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# Perturbed Graphical Metric Spaces and Fixed Point Theory

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Abstract. Taking into account the recent view of fixed point theory as expressed by Jalali and Samet [On Banach's fixed point theorem in perturbed metric spaces, J. Appl. Anal. Comput. 14 (2) (2025), 992–1001], we first introduce a perturbed metric space equipped with a graph and then present new concepts and notions related to this space. Next, we prove some fixed point theorems related to this new space. Several consequences and an example are also presented to demonstrate the effectiveness of the main results. Following the idea of this article, one can continue this new way to obtain fixed points of the mappings that do not satisfy classic contractions in such spaces endowed with a graph or a partial order.

## 1. Introduction

Finding the fixed point of a mapping has been an interesting topic for many years, but it became famous in 1922 when Banach [2] stated his famous principle, known as the Banach contraction mapping. Afterwards, many researchers have discussed the existence of uniqueness of fixed point of different contraction mappings in various metric spaces, which form the metric version of the fixed point theory (see[1, 7–9, 11, 13–16] and references therein). The method used in previous works is based the measure of the distance between two points, which can have errors due to different reasons. Although these errors are normally small and little, their accumulation can cause problems and as a result, they will be important. To solve this problem, many numerical works and finding the smallest error are done by researchers working on applied issues. For pure topics, recently, Jleli and Samet [12] have defined an attractive

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notion named perturbed metric spaces and then established an interesting generalization of Banach's fixed point theorem. Throughout this article, let $\mathcal{A}$, $\mathbb{N}$, and $\mathbb{R}$ be an arbitrary nonempty set, the set of positive integers and real numbers, respectively.

**Definition 1.1** ([12])**.** Presume that $\mathcal{W}, \mathcal{S} : \mathcal{A} \times \mathcal{A} \to [0, +\infty)$ are two arbitrary mappings. Then $\mathcal{W}$ is called a perturbed metric on $\mathcal{A}$ with respect to $\mathcal{S}$ when

$$\mathcal{W} - \mathcal{S} : \mathcal{A} \times \mathcal{A} \to [0, +\infty)$$
$$(\mathfrak{a}, \mathfrak{b}) \longmapsto \mathcal{W}(\mathfrak{a}, \mathfrak{b}) - \mathcal{S}(\mathfrak{a}, \mathfrak{b})$$

is a metric on $\mathcal{A}$; that is, the following are held:

(1) $(\mathcal{W} - \mathcal{S})(\mathfrak{a}, \mathfrak{b}) \geq 0$;
(2) $(\mathcal{W} - \mathcal{S})(\mathfrak{a}, \mathfrak{b}) = 0$ iff $\mathfrak{a} = \mathfrak{b}$;
(3) $(\mathcal{W} - \mathcal{S})(\mathfrak{a}, \mathfrak{b}) = (\mathcal{W} - \mathcal{S})(\mathfrak{b}, \mathfrak{a})$;
(4) $(\mathcal{W} - \mathcal{S})(\mathfrak{a}, \mathfrak{b}) \leq (\mathcal{W} - \mathcal{S})(\mathfrak{a}, \mathfrak{c}) + (\mathcal{W} - \mathcal{S})(\mathfrak{c}, \mathfrak{b})$

for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathcal{A}$. In this case, $\mathcal{S}$, $d = \mathcal{W} - \mathcal{S}$, and $(\mathcal{A}, \mathcal{W}, \mathcal{S})$ are a perturbed mapping, an exact metric, and a perturbed metric space, respectively.

**Example 1.2** ([12])**.** Assume $\mathcal{C}([0, 1])$ is the set of all $u : [0, 1] \to [0, 1]$ such that $u$ is continuous on $[0, 1]$ and define the mapping $\mathcal{W} : \mathcal{C}([0, 1]) \times \mathcal{C}([0, 1]) \to [0, +\infty)$ by

$$\mathcal{W}(u, v) = \int_0^1 |u(t) - v(t)| dt + (s(0) - v(0))^2$$

for $u, v \in \mathcal{C}([0, 1])$. Then $\mathcal{W}$ is a perturbed metric on $\mathcal{C}([0, 1])$ in respect of the perturbed mapping $\mathcal{S} : \mathcal{C}([0, 1]) \times \mathcal{C}([0, 1]) \to [0, +\infty)$ given by $\mathcal{S}(u, v) = (u(0) - v(0))^2$ for $u, v \in \mathcal{C}([0, 1])$. In this case, the mapping $d : \mathcal{C}([0, 1]) \times \mathcal{C}([0, 1]) \to [0, +\infty)$ considered by $d(u, v) = \int_0^1 |u(t) - v(t)| dt$ for $u, v \in \mathcal{C}([0, 1])$, is the exact metric. It should be mentioned that $\mathcal{W}$ is symmetric and $\mathcal{W}(u, v) = 0$ iff $u = v$, but $\mathcal{W}$ is not a metric on $\mathcal{C}([0, 1])$. It is enough to take constant functions $u_1 \equiv 0$, $u_2 \equiv \frac{1}{2}$ and $u_3 \equiv 1$. Then we have $\mathcal{W}(u_1, u_2) = \frac{3}{4}$, $\mathcal{W}(u_2, u_3) = \frac{3}{4}$ and $\mathcal{W}(u_1, u_3) = 2$ which imply that $\mathcal{W}(u_1, u_3) > \mathcal{W}(u_1, u_2) + \mathcal{W}(u_2, u_3)$. Therefore, the triangle inequality doesn't hold for $\mathcal{W}$.

More examples for this metric are introduced by Jleli and Samet. Also, it should be noted that a perturbed metric on $\mathcal{W}$ does not necessarily need to be a metric on $\mathcal{A}$. Now, we give some new examples as follows:

**Example 1.3.** Assume $\mathcal{B}(\mathcal{A})$ is the set of all real bounded functions on $\mathcal{A}$. Take $\mathfrak{a}_0 \in \mathcal{A}$ and define the mapping $\mathcal{W} : \mathcal{B}(\mathcal{A}) \times \mathcal{B}(\mathcal{A}) \to [0, +\infty)$

by

$$\mathcal{W}(f,g)=\sup_{\mathfrak{a}\in\mathcal{A}}|f(\mathfrak{a})-g(\mathfrak{a})|+(f(\mathfrak{a}_0)g(\mathfrak{a}_0))^2$$

for $f,g\in\mathcal{B}(\mathcal{A})$. Then $\mathcal{W}$ is a perturbed metric on $\mathcal{B}(\mathcal{A})$ regarding the perturbed mapping $\mathcal{S}:\mathcal{B}(\mathcal{A})\times\mathcal{B}(\mathcal{A})\to[0,+\infty)$ given by $\mathcal{S}(f,g)=(f(\mathfrak{a}_0)g(\mathfrak{a}_0))^2$ for $f,g\in\mathcal{B}(\mathcal{A})$. In this case, the exact metric is the mapping $d:\mathcal{B}(\mathcal{A})\times\mathcal{B}(\mathcal{A})\to[0,+\infty)$ defined by $d(f,g)=\sup_{\mathfrak{a}\in\mathcal{A}}|f(\mathfrak{a})-g(\mathfrak{a})|$ for $f,g\in\mathcal{B}(\mathcal{A})$. Clearly, $\mathcal{W}$ is symmetric, $\mathcal{W}(f,g)=0$ if and only if $f=g$ and either $f(\mathfrak{a}_0)=0$ or $g(\mathfrak{a}_0)=0$. However, $\mathcal{W}$ is not a metric on $\mathcal{C}([0,1])$. It is sufficient to take the constant functions $f\equiv 2$, $g\equiv 3$ and $h\equiv 1$. Then we have $\mathcal{W}(f,g)=37$, $\mathcal{W}(f,h)=5$ and $\mathcal{W}(h,g)=1$, which deduce that $\mathcal{W}(f,g)>\mathcal{W}(f,h)+\mathcal{W}(h,g)$. This shows that the triangle inequality is not valid for this $\mathcal{W}$.

**Example 1.4.** Let $n\in\mathbb{N}$, $\mathbb{M}_n(\mathbb{R})$ be a set of all square $n\times n$ matrices of real numbers and $(1,0,\dots,0)\in\mathbb{R}^n$. For $A,B\in\mathbb{M}_n(\mathbb{R})$, define the mapping $\mathcal{W}:\mathbb{M}_n(\mathbb{R})\times\mathbb{M}_n(\mathbb{R})\to[0,+\infty)$ by

$$\mathcal{W}(A,B)=\sup_{x\in\mathbb{R}^n,\|x\|=1}\left\|Ax^T-Bx^T\right\|+\left\|AB(1,0,\dots,0)^T\right\|,$$

where $x^T$ is a transpose of vector $x$. Then $\mathcal{W}$ is a perturbed metric on $\mathbb{M}_n(\mathbb{R})$ in respect of the perturbed mapping $\mathcal{S}:\mathbb{M}_n(\mathbb{R})\times\mathbb{M}_n(\mathbb{R})\to[0,+\infty)$ introduced by $\mathcal{S}(A,B)=\left\|AB(1,0,\dots,0)^T\right\|$ for $A,B\in\mathbb{M}_n(\mathbb{R})$. It is evident that the mapping $d:\mathbb{M}_n(\mathbb{R})\times\mathbb{M}_n(\mathbb{R})\to[0,+\infty)$, considered by

$$d(A,B)=\sup\left\{\left\|Ax^T-Bx^T\right\|:x\in\mathbb{R}^n,\|x\|=1\right\}$$

for $A,B\in\mathbb{M}_n(\mathbb{R})$, is the exact metric. Also, notice that $\mathcal{W}$ is not a metric on $\mathbb{M}_n(\mathbb{R})$.

Jleli and Samet mentioned some elementary properties of perturbed metric spaces and some topological concepts in such spaces as follows:

**Proposition 1.5** ([12])**.** *Assume $\mathcal{W},\mathcal{S},\mathcal{P}:\mathcal{A}\times\mathcal{A}\to[0,\infty)$ are three mappings and $\alpha>0$.*

1. *If $(\mathcal{A},\mathcal{W},\mathcal{S})$ and $(\mathcal{A},\mathcal{W},\mathcal{P})$ are two perturbed metric spaces, then so $\left(\mathcal{A},\mathcal{W},\frac{\mathcal{S}+\mathcal{P}}{2}\right)$ is.*
2. *If $(\mathcal{A},\mathcal{W},\mathcal{S})$ is a perturbed metric space, then so $(\mathcal{A},\alpha\mathcal{W},\alpha\mathcal{S})$ is.*

**Definition 1.6** ([12])**.** Suppose that $(\mathcal{A},\mathcal{W},\mathcal{S})$ is a perturbed metric space, $\{\mathfrak{a}_i\}$ is a sequence in $\mathcal{A}$ and $\mathcal{F}:\mathcal{W}\to\mathcal{W}$ is a mapping.

1. $\{\mathfrak{a}_i\}$ is a perturbed convergent (Cauchy) sequence in $(\mathcal{A},\mathcal{W},\mathcal{S})$ if $\{\mathfrak{a}_i\}$ is a convergent (Cauchy) sequence in $(\mathcal{A},d)$ in which $d$ is the exact metric.

(2) $(\mathcal{A},\mathcal{W},\mathcal{S})$ is a complete perturbed metric space if $(\mathcal{A},d)$ is a complete metric space.
(3) $\mathcal{F}$ is a perturbed continuous mapping whenever $\mathcal{F}$ is continuous in respect of $d$.

## 2. Main Results

Following the idea of Jachymski [10] and Fallahi and Aghanians [6], we define a perturbed metric space with a graph. In a given (not necessarily simple) graph $\mathcal{H}$, a link and a loop will be an edge of $\mathcal{H}$ with distinct ends and identical ends, respectively. Two or more links of $\mathcal{H}$ with the same pairs of ends are parallel edges. Assume $(\mathcal{A},\mathcal{W},\mathcal{S})$ is a perturbed metric space and $\mathcal{H}$ is a directed graph in which vertex set $V(\mathcal{H})=\mathcal{A}$ and edge set $\mathcal{E}(\mathcal{H})$ contains all loops (note that $\mathcal{H}$ may totally have uncountably many vertices). Additionally, presume $\mathcal{H}$ has no parallel edges. Then $(\mathcal{A},\mathcal{W},\mathcal{S})$ is called a perturbed metric space equipped with the graph $\mathcal{H}$. In addition, $\mathcal{H}^{-1}$ is the conversion of $\mathcal{H}$, i.e. a directed graph obtained from $\mathcal{H}$ by reversing the directions of the edges of $\mathcal{H}$, and $\widetilde{\mathcal{H}}$ is the undirected graph obtained from $\mathcal{H}$ by ignoring the directions of the edges $\mathcal{H}$. It is clear that $V\left(\mathcal{H}^{-1}\right)=V\left(\widetilde{\mathcal{H}}\right)=V(\mathcal{H})=\mathcal{A}$, $\mathcal{E}\left(\mathcal{H}^{-1}\right)=\{(\mathfrak{a},\mathfrak{b})\in\mathcal{A}\times\mathcal{A}:(\mathfrak{b},\mathfrak{a})\in\mathcal{E}(\mathcal{H})\}$ and $\mathcal{E}\left(\widetilde{\mathcal{H}}\right)=\mathcal{E}(\mathcal{H})\cup\mathcal{E}\left(\mathcal{H}^{-1}\right)$. In the sequel, assume $(\mathcal{A},\mathcal{W},\mathcal{S})$ is a perturbed metric space with the graph $\mathcal{H}$ and $\mathcal{F}:\mathcal{A}\to\mathcal{A}$ is a mapping. We start with some new definitions, transforming into a perturbed metric.

**Definition 2.1.** The mapping $\mathcal{F}$ is said to be an orbitally perturbed $\mathcal{H}$-continuous mapping on $\mathcal{A}$ if $\mathcal{F}^{r_i}\mathfrak{a}\to\mathfrak{b}$ implies $\mathcal{F}(\mathcal{F}^{r_i}\mathfrak{a})\to\mathcal{F}\mathfrak{b}$ for all $\mathfrak{a},\mathfrak{b}\in\mathcal{A}$ and all sequences $\{r_i\}$ of positive integers provided that $\left(\mathcal{F}^{r_i}\mathfrak{a},\mathcal{F}^{r_i+1}\mathfrak{a}\right)\in\mathcal{E}(\mathcal{H})$ for all $i\in\mathbb{N}$.

The next definition is allocated to convert a $\mathcal{H}$-quasi-contraction ([6, Definition 1]) in metric spaces with a graph to perturbed metric spaces endowed with a graph.

**Definition 2.2.** The mapping $\mathcal{F}$ is called a perturbed $\mathcal{H}$-quasi-contraction if

($\mathcal{Q}_1$) $\mathcal{F}$ keeps the edges of $\mathcal{H}$; that is, $(\mathfrak{a},\mathfrak{b})\in\mathcal{E}(\mathcal{H})$ implies $(\mathcal{F}\mathfrak{a},\mathcal{F}\mathfrak{b})\in\mathcal{E}(\mathcal{H})$ for all $\mathfrak{a},\mathfrak{b}\in\mathcal{A}$;
($\mathcal{Q}_2$) there is a $\alpha\in[0,1)$ provided that

$$\mathcal{W}(\mathcal{F}\mathfrak{a},\mathcal{F}\mathfrak{b})\leq\alpha\max\left\{\mathcal{W}(\mathfrak{a},\mathfrak{b}),\mathcal{W}\left(\mathfrak{a},\mathcal{F}\mathfrak{a}\right),\mathcal{W}(\mathfrak{b},\mathcal{F}\mathfrak{b}),\mathcal{W}(\mathfrak{a},\mathcal{F}\mathfrak{b}),\mathcal{W}(\mathfrak{b},\mathcal{F}\mathfrak{a})\right\}$$

for all $\mathfrak{a},\mathfrak{b}\in\mathcal{A}$ with $(\mathfrak{a},\mathfrak{b})\in\mathcal{E}(\mathcal{H})$.

Note that $\alpha$ in ($\mathcal{Q}_2$) is named a quasi-contractive constant of $\mathcal{F}$. Also, next proposition is an immediate consequence of Definition 2.2 and gives a simple procedure to construct new perturbed $\mathcal{H}$-quasi-contractions from older ones.

**Proposition 2.3.** *Presume that* $(\mathcal{A}, \mathcal{W}, \mathcal{S})$ *is a perturbed metric space endowed with* $\mathcal{H}$ *and* $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ *is a mapping.*

- *If* $\mathcal{F}$ *keeps the edges of* $\mathcal{H}$*,* $\mathcal{F}$ *keeps the edges of* $\mathcal{H}^{-1}$ *and* $\widetilde{\mathcal{H}}$*.*
- *If* $\mathcal{Q}_2$ *for the graph* $\mathcal{H}$ *is held by* $\mathcal{F}$*, then it is held for both the graphs* $\mathcal{H}^{-1}$ *and* $\widetilde{\mathcal{H}}$ *by* $\mathcal{F}$*.*
- *If* $\mathcal{F}$ *is a* $\mathcal{H}$*-quasi-contraction with* $\alpha \in [0, 1)$*, then* $\mathcal{F}$ *is a* $\mathcal{H}^{-1}$*-quasi-contraction and a* $\widetilde{\mathcal{H}}$*-quasi-contraction with* $\alpha$*.*

*Proof.* The proof of this is similar to what Fallahi and Aghanians [6, 2016] have done. □

The following notions will also be required in the sequel.
Take $\mathcal{C}_{\mathcal{F}}$ the set of all points $\mathfrak{a} \in \mathcal{A}$ so that $(\mathcal{F}^m\mathfrak{a}, \mathcal{F}^n\mathfrak{a})$ is an edge of $\widetilde{\mathcal{H}}$ for all $m, n \in \mathbb{N} \cup \{0\}$, i.e.

$$\mathcal{C}_{\mathcal{F}} = \left\{ \mathfrak{a} \in \mathcal{A} : (\mathcal{F}^m\mathfrak{a}, \mathcal{F}^n\mathfrak{a}) \in \mathcal{E}\left(\widetilde{\mathcal{H}}\right), \quad m, n = 0, 1, \dots \right\}.$$

For $\mathfrak{a} \in \mathcal{A}$ and $n \in \mathbb{N} \cup \{0\}$, the $n$-th orbit of $\mathfrak{a}$ under $\mathcal{F}$ is $\mathcal{O}(\mathfrak{a}; n) = \{\mathfrak{a}, \mathcal{F}\mathfrak{a}, \dots, \mathcal{F}^n\mathfrak{a}\}$. Let $\mathcal{B}$ be a subset of $\mathcal{A}$. The diameter of $\mathcal{B}$ in $\mathcal{A}$ is $diam(\mathcal{B}) = \sup\{\mathcal{W}(\mathfrak{a}, \mathfrak{b}) : \mathfrak{a}, \mathfrak{b} \in \mathcal{B}\}$.

To prove the existence of a fixed point for a perturbed $\mathcal{H}$-quasi-contraction in a perturbed metric space equipped with $\mathcal{H}$, we need next lemmas.

**Lemma 2.4.** *Assume* $(\mathcal{A}, \mathcal{W}, \mathcal{S})$ *is a perturbed metric space with the graph* $\mathcal{H}$ *and* $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ *is a perturbed* $\mathcal{H}$*-quasi-contraction. Then, for each* $\mathfrak{a} \in \mathcal{C}_{\mathcal{F}}$ *and each* $n \in \mathbb{N}$*, there exists a* $l \in \mathbb{N}$ *no more than* $n$ *so that* $diam\,(\mathcal{O}(\mathfrak{a}; i)) = \mathcal{W}(\mathfrak{a}, \mathcal{F}^l\mathfrak{a})$*.*

*Proof.* Let $\mathfrak{a} \in \mathcal{C}_{\mathcal{F}}$ and $i \in \mathbb{N}$ be given. If $diam(\mathcal{O}(\mathfrak{a}; i)) = 0$, then $\mathcal{O}(\mathfrak{a}; i)$ is singleton. Specifically, $\mathfrak{a}$ is a fixed point for $\mathcal{F}$ and $\mathcal{W}(\mathcal{F}^n\mathfrak{a}, \mathcal{F}^m\mathfrak{a}) = 0$ for all $n, m = 0, \dots, i$. Thus, the statement holds for any positive integer $l$ no more than $n$. Otherwise, as $\mathcal{O}(\mathfrak{a}; i)$ is a finite set, there are distinct nonnegative integers $n$ and $m$ no more that $i$ so that $diam(\mathcal{O}(\mathfrak{a}; i)) = \mathcal{W}(\mathcal{F}^n\mathfrak{a}, \mathcal{F}^m\mathfrak{a})$. Since $\mathcal{F}$ satisfies $(\mathcal{Q}_2)$ for the graph $\widetilde{\mathcal{H}}$, if both the integers $n$ and $m$ are positive, since $\left(\mathcal{F}^{n-1}\mathfrak{a}, \mathcal{F}^{m-1}\mathfrak{a}\right) \in \mathcal{E}\left(\widetilde{\mathcal{H}}\right)$, then

$$\begin{aligned} \mathcal{W}\left(\mathcal{F}^n\mathfrak{a}, \mathcal{F}^m\mathfrak{a}\right) &= \mathcal{W}\left(\mathcal{F}\mathcal{F}^{n-1}\mathfrak{a}, \mathcal{F}\mathcal{F}^{m-1}\mathfrak{a}\right) \\ &\leq \alpha \max\left\{\mathcal{W}\left(\mathcal{F}^{n-1}\mathfrak{a}, \mathcal{F}^{m-1}\mathfrak{a}\right), \mathcal{W}\left(\mathcal{F}^{n-1}\mathfrak{a}, \mathcal{F}^{m}\mathfrak{a}\right), \right. \\ &\quad \left. \mathcal{W}\left(\mathcal{F}^{m-1}\mathfrak{a}, \mathcal{F}^{m}\mathfrak{a}\right), \mathcal{W}\left(\mathcal{F}^{n-1}\mathfrak{a}, \mathcal{F}^{m}\mathfrak{a}\right), \mathcal{W}\left(\mathcal{F}^{m-1}\mathfrak{a}, \mathcal{F}^{n}\mathfrak{a}\right)\right\} \end{aligned} \tag{2.1}$$

$$\leq \alpha diam\left(\mathcal{O}(\mathfrak{a};i)\right).$$

Therefore,

$$diam\left(\mathcal{O}(\mathfrak{a};i)\right)=\mathcal{W}(\mathcal{F}^n\mathfrak{a},\mathcal{F}^m\mathfrak{a})\leq \alpha diam\left(\mathcal{O}(\mathfrak{a};i)\right),$$

which is a contradiction. Hence, either $n$ or $m$ must be zero and the proof in complete. □

**Lemma 2.5.** *Presume* $(\mathcal{A},\mathcal{W},\mathcal{S})$ *is a perturbed metric space with the graph* $\mathcal{H}$ *and* $\mathcal{F}:\mathcal{A}\to\mathcal{A}$ *is a perturbed* $\mathcal{H}$*-quasi-contraction. Then* $\{\mathcal{F}^i\mathfrak{a}\}$ *is a perturbed Cauchy sequence in* $(\mathcal{A},\mathcal{W},\mathcal{S})$ *for all* $\mathfrak{a}\in\mathcal{C}_{\mathcal{F}}$.

*Proof.* Let $\mathfrak{a}\in\mathcal{C}_{\mathcal{F}}$. If $i,j\in\mathbb{N}$ and $j\geq i\geq 2$, it follows from $\mathcal{F}^{j-1}\mathfrak{a}\in\mathcal{C}_{\mathcal{F}}$ and putting $n=j-i+1$ and $m=1$ in (2.1) that

$$\begin{aligned}\mathcal{W}\left(\mathcal{F}^j\mathfrak{a},\mathcal{F}^i\mathfrak{a}\right)&=\mathcal{W}\left(\mathcal{F}^{j-i+1}\mathcal{F}^{i-1}\mathfrak{a},\mathcal{F}\mathcal{F}^{i-1}\mathfrak{a}\right)\\&\leq \alpha diam\left(\mathcal{O}\left(\mathcal{F}^{i-1}\mathfrak{a};j-i+1\right)\right),\end{aligned}\tag{2.2}$$

where $\alpha\in[0,1)$ is a quasi-contractive constant of $\mathcal{F}$. Moreover, by Lemma 2.4, there is $k\in\mathbb{N}$ no more than $j-i+1$ so that

$$diam\left(\mathcal{O}\left(\mathcal{F}^{i-1}\mathfrak{a};j-i+1\right)\right)=\mathcal{W}\left(\mathcal{F}^{i-1}\mathfrak{a},\mathcal{F}^{k+i-1}\mathfrak{a}\right).\tag{2.3}$$

Now, it follows from $\mathcal{F}^{i-2}\mathfrak{a}\in\mathcal{C}_{\mathcal{F}}$ for $i\geq 2$ and so putting $n=1$ and $m=k+1$ in (2.1) that

$$\begin{aligned}\mathcal{W}\left(\mathcal{F}^{i-1}\mathfrak{a},\mathcal{F}^{k+i-1}\mathfrak{a}\right)&=\mathcal{W}\left(\mathcal{F}\mathcal{F}^{i-2}\mathfrak{a},\mathcal{F}^{k+1}\mathcal{F}^{i-2}\mathfrak{a}\right)\\&\leq \alpha diam\left(O\left(\mathcal{F}^{i-2}\mathfrak{a};j-i+2\right)\right).\end{aligned}\tag{2.4}$$

Putting $n=1$ and $m=k$ in Lemma 2.4 and taking the exact metric $d=\mathcal{W}-\mathcal{S}$, we conclude from (2.1) that

$$\begin{aligned}diam\left(\mathcal{O}(\mathfrak{a};i)\right)&=\mathcal{W}\left(\mathfrak{a},\mathcal{F}^k\mathfrak{a}\right)\\&=d\left(\mathfrak{a},\mathcal{F}^k\mathfrak{a}\right)+\mathcal{S}\left(\mathfrak{a},\mathcal{F}^k\mathfrak{a}\right)\\&\leq d\left(\mathfrak{a},\mathcal{F}\mathfrak{a}\right)+d\left(\mathcal{F}\mathfrak{a},\mathcal{F}^k\mathfrak{a}\right)+\mathcal{S}\left(\mathfrak{a},\mathcal{F}^k\mathfrak{a}\right)\\&=\mathcal{W}\left(\mathfrak{a},\mathcal{F}\mathfrak{a}\right)-\mathcal{S}\left(\mathfrak{a},\mathcal{F}\mathfrak{a}\right)+\mathcal{W}\left(\mathcal{F}\mathfrak{a},\mathcal{F}^k\mathfrak{a}\right)\\&\quad-\mathcal{S}\left(\mathcal{F}\mathfrak{a},\mathcal{F}^k\mathfrak{a}\right)+\mathcal{S}\left(\mathfrak{a},\mathcal{F}^k\mathfrak{a}\right)\\&\leq\mathcal{W}\left(\mathfrak{a},\mathcal{F}\mathfrak{a}\right)+\mathcal{W}\left(\mathcal{F}\mathfrak{a},\mathcal{F}^k\mathfrak{a}\right)+\mathcal{S}\left(\mathfrak{a},\mathcal{F}^k\mathfrak{a}\right)\\&\leq\mathcal{W}\left(\mathfrak{a},\mathcal{F}\mathfrak{a}\right)+\alpha diam\left(\mathcal{O}(\mathfrak{a};i)\right)+\mathcal{S}\left(\mathfrak{a},\mathcal{F}^k\mathfrak{a}\right),\end{aligned}$$

which implies that

$$diam\left(\mathcal{O}(\mathfrak{a};i)\right) \le \frac{1}{1-\alpha}\left(\mathcal{W}\left(\mathfrak{a},\mathcal{F}\mathfrak{a}\right)+\mathcal{S}\left(\mathfrak{a},\mathcal{F}^{k}\mathfrak{a}\right)\right). \tag{2.5}$$

Finally, combining (2.2)-(2.4) and using (2.5), we obtain

$$\begin{aligned} d\left(\mathcal{F}^{j}\mathfrak{a},\mathcal{F}^{i}\mathfrak{a}\right) &= \mathcal{W}\left(\mathcal{F}^{j}\mathfrak{a},\mathcal{F}^{i}\mathfrak{a}\right)-\mathcal{S}\left(\mathcal{F}^{j}\mathfrak{a},\mathcal{F}^{i}\mathfrak{a}\right)\\ &\le \mathcal{W}\left(\mathcal{F}^{j}\mathfrak{a},\mathcal{F}^{i}\mathfrak{a}\right)\\ &\le \alpha diam\left(O\left(\mathcal{F}^{i-1}\mathfrak{a};j-i+1\right)\right)\\ &= \alpha\mathcal{W}\left(\mathcal{F}^{i-1}\mathfrak{a},\mathcal{F}^{k+i-1}\mathfrak{a}\right)\\ &\le \alpha^{2} diam\left(O\left(\mathcal{F}^{i-2}\mathfrak{a};j-i+2\right)\right)\\ &\;\;\vdots\\ &\le \alpha^{i} diam\left(\mathcal{O}(\mathfrak{a};j)\right)\\ &\le \frac{\alpha^{i}}{1-\alpha}\left(\mathcal{W}\left(\mathfrak{a},\mathcal{F}\mathfrak{a}\right)+\mathcal{S}\left(\mathfrak{a},\mathcal{F}^{k}\mathfrak{a}\right)\right). \end{aligned} \tag{2.6}$$

Thus, we obtain

$$d\left(\mathcal{F}^{j}\mathfrak{a},\mathcal{F}^{i}\mathfrak{a}\right) \le \frac{\alpha^{i}}{1-\alpha}\left(\mathcal{W}\left(\mathfrak{a},\mathcal{F}\mathfrak{a}\right)+\mathcal{S}\left(\mathfrak{a},\mathcal{F}^{k}\mathfrak{a}\right)\right).$$

Letting $j,i\to\infty$, we have $d\left(\mathcal{F}^{j}\mathfrak{a},\mathcal{F}^{i}\mathfrak{a}\right)\to 0$. Hence, $\{\mathcal{F}^{i}\mathfrak{a}\}$ is a Cauchy sequence in $(\mathcal{A},d)$; that is, $\{\mathcal{F}^{i}\mathfrak{a}\}$ is a perturbed Cauchy sequence in the perturbed metric space $(\mathcal{A},\mathcal{W},\mathcal{S})$. □

Now, we are able to state main theorem of this paper, which is the existence of fixed points for perturbed $\mathcal{H}$-quasi-contraction mappings in complete perturbed metric spaces with a graph.

**Theorem 2.6.** *Assume $(\mathcal{A},\mathcal{W},\mathcal{S})$ is a complete perturbed metric space equipped with the graph $\mathcal{H}$, and $\mathcal{F}:\mathcal{A}\to\mathcal{A}$ is a perturbed $\mathcal{H}$-quasi-contraction and also, an orbitally perturbed $\mathcal{H}$-continuous mapping on $\mathcal{A}$. Then the restriction of $\mathcal{F}$ to $\mathcal{C}_{\mathcal{F}}$ has a fixed point. Specifically, if $\mathcal{F}$ is a perturbed $\mathcal{H}$-quasi-contraction, $\mathcal{F}$ possesses a fixed point in $\mathcal{A}$ iff $\mathcal{C}_{\mathcal{F}}\neq\emptyset$.*

*Proof.* If $\mathcal{C}_{\mathcal{F}}=\emptyset$, then there is nothing to prove. Hence, let $\mathcal{C}_{\mathcal{F}}\neq\emptyset$ and $\mathfrak{a}\in\mathcal{C}_{\mathcal{F}}$. Then it follows from $\left(\mathcal{F}^{j}\mathfrak{a},\mathcal{F}^{i}\mathfrak{a}\right)\in E(\widetilde{G})$ for all $j,i\in\mathbb{N}\cup\{0\}$ that $\mathcal{F}\mathfrak{a}\in\mathcal{C}_{\mathcal{F}}$. Thus, $\mathcal{C}_{\mathcal{F}}$ is $\mathcal{F}$-invariant, i.e. $\mathcal{F}(\mathcal{C}_{\mathcal{F}})\subseteq\mathcal{C}_{\mathcal{F}}$. Moreover, by Lemma 2.5, $\{\mathcal{F}^{i}\mathfrak{a}\}$ is a perturbed Cauchy sequence in the complete perturbed metric space $(\mathcal{A},\mathcal{W},\mathcal{S})$. Hence, there is $\mathfrak{a}^{\star}\in\mathcal{A}$ (depending on $\mathfrak{a}$) provided that $\mathcal{F}^{i}\mathfrak{a}\to\mathfrak{a}^{\star}$. We show that $\mathfrak{a}^{\star}$ is a fixed point for $\mathcal{F}$. To do this, note first that from $\mathfrak{a}\in\mathcal{C}_{\mathcal{F}}$, we obtain $(\mathcal{F}^{i}\mathfrak{a},\mathcal{F}^{i+1}\mathfrak{a})\in\mathcal{E}\left(\widetilde{\mathcal{H}}\right)$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\mathcal{F}$ is orbitally perturbed $\widetilde{\mathcal{H}}$-continuous on $\mathcal{A}$, then $\mathcal{F}^i\mathfrak{a} \to \mathfrak{a}^\star$ implies that $\mathcal{F}^{i+1}\mathfrak{a} = \mathcal{F}(\mathcal{F}^i\mathfrak{a}) \to \mathcal{F}\mathfrak{a}^\star$. Using the uniqueness of the limit, we get $\mathcal{F}\mathfrak{a}^\star = \mathfrak{a}^\star$. Finally, since $\mathcal{C}_\mathcal{F}$ contains all fixed points of $\mathcal{F}$, it follows that $\mathfrak{a}^\star \in \mathcal{C}_\mathcal{F}$. In conclusion, $\mathcal{F}\mid_{\mathcal{C}_\mathcal{F}}: \mathcal{C}_\mathcal{F} \to \mathcal{C}_\mathcal{F}$ possesses a fixed point. □

**Example 2.7.** Presume that $\mathcal{A} = \{\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4\}$ in which $\mathcal{R}_1 = \left(\frac{\sqrt{2}}{2}, 0, 0\right)$, $\mathcal{R}_2 = \left(0, \frac{\sqrt{2}}{2}, 0\right)$, $\mathcal{R}_3 = \left(0, 0, \frac{\sqrt{2}}{2}\right)$ and $\mathcal{R}_4 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ in $\mathbb{R}^3$. Using Euclidean norm in $\mathbb{R}^3$, we get $\|\mathcal{R}_i - \mathcal{R}_j\| = 1$ for $i \neq j$. Now, take the mapping $\mathcal{F} : \mathcal{A} \to \mathcal{S}$ by

$$\mathcal{F}\mathcal{R}_1 = \mathcal{R}_2, \qquad \mathcal{F}\mathcal{R}_2 = \mathcal{R}_3, \qquad \mathcal{F}\mathcal{R}_3 = \mathcal{R}_4, \qquad \mathcal{F}\mathcal{R}_4 = \mathcal{R}_4,$$

and consider the mapping $\mathcal{S} : \mathcal{A} \times \mathcal{A} \to [0, +\infty)$ by

$$\begin{array}{ll} \mathcal{S}(\mathcal{R}_1, \mathcal{R}_2) = \mathcal{S}(\mathcal{R}_2, \mathcal{R}_1) = 7, & \mathcal{S}(\mathcal{R}_1, \mathcal{R}_3) = \mathcal{S}(\mathcal{R}_3, \mathcal{R}_1) = 6, \\ \mathcal{S}(\mathcal{R}_1, \mathcal{R}_4) = \mathcal{S}(\mathcal{R}_4, \mathcal{R}_1) = 5, & \mathcal{S}(\mathcal{R}_2, \mathcal{R}_3) = \mathcal{S}(\mathcal{R}_2, \mathcal{R}_3) = 6, \\ \mathcal{S}(\mathcal{R}_2, \mathcal{R}_4) = \mathcal{S}(\mathcal{R}_4, \mathcal{R}_2) = 13, & \mathcal{S}(\mathcal{R}_3, \mathcal{R}_4) = \mathcal{S}(\mathcal{R}_4, \mathcal{R}_3) = 5, \\ \mathcal{S}(\mathcal{R}_i, \mathcal{R}_i) = 0, i = 1, 2, 3, 4. & \end{array}$$

Also, define $\mathcal{W} : \mathcal{A} \times \mathcal{A} \to [0, +\infty)$ by $\mathcal{W}(\mathcal{R}_i, \mathcal{R}_j) = \|\mathcal{R}_i - \mathcal{R}_j\| + \mathcal{S}(\mathcal{R}_i, \mathcal{R}_j)$ for $i, j \in \{1, 2, 3, 4\}$. Clearly $(\mathcal{A}, \mathcal{W}, \mathcal{S})$ is a perturbed metric space. In this case, the exact metric is the discrete metric $d : \mathcal{A} \times \mathcal{A} \to [0, +\infty)$ introduced by $d(\mathcal{R}_i, \mathcal{R}_j) = \|\mathcal{R}_i - \mathcal{R}_j\|$ for $i, j \in \{1, 2, 3, 4\}$. Observe that

$$d(\mathcal{R}_i, \mathcal{R}_j) = \begin{cases} 1 & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases}$$

By simple calculations, we get $\mathcal{W}(\mathcal{R}_2, \mathcal{R}_3) = 1 + \mathcal{S}(\mathcal{R}_2, \mathcal{R}_3) = 7$, $\mathcal{W}(\mathcal{R}_2, \mathcal{R}_4) = 1 + \mathcal{S}(\mathcal{R}_2, \mathcal{R}_4) = 14$, and $\mathcal{W}(\mathcal{R}_3, \mathcal{R}_4) = 1 + \mathcal{S}(\mathcal{R}_3, \mathcal{R}_4) = 5$ which deduce that

$$\mathcal{W}(\mathcal{R}_2, \mathcal{R}_4) > \mathcal{W}(\mathcal{R}_2, \mathcal{R}_3) + \mathcal{W}(\mathcal{R}_3, \mathcal{R}_4).$$

So $\mathcal{W}$ is not metric on $\mathcal{A}$. Further, for elements $\mathcal{R}_1$ and $\mathcal{R}_3$ and any $\alpha \in [0, 1)$, we get

$$\begin{aligned} \mathcal{W}(\mathcal{F}\mathcal{R}_1, \mathcal{F}\mathcal{R}_3) &= \|\mathcal{F}\mathcal{R}_1 - \mathcal{F}\mathcal{R}_3\| + \mathcal{S}(\mathcal{F}\mathcal{R}_1, \mathcal{F}\mathcal{R}_3) \\ &= \|\mathcal{R}_2 - \mathcal{R}_4\| + \mathcal{S}(\mathcal{R}_2, \mathcal{R}_4) \\ &= 14 \\ &> 7 \\ &= \|\mathcal{R}_1 - \mathcal{R}_3\| + \mathcal{S}(\mathcal{R}_1, \mathcal{R}_3), \end{aligned}$$

and

$$\begin{aligned} \mathcal{W}(\mathcal{F}\mathcal{R}_1, \mathcal{F}\mathcal{R}_3) &= \|\mathcal{F}\mathcal{R}_1 - \mathcal{F}\mathcal{R}_3\| + \mathcal{S}(\mathcal{F}\mathcal{R}_1, \mathcal{F}\mathcal{R}_3) \\ &= \|\mathcal{R}_2 - \mathcal{R}_4\| + \mathcal{S}(\mathcal{R}_2, \mathcal{R}_4) \end{aligned}$$

$$
\begin{aligned}
&= 14 > 8\alpha \\
&= \alpha \max \Big\{ \|\mathcal{R}_1 - \mathcal{R}_3\| + \mathcal{S}(\mathcal{R}_1, \mathcal{R}_3), \|\mathcal{R}_1 - \mathcal{F}R_1\| \\
&\quad + \mathcal{S}(\mathcal{R}_1, \mathcal{F}\mathcal{R}_1), \|\mathcal{R}_3 - \mathcal{F}\mathcal{R}_3\| + \mathcal{S}(\mathcal{R}_3, \mathcal{F}\mathcal{R}_3), \\
&\quad \|\mathcal{R}_1 - \mathcal{F}\mathcal{R}_3\| + \mathcal{S}(\mathcal{R}_1, \mathcal{F}\mathcal{R}_3), \|\mathcal{R}_3 - \mathcal{F}\mathcal{R}_1\| + \mathcal{S}(\mathcal{R}_3, \mathcal{F}\mathcal{R}_1) \Big\} \\
&= \alpha \max \{\mathcal{W}(\mathcal{R}_1, \mathcal{R}_3), \mathcal{W}(\mathcal{R}_1, \mathcal{F}\mathcal{R}_1), \mathcal{W}(\mathcal{R}_3, \mathcal{F}\mathcal{R}_3), \\
&\quad \mathcal{W}(\mathcal{R}_1, \mathcal{F}\mathcal{R}_3), \mathcal{W}(\mathcal{R}_3, \mathcal{F}\mathcal{R}_1)\}.
\end{aligned}
$$

So $\mathcal{F}$ and does not satisfy Definition 2.2 ($\mathcal{Q}_2$) and Theorem 3.1 [11] in the usual version. Now, consider the graph $\mathcal{H}^*$ by $V(\mathcal{H}^*) = \mathcal{A}$ and $\mathcal{E}(\mathcal{H}^*) = \{(\mathcal{R}_i, \mathcal{R}_i) : \mathcal{R}_i \in \mathcal{A}\} \cup \{(\mathcal{R}_1, \mathcal{R}_2), (\mathcal{R}_2, \mathcal{R}_1)\}$ and assume $(\mathcal{A}, \mathcal{W}, \mathcal{S})$ is equipped with the graph $\mathcal{H}^*$. It is evident $\mathcal{F}$ is orbitally perturbed $\widetilde{\mathcal{H}}$-continuous on $\mathcal{A}$. For each $\mathcal{R}_i \in \mathcal{A}$ for $i = 1, 2, 3, 4$ and constant $\alpha = \frac{7}{8}$, we get

$$
\mathcal{W}(\mathcal{F}\mathcal{R}_i, \mathcal{F}\mathcal{R}_i) = 0 \leq \alpha \max \{\mathcal{W}(\mathcal{R}_i, \mathcal{R}_i), \mathcal{W}(\mathcal{R}_i, \mathcal{F}\mathcal{R}_i)\}.
$$

Also,

$$
\begin{aligned}
\mathcal{W}(\mathcal{F}\mathcal{R}_1, \mathcal{F}\mathcal{R}_2) &= \|\mathcal{F}\mathcal{R}_1 - \mathcal{F}\mathcal{R}_2\| + \mathcal{S}(\mathcal{F}\mathcal{R}_2, \mathcal{F}\mathcal{R}_2) \\
&= 7 \\
&\leq \alpha \max \{\mathcal{W}(\mathcal{R}_1, \mathcal{R}_2), \mathcal{W}(\mathcal{R}_1, \mathcal{F}\mathcal{R}_1), \mathcal{W}(\mathcal{R}_2, \mathcal{F}\mathcal{R}_2), \\
&\quad \mathcal{W}(\mathcal{R}_1, \mathcal{F}\mathcal{R}_2), \mathcal{W}(\mathcal{R}_2, \mathcal{F}\mathcal{R}_1)\}.
\end{aligned}
$$

Hence, all hypotheses of Theorem 2.6 is held and $\mathcal{F}$ possesses a fixed point in $\mathcal{A}$. Clearly, $\mathcal{R}_4 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ is only fixed point of $\mathcal{F}$.

Several corollaries can be deduced from Theorem 2.6 by choosing proper contractions. First, suppose $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ is a perturbed Banach $\mathcal{H}$-contraction in sense of Jachymski [10, Definition 2.1] in a perturbed metric space. Then we have the following corollary.

**Corollary 2.8.** *Assume $(\mathcal{A}, \mathcal{W}, \mathcal{S})$ is a complete perturbed metric space equipped with the graph $\mathcal{H}$ and $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ is an orbitally perturbed $\mathcal{H}$-continuous on $\mathcal{A}$ and also, a perturbed Banach $\mathcal{H}$-contraction, i.e.*

- *$\mathcal{F}$ preserves the edges of $\mathcal{H}$;*
- *there exists a $\alpha \in [0, 1)$ so that $\mathcal{W}(\mathcal{F}\mathfrak{a}, \mathcal{F}\mathfrak{b}) \leq \alpha \mathcal{W}(\mathfrak{a}, \mathfrak{b})$ for each $\mathfrak{a}, \mathfrak{b} \in \mathcal{A}$ with $(\mathfrak{a}, \mathfrak{b}) \in \mathcal{E}(\mathcal{H})$.*

*Then the restriction of $\mathcal{F}$ to $\mathcal{C}_{\mathcal{F}}$ has a fixed point. Specifically, if $\mathcal{F}$ is a perturbed Banach $\mathcal{H}$-contraction mapping, $\mathcal{F}$ possesses a fixed point in $\mathcal{A}$ iff $\mathcal{C}_{\mathcal{F}} \neq \emptyset$.*

*Proof.* Let $(\mathfrak{a},\mathfrak{b}) \in \mathcal{E}(\mathcal{H})$. Then

$$\begin{aligned}
&\mathcal{W}(\mathcal{F}\mathfrak{a},\mathcal{F}\mathfrak{b})\\
&\quad\leq \alpha\mathcal{W}(\mathfrak{a},\mathfrak{b})\\
&\quad\leq \alpha\max\left\{\mathcal{W}(\mathfrak{a},\mathfrak{b}),\mathcal{W}\left(\mathfrak{a},\mathcal{F}\mathfrak{a}\right),\mathcal{W}(\mathfrak{b},\mathcal{F}\mathfrak{b}),\mathcal{W}(\mathfrak{a},\mathcal{F}\mathfrak{b}),\mathcal{W}(\mathfrak{b},\mathcal{F}\mathfrak{a})\right\}.
\end{aligned}$$

Therefore, $(\mathcal{Q}_2)$ is held for $\mathcal{F}$ and as a result, $\mathcal{F}$ is a perturbed $\mathcal{H}$-quasi-contraction. Applying Theorem 2.6, the assertion will be obtained. □

Next, presume $\mathcal{F}:\mathcal{A}\to\mathcal{A}$ is a perturbed Kannan $\mathcal{H}$-contraction mapping in the sense of Bojor [3, Definition 4], in a perturbed metric space. Then we obtain the following corollary.

**Corollary 2.9.** *Assume $(\mathcal{A},\mathcal{W},\mathcal{S})$ is a complete perturbed metric space equipped with the graph $\mathcal{H}$, and $\mathcal{F}:\mathcal{A}\to\mathcal{A}$ is an orbitally perturbed $\mathcal{H}$-continuous on $\mathcal{A}$ and also, a perturbed Kannan $\mathcal{H}$-contraction mapping, i.e.*

- *$\mathcal{F}$ preserves the edges of $\mathcal{H}$,*
- *there is $\alpha \in \left[0,\frac{1}{2}\right)$ so that $\mathcal{W}(\mathcal{F}\mathfrak{a},\mathcal{F}\mathfrak{b}) \leq \alpha[\mathcal{W}\left(\mathfrak{a},\mathcal{F}\mathfrak{a}\right)+\mathcal{W}(\mathfrak{b},\mathcal{F}\mathfrak{b})]$ for each $\mathfrak{a},\mathfrak{b}\in\mathcal{A}$ with $(\mathfrak{a},\mathfrak{b})\in\mathcal{E}(\mathcal{H})$.*

*Then the restriction of $\mathcal{F}$ to $\mathcal{C}_{\mathcal{F}}$ has a fixed point. Specifically, if $\mathcal{F}$ is a perturbed Kannan $\mathcal{H}$-contraction mapping, $\mathcal{F}$ possesses a fixed point in $\mathcal{A}$ iff $\mathcal{C}_{\mathcal{F}}\neq\emptyset$.*

*Proof.* If $(\mathfrak{a},\mathfrak{b}) \in \mathcal{E}(\mathcal{H})$, then

$$\begin{aligned}
&\mathcal{W}(\mathcal{F}\mathfrak{a},\mathcal{F}\mathfrak{b})\\
&\quad\leq \alpha[\mathcal{W}\left(\mathfrak{a},\mathcal{F}\mathfrak{a}\right)+\mathcal{W}(\mathfrak{b},\mathcal{F}\mathfrak{b})]\\
&\quad\leq 2\alpha\max\left\{\mathcal{W}\left(\mathfrak{a},\mathcal{F}\mathfrak{a}\right),\mathcal{W}(\mathfrak{b},\mathcal{F}\mathfrak{b})\right\}\\
&\quad\leq 2\alpha\max\left\{\mathcal{W}(\mathfrak{a},\mathfrak{b}),\mathcal{W}\left(\mathfrak{a},\mathcal{F}\mathfrak{a}\right),\mathcal{W}(\mathfrak{b},\mathcal{F}\mathfrak{b}),\mathcal{W}(\mathfrak{a},\mathcal{F}\mathfrak{b}),\mathcal{W}(\mathfrak{b},\mathcal{F}\mathfrak{a})\right\}
\end{aligned}$$

Thus, $(\mathcal{Q}_2)$ is held for $\mathcal{F}$, and $\mathcal{F}$ will be a perturbed $\mathcal{H}$-quasi-contraction. Now, using Theorem 2.6, we conclude that the restriction of $\mathcal{F}$ to $\mathcal{C}_{\mathcal{F}}$ possesses a fixed point. □

Now, suppose $\mathcal{F}:\mathcal{A}\to\mathcal{A}$ is a perturbed Ćirić-Reich-Rus $\mathcal{H}$-contraction mapping in the sense of Bojor [4, Definition 7] in a perturbed metric space. Hence we get the following corollary.

**Corollary 2.10.** *Assume $(\mathcal{A},\mathcal{W},\mathcal{S})$ is a complete perturbed metric space equipped with the graph $\mathcal{H}$, and $\mathcal{F}:\mathcal{A}\to\mathcal{A}$ is an orbitally perturbed $\mathcal{H}$-continuous on $\mathcal{A}$ and also, a perturbed Ćirić-Reich-Rus $\mathcal{H}$-contraction mapping, i.e.*

- *$\mathcal{F}$ preserves the edges of $\mathcal{H}$,*

- *there is a* $\vartheta, \zeta, \eta \geq 0$ *with* $\vartheta + \zeta + \eta \leq 1$ *so that*

$$\mathcal{W}(\mathcal{F}\mathfrak{a}, \mathcal{F}\mathfrak{b}) \leq \vartheta \mathcal{W}(\mathfrak{a}, \mathcal{F}\mathfrak{a}) + \zeta \mathcal{W}(\mathfrak{a}, \mathcal{F}\mathfrak{a}) + \eta \mathcal{W}(\mathfrak{b}, \mathcal{F}\mathfrak{b})$$

*for all* $\mathfrak{a}, \mathfrak{b} \in \mathcal{A}$ *with* $(\mathfrak{a}, \mathfrak{b}) \in \mathcal{E}(\mathcal{H})$.

*Then the restriction of* $\mathcal{F}$ *to* $\mathcal{C}_{\mathcal{F}}$ *has a fixed point. Specifically, if* $\mathcal{F}$ *is a perturbed Ćirić-Reich-Rus* $\mathcal{H}$*-contraction mapping,* $\mathcal{F}$ *possesses a fixed point in* $\mathcal{A}$ *iff* $\mathcal{C}_{\mathcal{F}} \neq \emptyset$.

*Proof.* Let $(\mathfrak{a}, \mathfrak{b}) \in \mathcal{E}(\mathcal{H})$. Then

$$\begin{aligned}\mathcal{W}(\mathcal{F}\mathfrak{a}, \mathcal{F}\mathfrak{b}) &\leq \vartheta \mathcal{W}(\mathfrak{a}, \mathcal{F}\mathfrak{a}) + \zeta \mathcal{W}(\mathfrak{a}, \mathcal{F}\mathfrak{a}) + \eta \mathcal{W}(\mathfrak{b}, \mathcal{F}\mathfrak{b}) \\ &\leq (\vartheta + \zeta + \eta) \max \{\mathcal{W}(\mathfrak{a}, \mathfrak{b}), \mathcal{W}(\mathfrak{a}, \mathcal{F}\mathfrak{a}), \mathcal{W}(\mathfrak{b}, \mathcal{F}\mathfrak{b}), \\ &\quad \mathcal{W}(\mathfrak{a}, \mathcal{F}\mathfrak{b}), \mathcal{W}(\mathfrak{b}, \mathcal{F}\mathfrak{a})\}.\end{aligned}$$

Hence, $(\mathcal{Q}_2)$ is held for $\mathcal{F}$ and so $\mathcal{F}$ is a perturbed Ćirić-Reich-Rus $\mathcal{H}$-contraction. Applying Theorem 2.6, the restriction of $\mathcal{F}$ to $\mathcal{C}_{\mathcal{F}}$ possesses a fixed point. □

Finally, presume $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ is a perturbed $\lambda$-generalized contraction mapping in the sense of Ćirić [5, Definition 2.1] in a perturbed metric space. Then we get next corollary.

**Corollary 2.11.** *Assume* $(\mathcal{A}, \mathcal{W}, \mathcal{S})$ *is a complete perturbed metric space equipped with the graph* $\mathcal{H}$*, and* $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ *is an orbitally perturbed* $\mathcal{H}$*-continuous on* $\mathcal{A}$ *and also, a perturbed* $\lambda$*-generalized contraction mapping, i.e.*

- $\mathcal{F}$ *keeps the edges of* $\mathcal{H}$*,*
- *there are four functions* $\gamma, \delta, \vartheta, \xi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ *with*

$$\sup \{\gamma(\mathfrak{a}, \mathfrak{b}) + \delta(\mathfrak{a}, \mathfrak{b}) + \vartheta(\mathfrak{a}, \mathfrak{b}) + 2\xi(\mathfrak{a}, \mathfrak{b}) : \mathfrak{a}, \mathfrak{b} \in \mathcal{A} \times \mathcal{A}\} = \lambda < 1$$

*so that*

$$\begin{aligned}\mathcal{W}(\mathcal{F}\mathfrak{a}, \mathcal{F}\mathfrak{b}) \leq{}& \gamma(\mathfrak{a}, \mathfrak{b})\mathcal{W}(\mathfrak{a}, \mathfrak{b}) + \delta(\mathfrak{a}, \mathfrak{b})\mathcal{W}(\mathfrak{a}, \mathcal{F}\mathfrak{a}) + \vartheta(\mathfrak{a}, \mathfrak{b})\mathcal{W}(\mathfrak{b}, \mathcal{F}\mathfrak{b}) \\ &+ \xi(\mathfrak{a}, \mathfrak{b}) (\mathcal{W}(\mathfrak{a}, \mathcal{F}\mathfrak{b}) + \mathcal{W}(\mathfrak{b}, \mathcal{F}\mathfrak{a}))\end{aligned}$$

*for all* $\mathfrak{a}, \mathfrak{b} \in \mathcal{A}$ *with* $(\mathfrak{a}, \mathfrak{b}) \in \mathcal{E}(\mathcal{H})$.

*Then the restriction of* $\mathcal{F}$ *to* $\mathcal{C}_{\mathcal{F}}$ *has a fixed point. Specifically, if* $\mathcal{F}$ *is a perturbed* $\lambda$*-generalized contraction mapping,* $\mathcal{F}$ *possesses a fixed point in* $\mathcal{A}$ *iff* $\mathcal{C}_{\mathcal{F}} \neq \emptyset$.

*Proof.* Let $(\mathfrak{a}, \mathfrak{b}) \in \mathcal{E}(\mathcal{H})$. Then

$$\begin{aligned}\mathcal{W}(\mathcal{F}\mathfrak{a}, \mathcal{F}\mathfrak{b}) &\leq \gamma(\mathfrak{a}, \mathfrak{b})\mathcal{W}(\mathfrak{a}, \mathfrak{b}) + \delta(\mathfrak{a}, \mathfrak{b})\mathcal{W}(\mathfrak{a}, \mathcal{F}\mathfrak{a}) + \vartheta(\mathfrak{a}, \mathfrak{b})\mathcal{W}(\mathfrak{b}, \mathcal{F}\mathfrak{b}) \\ &\quad + \xi(\mathfrak{a}, \mathfrak{b}) (\mathcal{W}(\mathfrak{a}, \mathcal{F}\mathfrak{b}) + \mathcal{W}(\mathfrak{b}, \mathcal{F}\mathfrak{a})) \\ &\leq (\gamma(\mathfrak{a}, \mathfrak{b}) + \delta(\mathfrak{a}, \mathfrak{b}) + \vartheta(\mathfrak{a}, \mathfrak{b}) + 2\xi(\mathfrak{a}, \mathfrak{b})) \\ &\quad \max \{\mathcal{W}(\mathfrak{a}, \mathfrak{b}), \mathcal{W}(\mathfrak{a}, \mathcal{F}\mathfrak{a}), \mathcal{W}(\mathfrak{b}, \mathcal{F}\mathfrak{b}), \mathcal{W}(\mathfrak{a}, \mathcal{F}\mathfrak{b}), \mathcal{W}(\mathfrak{b}, \mathcal{F}\mathfrak{a})\}\end{aligned}$$

$$\leq \lambda \max\{\mathcal{W}(\mathfrak{a},\mathfrak{b}), \mathcal{W}(\mathfrak{a},\mathcal{F}\mathfrak{a}), \mathcal{W}(\mathfrak{b},\mathcal{F}\mathfrak{b}), \mathcal{W}(\mathfrak{a},\mathcal{F}\mathfrak{b}), \mathcal{W}(\mathfrak{b},\mathcal{F}\mathfrak{a})\}$$

Therefore, $(\mathcal{Q}_2)$is held for $\mathcal{F}$ and so $\mathcal{F}$ is a perturbed $\lambda$-generalized contraction mapping. Hence, by Theorem 2.6, the restriction of $\mathcal{F}$ to $\mathcal{C}_{\mathcal{F}}$ possesses a fixed point. □

Some consequences of the main theorem can also be obtained by choosing particular graphs. Take $(\mathcal{A}, \sqsubseteq)$ a partially ordered set and define the graph $\mathcal{H}_1$ by $V(\mathcal{H}_1) = \mathcal{A}$ and $\mathcal{E}(\mathcal{H}_1) = \{(\mathfrak{a},\mathfrak{b}) \in \mathcal{A}\times\mathcal{A} : \mathfrak{a} \sqsubseteq \mathfrak{b}\}$. It is evident that $\mathcal{E}(\mathcal{H}_1)$ contains all loops. Consider $(\mathcal{A}, \mathcal{W}, \mathcal{S})$ with $\mathcal{H}_1$. The following is the partially ordered version of Theorem 2.6.

**Corollary 2.12.** *Assume $(\mathcal{A}, \mathcal{W}, \mathcal{S})$ is a complete perturbed metric space with the partial order $\sqsubseteq$, and $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ is a perturbed ordered quasi-contraction and is orbitally perturbed ordered-continuous on $\mathcal{A}$. Then the restriction of $\mathcal{F}$ to $\mathcal{C}_{\mathcal{F}}$ has a fixed point. Specifically, if $\mathcal{F}$ is perturbed ordered quasi-contraction, $\mathcal{F}$ possesses a fixed point in $\mathcal{A}$ iff $\mathcal{C}_{\mathcal{F}} \neq \emptyset$.*

For the next consequence, take $(\mathcal{A}, \sqsubseteq)$ a partially ordered set, and define graph $\mathcal{H}_2$ by $V(\mathcal{H}_2) = \mathcal{A}$ and $\mathcal{E}(\mathcal{H}_2) = \{(\mathfrak{a},\mathfrak{b}) \in \mathcal{A}\times\mathcal{A} : \mathfrak{a} \sqsubseteq \mathfrak{b} \text{ or } \mathfrak{b} \sqsubseteq \mathfrak{a}\}$. Setting $\mathcal{H} = \mathcal{H}_2$ in Theorem 2.6, we obtain another perturbed $\mathcal{H}$-quasi-contraction in complete perturbed partially ordered metric space.

**Corollary 2.13.** *Assume $(\mathcal{A}, \mathcal{W}, \mathcal{S})$ is a complete perturbed metric space with the partial order $\sqsubseteq$, and $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ is a perturbed $\mathcal{H}_2$-quasi-contraction and orbitally perturbed ordered-continuous on $\mathcal{A}$. Then the restriction of $\mathcal{F}$ to $\mathcal{C}_{\mathcal{F}}$ has a fixed point. Specifically, if $\mathcal{F}$ is perturbed ordered quasi-contraction, $\mathcal{F}$ possesses a fixed point in $\mathcal{A}$ iff $\mathcal{C}_{\mathcal{F}} \neq \emptyset$.*

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