1}^n m'_i; m'_i = m_i$

or  $m'_i \in \sum_{j=1}^{n_i} \left( \prod_{k=1}^{k_{ij}} x_{ijk} \right) m_i$  and  $b \in \sum_{i=1}^n m'_{\sigma(i)}$  where  $m'_{\sigma(i)} = m_{\sigma(i)}$  if

$m'_i = m_i$  and  $m'_{\sigma(i)} = D_{\sigma(i)} m_{\sigma(i)}$  if  $m'_i = \sum_{j=1}^{n_i} \left( \prod_{k=1}^{k_{ij}} x_{ijk} \right) m_i$  with

$$D_i = \sum_{j=1}^{n_i} A_{i\sigma_i(j)}, \quad A_{ij} = \prod_{k=1}^{k_{ij}} x_{ij\sigma_{ij}(k)}.$$

The relation  $\theta$  has reflective and symmetrical properties. Consider  $\theta^*$  as the transitive closure of the relation  $\theta$ . As a result, this relation naturally establishes a strongly regular relation [2] on both  $(M, +)$  and  $M$  as an  $H$ -hypermodule (proof in [3, Lemma 2.2]). A strongly regular relation has the property that the equivalence classes partition the set in such a way that the structure of the operation remains well-defined across these classes. Furthermore, the quotient  $M/\theta^*$  forms an Abelian group that is an  $H/\alpha^*$ -module, where  $H/\alpha^*$  is a commutative ring (proof in [3, Theorem 2.3]). Notably,  $\theta^*$  represents the minimal equivalence relation so that the (Abelian) quotient  $M/\theta^*$  forms an  $H/\alpha^*$ -module (a similar argument is presented in [3, Theorem 2.4]). In this construction, the operations  $\oplus$  and  $\odot$  on  $M/\theta^*$  are defined in the conventional way:

$$\begin{aligned} \theta^*(x) \oplus \theta^*(t) &= \theta^*(a) \text{ for every } a \in \theta^*(x) + \theta^*(t), \\ \alpha^*(r) \odot \theta^*(t) &= \theta^*(b) \text{ for every } b \in \alpha^*(r) \cdot \theta^*(t). \end{aligned}$$

**Definition 4.16** ([3, 19]). Let  $M$  be an  $H$ -hypermodule and  $C$  is a non-empty subset of  $M$ . We consider  $C$  as a  $\theta$ -part of  $M$  if for any  $n \in \mathbb{N}$  and  $\sigma \in \mathbb{S}_n$  and for all  $\{m'_1, \dots, m'_n\}$

$$\sum_{i=1}^n m'_i \cap C \neq \emptyset \quad \Rightarrow \quad \sum_{i=1}^n m'_{\sigma(i)} \subseteq C.$$

Also  $C$  is named to be a complete part of  $M$ , if  $\sigma$  is the identity ( $m'_i$  is the notation that has been defined in Definition 4.15).

**Theorem 4.17.** *Let  $M$  be a topological Krasner  $H$ -hypermodule so that any open subset of  $\mathcal{H}$  is a  $\theta$ -part. Then,  $(M/\theta^*, \oplus, \odot)$  is a topological  $H/\alpha^*$ -module.*

**Theorem 4.18.** *Let  $(M, +)$  be a topological Krasner hypermodule with topology  $\tau$  and  $W \in \tau$  so that  $W$  is a  $\theta$ -part. Since  $(M, +)$  is canonical hypergroup, it follows that  $W = \bigcup_{v \in W} \theta^*(v)$ .*

*Proof.* Obviously,  $W \subseteq \bigcup_{v \in W} \theta^*(v)$ . Suppose that  $x \in W$  and  $y \in \theta^*(x)$ . Then,  $\exists n \in \mathbb{N}, \exists (m'_1, \dots, m'_n) \in M^n, \exists (k_1, \dots, k_n) \in \mathbb{N}^n, \exists \sigma \in \mathbb{S}_n$  and  $\exists (x_{i1}, \dots, x_{ik_i}) \in H^{k_i}, \exists \sigma_{ij} \in \mathbb{S}_{k_{ij}}, \exists \sigma_i \in \mathbb{S}_{n_i}, (i = 1, \dots, n)$  so that

$$x \in \sum_{i=1}^n m'_i; m'_i = m_i \text{ or } m'_i \in \sum_{j=1}^{n_i} \left( \prod_{k=1}^{k_{ij}} x_{ijk} \right) m_i \text{ and } y \in \sum_{i=1}^n m'_{\sigma(i)}$$

where  $m'_{\sigma(i)} = m_{\sigma(i)}$  if  $m'_i = m_i$  and  $m'_{\sigma(i)} = D_{\sigma(i)} m_{\sigma(i)}$  if  $m'_i = \sum_{j=1}^{n_i} \left( \prod_{k=1}^{k_{ij}} x_{ijk} \right) m_i$  with

$$D_i = \sum_{j=1}^{n_i} A_{i\sigma_i(j)}, \quad A_{ij} = \prod_{k=1}^{k_{ij}} x_{ij\sigma_{ij}(k)}.$$

According to the property of the set  $W$ , it follows that  $\sum_{i=1}^n m'_{\sigma(i)} \subseteq W$

and also  $y \in \sum_{i=1}^n m'_{\sigma(i)}$ , thus  $y \in W$  and it follows  $\theta^*(x) \subseteq W$ . This means that,  $W = \bigcup_{v \in W} \theta^*(v)$ .  $\square$

**Lemma 4.19.** *Suppose that  $(M, +)$  is a Krasner hypermodule and  $\theta^*$  is the fundamental relation on  $M$ . Then,  $\mathcal{B} = \{\theta^*(\mathbf{z}) \mid \mathbf{z} \in M\}$  is a base for a topology on  $M$  and any open subset of  $M$  is a  $\theta$ -part.*

*Proof.* Since  $M = \bigcup_{\mathbf{z} \in M} \theta^*(\mathbf{z})$ , it follows that  $\mathcal{B}$  is a base for a topology on  $M$ . It is easy to see that any open subset of  $M$  is a  $\theta$ -part.  $\square$

In the continuation, we show the topology in the previous lemma by  $\tau_\theta$ .

**Theorem 4.20.** *Consider  $(M, +)$  as a Krasner hypermodule. If  $\theta^*$  is the fundamental relation on  $M$ , this implies that  $\tau_\theta$  is the finest topology on  $M$  so that  $M$  a topological Krasner hypermodule and any open subset of  $M$  is a  $\theta$ -part.*

*Proof.* Consider  $m_1, m_2 \in M$  so that  $m_1 + m_2 \subseteq S$  for some open subset  $S$  of  $M$ . So by Theorem 4.18, we know  $S = \bigcup_{u \in S} \theta^*(u)$ . Thus, there is  $u \in S$  so that  $m_1 + m_2 \subseteq \theta^*(u)$ . As a result,  $\theta^*(m_1) \oplus \theta^*(m_2) \subseteq \theta^*(u) \subseteq S$  and  $\theta^*(m_1)$  and  $\theta^*(m_2)$  are open subsets of  $M$  including  $m_1$  and  $m_2$ , respectively. Thus, the hyperoperation “+” is continuous. Let  $r \in H$  and  $m \in M$  so that  $r.m \in S$  for some subset  $S$  of  $M$  that is open. Thus, there is  $u \in S$  so that  $r.m \in \theta^*(u)$ . Thus,  $\theta^*(r) \odot \theta^*(m) \in \theta^*(u) \subseteq S$  and  $\theta^*(r)$  and  $\theta^*(m)$  are subsets of  $M$  including  $r$  and  $m$ , respectively, so that are open. Thus, the scalar product “.” is continuous. Now, consider  $\tau$  as a topology on  $M$  so that every open subset of  $(M, \tau)$  is a  $\theta$ -part and  $(M, +, \cdot, \tau)$  is a topological Krasner hypermodule. Let  $a \in S$  and  $S \in \tau$ . Then, by Theorem 4.18, we have  $S = \bigcup_{u \in S} \theta^*(u)$ . Thus,  $\theta^*(a) \subseteq S$  and  $\theta^*(a)$  is subset of  $(M, \tau_\theta)$  that is open. In this way, the desired result is achieved.  $\square$

By using the previous theorem, assume that  $(M, +)$  is a Krasner hypermodule and  $\theta^*$  is the fundamental relation on  $M$ . This shows that,  $\tau_\theta$  is considered as the collection of all possible unions of sets  $\theta^*(u)$  for subsets  $S$  of  $M$  and any element  $u$  of  $S$ , along with the empty set, is a topology on  $M$  and  $(M, +, \cdot, \tau_\theta)$  is a topological Krasner hypermodule.

**Theorem 4.21.** *Let  $(M, +, \tau_\theta)$  be a topological Krasner hypermodule and  $T_0$  space(a topological space is a  $T_0$  space if for any two distinct points, at least one of them has a neighborhood that does not include the other). Then,  $M$  is a module.*

*Proof.* We prove that  $|m_1 + m_2| = 1$  for all  $m_1, m_2 \in M$ . Suppose for the contradiction that  $a, b \in m_1 + m_2$  and  $a \neq b$ . Since  $M$  is  $T_0$ , it implies that there is an open subset  $S$  of  $M$  including exactly one of  $a$  or  $b$ . Let  $a \in S$  and  $b \notin S$ . Then, by Theorem 4.18,  $a \in \theta^*(u)$  for some  $u \in S$ . Also by Definition 4.15,  $\theta^*(b) = \theta^*(a)$ . Thus,  $b \in \theta^*(b) = \theta^*(a) = \theta^*(u)$ . As a result,  $b \in S$  and it is a contradiction. So  $|m_1 + m_2| = 1$ . Thus,  $M$  is a module.  $\square$

## 5. CONCLUSION

This article deals with one of the theories of hyperstructures, namely Krasner topological hypermodules. Then, their properties are mainly

studied in submodules. Topological hypermodules can play an important role in the study of topological spaces and their properties. One of the important subsets is bounded sets, whose definition is expressed in this type of hyperstructure. In the following, the necessary conditions for sets with special properties to be bounded in Krasner hypermodules are presented. Finally, the properties of bounded subhypermodules are studied, and also by defining a fundamental relation on hypermodules, we obtain the conditions under which a topological hypermodule can be transformed into a module by constructing a finest topology. As we continue we can to continue the study of bounded sets in Krasner topological hypermodules and take a look at how closed and bounded sets function in this structure. It will also be interesting to study compact sets and how connected and compact sets are related.

Topological Krasner hypermodules are valuable tools in fuzzy algebra due to their unique structural and topological properties. They provide a precise framework for modeling fuzzy systems. By combining with fuzzy algebra, these hypermodules help build richer algebraic systems that can represent different fuzzy states. Furthermore, they help define and study of operations on fuzzy sets and contribute to the development of fuzzy set theory.

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