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Advancements in Convex Analysis Through Inverse Cosine Function with Applications

Atika Imran[1], Muhammad Samraiz[2*] and Saima Naheed[3]

Abstract. In this article, we introduce a new class of convex functions called α-inverse cosine convex functions (α-ICCF), which extends the traditional classes. We analyze various algebraic and geometric properties by illustrating the graphs of several significant α-ICCF via visual representations. Utilizing this novel class, we derive the Hermite-Hadamard (HH) inequality and certain refinements for functions whose first derivative in absolute value is α-ICCF. The primary tools employed in deriving the main results include Hölder's inequality, Hölder-İşcan inequality and power-mean integral inequality. Our findings demonstrate that the approximations obtained using Hölder-İşcan and the improved power-mean integral inequality are superior to those derived from other methods. In particular, when $\alpha = 1$, the derived results will coincide with those of classical ICCF. This innovative concept of α-inverse cosine convexity opens new avenues for research, encouraging further exploration of such convexity classes.

1. Introduction

Convex analysis is a fundamental concept in mathematical optimization and related disciplines [4, 10, 23, 24]. The properties of convex functions facilitate their effective use in various fields. Due to their well-understood mathematical characteristics, convex functions are instrumental for optimization and analysis [22]. Optimization itself is

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a powerful tool in numerous technical and scientific domains [16], including game theory [28], economics [8] and machine learning [29]. It provides a reliable and effective mathematical basis for understanding and investigating problems related to convex sets and functions. For further applications, the reader is referred to [18].

The concept of convexity has significantly expanded and innovated various mathematical and scientific fields, particularly in the study of inequalities. Numerous inequalities for convex functions have been established by researchers, including Jensen's [22], Hardy [20], Gagliardo-Nirenberg [1], Ostrowski [5], Loomis-Whitney [17] and Olsen [15] inequalities. A well-known result stemming from the work of Hermite and Hadamard is the Hermite-Hadamard (HH) inequality

$$\grave{\Theta}\left(\frac{\theta+\vartheta}{2}\right) \leq \frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta} \grave{\Theta}(\varsigma)d\varsigma \leq \frac{\grave{\Theta}(\theta)+\grave{\Theta}(\vartheta)}{2}, \tag{1.1}$$

which specifies the necessary and sufficient conditions for a function to be convex. The applications and generalized versions of this inequality have been extensively studied, making it a prominent and widely recognized inequality in analysis. The HH inequality for convex functions has been improved through various refinements. Samraiz *et al.* [26] examined a novel class of HH type integral inequalities utilizing n-polynomial exponential type s-convex functions and established new bounds for specific means of positive real numbers as applications. Kadakal [12] presented the idea of harmonic trigonometrically convex functions and established two HH type inequalities for this class of functions. Samraiz *et al.* [27] explored a new class of HH inequalities for functions with h-convex absolute derivatives and provided error estimates based on difference means. Bakht and Anwar [3] introduced α-exponential type convexity and developed novel versions of the HH and Ostrowski-type inequalities. For further refinements of HH inequality, refer to [21, 30–32].

Inspired by recent advancements in convexity theory, where researchers have introduced novel convexities using various transcendental functions such as the exponential function, the logarithm and trigonometric functions [2, 6, 11, 13, 19], this study presents a new class of convexity termed as α-inverse cosine convex functions (α-ICCF). This novel convexity generalizes the traditional inverse cosine convex functions (ICCF) [25] for $\alpha = 1$, providing a broader framework for theoretical exploration and practical applications. By employing this novel α-ICCF convexity, HH and related inequalities are investigated as applications. Some examples related to the unique approach to convexity explored in this study, which is seldom addressed in convexity theory. The fundamental concepts used in this work are outlined below.

Definition 1.1. A function $\grave{\Theta} : \grave{\mathbb{I}} \to \mathbb{R}$ is deemed convex if it satisfies the inequality

$$\grave{\Theta}(\tau\theta + (1-\tau)\vartheta) \le \tau\grave{\Theta}(\theta) + (1-\tau)\grave{\Theta}(\vartheta),$$

for all $\theta, \vartheta \in \grave{\mathbb{I}}$ and $\tau \in [0,1]$. The function $\grave{\Theta}$ is classified to be concave if this inequality reverses.

Definition 1.2 ([25])**.** Assuming an interval $\grave{\mathbb{I}} \subseteq \mathbb{R}$. A function $\grave{\Theta} : \grave{\mathbb{I}} \to \mathbb{R}$ is deemed inverse cosine convex function if it satisfies the inequality
(1.2)

$$\grave{\Theta}(\tau\theta + (1-\tau)\vartheta) \le \left(\frac{2}{\pi}\arccos(1-\tau)\right)\grave{\Theta}(\theta) + \left(\frac{2}{\pi}\arccos(\tau)\right)\grave{\Theta}(\vartheta),$$

for every $\theta, \vartheta \in \grave{\mathbb{I}},\ \tau \in [0,1]$.

Definition 1.3 ([12])**.** A non-negative function $\grave{\Theta} : \grave{\mathbb{I}} \to \mathbb{R}$ is deemed an inverse trigonometrically convex if it satisfies the inequality

$$\grave{\Theta}(\tau\theta + (1-\tau)\vartheta) \le \left(\frac{2}{\pi}\arcsin(\tau)\right)\grave{\Theta}(\theta) + \left(\frac{2}{\pi}\arccos(\tau)\right)\grave{\Theta}(\vartheta),$$

for every $\theta, \vartheta \in \grave{\mathbb{I}}$ and $\tau \in [0,1]$.

Dragomir et al. introduced the following lemma in [7].

Lemma 1.4. *Assume a differentiable map $\grave{\Theta} : \grave{\mathbb{I}}^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$, $\theta, \vartheta \in \grave{\mathbb{I}}^{\circ}$ with $\theta < \vartheta$. If $\grave{\Theta}' \in \mathbb{L}[\theta, \vartheta]$, then the following equality holds.*
(1.3)

$$\frac{\grave{\Theta}(\theta) + \grave{\Theta}(\vartheta)}{2} - \frac{1}{\vartheta - \theta}\int_{\theta}^{\vartheta} \grave{\Theta}(\varsigma)d\varsigma = \frac{\vartheta - \theta}{2}\int_{0}^{1}(1-2\tau)\grave{\Theta}'(\tau\theta + (1-\tau)\vartheta)d\tau.$$

In [9], İşcan refined the Hölder's integral inequality in the following way.

Theorem 1.5. *Assuming that $\mathfrak{p} > 1$ with $\frac{1}{\mathfrak{p}} + \frac{1}{\mathfrak{q}} = 1$. If we consider the real functions $\grave{\Theta}$ and $\grave{\Xi}$ defined on the interval $[\theta, \vartheta]$, alongside the integrable functions $\left|\grave{\Theta}\right|^{\mathfrak{p}}$ and $\left|\grave{\Xi}\right|^{\mathfrak{q}}$, then*

$$\begin{aligned}
&\int_{\theta}^{\vartheta} \left|\grave{\Theta}(\varsigma)\grave{\Xi}(\varsigma)\right| d\varsigma \\
&\quad \le \frac{1}{\vartheta - \theta}\left\{\left(\int_{\theta}^{\vartheta}(\vartheta - \varsigma)\left|\grave{\Theta}(\varsigma)\right|^{\mathfrak{p}} d\varsigma\right)^{\frac{1}{\mathfrak{p}}}\left(\int_{\theta}^{\vartheta}(\vartheta - \varsigma)\left|\grave{\Xi}(\varsigma)\right|^{\mathfrak{q}} d\varsigma\right)^{\frac{1}{\mathfrak{q}}}\right. \\
&\qquad \left. + \left(\int_{\theta}^{\vartheta}(\varsigma - \theta)\left|\grave{\Theta}(\varsigma)\right|^{\mathfrak{p}} d\varsigma\right)^{\frac{1}{\mathfrak{p}}}\left(\int_{\theta}^{\vartheta}(\varsigma - \theta)\left|\grave{\Xi}(\varsigma)\right|^{\mathfrak{q}} d\varsigma\right)^{\frac{1}{\mathfrak{q}}}\right\}.
\end{aligned}$$

The Hölder-İşcan integral inequality provides improved approximations compared to the Hölder integral inequality. This enhancement leads to a refinement of the power-mean integral inequality, known as the improved power-mean integral inequality, as presented in the following theorem [14].

Theorem 1.6. *Assuming that* $\mathfrak{q} > 1$*. If we consider the real functions* $\grave{\Theta}$ *and* $\grave{\Xi}$ *defined on the interval* $[\theta, \vartheta]$*, alongside the integrable functions* $\left|\grave{\Theta}\right|$ *and* $\left|\grave{\Theta}\right|\left|\grave{\Xi}\right|^{\mathfrak{q}}$*, then*

$$\begin{aligned}
&\int_{\theta}^{\vartheta}\left|\grave{\Theta}(\varsigma)\grave{\Xi}(\varsigma)\right| d\varsigma \\
&\quad \leq \frac{1}{\vartheta-\theta}\Bigg\{\left(\int_{\theta}^{\vartheta}(\vartheta-\varsigma)\left|\grave{\Theta}(\varsigma)\right| d\varsigma\right)^{1-\frac{1}{\mathfrak{q}}}\left(\int_{\theta}^{\vartheta}(\vartheta-\varsigma)\left|\grave{\Theta}(\varsigma)\right|\left|\grave{\Xi}(\varsigma)\right|^{\mathfrak{q}} d\varsigma\right)^{\frac{1}{\mathfrak{q}}} \\
&\qquad +\left(\int_{\theta}^{\vartheta}(\varsigma-\theta)\left|\grave{\Theta}(\varsigma)\right| d\varsigma\right)^{1-\frac{1}{\mathfrak{q}}}\left(\int_{\theta}^{\vartheta}(\varsigma-\theta)\left|\grave{\Theta}(\varsigma)\right|\left|\grave{\Xi}(\varsigma)\right|^{\mathfrak{q}} d\varsigma\right)^{\frac{1}{\mathfrak{q}}}\Bigg\}.
\end{aligned}$$

2. α-Inverse Cosine Convex Functions and Their Properties

In this section, we explore a unique class of convexity called α-inverse cosine convex functions (α-ICCF). We investigate the distinctive characteristics and mathematical properties of this introduced class. We aim to examine the behavior and applications of these functions while drawing connections to established convexities.

Definition 2.1. Assuming an interval $\grave{\mathbb{I}} \subseteq \mathbb{R}$ and $\alpha \in (0, \infty)$. A function $\grave{\Theta} : \grave{\mathbb{I}} \to \mathbb{R}$ is termed α-inverse cosine convex function if the inequality

(2.1)
$$\grave{\Theta}(\tau\theta + (1-\tau)\vartheta) \leq \left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha}\grave{\Theta}(\theta) + \left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha}\grave{\Theta}(\vartheta)$$

is valid for every $\theta, \vartheta \in \grave{\mathbb{I}},\ \tau \in [0,1]$.

Remark 2.2. For $\alpha = 1$, α-ICCF becomes classical ICCF defined in [25].

We examine the links that exist between the class of convex functions (CF) and the class of α-ICCF.

Proposition 2.3. *Every nonnegative* α*-ICCF is CF for* $\alpha \in [2, \infty)$ *and every nonnegative CF is* α*-ICCF for* $\alpha \in (0, 1]$*, whereas the link between the classes* α*-ICCF and CF is ambiguous in* $(1, 2)$*.*

Proof. The relations $\left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha_2} \leq \tau \leq \frac{2}{\pi}\arccos(1-\tau)^{\alpha_1}$ and $\left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha_2} \leq 1-\tau \leq \frac{2}{\pi}\arccos(\tau)^{\alpha_1}$ can be observed from Figures

1, 2 for $\alpha_2 \in [2, \infty)$, $\alpha_1 \in (0, 1]$ and $\tau \in [0, 1]$. Let $\theta, \vartheta \in \grave{\mathbb{I}}$, $\alpha_2 \in [2, \infty)$, $\alpha_1 \in (0, 1]$ and $\tau \in [0, 1]$, then we have

$$\begin{aligned}
&\grave{\Theta}(\tau\theta + (1-\tau)\vartheta) \\
&\quad \leq \left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha_2} \grave{\Theta}(\theta) + \left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha_2} \grave{\Theta}(\vartheta) \\
&\quad \leq \tau\grave{\Theta}(\theta) + (1-\tau)\grave{\Theta}(\vartheta) \\
&\quad \leq \left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha_1} \grave{\Theta}(\theta) + \left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha_1} \grave{\Theta}(\vartheta).
\end{aligned}$$

In $(1, 2)$, the ambiguity of the relation between the classes α-ICCF and CF is shown via Figure 3 by taking $\alpha = 1.7$.

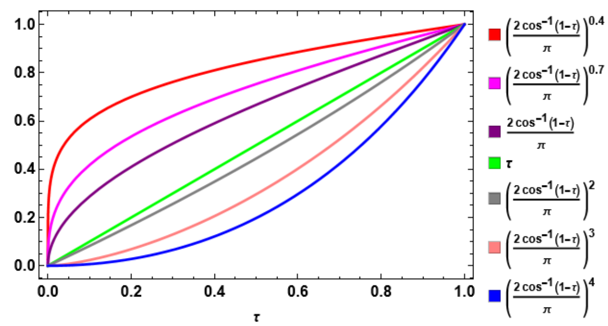

FIGURE 1. The visualizations in Figure 1 provide a graphical overview of the described connections across various values of $\alpha = 0.4, 0.7, 1, 2, 3, 4$, where $\tau \in [0, 1]$.

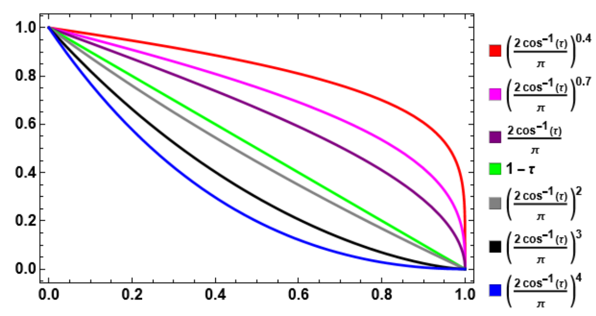

FIGURE 2. The visualizations in Figure 2 provide a graphical overview of the described connections across various values of $\alpha = 0.4, 0.7, 1, 2, 3, 4$, where $\tau \in [0, 1]$.

□

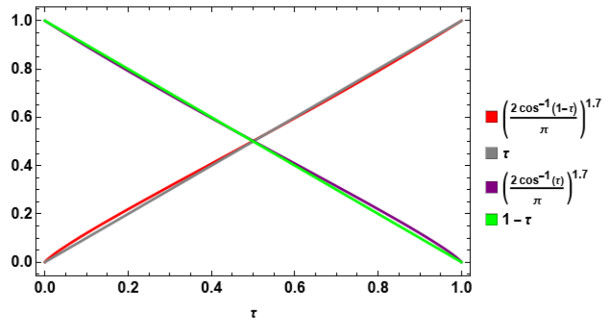

FIGURE 3. The visualizations in Figure 3 provide a graphical overview of the ambiguous relation between the classes α-ICCF and CF by taking $\alpha = 1.7$, where $\tau \in [0,1]$.

Proposition 2.4. *Every nonnegative α_2-ICCF is α_1-ICCF for $\alpha_1, \alpha_2 \in (0,\infty)$ with $\alpha_1 \leq \alpha_2$.*

Proof. The relations $\left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha_2} \leq \left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha_1}$ and $\left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha_2} \leq \left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha_1}$ can be observed from Figures 1, 2 for $\alpha_1, \alpha_2 \in (0,\infty)$ with $\alpha_1 \leq \alpha_2$ and $\tau \in [0,1]$. Let $\theta, \vartheta \in \grave{\mathbb{I}}$, $\alpha_1, \alpha_2 \in (0,\infty)$ with $\alpha_1 \leq \alpha_2$ and $\tau \in [0,1]$, then we have

$$\begin{aligned}
&\grave{\Theta}(\tau\theta + (1-\tau)\vartheta) \\
&\quad \leq \left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha_2} \grave{\Theta}(\theta) + \left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha_2} \grave{\Theta}(\vartheta) \\
&\quad \leq \left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha_1} \grave{\Theta}(\theta) + \left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha_1} \grave{\Theta}(\vartheta).
\end{aligned}$$

□

The examples of α-ICCF and their geometrical interpretations are presented below.

Example 2.5. A function $\grave{\Theta} : \grave{\mathbb{I}} \subseteq \mathbb{R} \to \mathbb{R}, \grave{\Theta}(\varsigma) = \grave{c} < 0$ is an α-inverse cosine convex function for $\alpha \in [2,\infty)$. By using the fact $\left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha} + \left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha} \leq 1$ for all $\tau \in [0,1]$ and $\alpha \in [2,\infty)$, Figure 4 clearly shows $\grave{\Theta}(\varsigma) = -1$ is an α-ICCF on $[0,3]$.

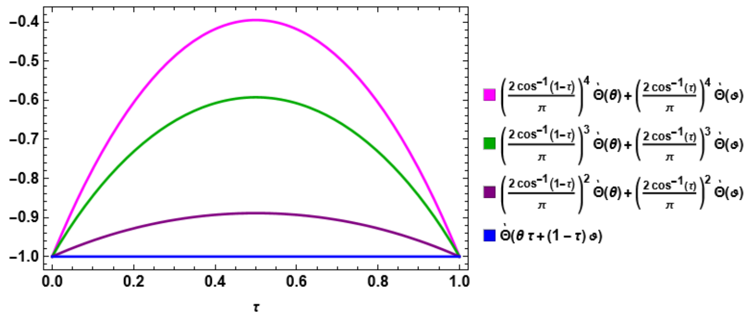

FIGURE 4. Figure 4 shows that constant function $\grave{\Theta}(\varsigma) = -1$ is α-ICCF on $[0, 3]$ for $\alpha \in [2, \infty)$.

This upcoming remark illustrates the broader scope and increased flexibility of the α-ICCF framework compared to the classical ICCF.

Remark 2.6. There exist functions that satisfy the α-ICCF condition for $\alpha \in [2, \infty)$ but do not qualify as ICCF (for $\alpha = 1$) under the original definition. For instance, in Example 2.5 of this manuscript, the function $\grave{\Theta}(\varsigma) = -1$ is shown to be an α-ICCF on the interval $[0, 3]$ for all $\alpha \in [2, \infty)$, yet it does not belong to the ICCF class (for $\alpha = 1$), as demonstrated in the Figure 5.

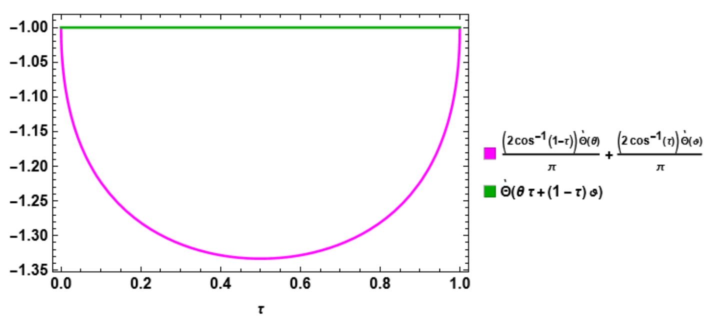

FIGURE 5. Figure 5 shows that constant function $\grave{\Theta}(\varsigma) = -1$ is not ICCF on $[0, 3]$.

Example 2.7. It can be observed from Figure 1 that the relations $\tau - \left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha} \geq 0$ and $(1-\tau) - \left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha} \geq 0$ exists for $\tau \in [0, 1]$ and $\alpha \in [2, \infty)$. So that the linear function $\grave{\Theta} : \grave{\mathbb{I}} \subseteq (-\infty, 0) \to \mathbb{R}, \grave{\Theta}(\varsigma) = \varsigma$ is an α-inverse cosine convex function for $\alpha \in [2, \infty)$. Graphical view is shown in Figure 6 for $[-3, 0]$.

In forthcoming discussions, we delve into the properties of α-ICCF, shedding light on their significance and potential applications.

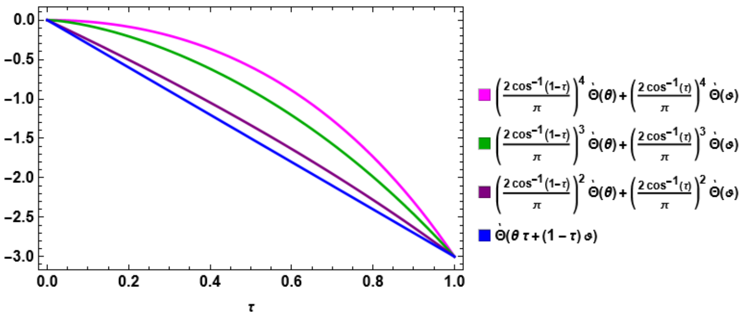

FIGURE 6. Figure 6 exemplifies that identity functions $\grave{\Theta}(\varsigma) = \varsigma$ is α-ICCF on $[-3, 0]$ for $\alpha \in [2, \infty)$.

Theorem 2.8. *Consider $\grave{\Theta} : \grave{\mathbb{I}} \to \grave{\mathbb{J}}$ and nondecreasing function $\grave{\Xi} : \grave{\mathbb{J}} \to \mathbb{R}$. If $\grave{\Theta} \in CF$ and $\grave{\Xi} \in \alpha$-ICCF, then $\grave{\Xi} \circ \grave{\Theta} \in \alpha$-ICCF for $\alpha \in (0, \infty)$.*

Proof. For $\theta, \vartheta \in \grave{\mathbb{I}}$, $\alpha \in (0, \infty)$ and $\tau \in [0, 1]$, we can write

$$\begin{aligned}\left(\grave{\Xi} \circ \grave{\Theta}\right)(\tau\theta + (1-\tau)\vartheta) &= \grave{\Xi}\left(\grave{\Theta}(\tau\theta + (1-\tau)\vartheta)\right) \\ &\leq \grave{\Xi}\left(\tau\grave{\Theta}(\theta) + (1-\tau)\grave{\Theta}(\vartheta)\right) \\ &\leq \left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha} \grave{\Xi}\left(\grave{\Theta}(\theta)\right) \\ &\quad + \left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha} \grave{\Xi}\left(\grave{\Theta}(\vartheta)\right).\end{aligned}$$

The proof of the theorem is now concluded. □

Theorem 2.9. *Let $\grave{\Theta}, \grave{\Xi} : \grave{\mathbb{I}} \subseteq \mathbb{R} \to \mathbb{R}$ and $\alpha \in (0, \infty)$. If $\grave{\Theta}$, $\grave{\Xi} \in \alpha$-ICCF, then*

(i) $\grave{\Theta} + \grave{\Xi} \in \alpha$*-ICCF,*
(ii) *For $\grave{c} \in \mathbb{R}$ ($\grave{c} \geq 0$), $\grave{c}\,\grave{\Theta} \in \alpha$-ICCF.*

Proof. (i) Let $\grave{\Theta}$, $\grave{\Xi} \in \alpha$-ICCF, then

$$\begin{aligned}&\left(\grave{\Theta} + \grave{\Xi}\right)(\tau\theta + (1-\tau)\vartheta) \\ &= \grave{\Theta}(\tau\theta + (1-\tau)\vartheta) + \grave{\Xi}(\tau\theta + (1-\tau)\vartheta) \\ &\leq \left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha} \grave{\Theta}(\theta) + \left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha} \grave{\Theta}(\vartheta) \\ &\quad + \left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha} \grave{\Xi}(\theta) + \left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha} \grave{\Xi}(\vartheta) \\ &= \left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha} \left(\grave{\Theta} + \grave{\Xi}\right)(\theta) + \left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha} \left(\grave{\Theta} + \grave{\Xi}\right)(\vartheta).\end{aligned}$$

(ii) Let $\grave{\Theta}$, $\grave{\Xi} \in \alpha$-ICCF and $\grave{c} \in \mathbb{R}$ be non-negative, then

$$\begin{aligned}
&\left(\grave{c}\,\grave{\Theta}\right)(\tau\theta + (1-\tau)\vartheta) \\
&\quad \leq \grave{c}\left[\left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha}\grave{\Theta}(\theta) + \left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha}\grave{\Theta}(\vartheta)\right] \\
&\quad = \left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha}\left(\grave{c}\,\grave{\Theta}\right)(\theta) + \left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha}\left(\grave{c}\,\grave{\Theta}\right)(\vartheta).
\end{aligned}$$

This proved that the sum and positive scalar multiple of α-ICCF are also α-ICCF. □

Theorem 2.10. *Let $\grave{\Theta}_{\kappa} : [\theta, \vartheta] \to \mathbb{R}$ be a family of α-inverse cosine convex functions for $\alpha \in (0, \infty)$ and let $\grave{\Theta}(\varsigma) = \sup_{\kappa} \grave{\Theta}_{\kappa}(\varsigma)$. If $\grave{\mathbb{J}} = \{\mu \in [\theta, \vartheta] : \grave{\Theta}(\mu) < \infty\}$ with $\grave{\mathbb{J}} \neq \emptyset$, then $\grave{\mathbb{J}}$ is an interval and $\grave{\Theta} \in \alpha$-ICCF$(\grave{\mathbb{J}})$.*

Proof. We assume that $\tau \in [0, 1]$ and that $\theta, \vartheta \in \grave{\mathbb{J}}$ are arbitrary. Then

$$\begin{aligned}
&\grave{\Theta}(\tau\theta + (1-\tau)\vartheta) \\
&\quad = \sup_{\kappa} \grave{\Theta}_{\kappa}(\tau\theta + (1-\tau)\vartheta) \\
&\quad \leq \sup_{\kappa}\left[\left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha}\grave{\Theta}_{\kappa}(\theta) + \left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha}\grave{\Theta}_{\kappa}(\vartheta)\right] \\
&\quad \leq \left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha}\sup_{\kappa}\grave{\Theta}_{\kappa}(\theta) + \left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha}\sup_{\kappa}\grave{\Theta}_{\kappa}(\vartheta) \\
&\quad = \left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha}\grave{\Theta}(\theta) + \left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha}\grave{\Theta}(\vartheta) \\
&\quad < \infty.
\end{aligned}$$

This shows that $\grave{\mathbb{J}}$ is an interval and $\grave{\Theta} \in \alpha$-ICCF$(\grave{\mathbb{J}})$ for $\alpha \in (0, \infty)$. □

Theorem 2.11. *If $\grave{\Theta} : [\theta, \vartheta] \to \mathbb{R}$ is an α-inverse cosine convex function then $\grave{\Theta}$ is bounded on $[\theta, \vartheta]$ for $\alpha \in (0, \infty)$.*

Proof. Consider $\mathfrak{M} = \max\{\grave{\Theta}(\theta), \grave{\Theta}(\vartheta)\}$ and let ς be an arbitrary point in the interval $[\theta, \vartheta]$. Then, there exists a $\tau \in [0, 1]$ such that $\varsigma = \tau\theta + (1-\tau)\vartheta$. Thus by using the facts $\left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha} \leq 1$ and $\left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha} \leq 1$ for $\alpha \in (0, \infty)$, we have

$$\begin{aligned}
\grave{\Theta}(\varsigma) &= \grave{\Theta}(\tau\theta + (1-\tau)\vartheta) \\
&\leq \left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha}\grave{\Theta}(\theta) + \left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha}\grave{\Theta}(\vartheta) \\
&\leq 2\mathfrak{M} \\
&= \mho.
\end{aligned}$$

Also, for every $\varsigma \in [\theta, \vartheta]$ there exists a $\wp \in \left[0, \frac{\vartheta-\theta}{2}\right]$ such that $\varsigma = \frac{\theta+\vartheta}{2} + \wp$ or $\varsigma = \frac{\theta+\vartheta}{2} - \wp$. Without loss of generality we can assume $\varsigma = \frac{\theta+\vartheta}{2} + \wp$. So, we have

$$\begin{aligned}\grave{\Theta}\left(\frac{\theta+\vartheta}{2}\right) &= \grave{\Theta}\left(\frac{1}{2}\left[\frac{\theta+\vartheta}{2}+\wp\right]+\frac{1}{2}\left[\frac{\theta+\vartheta}{2}-\wp\right]\right) \\ &\leq \left(\frac{2}{3}\right)^{\alpha}\left(\grave{\Theta}(\varsigma)+\grave{\Theta}\left(\frac{\theta+\vartheta}{2}-\wp\right)\right).\end{aligned}$$

Using $\mho$ as the upper bound, we get

$$\grave{\Theta}(\varsigma) \geq \left(\frac{3}{2}\right)^{\alpha}\grave{\Theta}\left(\frac{\theta+\vartheta}{2}\right)-\grave{\Theta}\left(\frac{\theta+\vartheta}{2}-\wp\right) \geq \left(\frac{3}{2}\right)^{\alpha}\grave{\Theta}\left(\frac{\theta+\vartheta}{2}\right)-\mho=\ell,$$

for $\alpha \in (0, \infty)$. Hence the required result is proved. □

3. Classical form of HH Inequality Involving Novel Convexity

In this section, we present the classical form of the Hermite-Hadamard (HH) inequality tailored for α-ICCF. Throughout our discussion, $\mathbb{L}[\theta, \vartheta]$ denotes the class of Lebesgue integrable functions defined over the interval $[\theta, \vartheta]$.

Theorem 3.1. *Let $\alpha \in (0, \infty)$. Assuming an α-inverse cosine convex function $\grave{\Theta} : [\theta, \vartheta] \to \mathbb{R}$. If $\theta < \vartheta$ and $\grave{\Theta} \in \mathbb{L}[\theta, \vartheta]$, then the HH integral inequality*

$$\begin{aligned}\frac{3^{\alpha}}{2^{\alpha+1}}\grave{\Theta}\left(\frac{\theta+\vartheta}{2}\right) &\leq \frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\grave{\Theta}(\varsigma)d\varsigma \\ &\leq \left[\grave{\Theta}(\theta)+\grave{\Theta}(\vartheta)\right]\int_{0}^{1}\left(\frac{2}{\pi}\arccos(\varsigma)\right)^{\alpha}d\varsigma\end{aligned} \tag{3.1}$$

holds.

Proof. Initially, we used the property of the α-ICCF to acquire

$$\begin{aligned}\grave{\Theta}\left(\frac{\theta+\vartheta}{2}\right) &= \grave{\Theta}\left(\frac{1}{2}\left[\tau\theta+(1-\tau)\vartheta\right]+\frac{1}{2}\left[(1-\tau)\theta+\tau\vartheta\right]\right) \\ &\leq \left(\frac{2}{3}\right)^{\alpha}\grave{\Theta}(\tau\theta+(1-\tau)\vartheta)+\left(\frac{2}{3}\right)^{\alpha}\grave{\Theta}((1-\tau)\theta+\tau\vartheta).\end{aligned}$$

Integrating with respect to $\tau \in [0, 1]$, we have

$$\begin{aligned}\grave{\Theta}\left(\frac{\theta+\vartheta}{2}\right) \leq &\left(\frac{2}{3}\right)^{\alpha}\int_{0}^{1}\grave{\Theta}(\tau\theta+(1-\tau)\vartheta)d\tau \\ &+\left(\frac{2}{3}\right)^{\alpha}\int_{0}^{1}\grave{\Theta}((1-\tau)\theta+\tau\vartheta)d\tau\end{aligned}$$

$$= 2\left(\frac{2}{3}\right)^{\alpha}\frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\grave{\Theta}(\varsigma)d\varsigma.$$

Now, if the variable is changed as $\mu = \tau\theta + (1-\tau)\vartheta$, then the property of the α-inverse cosine convexity of $\grave{\Theta}$ leads to

$$\begin{aligned}\frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\grave{\Theta}(\mu)d\mu &= \int_{0}^{1}\grave{\Theta}(\tau\theta+(1-\tau)\vartheta)d\tau \\ &\leq \int_{0}^{1}\left[\left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha}\grave{\Theta}(\theta)\right. \\ &\quad \left. + \left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha}\grave{\Theta}(\vartheta)\right]d\tau \\ &= \left[\grave{\Theta}(\theta)+\grave{\Theta}(\vartheta)\right]\int_{0}^{1}\left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha}d\tau.\end{aligned}$$

The proof of the theorem is now concluded. □

Remark 3.2. For $\alpha = 1$, (3.1) will reduce to HH inequaity for classical *ICCF* [25].

Example 3.3. Consider an 2-ICCF $\grave{\Theta}(\varsigma) = |\varsigma|$ on $[-1, 1]$. Rewriting the HH inequality (3.1) as

(3.2)

$$\frac{9}{8}\grave{\Theta}\left(\frac{\theta+\vartheta}{2}\right)\times(\vartheta-\theta) \leq \int_{\theta}^{\vartheta}\grave{\Theta}(\varsigma)d\varsigma \leq \frac{4(\pi-2)}{\pi^2}\left[\grave{\Theta}(\theta)+\grave{\Theta}(\vartheta)\right]\times(\vartheta-\theta),$$

which is true since $0 \leq 1 \leq \frac{16(\pi-2)}{\pi^2}$. The findings in Theorem 3.1 are further supported when considering inequality (3.2) within the context of areas, as depicted in Figure 7.

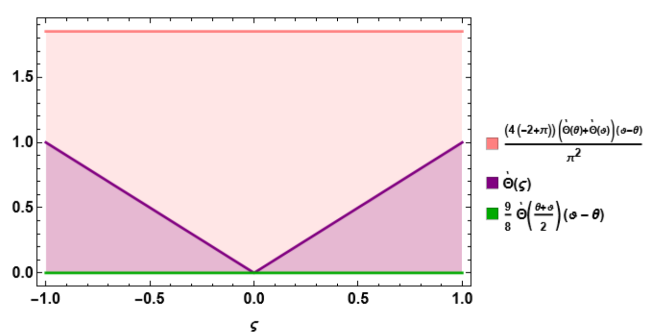

FIGURE 7. The Figure 7 presents a 2D graphical depiction of (3.2), showcasing the relationship in terms of areas on $[-1, 1]$ for $\grave{\Theta}(\varsigma) = |\varsigma|$.

The upcoming remark indicates that the HH inequality for α-ICCF is more refined than ICCF, demonstrating the enhanced sharpness achieved through the α-ICCF framework.

Remark 3.4. For functions that belong to both ICCF and α-ICCF classes for multiple values of α, the associated inequalities become increasingly refined as α increases. The function presented in Example 3.3 and also discussed in Example 3.1 of [25], qualifies as both an ICCF and a 2-ICCF. But the left-hand side of the HH inequality 3.2 is more refined than the corresponding HH inequality in [25] (equation (3.2)) as

$$\int_\theta^\vartheta \grave{\Theta}(\varsigma)d\varsigma \leq \frac{4(\pi-2)}{\pi^2}\left[\grave{\Theta}(\theta)+\grave{\Theta}(\vartheta)\right]\times(\vartheta-\theta) \leq \frac{2}{\pi}\left[\grave{\Theta}(\theta)+\grave{\Theta}(\vartheta)\right]\times(\vartheta-\theta).$$

4. Some Novel Refined HH Type Inequalities for α-ICCF

Within this section, our focus lies on crafting novel refinements of the Hermite-Hadamard (HH) integral inequalities tailored for functions whose first derivative, when raised to a certain power, exhibits α-ICCF properties. Furthermore, we demonstrate that the Hölder-İşcan integral inequality presents an enhanced approach compared to Hölder's integral inequality.

Theorem 4.1. *Assume a differentiable mapping $\grave{\Theta}:\grave{\mathbb{I}}\subseteq\mathbb{R}\to\mathbb{R}$ on $\grave{\mathbb{I}}^\circ$, $\theta,\vartheta\in\grave{\mathbb{I}}^\circ$ with $\theta<\vartheta$ and $\grave{\Theta}'\in\mathbb{L}[\theta,\vartheta]$. If $\left|\grave{\Theta}'\right|\in\alpha$-ICCF with $\alpha\in(0,\infty)$, then the following integral inequality*

$$\left|\frac{\grave{\Theta}(\theta)+\grave{\Theta}(\vartheta)}{2}-\frac{1}{\vartheta-\theta}\int_\theta^\vartheta\grave{\Theta}(\varsigma)d\varsigma\right| \leq (\vartheta-\theta)\mathbb{A}\left(\left|\grave{\Theta}'(\theta)\right|,\left|\grave{\Theta}'(\vartheta)\right|\right)\int_0^1|1-2\varsigma|\left(\frac{2}{\pi}\arccos(\varsigma)\right)^\alpha d\varsigma, \tag{4.1}$$

is valid, where $\mathbb{A}$ is the arithmetic mean.

Proof. Utilizing Lemma 1.4 alongside the given inequality

$$\left|\grave{\Theta}'(\tau\theta+(1-\tau)\vartheta)\right| \leq \left(\frac{2}{\pi}\arccos(1-\tau)\right)^\alpha\left|\grave{\Theta}'(\theta)\right|+\left(\frac{2}{\pi}\arccos(\tau)\right)^\alpha\left|\grave{\Theta}'(\vartheta)\right|,$$

we get

$$\left|\frac{\grave{\Theta}(\theta)+\grave{\Theta}(\vartheta)}{2}-\frac{1}{\vartheta-\theta}\int_\theta^\vartheta\grave{\Theta}(\varsigma)d\varsigma\right| \leq \frac{\vartheta-\theta}{2}\int_0^1|1-2\tau|\left|\grave{\Theta}'(\tau\theta+(1-\tau)\vartheta)\right|d\tau$$

$$\leq \frac{\vartheta-\theta}{2}\int_0^1 |1-2\tau|\left[\left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha}\left|\grave{\Theta}'(\theta)\right| + \left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha}\left|\grave{\Theta}'(\vartheta)\right|\right]d\tau$$

$$= \frac{\vartheta-\theta}{2}\left[\left|\grave{\Theta}'(\theta)\right|\int_0^1 |1-2\tau|\left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha}d\tau + \left|\grave{\Theta}'(\vartheta)\right|\int_0^1 |1-2\tau|\left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha}d\tau\right]$$

$$= (\vartheta-\theta)\mathbb{A}\left(\left|\grave{\Theta}'(\theta)\right|, \left|\grave{\Theta}'(\vartheta)\right|\right)\int_0^1 |1-2\tau|\left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha}d\tau,$$

where

$$\int_0^1 |1-2\tau|\left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha}d\tau = \int_0^1 |1-2\tau|\left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha}d\tau.$$

The proof of the theorem is now concluded. □

Remark 4.2. For $\alpha = 1$, (4.1) will reduce to the inequaity of Theorem 4.1 in [25].

Theorem 4.3. *Assume a differentiable mapping* $\grave{\Theta} : \grave{\mathbb{I}} \subseteq \mathbb{R} \to \mathbb{R}$ *on* $\grave{\mathbb{I}}^{\circ}$, $\theta, \vartheta \in \grave{\mathbb{I}}^{\circ}$ *with* $\theta < \vartheta$, $\mathfrak{q} > 1$ *and* $\grave{\Theta}' \in \mathbb{L}[\theta, \vartheta]$. *If* $\left|\grave{\Theta}'\right|^{\mathfrak{q}} \in \alpha$*-ICCF with* $\alpha \in (0, \infty)$*, then the following integral inequality*

$$\left|\frac{\grave{\Theta}(\theta)+\grave{\Theta}(\vartheta)}{2} - \frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\grave{\Theta}(\varsigma)d\varsigma\right| \leq (\vartheta-\theta)\left(\frac{1}{2(\mathfrak{p}+1)}\right)^{\frac{1}{\mathfrak{p}}}\mathbb{A}^{\frac{1}{\mathfrak{q}}}\left(\left|\grave{\Theta}'(\theta)\right|^{\mathfrak{q}}, \left|\grave{\Theta}'(\vartheta)\right|^{\mathfrak{q}}\right) \times \left(\int_0^1\left(\frac{2}{\pi}\arccos(\varsigma)\right)^{\alpha}d\varsigma\right)^{\frac{1}{\mathfrak{q}}}, \tag{4.2}$$

is valid, where $\mathbb{A}$ *is the arithmetic mean and* $\frac{1}{\mathfrak{p}} + \frac{1}{\mathfrak{q}} = 1$.

Proof. By employing Lemma 1.4 and Hölder's integral inequality in conjunction with the inequality

$$\left|\grave{\Theta}'(\tau\theta+(1-\tau)\vartheta)\right|^{\mathfrak{q}} \leq \left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha}\left|\grave{\Theta}'(\theta)\right|^{\mathfrak{q}} + \left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha}\left|\grave{\Theta}'(\vartheta)\right|^{\mathfrak{q}},$$

yields

$$\begin{aligned}
&\left|\frac{\grave{\Theta}(\theta)+\grave{\Theta}(\vartheta)}{2}-\frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\grave{\Theta}(\varsigma)d\varsigma\right| \\
&\quad\le \frac{\vartheta-\theta}{2}\int_0^1|1-2\tau|\left|\grave{\Theta}'(\tau\theta+(1-\tau)\vartheta)\right|d\tau \\
&\quad\le \frac{\vartheta-\theta}{2}\left(\int_0^1|1-2\tau|^{\mathfrak{p}}d\tau\right)^{\frac{1}{\mathfrak{p}}}\left(\int_0^1\left|\grave{\Theta}'(\tau\theta+(1-\tau)\vartheta)\right|^{\mathfrak{q}}d\tau\right)^{\frac{1}{\mathfrak{q}}} \\
&\quad\le \frac{\vartheta-\theta}{2}\left(\frac{1}{\mathfrak{p}+1}\right)^{\frac{1}{\mathfrak{p}}} \\
&\qquad\times\left(\int_0^1\left[\left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha}\left|\grave{\Theta}'(\theta)\right|^{\mathfrak{q}}+\left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha}\left|\grave{\Theta}'(\vartheta)\right|^{\mathfrak{q}}\right]d\tau\right)^{\frac{1}{\mathfrak{q}}} \\
&\quad= \frac{\vartheta-\theta}{2}2^{\frac{1}{\mathfrak{q}}}\left(\frac{1}{\mathfrak{p}+1}\right)^{\frac{1}{\mathfrak{p}}}\mathbb{A}^{\frac{1}{\mathfrak{q}}}\left(\left|\grave{\Theta}'(\theta)\right|^{\mathfrak{q}},\left|\grave{\Theta}'(\vartheta)\right|^{\mathfrak{q}}\right)\left(\int_0^1\left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha}d\tau\right)^{\frac{1}{\mathfrak{q}}},
\end{aligned}$$

where

$$\int_0^1|1-2\tau|^{\mathfrak{p}}d\tau=\frac{1}{\mathfrak{p}+1},$$
$$\int_0^1\left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha}d\tau=\int_0^1\left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha}d\tau.$$

The proof of the theorem is now concluded. □

Remark 4.4. For $\alpha=1$, (4.2) will reduce to the inequaity of Theorem 4.2 in [25].

Example 4.5. Consider an 3-ICCF $\grave{\Theta}(\varsigma)=e^{\varsigma}$ on $[0,3]$. The visual representation presented in Figure 8, corresponding to equation (4.2), offers validation of Theorem 4.3 for $2\le\mathfrak{q}\le 10$.

Theorem 4.6. *Assume a differentiable mapping* $\grave{\Theta}:\grave{\mathbb{I}}\subseteq\mathbb{R}\to\mathbb{R}$ *on* $\grave{\mathbb{I}}^{\circ}$, $\theta,\vartheta\in\grave{\mathbb{I}}^{\circ}$ *with* $\theta<\vartheta$, $\mathfrak{q}\ge 1$ *and* $\grave{\Theta}'\in\mathbb{L}[\theta,\vartheta]$. *If* $\left|\grave{\Theta}'\right|^{\mathfrak{q}}\in\alpha$*-ICCF with* $\alpha\in(0,\infty)$, *then the following integral inequality*

(4.3)
$$\begin{aligned}
&\left|\frac{\grave{\Theta}(\theta)+\grave{\Theta}(\vartheta)}{2}-\frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\grave{\Theta}(\varsigma)d\varsigma\right| \\
&\quad\le\frac{\vartheta-\theta}{2^{2-\frac{2}{\mathfrak{q}}}}\mathbb{A}^{\frac{1}{\mathfrak{q}}}\left(\left|\grave{\Theta}'(\theta)\right|^{\mathfrak{q}},\left|\grave{\Theta}'(\vartheta)\right|^{\mathfrak{q}}\right)\left(\int_0^1|1-2\varsigma|\left(\frac{2}{\pi}\arccos(\varsigma)\right)^{\alpha}d\varsigma\right)^{\frac{1}{\mathfrak{q}}},
\end{aligned}$$

is valid, where $\mathbb{A}$ *is the arithmetic mean.*

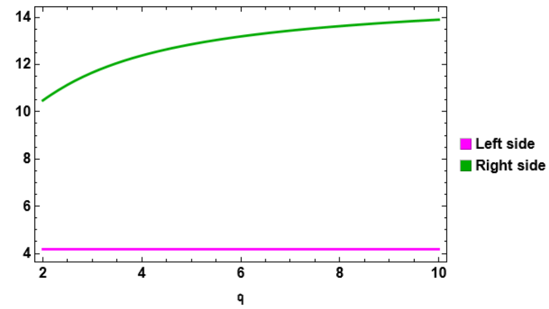

FIGURE 8. The Figure 8 provides a visual depiction of the 2D representation of equation (4.2) for $2 \leq \mathfrak{q} \leq 10$, with $\grave{\Theta}(\varsigma) = e^{\varsigma}$.

Proof. By employing Lemma 1.4 and power mean integral inequality in conjunction with the convexity of α-ICCF $\left|\grave{\Theta}'\right|^{\mathfrak{q}}$ for $q > 1$, we obtain

$$\begin{aligned}
&\left|\frac{\grave{\Theta}(\theta)+\grave{\Theta}(\vartheta)}{2}-\frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\grave{\Theta}(\varsigma)d\varsigma\right| \\
&\quad\leq \frac{\vartheta-\theta}{2}\int_0^1|1-2\tau|\left|\grave{\Theta}'(\tau\theta+(1-\tau)\vartheta)\right|d\tau \\
&\quad\leq \frac{\vartheta-\theta}{2}\left(\int_0^1|1-2\tau|d\tau\right)^{1-\frac{1}{\mathfrak{q}}}\left(\int_0^1|1-2\tau|\left|\grave{\Theta}'(\tau\theta+(1-\tau)\vartheta)\right|^{\mathfrak{q}}d\tau\right)^{\frac{1}{\mathfrak{q}}} \\
&\quad\leq \frac{\vartheta-\theta}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{\mathfrak{q}}}\left(\int_0^1|1-2\tau|\left[\left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha}\left|\grave{\Theta}'(\theta)\right|^{\mathfrak{q}}\right.\right. \\
&\qquad\left.\left.+\left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha}\left|\grave{\Theta}'(\vartheta)\right|^{\mathfrak{q}}\right]d\tau\right)^{\frac{1}{\mathfrak{q}}} \\
&\quad= \frac{\vartheta-\theta}{2^{2-\frac{2}{\mathfrak{q}}}}\mathbb{A}^{\frac{1}{\mathfrak{q}}}\left(\left|\grave{\Theta}'(\theta)\right|^{\mathfrak{q}},\left|\grave{\Theta}'(\vartheta)\right|^{\mathfrak{q}}\right)\left(\int_0^1|1-2\tau|\left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha}d\tau\right)^{\frac{1}{\mathfrak{q}}}.
\end{aligned}$$

This completes the proof of the result. □

Remark 4.7. For $\alpha = 1$, (4.3) will reduce to the inequaity of Theorem 4.3 in [25].

Corollary 4.8. *We conclude Theorem 4.1 under the assumption of Theorem 4.6 with* $\mathfrak{q} = 1$.

Example 4.9. Consider an 4-ICCF $\grave{\Theta}(\varsigma) = \varsigma^3$ on $[0, 3]$. The visual representation presented in Figure 9, corresponding to equation (4.3), offers validation of Theorem 4.6 for $1 \leq \mathfrak{q} \leq 10$.

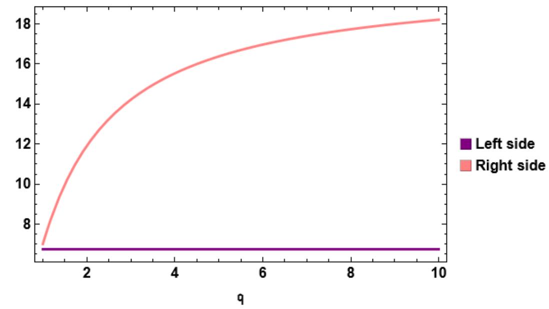

FIGURE 9. The Figure 9 provides a visual depiction of the 2D representation of equation (4.3) for $1 \leq \mathfrak{q} \leq 10$, with $\grave{\Theta}(\varsigma) = \varsigma^3$.

Theorem 4.10. *Assume a differentiable mapping* $\grave{\Theta} : \grave{\mathbb{I}} \subseteq \mathbb{R} \to \mathbb{R}$ *on* $\grave{\mathbb{I}}^\circ$, $\theta, \vartheta \in \grave{\mathbb{I}}^\circ$ *with* $\theta < \vartheta$, $\mathfrak{q} > 1$ *and* $\grave{\Theta}' \in \mathbb{L}[\theta, \vartheta]$. *If* $\left|\grave{\Theta}'\right|^{\mathfrak{q}} \in \alpha$*-ICCF with* $\alpha \in (0, \infty)$, *then the following integral inequality*

(4.4)

$$\begin{aligned}
&\left|\frac{\grave{\Theta}(\theta) + \grave{\Theta}(\vartheta)}{2} - \frac{1}{\vartheta - \theta}\int_\theta^\vartheta \grave{\Theta}(\varsigma)d\varsigma\right| \\
&\quad \leq \frac{\vartheta - \theta}{2}\left(\frac{1}{2(\mathfrak{p}+1)}\right)^{\frac{1}{\mathfrak{p}}}\left[\left(\left|\grave{\Theta}'(\theta)\right|^{\mathfrak{q}}\int_0^1 (1-\varsigma)\left(\frac{2}{\pi}\arccos(1-\varsigma)\right)^\alpha d\varsigma\right.\right. \\
&\quad\quad \left. + \left|\grave{\Theta}'(\vartheta)\right|^{\mathfrak{q}}\int_0^1 (1-\varsigma)\left(\frac{2}{\pi}\arccos(\varsigma)\right)^\alpha d\varsigma\right)^{\frac{1}{\mathfrak{q}}} \\
&\quad\quad + \left(\left|\grave{\Theta}'(\theta)\right|^{\mathfrak{q}}\int_0^1 \varsigma\left(\frac{2}{\pi}\arccos(1-\varsigma)\right)^\alpha d\varsigma\right. \\
&\quad\quad \left.\left. + \left|\grave{\Theta}'(\vartheta)\right|^{\mathfrak{q}}\int_0^1 \varsigma\left(\frac{2}{\pi}\arccos(\varsigma)\right)^\alpha d\varsigma\right)^{\frac{1}{\mathfrak{q}}}\right],
\end{aligned}$$

is valid, where $\frac{1}{\mathfrak{p}} + \frac{1}{\mathfrak{q}} = 1$.

Proof. By employing Lemma 1.4 and Hölder-İşcan integral inequality in conjunction with the convexity of α-ICCF $\left|\grave{\Theta}'\right|^{\mathfrak{q}}$, we obtain

$$\begin{aligned}
&\left|\frac{\grave{\Theta}(\theta) + \grave{\Theta}(\vartheta)}{2} - \frac{1}{\vartheta - \theta}\int_\theta^\vartheta \grave{\Theta}(\varsigma)d\varsigma\right| \\
&\quad \leq \frac{\vartheta - \theta}{2}\int_0^1 |1 - 2\tau|\left|\grave{\Theta}'(\tau\theta + (1-\tau)\vartheta)\right| d\tau
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\vartheta-\theta}{2}\left(\int_0^1(1-\tau)|1-2\tau|^{\mathfrak{p}}d\tau\right)^{\frac{1}{\mathfrak{p}}} \\
&\times\left(\int_0^1(1-\tau)\left|\grave{\Theta}'(\tau\theta+(1-\tau)\vartheta)\right|^{\mathfrak{q}}d\tau\right)^{\frac{1}{\mathfrak{q}}} \\
&+\frac{\vartheta-\theta}{2}\left(\int_0^1\tau|1-2\tau|^{\mathfrak{p}}d\tau\right)^{\frac{1}{\mathfrak{p}}}\left(\int_0^1\tau\left|\grave{\Theta}'(\tau\theta+(1-\tau)\vartheta)\right|^{\mathfrak{q}}d\tau\right)^{\frac{1}{\mathfrak{q}}} \\
&\leq \frac{\vartheta-\theta}{2}\left(\int_0^1(1-\tau)|1-2\tau|^{\mathfrak{p}}d\tau\right)^{\frac{1}{\mathfrak{p}}} \\
&\times\left(\int_0^1(1-\tau)\left[\left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha}\left|\grave{\Theta}'(\theta)\right|^{\mathfrak{q}}\right.\right. \\
&\left.\left.+\left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha}\left|\grave{\Theta}'(\vartheta)\right|^{\mathfrak{q}}\right]d\tau\right)^{\frac{1}{\mathfrak{q}}} \\
&+\frac{\vartheta-\theta}{2}\left(\int_0^1\tau|1-2\tau|^{\mathfrak{p}}d\tau\right)^{\frac{1}{\mathfrak{p}}} \\
&\times\left(\int_0^1\tau\left[\left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha}\left|\grave{\Theta}'(\theta)\right|^{\mathfrak{q}}\right.\right. \\
&\left.\left.+\left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha}\left|\grave{\Theta}'(\vartheta)\right|^{\mathfrak{q}}\right]d\tau\right)^{\frac{1}{\mathfrak{q}}} \\
&=\frac{\vartheta-\theta}{2}\left(\frac{1}{2(\mathfrak{p}+1)}\right)^{\frac{1}{\mathfrak{p}}}\left(\left|\grave{\Theta}'(\theta)\right|^{\mathfrak{q}}\int_0^1(1-\tau)\left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha}d\tau\right. \\
&\left.+\left|\grave{\Theta}'(\vartheta)\right|^{\mathfrak{q}}\int_0^1(1-\tau)\left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha}d\tau\right)^{\frac{1}{\mathfrak{q}}} \\
&+\frac{\vartheta-\theta}{2}\left(\frac{1}{2(\mathfrak{p}+1)}\right)^{\frac{1}{\mathfrak{p}}}\left(\left|\grave{\Theta}'(\theta)\right|^{\mathfrak{q}}\int_0^1\tau\left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha}d\tau\right. \\
&\left.+\left|\grave{\Theta}'(\vartheta)\right|^{\mathfrak{q}}\int_0^1\tau\left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha}d\tau\right)^{\frac{1}{\mathfrak{q}}},
\end{aligned}
$$

where

$$\int_0^1(1-\tau)|1-2\tau|^{\mathfrak{p}}d\tau=\int_0^1\tau|1-2\tau|^{\mathfrak{p}}d\tau=\frac{1}{2(\mathfrak{p}+1)}.$$

The proof of the theorem is now concluded. □

Remark 4.11. For $\alpha=1$, (4.4) will reduce to the inequaity of Theorem 4.4 in [25].

Example 4.12. Consider an 5-ICCF $\grave{\Theta}(\varsigma) = \varsigma^5$ on $[0, 3]$. The visual representation presented in Figure 10, corresponding to equation (4.4), offers validation of Theorem 4.6 for $2 \leq \mathfrak{q} \leq 10$.

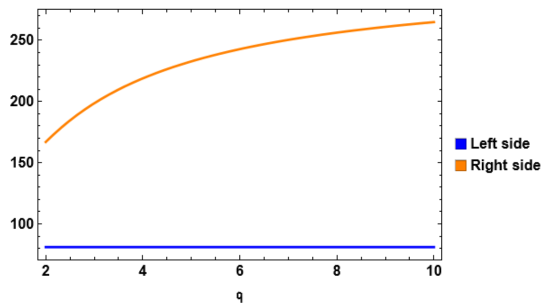

FIGURE 10. The Figure 10 provides a visual depiction of the 2D representation of equation (4.4) across the range $2 \leq \mathfrak{q} \leq 10$, with $\grave{\Theta}(\varsigma) = \varsigma^5$.

The subsequent proposition showcases that the Hölder-İşcan integral inequality yields superior approximations when contrasted with the Hölder integral inequality for α-ICCF functions.

Proposition 4.13. *The inequality* (4.4) *provides superior estimations compared to* (4.2)*.*

Proof. The fact that $h : [0, \infty) \to \mathbb{R}$, $h(\varsigma) = \varsigma^{\varrho}$, is a concave function for $0 < \varrho \leq 1$, we have

$$h\left(\frac{\omega + \varpi}{2}\right) = \left(\frac{\omega + \varpi}{2}\right)^{\varrho} \geq \frac{h(\omega) + h(\varpi)}{2} = \frac{\omega^{\varrho} + \varpi^{\varrho}}{2},$$

for all $\omega, \varpi \geq 0$. From here, we get

$$\begin{aligned}
&\frac{\vartheta - \theta}{2}\left(\frac{1}{2(\mathfrak{p}+1)}\right)^{\frac{1}{\mathfrak{p}}}\left(\left|\grave{\Theta}'(\theta)\right|^{\mathfrak{q}}\int_0^1 (1-\varsigma)\left(\frac{2}{\pi}\arccos(1-\varsigma)\right)^{\alpha} d\varsigma\right. \\
&\quad \left. + \left|\grave{\Theta}'(\vartheta)\right|^{\mathfrak{q}}\int_0^1 (1-\varsigma)\left(\frac{2}{\pi}\arccos(\varsigma)\right)^{\alpha} d\varsigma\right)^{\frac{1}{\mathfrak{q}}} \\
&\quad + \frac{\vartheta - \theta}{2}\left(\frac{1}{2(\mathfrak{p}+1)}\right)^{\frac{1}{\mathfrak{p}}}\left(\left|\grave{\Theta}'(\theta)\right|^{\mathfrak{q}}\int_0^1 \varsigma\left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha} d\tau\right. \\
&\quad \left. + \left|\grave{\Theta}'(\vartheta)\right|^{\mathfrak{q}}\int_0^1 \varsigma\left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha} d\varsigma\right)^{\frac{1}{\mathfrak{q}}} \\
&\leq \frac{\vartheta - \theta}{2}\left(\frac{1}{2(\mathfrak{p}+1)}\right)^{\frac{1}{\mathfrak{p}}} 2\left[\frac{1}{2}\left(\left|\grave{\Theta}'(\theta)\right|^{\mathfrak{q}}\int_0^1 \left(\frac{2}{\pi}\arccos(1-\varsigma)\right)^{\alpha} d\varsigma\right.\right.
\end{aligned}$$

$$+\left|\grave{\Theta}'(\vartheta)\right|^{\mathfrak{q}}\int_0^1\left(\frac{2}{\pi}\arccos(\varsigma)\right)^{\alpha}d\varsigma\Big)\Big]^{\frac{1}{\mathfrak{q}}}$$
$$=(\vartheta-\theta)\left(\frac{1}{2(\mathfrak{p}+1)}\right)^{\frac{1}{\mathfrak{p}}}\mathbb{A}^{\frac{1}{\mathfrak{q}}}\left(\left|\grave{\Theta}'(\theta)\right|^{\mathfrak{q}},\left|\grave{\Theta}'(\vartheta)\right|^{\mathfrak{q}}\right)$$
$$\times\left(\int_0^1\left(\frac{2}{\pi}\arccos(\varsigma)\right)^{\alpha}d\varsigma\right)^{\frac{1}{\mathfrak{q}}}.$$

This indicates that the inequality (4.4) produces better approximations compared to the inequality (4.2). ☐

Theorem 4.14. *Assume a differentiable mapping* $\grave{\Theta}:\grave{\mathbb{I}}\subseteq\mathbb{R}\to\mathbb{R}$ *on* $\grave{\mathbb{I}}^{\circ}$, $\theta,\vartheta\in\grave{\mathbb{I}}^{\circ}$ *with* $\theta<\vartheta$, $\mathfrak{q}\geq 1$ *and* $\grave{\Theta}'\in\mathbb{L}[\theta,\vartheta]$. *If* $\left|\grave{\Theta}'\right|^{\mathfrak{q}}\in\alpha$*-ICCF with* $\alpha\in(0,\infty)$*, then the following integral inequality*

(4.5)
$$\left|\frac{\grave{\Theta}(\theta)+\grave{\Theta}(\vartheta)}{2}-\frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\grave{\Theta}(\varsigma)d\varsigma\right|$$
$$\leq\frac{\vartheta-\theta}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{\mathfrak{q}}}\left[\left(\left|\grave{\Theta}'(\theta)\right|^{\mathfrak{q}}\int_0^1(1-\varsigma)|1-2\varsigma|\left(\frac{2}{\pi}\arccos(1-\varsigma)\right)^{\alpha}d\varsigma\right.\right.$$
$$\left.+\left|\grave{\Theta}'(\vartheta)\right|^{\mathfrak{q}}\int_0^1(1-\varsigma)|1-2\varsigma|\left(\frac{2}{\pi}\arccos(\varsigma)\right)^{\alpha}d\varsigma\right)^{\frac{1}{\mathfrak{q}}}$$
$$+\left(\left|\grave{\Theta}'(\theta)\right|^{\mathfrak{q}}\int_0^1\varsigma|1-2\varsigma|\left(\frac{2}{\pi}\arccos(1-\varsigma)\right)^{\alpha}d\varsigma\right.$$
$$\left.\left.+\left|\grave{\Theta}'(\vartheta)\right|^{\mathfrak{q}}\int_0^1\varsigma|1-2\varsigma|\left(\frac{2}{\pi}\arccos(\varsigma)\right)^{\alpha}d\varsigma\right)^{\frac{1}{\mathfrak{q}}}\right],$$

is valid.

Proof. Presuming $\mathfrak{q}>1$. Through the utilization of Lemma 1.4 and the improved power mean integral inequality in conjunction with the convexity property of α-ICCF $\left|\grave{\Theta}'\right|^{\mathfrak{q}}$, we derive

$$\left|\frac{\grave{\Theta}(\theta)+\grave{\Theta}(\vartheta)}{2}-\frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\grave{\Theta}(\varsigma)d\varsigma\right|$$
$$\leq\frac{\vartheta-\theta}{2}\int_0^1|1-2\tau|\left|\grave{\Theta}'(\tau\theta+(1-\tau)\vartheta)\right|d\tau$$
$$\leq\frac{\vartheta-\theta}{2}\left(\int_0^1(1-\tau)|1-2\tau|d\tau\right)^{1-\frac{1}{\mathfrak{q}}}$$

$$\times\left(\int_0^1 (1-\tau)|1-2\tau|\left|\grave{\Theta}'(\tau\theta+(1-\tau)\vartheta)\right|^{\mathfrak{q}} d\tau\right)^{\frac{1}{\mathfrak{q}}}$$

$$+\frac{\vartheta-\theta}{2}\left(\int_0^1 \tau|1-2\tau|d\tau\right)^{1-\frac{1}{\mathfrak{q}}}\left(\int_0^1 \tau|1-2\tau|\left|\grave{\Theta}'(\tau\theta+(1-\tau)\vartheta)\right|^{\mathfrak{q}} d\tau\right)^{\frac{1}{\mathfrak{q}}}$$

$$\leq \frac{\vartheta-\theta}{2}\left(\int_0^1 (1-\tau)|1-2\tau|d\tau\right)^{1-\frac{1}{\mathfrak{q}}}$$

$$\times\left(\int_0^1 (1-\tau)|1-2\tau|\left[\left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha}\left|\grave{\Theta}'(\theta)\right|^{\mathfrak{q}}\right.\right.$$

$$\left.\left.+\left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha}\left|\grave{\Theta}'(\vartheta)\right|^{\mathfrak{q}}\right]d\tau\right)^{\frac{1}{\mathfrak{q}}}+\frac{\vartheta-\theta}{2}\left(\int_0^1 \tau|1-2\tau|d\tau\right)^{1-\frac{1}{\mathfrak{q}}}$$

$$\times\left(\int_0^1 \tau|1-2\tau|\left[\left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha}\left|\grave{\Theta}'(\theta)\right|^{\mathfrak{q}}\right.\right.$$

$$\left.\left.+\left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha}\left|\grave{\Theta}'(\vartheta)\right|^{\mathfrak{q}}\right]d\tau\right)^{\frac{1}{\mathfrak{q}}}$$

$$=\frac{\vartheta-\theta}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{\mathfrak{q}}}\left(\left|\grave{\Theta}'(\theta)\right|^{\mathfrak{q}}\int_0^1 (1-\tau)|1-2\tau|\left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha}d\tau\right.$$

$$\left.+\left|\grave{\Theta}'(\vartheta)\right|^{\mathfrak{q}}\int_0^1 (1-\tau)|1-2\tau|\left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha}d\tau\right)^{\frac{1}{\mathfrak{q}}}$$

$$+\frac{\vartheta-\theta}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{\mathfrak{q}}}\left(\left|\grave{\Theta}'(\theta)\right|^{\mathfrak{q}}\int_0^1 \tau|1-2\tau|\left(\frac{2}{\pi}\arccos(1-\tau)\right)^{\alpha}d\tau\right.$$

$$\left.+\left|\grave{\Theta}'(\vartheta)\right|^{\mathfrak{q}}\int_0^1 \tau|1-2\tau|\left(\frac{2}{\pi}\arccos(\tau)\right)^{\alpha}d\tau\right)^{\frac{1}{\mathfrak{q}}},$$

where

$$\int_0^1 (1-\tau)|1-2\tau|d\tau=\int_0^1 \tau|1-2\tau|d\tau=\frac{1}{4}.$$

The proof of the theorem is now concluded. □

Remark 4.15. For $\alpha = 1$, (4.5) will reduce to the inequaity of Theorem (4.5) in [25].

Example 4.16. Consider an 6-ICCF $\grave{\Theta}(\varsigma) = e^{\varsigma^3}$ on $[0, 3]$. The visual representation presented in Figure 11, corresponding to equation (4.5), offers validation of Theorem 4.14 for $1 \leq \mathfrak{q} \leq 10$.

The following proposition illustrates that the improved power-mean integral inequality provides superior approximations compared to the power-mean integral inequality.

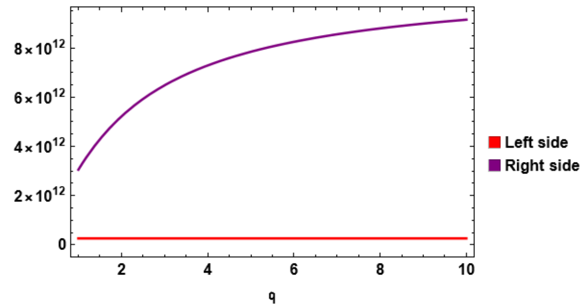

FIGURE 11. The Figure 11 provides a visual depiction of the 2D representation of equation (4.5) across the range $1 \leq \mathfrak{q} \leq 10$, with $\grave{\Theta}(\varsigma) = e^{\varsigma^3}$.

Proposition 4.17. *The inequality* (4.5) *provides superior estimates compared to inequality* (4.3)*.*

Proof. Similar to the approach in Proposition 4.13, we obtain

$$\begin{aligned}
&\frac{\vartheta-\theta}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{\mathfrak{q}}}\left[\left(\left|\grave{\Theta}'(\theta)\right|^{\mathfrak{q}}\int_0^1(1-\varsigma)|1-2\varsigma|\left(\frac{2}{\pi}\arccos(1-\varsigma)\right)^{\alpha}d\varsigma\right.\right.\\
&\qquad\left.+\left|\grave{\Theta}'(\vartheta)\right|^{\mathfrak{q}}\int_0^1(1-\varsigma)|1-2\varsigma|\left(\frac{2}{\pi}\arccos(\varsigma)\right)^{\alpha}d\varsigma\right)^{\frac{1}{\mathfrak{q}}}\\
&\qquad+\left(\left|\grave{\Theta}'(\theta)\right|^{\mathfrak{q}}\int_0^1\varsigma|1-2\varsigma|\left(\frac{2}{\pi}\arccos(1-\varsigma)\right)^{\alpha}d\varsigma\right.\\
&\qquad\left.\left.+\left|\grave{\Theta}'(\vartheta)\right|^{\mathfrak{q}}\int_0^1\varsigma|1-2\varsigma|\left(\frac{2}{\pi}\arccos(\varsigma)\right)^{\alpha}d\varsigma\right)^{\frac{1}{\mathfrak{q}}}\right]\\
&\leq\frac{\vartheta-\theta}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{\mathfrak{q}}}2\left[\frac{1}{2}\left(\left|\grave{\Theta}'(\theta)\right|^{\mathfrak{q}}\int_0^1|1-2\varsigma|\left(\frac{2}{\pi}\arccos(1-\varsigma)\right)^{\alpha}d\varsigma\right.\right.\\
&\qquad\left.\left.+\left|\grave{\Theta}'(\vartheta)\right|^{\mathfrak{q}}\int_0^1|1-2\varsigma|\left(\frac{2}{\pi}\arccos(\varsigma)\right)^{\alpha}d\varsigma\right)\right]^{\frac{1}{\mathfrak{q}}}\\
&=\frac{\vartheta-\theta}{2^{2-\frac{2}{\mathfrak{q}}}}\mathbb{A}^{\frac{1}{\mathfrak{q}}}\left(\left|\grave{\Theta}'(\theta)\right|^{\mathfrak{q}},\left|\grave{\Theta}'(\vartheta)\right|^{\mathfrak{q}}\right)\left(\int_0^1|1-2\varsigma|\left(\frac{2}{\pi}\arccos(\varsigma)\right)^{\alpha}d\varsigma\right)^{\frac{1}{\mathfrak{q}}}.
\end{aligned}$$

This observation suggests that inequality (4.5) offers superior approximations compared to inequality (4.3). □

5. Conclusion

In this study, we introduced the novel class of convex functions named α-ICCF. This extends the traditional class of ICCF. We explored their algebraic and geometric properties, providing graphical representations to enhance readers' understanding. Using α-ICCF, we derived the HH inequality and its refinements for functions whose first derivative in absolute value is α-ICCF, employing integral inequalities such as Hölder, Hölder-İşcan and power-mean integral inequalities. The presented results indicate that the newly introduced class of convex functions produces more refined and generalized outcomes in comparison with ICCF. These inequalities form the backbone of various theoretical and applied areas in mathematics, providing essential tools for analysis, approximation and the study of integrals. Our findings show that Hölder-İşcan and improved power-mean integral inequalities offer more precise approximations compared to other methods. When $\alpha = 1$, our results align with those of classical ICCF, demonstrating the robustness and versatility of the α-ICCF framework. This innovative concept expands the theoretical landscape of convexity, opening new research avenues and potential applications in various fields. We hope this work will inspire further investigation into α-inverse cosine convexity, fostering deeper insights and developments in convex analysis and related disciplines.

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