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New Sequence Spaces Generated via a Matrix Defined by Schröder and Catalan Numbers

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Abstract. This study introduces the sequence spaces $c(S^\star)$ and $c_0(S^\star)$, defined as the domain of the matrix $S^\star$ constructed from a hybrid structure involving Schröder and Catalan numbers. A comprehensive investigation is conducted into the fundamental topological and structural properties of these sequence spaces, such as completeness and their relationships and embedding within classical sequence spaces. Furthermore, the dual space structures corresponding to these newly defined spaces are thoroughly characterized. In the final sections, various classes of matrix transformations and compact linear operators acting on these sequence spaces are examined, emphasizing their significance in functional analysis.

1. Introduction

The construction and analysis of sequence spaces have been a focal point of modern functional analysis, particularly through the lens of summability theory and matrix transformations. Traditional sequence spaces, including ℓ_p, c, c_0 and ℓ_∞ have been extensively investigated due to their rich structure and applications. Recently, attention has shifted toward the creation of new sequence spaces using infinite matrices derived from well-known integer sequences.

Among these sequences, the Schröder numbers and the Catalan numbers stand out for their deep combinatorial significance and their elegant recursive structures. Originally emerging in the enumeration of lattice

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paths, polygon triangulations and tree structures, these sequences encode powerful combinatorial identities and have found applications in areas ranging from algebraic combinatorics to formal language theory.

The large Schröder numbers, denoted by the sequence $(S_i)_{i\geq 0}$, represent the total number of lattice paths from $(0,0)$ to (i,i) confined to the region on or below the main diagonal $y=x$, with permissible steps consisting of horizontal $(1,0)$, vertical $(0,1)$ and diagonal $(1,1)$ moves. These numbers demonstrate rapid growth and satisfy complex recursive relations. Similarly, the Catalan numbers $(C_i)_{i\geq 0}$, which are fundamental in combinatorics, count various combinatorial objects such as well-formed parenthetical expressions, rooted binary trees and Dyck paths.

The synergy between such integer sequences and sequence space theory has led researchers to define new matrix domains by embedding these sequences into the structure of infinite matrices. Pioneering works in this direction have used Fibonacci, Lucas, Padovan, Bell and other sequences to construct new normed or paranormed spaces with interesting topological and algebraic properties.

The primary objective of this work is to introduce novel sequence spaces $c(S^\star)$ and $c_0(S^\star)$ generated by the Schröder-Catalan matrix $S^\star$. Emphasis is placed on investigating their intrinsic structural features, including aspects like regularity, completeness and their relationships with classical sequence spaces. Additionally, the dual spaces are identified, accompanied by detailed analyses of specific classes of matrix operators and compact linear mappings defined on these newly formed spaces.

2. Preliminaries

A sequence space is defined as a subset of the space ω, which contains all real-valued sequences. In this work, the standard sequence spaces that will be used are described as follows:

$$c := \left\{ u = (u_j) \in \omega : \lim_{j\to\infty} u_j \text{ exists} \right\},$$

$$c_0 := \left\{ u = (u_j) \in \omega : \lim_{j\to\infty} u_j = 0 \right\},$$

$$\ell_p := \left\{ u = (u_j) \in \omega : \sum_{j=0}^{\infty} |u_j|^p < \infty \right\},$$

$$\ell_\infty := \left\{ u = (u_j) \in \omega : \sup_{j\in\mathbb{N}} |u_j| < \infty \right\},$$

$$bs := \left\{ u = (u_j) \in \omega : \left(\sum_{j=0}^{i} u_j \right)_{i=0}^{\infty} \in \ell_\infty \right\},$$
$$cs := \left\{ u = (u_j) \in \omega : \left(\sum_{j=0}^{i} u_j \right)_{i=0}^{\infty} \in c \right\},$$

where $1 \leq p < \infty$ and $\mathbb{N} = \{0, 1, 2, 3, \dots\}$.

A BK-space is defined as a Banach space in which all coordinate functionals are continuous. Notable instances of BK-spaces include the classical sequence spaces c, c_0 and ℓ_∞, each endowed with the supremum norm given by

$$\|u\|_{\ell_\infty} = \sup_{j \in \mathbb{N}} |u_j|.$$

Additionally, for $1 \leq p < \infty$, the space ℓ_p also constitutes a BK-space with the norm

$$\|u\|_{\ell_p} = \left(\sum_{j=0}^{\infty} |u_j|^p \right)^{1/p}.$$

Let $F = (f_{ij})_{i,j \in \mathbb{N}}$ be a real-valued infinite matrix and let F_i denote the i-th row of F. Given a sequence $u = (u_j)$, the transformation associated with the matrix F, called the F-transform, is defined by

$$(Fu)_i = \sum_{j=0}^{\infty} f_{ij} u_j,$$

provided that the series converges for every $i \in \mathbb{N}$.

The dual spaces associated with a sequence space D are given by:

$$D^\alpha := \left\{ a = (a_j) \in \omega : \sum_{j=0}^{\infty} |a_j u_j| < \infty \text{ for all } u \in D \right\},$$
$$D^\beta := \left\{ a = (a_j) \in \omega : \left(\sum_{j=0}^{i} a_j u_j \right)_{i=0}^{\infty} \text{ converges for every } u \in D \right\},$$
$$D^\gamma := \left\{ a = (a_j) \in \omega : \sup_{i \in \mathbb{N}} \left| \sum_{j=0}^{i} a_j u_j \right| < \infty \text{ for all } u \in D \right\}.$$

Two normed vector spaces D and P are said to be linearly isomorphic, denoted by $D \cong P$, if there exists a bijective linear isometry between them. For a comprehensive background on summability theory and matrix transformations, one may consult [2, 4, 29, 33].

Next, we briefly review the definitions and essential properties of two important number sequences utilized in the construction of our matrix $S^{\star}$.

The large Schröder numbers $(S_i)_{i\geq 0}$ are defined recursively by:

$$S_{i+1} = S_i + \sum_{j=0}^{i} S_j S_{i-j}, \quad S_0 = 1, \quad i \in \mathbb{N},$$

and asymptotic ratio:

$$\lim_{i\to\infty} \frac{S_{i+1}}{S_i} = 3 + 2\sqrt{2}.$$

The Catalan numbers $(C_i)_{i\geq 0}$ are given by the explicit formula:

$$C_i = \frac{1}{i+1}\binom{2i}{i}, \qquad C_0 = 1.$$

Additionally, these numbers satisfy the recursive relations:

$$C_{i+1} = \sum_{j=0}^{i} C_j C_{i-j}, \qquad C_{i+1} = \frac{4i+2}{i+2} C_i,$$

and their growth is characterized by:

$$\lim_{i\to\infty} \frac{C_{i+1}}{C_i} = 4.$$

An important formula connecting these two sequences is:

$$S_i = \sum_{j=0}^{i} \binom{2i-j}{j} C_{i-j}.$$

For more detailed information on these sequences, refer to the literature such as [5, 17, 32, 35–37].

The development of new sequence spaces through the use of infinite matrices has become a significant topic in functional analysis. Recently, a modern trend in this area focuses on constructing such matrices based on well-established integer sequences. Notably, the pioneering studies by Kara and Başarır [21, 22] introduced novel normed and paranormed sequence spaces derived from the Fibonacci sequence, which have inspired further research in this direction. This methodology enables the creation of systematically structured sequence spaces by utilizing various special numerical sequences. As a result, a variety of integer sequences-including Fibonacci [6–9], Lucas [3, 23, 24], Padovan [41], Leonardo [42], Catalan [18, 19, 26], Bell [25], Schröder [10, 11], Mersenne [13] and Motzkin [14, 16] have been applied to define new infinite matrices and associated sequence spaces. Alongside their construction, various key properties of these newly formed sequence spaces have been investigated.

3. Schröder–Catalan Sequence Spaces $c(S^\star)$ and $c_0(S^\star)$

In this part, the BK-spaces $c(S^\star)$ and $c_0(S^\star)$ are defined using the infinite Schröder–Catalan matrix; their linear isomorphisms with the classical sequence spaces c and c_0 are demonstrated. Furthermore, the Schauder bases of these newly introduced sequence spaces are presented and their inclusion relations are analyzed.

The matrix $S^\star = (s^\star_{ij})_{i,j\in\mathbb{N}}$, known as the Schröder–Catalan matrix, is introduced by Erdem in [15] through a combination of Schröder and Catalan numbers. It is defined by

$$s^\star_{ij} := \begin{cases} \binom{2i-j}{j}\dfrac{C_{i-j}}{S_i}, & \text{if } 0 \le j \le i, \\ \\ 0, & \text{if } j > i, \end{cases}$$

where C_{i-j} and S_i represent the $(i-j)$-th Catalan number and the i-th Schröder number, respectively.

For any $u \in \omega$, the $S^\star$-transform is expressed as:

$$\nu_i := (S^\star u)_i = \frac{1}{S_i}\sum_{j=0}^{i}\binom{2i-j}{j}C_{i-j}u_j. \tag{3.1}$$

The inverse matrix of $S^\star$, denoted by $(s^\star_{ij})^{-1}$, is calculated in [15] as:

$$s^{\star\,-1}_{ij} := \begin{cases} (-1)^{i-j}\binom{2i}{i+j}\dfrac{(2j+1)S_j}{(i+j+1)C_i}, & \text{if } 0 \le j \le i, \\ \\ 0, & \text{if } j > i. \end{cases}$$

Recall that an infinite matrix is termed regular if it transforms every convergent sequence into another convergent sequence with the same limit. We now present a pertinent lemma derived from this definition that is essential for our investigation.

Lemma 3.1 ([15])**.** *$S^\star$ is regular.*

The sequence spaces $c(S^\star)$ and $c_0(S^\star)$ are constructed as the collections of all sequences whose $S^\star$-transforms converge and converge to zero, respectively, as detailed below:

$$c(S^\star) = \left\{ u = (u_j) \in \omega : \lim_{i\to\infty}\left(\frac{1}{S_i}\sum_{j=0}^{i}\binom{2i-j}{j}C_{i-j}u_j\right) \text{ exists} \right\},$$

$$c_0(S^\star) = \left\{ u = (u_j) \in \omega : \lim_{i\to\infty}\left(\frac{1}{S_i}\sum_{j=0}^{i}\binom{2i-j}{j}C_{i-j}u_j\right) = 0 \right\}.$$

It follows that $c(S^\star)$ and $c_0(S^\star)$ are, in fact, the matrix domains determined by $S^\star$ over the classical spaces c and c_0. Furthermore, as noted in [40], the matrix domain D_F retains the BK-space property whenever F is triangular and D is a BK-space.

Theorem 3.2. *Endowed with the norm*

$$\|u\| = \sup_{i\in\mathbb{N}} \left| \frac{1}{S_i} \sum_{j=0}^{i} \binom{2i-j}{j} C_{i-j} u_j \right|,$$

the sequence spaces $c(S^\star)$ and $c_0(S^\star)$ are BK-spaces.

Theorem 3.3. *There exists a linear isomorphism between the space $c(S^\star)$ and the classical space c, as well as between $c_0(S^\star)$ and c_0; that is,*

$$c(S^\star) \cong c \quad \textit{and} \quad c_0(S^\star) \cong c_0.$$

Proof. The isomorphism $c(S^\star) \cong c$ is proved; the case $c_0(S^\star) \cong c_0$ is similar.

Define the linear operator $T : c(S^\star) \to c$ by

$$T(u)_i = \left(\frac{1}{S_i} \sum_{j=0}^{i} \binom{2i-j}{j} C_{i-j} u_j \right)_{i=0}^{\infty}.$$

It is straightforward to verify that T is linear.

Assuming $T(u) = 0$, then for every $i \geq 0$,

$$\frac{1}{S_i} \sum_{j=0}^{i} \binom{2i-j}{j} C_{i-j} u_j = 0.$$

This implies $u_j = 0$ for all j, hence $u = 0$. Therefore, T is injective.

For any $\nu = (\nu_i) \in c$, define $u = (u_j)$ by the inverse transformation

$$u_j = \left(S^{\star -1} \nu \right)_j.$$

A direct calculation yields

$$\frac{1}{S_i} \sum_{j=0}^{i} \binom{2i-j}{j} C_{i-j} u_j = \nu_i,$$

so $T(u) = \nu$. Thus, T is surjective.

Furthermore, T preserves the supremum norm:

$$\|u\|_{c(S^\star)} = \sup_{i\geq 0} \left| \frac{1}{S_i} \sum_{j=0}^{i} \binom{2i-j}{j} C_{i-j} u_j \right| = \|T(u)\|_\infty.$$

Hence, T is an isometric isomorphism from $c(S^\star)$ onto c. ◻

Theorem 3.4. *The inclusion $c_0 \subset c_0(S^\star)$ holds.*

Proof. Consider a sequence $u = (u_i)$ in c_0 for which the limit $\lim_{i\to\infty} u_i = 0$. By the linearity of the $S^\star$-transform, we get:

$$(S^\star u)_i = \frac{1}{S_i}\sum_{j=0}^{i}\binom{2i-j}{j}C_{i-j}u_j.$$

By using Lemma 3.1, it follows that

$$\lim_{i\to\infty}(S^\star u)_i = 0.$$

Thus, $u \in c_0(S^\star)$, proving the inclusion $c_0 \subset c_0(S^\star)$. □

Theorem 3.5. *The inclusion $c_0(S^\star) \subset c(S^\star)$ holds.*

Proof. The proof proceeds in a manner analogous to that of Theorems 3.3 and 3.4. □

Consider a normed sequence space $(D, \|\cdot\|)$ and a sequence $(\sigma_i) \subset D$. The sequence (σ_i) is said to form a Schauder basis for D if, for every element $u \in D$, there exists a unique sequence of scalars (τ_i) satisfying

$$\left\| u - \sum_{j=0}^{i}\tau_j\sigma_j \right\| \xrightarrow[i\to\infty]{} 0.$$

In such a case, the element u can be represented as

$$u = \sum_{j=0}^{\infty}\tau_j\sigma_j.$$

Note that when $F = (f_{ij})_{i,j\in\mathbb{N}}$ is a triangle matrix, the existence of a basis in the matrix domain D_F is equivalent to the existence of a basis in the original normed space D [20]. This fact leads to the following theorem:

Theorem 3.6. *Define the sequence $\sigma^{(j)} = (\sigma_i^{(j)})_{i\in\mathbb{N}}$ by*

$$\sigma_i^{(j)} := \begin{cases} (-1)^{i-j}\frac{2j+1}{i+j+1}\binom{2i}{i+j}\frac{S_j}{C_i}, & \text{if } 0 \le j \le i,\\ 0, & \text{if } j > i.\end{cases}$$

Then, $\{\sigma^{(j)}\}_{j\in\mathbb{N}}$ constitutes a Schauder basis for the sequence spaces $c_0(S^\star)$ and $c(S^\star)$. Moreover, the following statements hold:

(i) *The system $\{\sigma^{(j)}\}$ forms a basis for $c_0(S^\star)$ and any element $u \in c_0(S^\star)$ admits a unique series expansion of the form*

$$u = \sum_{j=0}^{\infty}\tau_j\sigma^{(j)}.$$

(ii) *For the space* $c(S^\star)$*, the family* $\{e, \sigma^{(j)} : j \in \mathbb{N}\}$ *forms a basis. Each* $u \in c(S^\star)$ *can be uniquely represented as*

$$u = le + \sum_{j=0}^{\infty} (\tau_j - l)\sigma^{(j)},$$

where the coefficients satisfy $\tau_j = (S^\star u)_j \to l$ *as* $j \to \infty$.

4. Dual Spaces of Types α, β and γ

In this section, we focus on identifying the α-, β- and γ-dual spaces corresponding to the newly introduced sequence spaces. To support the main results, we begin by stating several fundamental lemmas that will serve as the foundation for the forthcoming theorems.

Lemma 4.1 ([38])**.** *The following statements hold:*

(i) $D = (d_{ij}) \in (c_0 : \ell_1) = (c : \ell_1)$ *iff*

$$\sup_{K \in \mathcal{T}} \sum_{i=0}^{\infty} \left| \sum_{j \in K} d_{ij} \right| < \infty,$$

where $\mathcal{T}$ *expresses the class of all finite subsets of* $\mathbb{N}$.

(ii) $D = (d_{ij}) \in (c_0 : c)$ *iff there exist* $\varkappa_j \in \mathbb{C}$ *such that*

$$\lim_{i \to \infty} d_{ij} = \varkappa_j \quad \text{for each } j \in \mathbb{N}, \tag{4.1}$$

and

$$\sup_{i \in \mathbb{N}} \sum_{j=0}^{\infty} |d_{ij}| < \infty. \tag{4.2}$$

(iii) $D = (d_{ij}) \in (c : c)$ *iff the conditions* (4.1)*,* (4.2) *and*

$$\lim_{i \to \infty} \sum_{j=0}^{\infty} d_{ij} \text{ exists.}$$

(iv) $D = (d_{ij}) \in (c_0 : \ell_\infty) = (c : \ell_\infty)$ *iff* (4.2) *holds.*

Theorem 4.2. *Consider the set*

$$\mathcal{A} = \left\{ a = (a_j) \in \omega : \sup_{K \in \mathcal{T}} \sum_{i=0}^{\infty} \left| \sum_{j \in K} (-1)^{i-j} \binom{2i}{i+j} \frac{(2j+1)S_j}{(i+j+1)C_i} a_j \right| < \infty \right\}.$$

Then,

$$\{c_0(S^\star)\}^\alpha = \{c(S^\star)\}^\alpha = \mathcal{A}.$$

Proof. Let $D = (d_{ij})$ be defined by

$$d_{ij} := \begin{cases} (-1)^{i-j}\binom{2i}{i+j}\dfrac{(2j+1)S_j}{(i+j+1)C_i}a_i, & \text{if } 0 \le j \le i, \\ 0, & \text{if } j > i, \end{cases} \tag{4.3}$$

For a sequence $u \in c_0(S^\star)$ and by employing equation (3.1), we obtain the following relation:

$$\begin{aligned} a_i u_i &= a_i \left(\sum_{j=0}^{i} (-1)^{i-j}\binom{2i}{i+j}\frac{(2j+1)S_j}{(i+j+1)C_i}\nu_j \right) \\ &= \sum_{j=0}^{i} \left((-1)^{i-j}\binom{2i}{i+j}\frac{(2j+1)S_j}{(i+j+1)C_i}a_i \right)\nu_j \\ &= (D\nu)_i, \end{aligned} \tag{4.4}$$

where $D = (d_{ij})$ is the matrix defined in (4.3). Hence, equation (4.4) yields that $au = (a_i u_i) \in \ell_1$ iff $D\nu \in \ell_1$ for $\nu \in c_0$. Therefore, $a \in (c_0(S^\star))^\alpha$ iff $D \in (c_0 : \ell_1)$. We conclude that $(c_0(S^\star))^\alpha = \mathcal{A}$.

The remainder of the proof can be established using analogous arguments. □

Theorem 4.3. *Consider the sets $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ defined by:*

$$\mathcal{B}_1 = \left\{ a \in \omega : \forall j \in \mathbb{N}, \lim_{i\to\infty} \sum_{k=j}^{i} (-1)^{k-j}\binom{2k}{k+j}\frac{(2j+1)S_j}{(k+j+1)C_k}a_k \text{ exists} \right\},$$

$$\mathcal{B}_2 = \left\{ a \in \omega : \sup_{i\in\mathbb{N}} \sum_{j=0}^{\infty} \left| \sum_{k=j}^{i} (-1)^{k-j}\binom{2k}{k+j}\frac{(2j+1)S_j}{(k+j+1)C_k}a_k \right| < \infty \right\},$$

$$\mathcal{B}_3 = \left\{ a \in \omega : \lim_{i\to\infty} \sum_{j=0}^{\infty} \sum_{k=j}^{i} (-1)^{k-j}\binom{2k}{k+j}\frac{(2j+1)S_j}{(k+j+1)C_k}a_k \text{ exists} \right\}.$$

Then,

(i) $(c_0(S^\star))^\beta = \mathcal{B}_1 \cap \mathcal{B}_2$,
(ii) $(c(S^\star))^\beta = \mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3$,
(iii) $(c_0(S^\star))^\gamma = (c(S^\star))^\gamma = \mathcal{B}_2$.

Proof. (i) Consider $\Delta = (\delta_{ij})_{i,j\in\mathbb{N}}$ defined by

$$\delta_{ij} := \begin{cases} \displaystyle\sum_{k=j}^{i} (-1)^{k-j}\binom{2k}{k+j}\frac{(2j+1)S_j}{(k+j+1)C_k}a_k, & \text{if } 0 \le j \le i, \\ 0, & \text{if } j > i. \end{cases}$$

By taking $u \in c_0(S^\star)$ with (3.1), we obtain:

$$\begin{aligned}\psi_i &= \sum_{j=0}^{i} a_j u_j \\ &= \sum_{j=0}^{i} a_j \left(\sum_{k=0}^{j} (-1)^{j-k} \binom{2j}{j+k} \frac{(2k+1)S_k}{(j+k+1)C_j} \nu_k \right) \\ &= \sum_{j=0}^{i} \left(\sum_{k=j}^{i} (-1)^{k-j} \binom{2k}{k+j} \frac{(2j+1)S_j}{(k+j+1)C_k} a_k \right) \nu_j \\ &= (\Delta\nu)_i. \end{aligned} \tag{4.5}$$

Therefore, from equation (4.5), $au \in cs$ for every $u \in c_0(S^\star)$ iff $\psi = (\psi_i) \in c$ for every $\nu \in c_0$. This leads to the conclusion that $a \in \{c_0(S^\star)\}^\beta$ iff $\Delta \in (c_0 : c)$. According to Lemma 4.1, the desired result follows.

The remaining cases can be handled using similar arguments. □

5. Matrix Transformations

Let λ denote either of the classical sequence spaces c_0 or c and let μ be one of the spaces c_0, c, ℓ_∞, or ℓ_1. In this part, we investigate necessary and sufficient conditions for a matrix transformation from $\lambda(S^\star)$ to μ and from μ to $\lambda(S^\star)$.

Theorem 5.1. *Let $\mathcal{N}^i = \left(n^i_{mj}\right)_{i,m,j\in\mathbb{N}}$ and $\mathcal{N} = (n_{ij})_{i,j\in\mathbb{N}}$ be two infinite matrices defined by*

$$n^i_{mj} := \begin{cases} \sum_{k=j}^{m} (-1)^{k-j} \binom{2k}{k+j} \frac{(2j+1)S_j}{(k+j+1)C_k} f_{ik}, & \text{if } 0 \le j \le m, \\ 0, & \text{if } j > m. \end{cases}$$

and

$$n_{ij} = \sum_{k=j}^{\infty} (-1)^{k-j} \binom{2k}{k+j} \frac{(2j+1)S_j}{(k+j+1)C_k} f_{ik}. \tag{5.1}$$

In this case, for any $D \in \{c, c_0\}$ and $P \in \omega$, the matrix $F = (f_{ij})$ defines a transformation from $D(S^\star)$ into P, i.e., $F \in (D(S^\star) : P)$, iff $\mathcal{N}^i \in (D : c)$ for each i and $\mathcal{N} \in (D : P)$.

Proof. Suppose that $F = (f_{ij}) \in (D(S^\star) : P)$ and let $u \in D(S^\star)$. Then, by the structure of the sequence u, we have

$$\sum_{j=0}^{m} f_{ij} u_j = \sum_{j=0}^{m} f_{ij} \left(\sum_{k=0}^{j} (-1)^{j-k} \binom{2j}{j+k} \frac{(2k+1)S_k}{(j+k+1)C_j} \nu_k \right) \tag{5.2}$$

$$= \sum_{j=0}^{m} \left(\sum_{k=j}^{m} (-1)^{k-j} \binom{2k}{k+j} \frac{(2j+1)S_j}{(k+j+1)C_k} f_{ik} \right) \nu_j$$
$$= \sum_{j=0}^{m} n^i_{mj} \nu_j$$

for all $m, i \in \mathbb{N}$. Since the left-hand side converges (i.e., Fu exists), it follows that $\mathcal{N}^i \in (D : c)$. Now, by taking the limit as $m \to \infty$ in equation (5.2), we get $Fu = \mathcal{N}\nu$. Because $Fu \in P$ by assumption, it must be that $\mathcal{N}\nu \in P$. Hence, $\mathcal{N} \in (D : P)$.

Conversely, assume that $\mathcal{N}^i \in (D : c)$ and $\mathcal{N} \in (D : P)$. Then each row $(n_{ij})_{j\in\mathbb{N}}$ belongs to D^β, implying that $(f_{ij})_{j\in\mathbb{N}} \in (D(S^\star))^\beta$. Therefore, the matrix F defines a transformation on the space $D(S^\star)$ and Fu exists for all $u \in D(S^\star)$. By passing to the limit in (5.2) as $m \to \infty$, we once again obtain $Fu = \mathcal{N}\nu$ and since $\mathcal{N} \in (D : P)$, this implies that $Fu \in P$, i.e., $F \in (D(S^\star) : P)$. □

Corollary 5.2. (i) $F \in (c_0(S^\star) : c_0)$ *iff the following conditions are satisfied:*

$$\sup_{m\in\mathbb{N}} \sum_{j=0}^{\infty} \left| n^{(i)}_{mj} \right| < \infty, \tag{5.3}$$

$$\lim_{m\to\infty} n^{(i)}_{mj} \text{ exists for all } j \in \mathbb{N}, \tag{5.4}$$

$$\sup_{i\in\mathbb{N}} \sum_{j=0}^{\infty} |n_{ij}| < \infty, \tag{5.5}$$

$$\lim_{i\to\infty} n_{ij} = 0 \quad \text{for each } j \in \mathbb{N}. \tag{5.6}$$

(ii) $F \in (c_0(S^\star) : c)$ *iff* (5.3), (5.4) *and* (5.5) *hold together with the condition:*

$$\lim_{i\to\infty} n_{ij} \text{ exists.} \tag{5.7}$$

(iii) $F \in (c_0(S^\star) : \ell_\infty)$ *iff all of the conditions* (5.3) *through* (5.5) *are satisfied.*

(iv) $F \in (c_0(S^\star) : \ell_1)$ *iff* (5.3) *and* (5.4) *hold and in addition:*

$$\sup_{K\in\mathcal{T}} \sum_{i=0}^{\infty} \left| \sum_{j\in K} n_{ij} \right| < \infty, \tag{5.8}$$

is satisfied.

Corollary 5.3. (i) *$F \in (c(S^\star) : c_0)$ iff (5.3), (5.4), (5.5) and (5.6) hold together with the conditions:*

$$\lim_{m\to\infty} \sum_{j=0}^{\infty} n_{mj}^{(i)} \text{ exists,} \tag{5.9}$$

$$\lim_{i\to\infty} \sum_{j=0}^{\infty} n_{ij} = 0. \tag{5.10}$$

(ii) *$F \in (c(S^\star) : c)$ iff (5.3), (5.4), (5.6), (5.7), (5.9) and (5.10) hold.*

(iii) *$F \in (c(S^\star) : \ell_\infty)$ iff (5.3), (5.4), (5.5) and (5.9) hold.*

(iv) *$F \in (c(S^\star) : \ell_1)$ iff (5.3), (5.4) and (5.8) hold.*

Corollary 5.4. (i) *$F \in (\ell_1 : c_0(S^\star))$ iff the following conditions are satisfied:*

$$\lim_{i\to\infty} \tilde{f}_{ij} = 0, \quad j = 0, 1, 2, ..., \tag{5.11}$$

$$\sup_{i,j\in\mathbb{N}} |\tilde{f}_{ij}| < \infty. \tag{5.12}$$

(ii) *$F \in (c_0 : c_0(S^\star))$ iff (5.11) holds together with the condition*

$$\sup_{i\in\mathbb{N}} \sum_{j=0}^{\infty} |\tilde{f}_{ij}| < \infty. \tag{5.13}$$

(iii) *$F \in (c : c_0(S^\star))$ iff (5.11) and (5.13) hold together with the condition*

$$\lim_{i\to\infty} \sum_{j=0}^{\infty} \tilde{f}_{ij} = 0.$$

(iv) *$F \in (\ell_\infty : c_0(S^\star))$ iff (5.11) holds together with the condition*

$$\lim_{i\to\infty} \sum_{j=0}^{\infty} |\tilde{f}_{ij}| = 0.$$

Corollary 5.5. (i) *$F \in (\ell_1 : c(S^\star))$ iff (5.12) holds together with the condition*

$$\lim_{i\to\infty} \tilde{f}_{ij} \text{ exists for all } j \in \mathbb{N}. \tag{5.14}$$

(ii) *$F \in (c_0 : c(S^\star))$ iff (5.13) and (5.14) hold.*

(iii) *$F \in (c : c_0(S^\star))$ iff (5.13) and (5.14) hold together with the condition*

$$\lim_{i\to\infty} \sum_{j=0}^{\infty} \tilde{f}_{ij} \text{ exists.}$$

(iv) $F \in (\ell_\infty : c_0(S^\star))$ *iff* (5.13) *holds together with the condition*

$$\lim_{i\to\infty} \sum_{j=0}^{\infty} |\tilde{f}_{ij}| = \sum_{j=0}^{\infty} \left| \lim_{i\to\infty} \tilde{f}_{ij} \right|.$$

6. Compact Operators on the Space $c_0(S^\star)$

It is qualified as a certain class of compact operators on new described sequence space $c_0(S^\star)$ in this part. Before characterizing the compact operators on the newly defined space $c_0(S^\star)$, we begin by introducing some essential notation and preliminary concepts related to normed spaces.

Let Π be a BK-space containing Ω and consider the unit sphere of Π, denoted by $\mathcal{D}_\Pi$. For any sequence $u = (u_i) \in \omega$, define the functional

$$\|u\|_\Pi^* := \sup_{z\in\mathcal{D}_\Pi} \left| \sum_{i=0}^{\infty} u_i z_i \right|,$$

provided that this supremum exists and is finite. Under these conditions, it follows that u belongs to the β-dual of the space Π, that is, $u \in \Pi^\beta$.

Lemma 6.1 ([27])**.** *The β-duals of the classical spaces ℓ_∞, c and c_0 all coincide with ℓ_1, that is,*

$$\ell_\infty^\beta = c^\beta = c_0^\beta = \ell_1.$$

Moreover, for any sequence u, the norm $\|u\|_\Pi^*$ equals the ℓ_1-norm, i.e., $\|u\|_\Pi^* = \|u\|_{\ell_1}$, where Π is one of the spaces in $\{\ell_\infty, c, c_0\}$.

Let $\mathfrak{B}(\Pi : \Psi)$ denote the set of all bounded (i.e., continuous) linear operators from the BK-space Π into the BK-space Ψ.

Lemma 6.2 ([28])**.** *If Π and Ψ are BK-spaces and $F \in (\Pi : \Psi)$, then the associated linear mapping $\mathcal{L}_F$, defined by $\mathcal{L}_F(u) = Fu$ for every $u \in \Pi$, belongs to $\mathfrak{B}(\Pi : \Psi)$.*

Lemma 6.3 ([28])**.** *Assume Π is a BK-space that contains all finitely nonzero sequences (i.e., $\Omega \subset \Pi$). If $F \in (\Pi : \Psi)$ with $\Psi \in \{\ell_\infty, c, c_0\}$, then the operator norm of $\mathcal{L}_F$ is given by*

$$\|\mathcal{L}_F\| = \|F\|_{(\Pi:\Psi)} = \sup_{i\in\mathbb{N}} \|F_i\|_\Pi^*,$$

where F_i denotes the i-th row of the matrix F.

Let A be a bounded subset of a metric space Π. The Hausdorff measure of noncompactness of A, denoted by $\chi(A)$, is defined as

$$\chi(A) = \inf \left\{ \varepsilon > 0 : A \subset \bigcup_{j=1}^{i} B(u_j, r_j),\ u_j \in \Pi,\ r_j < \varepsilon,\ i \in \mathbb{N} \right\},$$

where $B(u_j, r_j)$ refers to the open ball in Π centered at u_j with radius r_j. Further elaboration on this notion can be found in [28] and the references therein.

Theorem 6.4. *Let $A \subset c_0$ be bounded and for each $j \in \mathbb{N}$, define the truncation operator $\lambda_j : c_0 \to c_0$ by*

$$\lambda_j(u) = (u_0, u_1, \ldots, u_j, 0, 0, \ldots)$$

for all $u = (u_k) \in c_0$. Then, for the identity operator I on c_0, the measure of noncompactness is given by

$$\chi(A) = \lim_{j\to\infty} \sup_{u\in A} \|(I - \lambda_j)(u)\|_{c_0}.$$

A linear mapping $\mathcal{L} : \Pi \to \Psi$ is said to be a compact operator if it maps the unit ball in Π to a relatively compact subset in Ψ. In other words, for any bounded sequence $(u^{(n)}) \subset \Pi$, the image sequence $\big(\mathcal{L}(u^{(n)})\big)$ has a convergent subsequence in Ψ.

We denote the Hausdorff measure of noncompactness of $\mathcal{L}$ by $\|\mathcal{L}\|_\chi := \chi(\mathcal{L}(A_\Pi))$, where A_Π is the closed unit ball of Π. It is known that $\mathcal{L}$ is compact if and only if $\|\mathcal{L}\|_\chi = 0$.

We can consider $z = (z_i)$, $\tau = (\tau_i) \subset \omega$ connected to each other by the relation

$$\tau_i = \sum_{m=j}^{\infty} (-1)^{m-j} \binom{2m}{m+j} \frac{(2j+1)S_j}{(m+j+1)C_m} z_m.$$

Lemma 6.5. *Consider that $z = (z_i) \in (c_0(S^\star))^\beta$. In that case, $\tau = (\tau_i) \in \ell_1$ and*

$$\sum_{i=0}^{\infty} u_i z_i = \sum_{i=0}^{\infty} \nu_i \tau_i \tag{6.1}$$

for all $u = (u_i) \in c_0(S^\star)$.

Lemma 6.6. *The statement*

$$\|z\|^*_{c_0(S^\star)} = \sum_{i=0}^{\infty} |\tau_i| < \infty$$

holds for each $z = (z_i) \in (c_0(S^\star))^\beta$.

Proof. It is achieved from Lemma 6.5 that $\tau = (\tau_i) \in \ell_1$ and equation (6.1) holds for $z = (z_j) \in (c_0(S^\star))^\beta$. Since $\|u\|_{c_0(S^\star)} = \|\nu\|_{c_0}$, we reach that "$u \in \mathcal{D}_{c_0(S^\star)}$ iff $\nu \in \mathcal{D}_{c_0}$". Thus, we can write the equality

$$\|z\|^*_{c_0(S^\star)} = \sup_{u\in\mathcal{D}_{c_0(S^\star)}} \left|\sum_{i=0}^{\infty} z_i u_i\right| = \sup_{\nu\in\mathcal{D}_{c_0}} \left|\sum_{i=0}^{\infty} \tau_i \nu_i\right| = \|\tau\|^*_{c_0}.$$

By the aid of Lemma 6.1, it follows that

$$\|z\|^*_{c_0(S^\star)} = \|\tau\|^*_{c_0} = \|\tau\|_{\ell_1} = \sum_{i=0}^{\infty} |\tau_i| < \infty,$$

which is the desired result. □

In the forthcoming analysis, we make use of the matrices $\mathcal{N} = (n_{ij})$ and $F = (f_{ij})$, which are connected through the identity stated in equation (5.1), under the presumption that all involved series are convergent.

Lemma 6.7. *Suppose $\Psi \subset \omega$ and let $F = (f_{ij})$ be an infinite matrix. If the matrix transformation F maps $c_0(S^\star)$ into Ψ, i.e., $F \in (c_0(S^\star) : \Psi)$, then the corresponding matrix $\mathcal{N}$ defines a transformation from c_0 into Ψ and the identity $Fu = \mathcal{N}\nu$ holds for every $u \in c_0(S^\star)$.*

Proof. This result directly follows from Lemma 6.5; hence, the proof is omitted. □

Lemma 6.8. *If $F \in (c_0(S^\star) : \Psi)$, then it is achieved that*

$$\|\mathcal{L}_F\| = \|F\|_{(c_0(S^\star):\Psi)} = \sup_{i\in\mathbb{N}} \left(\sum_{j=0}^{\infty} |\mathcal{N}_{ij}| \right) < \infty,$$

where $\Psi \in \{\ell_\infty, c, c_0\}$.

Lemma 6.9 ([30])**.** *Consider Π as a BK-space. The statements below are valid:*

(i) *If $F \in (\Pi : \ell_\infty)$, then the Hausdorff measure of non-compactness of $\mathcal{L}_F$ satisfies*

$$0 \le \|\mathcal{L}_F\|_\chi \le \limsup_{i\to\infty} \|F_i\|^*_\Pi.$$

(ii) *For $F \in (\Pi : c_0)$, one has the equality*

$$\|\mathcal{L}_F\|_\chi = \limsup_{i\to\infty} \|F_i\|^*_\Pi.$$

(iii) *If the space Π either possesses the AK-property or equals ℓ_∞ and $F \in (\Pi : c)$, then the inequality*

$$\frac{1}{2} \limsup_{i\to\infty} \|F_i - f\|^*_\Pi \le \|\mathcal{L}_F\|_\chi \le \limsup_{i\to\infty} \|F_i - f\|^*_\Pi$$

holds, where the sequence $f = (f_j)$ is given by $f_j = \lim_{i\to\infty} f_{ij}$ for each $j \in \mathbb{N}$.

Lemma 6.10 ([30]). *Let Π be a BK-space satisfying $\Omega \subset \Pi$ and let $F \in (\Pi : \ell_1)$. Then the following estimate for the Hausdorff measure of non-compactness holds:*

$$\lim_{j\to\infty} \sup_{E\in\mathcal{E}_j} \left\| \sum_{i\in E} F_i \right\|_{\Pi}^{*} \leq \|\mathcal{L}_F\|_{\chi} \leq 4 \lim_{j\to\infty} \sup_{E\in\mathcal{E}_j} \left\| \sum_{i\in E} F_i \right\|_{\Pi}^{*},$$

where $\mathcal{E}_j$ denotes the collection of all finite subsets of $\mathbb{N}$ whose elements exceed j. Moreover, $\mathcal{L}_F$ is compact if and only if

$$\lim_{j\to\infty} \sup_{E\in\mathcal{E}_j} \left\| \sum_{i\in E} F_i \right\|_{\Pi}^{*} = 0.$$

Theorem 6.11. (i) *For $F \in (c_0(S^\star) : \ell_\infty)$, we have*

$$0 \leq \|\mathcal{L}_F\|_{\chi} \leq \limsup_i \sum_{j=0}^{\infty} |n_{ij}|.$$

(ii) *For $F \in (c_0(S^\star) : c)$,*

$$\frac{1}{2} \limsup_i \sum_{j=0}^{\infty} |n_{ij} - n_j| \leq \|\mathcal{L}_F\|_{\chi} \leq \limsup_i \sum_{j=0}^{\infty} |n_{ij} - n_j|.$$

(iii) *For $F \in (c_0(S^\star) : c_0)$,*

$$\|\mathcal{L}_F\|_{\chi} = \limsup_i \sum_{j=0}^{\infty} |n_{ij}|.$$

(iv) *For $F \in (c_0(S^\star) : \ell_1)$,*

$$\lim_j \|F\|^{(j)}_{(c_0(S^\star):\ell_1)} \leq \|\mathcal{L}_F\|_{\chi} \leq 4 \cdot \lim_j \|F\|^{(j)}_{(c_0(S^\star):\ell_1)},$$

where

$$\|F\|^{(j)}_{(c_0(S^\star):\ell_1)} = \sup_{E\in\mathcal{E}_j} \left(\sum_{j=0}^{\infty} \left| \sum_{i\in E} n_{ij} \right| \right),$$

with $n = (n_j)$ and $n_j = \lim_i n_{ij}$ for all $j \in \mathbb{N}$.

Proof. (i) Let $F \in (c_0(S^\star) : \ell_\infty)$. It is seen that $F_i \in (c_0(S^\star))^{\beta}$ because $\sum_{j=0}^{\infty} f_{ij} u_j$ converges for all $i \in \mathbb{N}$. From Lemma 6.6, we reach that

$$\|F_i\|^{*}_{c_0(S^\star)} = \|\mathcal{N}_i\|^{*}_{c_0} = \|\mathcal{N}_i\|_{\ell_1} = \left(\sum_{j=0}^{\infty} |n_{ij}| \right).$$

From Lemma 6.9 (i), it follows that

$$0 \le \|\mathcal{L}_F\|_\chi \le \limsup_i \sum_{j=0}^{\infty} |n_{ij}|.$$

(ii) From Lemma 6.7, we have $\mathcal{N} \in (c_0 : c)$ because $F \in (c_0(S^\star) : c)$. Thus, from Lemma 6.9 (iii),

$$\frac{1}{2} \limsup_i \|\mathcal{N}_i - n\|_{c_0}^* \le \|\mathcal{L}_F\|_\chi \le \limsup_i \|\mathcal{N}_i - n\|_{c_0}^*.$$

Consequently, by Lemma 6.1, we have

$$\|\mathcal{N}_i - n\|_{c_0}^* = \|\mathcal{N}_i - n\|_{\ell_1} = \sum_{j=0}^{\infty} |n_{ij} - n_j|, \quad \text{for each } i \in \mathbb{N}.$$

(iii) Let $F \in (c_0(S^\star) : c_0)$. From the relation $\|F_i\|_{c_0(S^\star)}^* = \|\mathcal{N}_i\|_{c_0}^* = \|\mathcal{N}_i\|_{\ell_1} = \sum_{j=0}^{\infty} |n_{ij}|$ and Lemma 6.9 (ii), it is concluded that

$$\|\mathcal{L}_F\|_\chi = \limsup_i \sum_{j=0}^{\infty} |n_{ij}|.$$

(iv) Let $F \in (c_0(S^\star) : \ell_1)$. By Lemma 6.7, we have $\mathcal{N} \in (c_0 : \ell_1)$. By Lemma 6.10, it is achieved that

$$\lim_j \left(\sup_{E \in \mathcal{E}_j} \left\| \sum_{i \in E} \mathcal{N}_i \right\|_{c_0}^* \right) \le \|\mathcal{L}_F\|_\chi \le 4 \lim_j \left(\sup_{E \in \mathcal{E}_j} \left\| \sum_{i \in E} \mathcal{N}_i \right\|_{c_0}^* \right).$$

Furthermore, Lemma 6.1 implies that

$$\left\| \sum_{i \in E} \mathcal{N}_i \right\|_{c_0}^* = \left\| \sum_{i \in E} \mathcal{N}_i \right\|_{\ell_1} = \left(\sum_{j=0}^{\infty} \left| \sum_{i \in E} n_{ij} \right| \right),$$

which is the desired result.

□

Thus, making use of the theorem given above, we can give the following result.

Corollary 6.12. (i) *For $F \in (c_0(S^\star) : \ell_\infty)$, the operator $\mathcal{L}_F$ is compact iff*

$$\lim_i \sum_{j=0}^{\infty} |n_{ij}| = 0.$$

(ii) *For $F \in (c_0(S^\star) : c)$, the operator $\mathcal{L}_F$ is compact iff*

$$\lim_i \sum_{j=0}^{\infty} |n_{ij} - n_j| = 0.$$

(iii) *For* $F \in (c_0(S^\star) : c_0)$*, the operator* $\mathcal{L}_F$ *is compact iff*

$$\lim_i \sum_{j=0}^{\infty} |n_{ij}| = 0.$$

(iv) *For* $F \in (c_0(S^\star) : \ell_1)$*, the operator* $\mathcal{L}_F$ *is compact iff*

$$\lim_j \|F\|^{(j)}_{(c_0(S^\star):\ell_1)} = 0,$$

where $\|F\|^{(j)}_{(c_0(S^\star):\ell_1)} = \sup_{E\in\mathcal{E}_j} \left(\sum_{j=0}^{\infty} \left|\sum_{i\in E} n_{ij}\right|\right)$ *for all* $i \in \mathbb{N}$.

7. Conclusion

In this paper, we have investigated matrix domains $c(S^\star)$ and $c_0(S^\star)$ of $S^\star$ within c and c_0, respectively. After that, we have rigorously shown that the associated sequence spaces form BK-spaces which are linearly isomorphic to c_0 and c.

We have provided Schauder bases for these new spaces and characterized their α-, β- and γ-duals in terms of explicit summation identities involving Schröder and Catalan numbers. Furthermore, we established necessary and sufficient conditions for various classes of infinite matrices between these newly constructed sequence spaces and classical sequence spaces.

Additionally, compactness criteria of such matrix operators have been obtained by employing the Hausdorff measure of noncompactness, which revealed insightful operator-theoretic behavior in these functional settings. The results obtained herein not only extend the theory of sequence spaces and matrix transformations but also contribute to the application of special number sequences such as Schröder and Catalan numbers in functional analysis and operator theory.

This study may motivate further research on constructing new sequence spaces and operators via other combinatorial sequences, which can potentially uncover new connections within summability and sequence spaces theory.

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