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Discrete and Continuous Multiplicative Differential Equations and Applications in Solving Non−Linear Difference and Differential Equations

Mohammad Jahanshahi[1]*, Nihan Aliyev[2] and Hamid Dehghani[3]

Abstract. Boundary and initial value problems, including nonlinear difference equations and nonlinear differential equations, are the mathematical models of many physics and engineering problems and natural phenomena. Usually, due to the lack of a solid theory for solving these types of equations, these equations are solved by using numerical and approximate methods. In this paper, first some elementary and basic definitions and concepts of discrete and continuous multiplicative calculus are given. Next we apply some ideas and methods to obtain invariant functions with respect to their associated derivative. These invariant functions are used to solve several types of nonlinear difference and differential equations that have appeared in natural sciences and physical problems. After that, these methods are expanded for solving nonlinear difference and differential equations through discrete and continuous multiplicative differential equations. Finally, some applications of multiplicative forms of differential equations are given which simplify numerical methods for solving nonlinear biological problems and exponential approximations for nonlinear functions.

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1. Introduction

Various calculus in mathematics are created and developed to investigate and solve boundary and initial value problems that arise from modeling physics and engineering problems. Ordinary calculus (Newtonian calculus) and generalized calculus (related to distributions) and

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fractional calculus are among them. The nascent multiplicative calculus has a particular importance. The importance of this calculus is due to the provision of analytical solutions to many nonlinear equations in mathematics, physics and engineering.

Additive calculus (or classical calculus) introduced in the 17th century is sometimes called Newtonian calculus. This calculus and its beautiful born (Differential Equations) have modeled and solved many physical and engineering problems in the 20th, 19th and 18th centuries. The problems solved by this continuous calculus appear in continuous spaces with continuous variables.[5, 15]

Two famous non-Newtonian calculus: multiplicative calculus and bigeometric calculus, introduced by Grossman and Katz in 1972 [8–10]. In these calculations, addition and subtraction are changed to multiplication and division. Recently, it has been shown that the non-Newtonian calculus (multiplicative) is more suitable for some problems than the ordinary Newtonian calculus (additive). For example, in statistics, finance, economics, biology, demography, pattern recognition in images, signal processing, thermostats and quantum information theory [2, 3, 16, 19, 20].

On the other hand,N. Aliyev et al.studied additive calculus in discrete mode. They introduced the additive discrete calculus and presented some basic formulas for the discrete additive derivative and integral.[1] Then N. Aliyev and M. Jahanshahi et al presented and extended multiplication calculus in discrete and continuous cases [11, 12] .

In recent years, some of Turkish mathematicians extended and applied the multiplicative analysis to the fixed point theory, sequence spaces and complex case [4, 6, 17, 21].

In this paper, we introduce and use some invariant functions for discrete and continuous multiplicative derivatives. Using these invariant multiplicative functions, we solve linear and nonlinear difference and differential equations via multiplicative calculus. Also, we will expand analytical and numerical methods to solve nonlinear and non-homogeneous differential equations through multiplicative calculations.

This paper is organized into 4 parts. At the first part, basic definitions of discrete and continuous multiplicative derivative and integral are given. In the secend part, continuous and discrete multiplicative differential equations are introdued. In the third part, nonlinear difference and differential equations are solved via multiplicative differential equations. In the fourth part, some applications and numerical methods are presented for solving some practial nonlinear biological problems.

2. Schematic of Newtonian and Non-Newtonian Calculus

Classical definition of integral operation and derivative operation is basis of ordinary continuous additive calculus which is also known as Newtonian calculus. Actually, the operation of limit in definition of integral and derivative such that they are related to the continuity.

In classical calculus we saw that the operation of integration was defined via multiplication and addition and then limitation. Also, we saw that ordinary derivative operation has been done by the two inverse operations division and subtraction and then limitation.

In multiplicative calculus, we will show that multiplicative integral is defined by the two direct operations: multiplication and power. Also continuous derivative multiplicative is basically defined by the two inverse operations: n th root and division, then the operation of limit is being used.

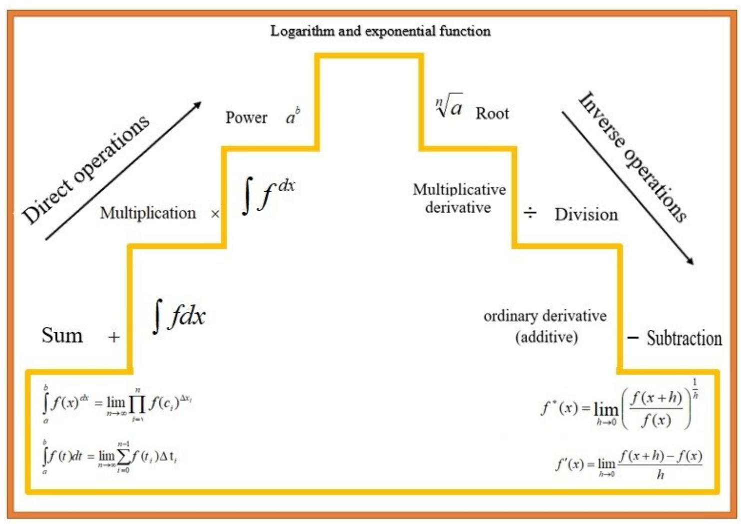

Figure 1. Logarithm and exponential function

3. Basic Definitions and Formulas of Multiplicative Calculus

Definition 3.1. Suppose the function $f : A \subset \mathbb{R}^+ \to \mathbb{R}$ is a positive value function, similar to the expression of the definition of the ordinary derivative of the function:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

The continuous multiplicative derivative is defined with the following limit:

$$f^*(x) = \lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{\frac{1}{h}}.$$

If this limit exists, then the function is multiplicatively differentiable and we denote it by $f^*(x)$.

Now we try to get a practical formula to calculate the multiplicative derivative of the function. For this, according to the above definition, we have:

$$\begin{aligned} f^*(x) &= \lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{\frac{1}{h}} \\ &= \lim_{h\to 0}\left(\frac{f(x+h)}{f(x)} - \frac{f(x)}{f(x)} + 1\right)^{\frac{1}{h}} \\ &= \lim_{h\to 0}\left(1 + \frac{f(x+h)-f(x)}{f(x)}\right)^{\frac{f(x)}{f(x+h)-f(x)}\frac{f(x+h)-f(x)}{h}\frac{1}{f(x)}} \\ &= e^{\frac{f'(x)}{f(x)}} \\ &= e^{(Lnf)'}. \end{aligned}$$

From the above relation, we can determine the relation of between additive and multiplicative derivative:

$$f'(x) = f(x)\cdot Ln(f^*(x)).$$

For the second order multiplicative derivative, we have:

$$f^{**}(x) = e^{(Lnf^*)'(x)} = e^{(Lnf)''(x)}.$$

By mathematical induction, we can calculate the arbitrary n'th order.

$$f^{(*n)}(x) = e^{(Lnf)^{(n)}(x)}, \quad n = 0, 1, 2, \ldots$$

Theorem 3.2 ([2, 11, 12])**.** *The following basic formulas for continuous multiplicative differentiable functions f, g hold:*

(i) $(cf)^*(x) = f^*(x)$;
(ii) $(f\cdot g)^*(x) = f^*(x)\cdot g^*(x)$;
(iii) $(\frac{f}{g})^*(x) = \frac{f^*(x)}{g^*(x)}$;
(iv) $((f)^g)^*(x) = f^*(x)^{g(x)}\cdot f(x)^{g'(x)}$;
(v) $(fog)^*(x) = f^*(g(x))^{g'(x)}$;

Remark 3.3 ([1, 11, 12])**.** The invariant function of the continuous multiplicative derivative is:

$$y = e^{e^{\lambda x}}.$$

3.1. **Definition of Continuous Multiplicative Integral.** Similar to the definition of additive integral and the use of the Riemann sum:

$$S(f,p)=\sum_{i=1}^{n} f(c_i)(x_i-x_{i-1}), \qquad \int_a^b f(x)\mathrm{d}x=\lim_{h\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x_i.$$

We will have the definition of multiplicative integral as follows:

$$\begin{aligned}\int_a^b f(x)^{\mathrm{d}x} &= \lim_{h\to\infty}\prod_{i=1}^{n} f(c_i)^{\Delta x_i}\\ &= e^{\lim_{h\to\infty} Ln\left(f(c_1)^{\Delta x_1}\cdot f(c_2)^{\Delta x_2}\cdot\dots\cdot f(c_n)^{\Delta x_n}\right)}\\ &= e^{\lim_{h\to\infty}\left(Lnf(c_1)^{\Delta x_1}+Lnf(c_2)^{\Delta x_2}+\dots+Lnf(c_n)^{\Delta x_n}\right)}\\ &= e^{\lim_{h\to\infty}\sum_1^n Lnf(c_i)\cdot\Delta x_i}\\ &= e^{\int_a^b Lnf(x)\mathrm{d}x}.\end{aligned}$$

Theorem 3.4 ([2, 11, 12])**.** *Basic formulas for continuous multiplicative integrable functions f, g:*

(i) $\int_a^b (f(x)\cdot g(x))^{\mathrm{d}x}=\int_a^b f(x)^{\mathrm{d}x}\cdot\int_a^b g(x)^{\mathrm{d}x}$;
(ii) $\int_a^b f(x)^{\mathrm{d}x}=\int_a^c f(x)^{\mathrm{d}x}\cdot\int_c^b f(x)^{\mathrm{d}x}$;
(iii) $\int_a^b (\frac{f(x)}{g(x)})^{\mathrm{d}x}=\frac{\int_a^b f(x)^{\mathrm{d}x}}{\int_a^b g(x)^{\mathrm{d}x}}$;
(iv) $\int_a^b (f(x)^p)^{\mathrm{d}x}=\left(\int_a^b f(x)^{\mathrm{d}x}\right)^p$;

3.2. **Definitions of Discrete Multiplicative Derivative.** Suppose the function f is defined as follows:

$$f: A\subset\mathbb{Z}\longrightarrow\mathbb{R},\quad x\in A\subset\mathbb{Z}.$$

Then the discrete multiplicative derivative of the function f is defined by:

$$f^{[1]}(x)=\frac{f(x+1)}{f(x)}.$$

where the notation[1] shows the discrete multiplicative derivative.

Remark 3.5. The discrete and continuous multiplicative derivative of a constant function is equal to unity. (The ordinary derivative of the constant function is equal to zero.)

Theorem 3.6 ([11, 12])**.** *The following relations hold.*

(i) $(f(x)\cdot g(x))^{[1]}= f^{[1]}(x)\cdot g^{[1]}(x)$;
(ii) $\left(\frac{f(x)}{g(x)}\right)^{[1]}=\frac{f^{[1]}(x)}{g^{[1]}(x)}$;
(iii) $(f^n(x))^{[1]}=\left(f^{[1]}(x)\right)^n$;

Remark 3.7 ([1, 11, 12])**.** The invariant function of the discrete multiplicative derivative is:

$$f(x) = c^{2^x}, \qquad c = \text{constant}.$$

For this, we consider

$$\left(c^{2^x}\right)^{[1]} = \frac{c^{2^{x+1}}}{c^{2^x}} = c^{2^{x+1}-2^x} = c^{2^x(2-1)} = c^{2^x}.$$

When we want to use it for solving discrete multiplicative differential equations, we use the parametric form of this function as follows:

$$y(k) = c^{(1+\lambda)^k}, \qquad y^{[1]}(k) = c^{\lambda(1+\lambda)^k}, \qquad y^{[11]}(k) = c^{\lambda^2(1+\lambda)^k}$$

4. Discrete Multiplicative Difference Equations

In this section, we introduce discrete multiplicative difference equations. We will use these equations to solve nonlinear difference equations.

$$\left(y^{[n]}(k)\right)^{\alpha_n} \cdot \left(y^{[n-1]}(k)\right)^{\alpha_{n-1}} \cdot \cdots \cdot \left(y^{[1]}(k)\right)^{\alpha_1} \cdot (y(k))^{\alpha_0} = 1 \tag{4.1}$$

with initial conditions:

$$y^{[n-1]}(k_0) = {y_0}^{n-1}, \ldots, y(k_0) = y_0$$

We use the invariant function to solve the difference equation 4.1. As a result, we have:

$$\left(c^{\lambda^n(1+\lambda)^k}\right)^{\alpha_n} \cdot \left(c^{\lambda^{n-1}(1+\lambda)^k}\right)^{\alpha_{n-1}} \cdots \left(c^{\lambda(1+\lambda)^k}\right)^{\alpha_1} \cdot \left(c^{(1+\lambda)^k}\right)^{\alpha_0} = 1$$

$$\left(c^{\alpha_n\lambda^n(1+\lambda)^k+\alpha_{n-1}\lambda^{n-1}(1+\lambda)^k+\cdots+\alpha_1\lambda(1+\lambda)^k+\alpha_0(1+\lambda)^k}\right) = 1$$

$$(1+\lambda)^k\left(\alpha_n\lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda + \alpha_0\right) = 0$$

Therefore, the corresponding characteristic equation will be as follows.

$$\alpha_n\lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda + \alpha_0 = 0.$$

In order to explain the method of solving the above equations, we consider discrete second-order equations as follows:

$$\left(y^{[11]}(k)\right)^a \cdot \left(y^{[1]}(k)\right)^b (y(k))^c = 1. \tag{4.2}$$

Using the concept of discrete multiplicative derivative, we have:

$$y^{[1]}(k) = \frac{y(k+1)}{y(k)} = \frac{y_{k+1}}{y_k}, \qquad y^{[11]}(k) = \frac{\frac{y_{k+2}}{y_{k+1}}}{\frac{y_{k+1}}{y_k}} = \frac{y_{k+2}\cdot y_k}{y_{k+1}^2}.$$

By inserting these relationships in equation 4.2

$$\left(\frac{y_{k+2}\cdot y_k}{y_{k+1}^2}\right)^a \cdot \left(\frac{y_{k+1}}{y_k}\right)^b (y(k))^c = 1$$

and

$$(4.3) \qquad {y^a}_{k+2} \cdot {y^{b-2a}}_{k+1} \cdot {y^{a-b+c}}_k = 1.$$

Now we consider the general form of the following nonlinear second-order difference equations:

$$(4.4) \qquad {y^\alpha}_{k+2} \cdot {y^\beta}_{k+1} \cdot {y^\gamma}_k = 1.$$

where α, β, γ are real constants.

By comparing the equations 4.3 and 4.4 we arrive at the following algebraic system:

$$\begin{cases} \alpha = a \\ \beta = b - 2a \\ \gamma = a - b + c. \end{cases}$$

Because the values α, β, γ in the equation are known, then the coefficients of following characteristic equation are determined from the solution of the algebraic system. According to the general form of the equation 4.2, we will have:

$$a\lambda^2 + b\lambda + c = 0.$$

By finding λ_1, λ_2 the roots of the above equation, the linear independent solutions will be as follows.

$$y_1(k) = {C_1}^{(1+\lambda_1)^k}, \qquad y_2(k) = {C_2}^{(1+\lambda_2)^k}.$$

Example 4.1. We solve the following second order nonlinear difference equation:

$$y_{n+2} = y_n^2.$$

Solution: We have

$$\begin{aligned} &{y^\alpha}_{n+2} \cdot {y^\beta}_{n+1} \cdot {y^\gamma}_n = 1 \\ &y_{n+2} = y_n^2 \\ &y_{n+2} \cdot^0 \cdot {y_n}^{-2} = 1. \end{aligned}$$

and

$$\begin{cases} \alpha = a \\ \beta = b - 2a \\ \gamma = a - b + c \end{cases} \Rightarrow \begin{cases} 1 = a \\ 0 = b - 2a \\ -2 = a - b + c \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = 2 \\ c = -1. \end{cases}$$

Also

$$a\lambda^2 + b\lambda + c = 0 \quad \Rightarrow \quad \lambda^2 + 2\lambda - 1 = 0 \quad \Rightarrow \quad \begin{cases} \lambda_1 = -1 + \sqrt{2} \\ \lambda_1 = -1 - \sqrt{2}. \end{cases}$$

Then

$$\begin{cases} y_1(n) = c_1^{\left(1+(-1+\sqrt{2})\right)^n} \\ y_2(n) = c_2^{\left(1+(-1-\sqrt{2})\right)^n} \end{cases} \Rightarrow \begin{cases} y_1(n) = c_1^{\left(\sqrt{2}\right)^n} \\ y_2(n) = c_1^{\left(-\sqrt{2}\right)^n}. \end{cases}$$

Below we show that the above solutions satisfy in the nonlinear difference equation.

$$\begin{aligned} y_{n+2} &= c_1^{\left(\sqrt{2}\right)^{n+2}} \\ &= c_1^{2\left(\sqrt{2}\right)^n} \\ &= \left(c_1^{\left(\sqrt{2}\right)^n}\right)^2 \\ &= y_n{}^2. \end{aligned}$$

It is easy to see that the other root satisfies in the given equation.

Example 4.2. Consider the following nonlinear difference equation.

$$y_{n+2} = \sqrt{y_{n+1}} \cdot y_n^3. \tag{4.5}$$

Solution: We write this equation in the form:

$$y_{n+2} \cdot y_{n+1}{}^{-\frac{1}{2}} \cdot y_n{}^{-3} = 1.$$

$$\begin{cases} \alpha = a \\ \beta = b - 2a \\ \gamma = a - b + c. \end{cases} \Rightarrow \begin{cases} 1 = a \\ -\frac{1}{2} = b - 2a \\ -3 = a - b + c \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = \frac{3}{2} \\ c = -\frac{5}{2}. \end{cases}$$

Therefore, its characteristic equation will be as follows:

$$a\lambda^2 + b\lambda + c = 0 \quad \Rightarrow \quad \lambda^2 + \frac{3}{2}\lambda - \frac{5}{2} = 0 \quad \Rightarrow \quad \begin{cases} \lambda_1 = 1 \\ \lambda_1 = -\frac{5}{2}. \end{cases}$$

then

$$\begin{cases} y_1(n) = c_1^{(1+1)^n} \\ y_2(n) = c_2^{\left(1+(-\frac{5}{2})\right)^n} \end{cases} \Rightarrow \begin{cases} y_1(n) = c_1^{(2)^n} \\ y_2(n) = c_1^{\left(-\frac{3}{2}\right)^n}. \end{cases}$$

We show that this Solution satisfies to the equation 4.5

$$\begin{aligned} y_{n+2} &= c_1^{2^{n+2}} \\ &= c_1^{2^n \cdot 4} \\ &= \left(c_1^{2^n}\right)^4 \\ &= y_n{}^4 \end{aligned}$$

and

$$y_{n+1} = c_1^{2^{n+1}}$$

$$= {c_1}^{2^n \cdot 2}$$
$$= \left({c_1}^{2^n}\right)^2$$
$$= {y_n}^2.$$

Now we put the above values in the equation.

$${y_n}^4 \cdot {y_n}^{-1} \cdot {y_n}^{-3} \cdot = 1.$$

The last relation shows this solution satisfies in the given difference equation. One can shows the other root satisfies in the given equation.

If the general second order discrete multiplicative differential equation is given with initial conditions, then its solution can be obtained by above-mentioned method.

5. Continuous Multiplicative Differential Equations

We consider the general form of $n{\cdot}th$ order continuous multiplicative differential equations as follows:

$$\left(y^{[n]}(x)\right)^{a_n} \cdot \left(y^{[n-1]}(x)\right)^{a_{n-1}} \cdots \left(y^{[1]}(x)\right)^{a_1} \cdot (y(x))^{a_0} = 1. \tag{5.1}$$

with initial conditions

$$y^{[n-1]}(x_0) = {y_0}^{n-1}, \ldots, y(x_0) = y_0.$$

For solving this equation, we use the invaryant function which introduced in the previous section.

Remark 5.1. For this we recall the invaryant function for the ordinary derivative:

$$y = e^{\lambda x}$$

Then

$$y^n = \lambda^n \cdot e^{\lambda x}.$$

It is similar to this relation, in continuous multiplicative case we have:

$$y = e^{e^{\lambda x}}$$

Then

$$y^{[n]} = \lambda^n \cdot e^{\lambda x}$$

By inserting into the equation 5.1, we have:

$$\left(e^{\lambda^n \cdot e^{\lambda x}}\right)^{a_n} \cdot \left(e^{\lambda^{n-1} \cdot e^{\lambda x}}\right)^{a_{n-1}} \cdots \left(e^{\lambda \cdot e^{\lambda x}}\right)^{a_1} \cdot \left(e^{e^{\lambda x}}\right)^{a_0} = 1.$$

Then

$$e^{\left(a_n \cdot \lambda^n \cdot e^{\lambda x} + a_{n-1} \cdot \lambda^{n-1} \cdot e^{\lambda x} + \cdots + a_1 \cdot \lambda \cdot e^{\lambda x} + a_0 \cdot e^{\lambda x}\right)} = 1.$$
$$a_n \cdot \lambda^n \cdot e^{\lambda x} + a_{n-1} \cdot \lambda^{n-1} \cdot e^{\lambda x} + \cdots + a_1 \cdot \lambda \cdot e^{\lambda x} + a_0 \cdot e^{\lambda x} = 0.$$

$$e^{\lambda x} \cdot \left(a_n \cdot \lambda^n + a_{n-1} \cdot \lambda^{n-1} + \cdots + a_1 \cdot \lambda + a_0\right) = 0.$$

Which $e^{\lambda x}$ cannot be zero because it will give a trivial solution. So

$$a_n \cdot \lambda^n + a_{n-1} \cdot \lambda^{n-1} + \cdots + a_1 \cdot \lambda + a_0 = 0.$$

We recall this equation as characteristic equation of the main equation. By finding values $\lambda_1, \lambda_2, \ldots, \lambda_n$ the linear independent solutions will be as follows.

$$y_1(x) = \left(e^{e^{\lambda_1 x}}\right)^{c_1}, y_2(x) = \left(e^{e^{\lambda_2 x}}\right)^{c_2}, \ldots, y_n(x) = \left(e^{e^{\lambda_n x}}\right)^{c_n}.$$

As a result, the general solution will be as follows:

$$y(x) = \left(e^{e^{\lambda_1 x}}\right)^{c_1} \cdot \left(e^{e^{\lambda_2 x}}\right)^{c_2} \cdots \left(e^{e^{\lambda_n x}}\right)^{c_n}.$$

where $c_1, c_2, \cdots, c_n$are constants.

Similar to the discrete case, for simplicity, we consider the following second order multiplicative differential equation

$$\begin{aligned} & y^{[11]}(x) \cdot \left(y^{[1]}(x)\right)^a \cdot (y(x))^b = 1. \\ & \lambda^2 + a\lambda + b = 0 \\ & \lambda = \frac{-a \pm \sqrt{b^2 - 4}}{2}. \end{aligned}$$

Example 5.2. Consider the following nonlinear differential equation

$$yy'' - y'^2 = y^2 \cdot e^x. \tag{5.2}$$

We can rewrite this equation as:

$$\frac{yy'' - y'^2}{y^2} = e^x.$$

Then

$$y^{**}(x) = e^x.$$

To solve the given equation, it is enough to integrate two times in the sense of multiplicative type, that is:

$$\begin{aligned} y^*(x) &= e^{\int^x \ln e^t \mathrm{d}t} \\ &= e^{\int^x t \mathrm{d}t} \\ &= c_1 e^x. \end{aligned}$$

Then

$$\begin{aligned} y(x) &= c_2 \int^x c_1 e^{t \mathrm{d}t} \\ &= c_2 e^{\int^x \ln c_1 {}^{e^t} \mathrm{d}t} \end{aligned}$$

$$= c_2 e^{\int^x e^t \ln c_1 \mathrm{d}t}$$
$$= c_2 e^{\ln c_1 e^x}.$$

Note that the nonlinear equation 5.2 cannot be solved by classical methods, while by using continuous multiplicative case, not only is an analytical solution given, but we can obtain its graph.

Example 5.3. Consider the nonlinear differential equation

$$\frac{yy'' - y'^2}{y^2} = \sin x.$$

For solving this equation, we also write:

$$e^{\frac{yy''-y'^2}{y^2}} = e^{\sin x}$$

Then

$$y^{**}(x) = e^{\sin x}$$

to solve the resulted multiplicative equation, two-time multiplicative integration should have been used:

$$y^*(x) = c_1 e^{\int^x \ln e^{\sin t} \mathrm{d}t} = c_1 e^{-\cos x}.$$

And we have:

$$\begin{aligned} y(x) &= c_2 e^{\int \ln c_1 e^{-\cos x} \mathrm{d}t} \\ &= c_2 e^{\int (\ln c_1 - \cos x) \mathrm{d}x} \\ &= c_2 e^{(x \ln c_1 - \sin x)} \\ &= c_2 e^{x \ln c_1} e^{-\sin x}. \end{aligned}$$

6. Applications in Numerical Calculations

6.1. **Multiplicative Model for The Problem of Hunting and Hunter (Hunter-Prey Problem).** As we know, the mathematical model of this problem in ordinary calculus (continuous additive calculus) is as follows:

$$\begin{cases} \frac{dx}{dt} = \alpha x - \beta xy, \quad x(0) = x_0, \\ \\ \frac{dy}{dt} = \sigma y + \gamma xy, \quad y(0) = y_0. \end{cases}$$

In which $x(t)$ is the population of hunting and $y(t)$ is the population of hunter. Numerical constants $\alpha, \beta, \sigma, \gamma$ depend on the isolated environment and the prey and predator populations of the problem.

The above system of ordinary differential equations can be written as follows, by using the multiplicative model due to the appearance of product sentences xy in the above system of differential equations.

(Choose for simplicity$\alpha = \beta = \sigma = \gamma = 1$)

$$\begin{cases} \frac{x'}{x} = 1 - y \\ \\ \frac{y'}{y} = 1 + x \end{cases} \Rightarrow \begin{cases} e^{\frac{x'}{x}} = e^{1-y} \\ \\ e^{\frac{y'}{y}} = e^{1+x} \end{cases} \Rightarrow \begin{cases} x^*(t) = e^{1-y(t)} \\ \\ y^*(t) = e^{1+x(t)}. \end{cases}$$

The solutions of this system of multiplicative differential equations can be written as a system of integral equations of the first type of Volterra by applying multiplicative integral and given initial conditions:

$$x(t) = x(0) \cdot e^{\int_0^t \ln e^{(1-y(\tau))} \mathrm{d}\tau}.$$
$$y(t) = y(0) \cdot e^{\int_0^t \ln e^{(1+x(\tau))} \mathrm{d}\tau}.$$

Now, to find approximate solutions and use the method of successive approximations, we make the following functional sequences as

$$x_n(t) = x(0) \cdot e^{\int_0^t (1-y_{n-1}(\tau))\mathrm{d}\tau}.$$
$$y_n(t) = y(0) \cdot e^{\int_0^t (1+x_{n-1}(\tau))\mathrm{d}\tau}.$$

It is clear that accurate approximate solutions can be obtained from the above sequences, which are much simpler than the analytical and approximate methods presented in various articles and books to solve the hunting and predator problem and we get better solutions with fewer repetitions.

The mathematics of hunting and predator problem can be extended to other phenomena such as the relationship between producer and consumer, police and criminals, diseases and humans.

with initial values $x(0) = 1, y(0) = 2$ we arrive at the following relations:

$$x_n(t) = x(0) \cdot e^{\int_0^t (1-y_{n-1}(\tau))\mathrm{d}\tau}.$$
$$x_1(t) = e^{\int_0^t (1-2)\mathrm{d}t} = e^{-t}$$
$$x_2(t) = e^{\int_0^t \left(1-e^{t-e^{-t}}\right)\mathrm{d}t} = e^{t-\frac{1}{2}e^{2t}}$$
$$x_3(t) = e^{\int_0^t (1-e^{2t})\mathrm{d}t} = \dots$$

$$y_n(t) = y(0) \cdot e^{\int_0^t (1+x_{n-1}(\tau))\mathrm{d}\tau}.$$
$$y_1(t) = e^{\int_0^t (1+1)\mathrm{d}t} = 2e^{2t}$$
$$y_2(t) = e^{\int_0^t (1+e^{-t})\mathrm{d}t} = 2e^{t-e^{-t}}$$
$$y_3(t) = e^{\int_0^t \left(1+e^{t-\frac{1}{2}e^{2t}}\right)\mathrm{d}t} = \dots$$

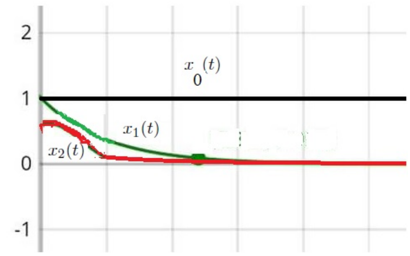

FIGURE 2. $x(t)$

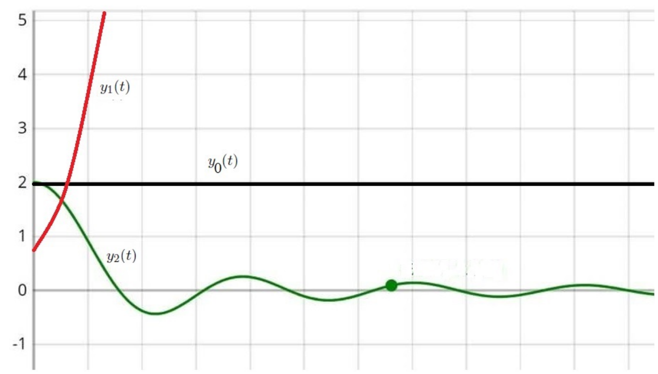

FIGURE 3. $y(t)$

As seen in Figures 2, 3, the initial approximations $x_0(t)$, $x_1(t)$ and $x_2(t)$, as well as $y_0(t)$, $y_1(t)$ and $y_2(t)$, are plotted in separate systems. The improved and subsequent approximations can be obtained in the form of integral expressions. However, in order to represent these approximations graphically, it is necessary to compute convolution integrals, which lead to ordinary integrals such as $\int e^{x^2+\frac{1}{2}x}\mathrm{d}x$ and $\int \frac{1}{x}e^{x^2+\frac{1}{2}x}\mathrm{d}x$.

The integrands of these integrals can be approximated through interpolation, allowing for better approximations in solving the predator-prey problem.

We know that the hunter-prey problem is a nonlinear system and so far no analytical and exact solution has been provided for it and researchers and authors calculate its solutions approximately. Also, since there is no exact analytical solution, the degree of closeness of the solutions and their comparison in terms of accuracy and error used is impossible practically. [18], [13] 6.2. **Exponential Approxmation for Non-Linear Functions.** In this section, we give the linear approximation as exponential approximation for nonlinear functions. Similarly, suppose $x(t)$ is a positive and differentiable function at a point $t = a$, then its linear approximation and exponential approximation are as follows:

$$L(t) = x(a) + x'(a)(t-a), \qquad E(t) = x(a).x^*(a)^{t-a}$$

In fact, this approximation is multiplicative analogy for linear approximation in additive calculus. As we see in Stanly [14] and the works of D. Filip and C. Piatecki [7], for the function $x(t) = \frac{1}{t}$ at the point $t = 2$, linear approximation in additive calculus and exponential approximation in multiplicative calculus are given by:

$$L(t) = x(2) + x'(2)(t-2) = 1 - \frac{1}{4}t$$
$$E(t) = x(2).x^*(2)^{t-2} = \frac{1}{2}e^{-\frac{1}{2}(t-2)} = \frac{1}{2}e^{-\frac{t}{2}+1}$$

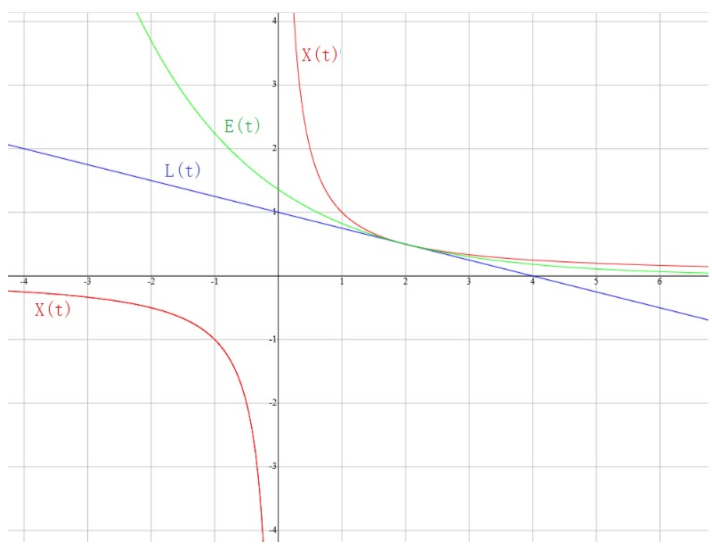

FIGURE 4. $x(t) = \frac{1}{t}$at the point$t = 2$

It is very interesting that the exponential approximation is a more near than the linear one. As it can be seen in the following Figure 4 .

CONCLUSION

As seen in this article, some invariant functions for discrete and continuous multiplicative derivatives were introduced. By using these invariant multiplicative functions, linear and nonlinear difference and differential equations were solved. Analytical and numerical methods were also developed to solve linear and nonlinear differential equations through multiplicative calculations.

The efficiency of multiplicative calculus in analytically solving nonlinear equations and its other applications in other fields of modeling physics and engineering problems, it is expected that this calculus and multiplicative difference and differential equations will play a greater role in phenomena in which changes occur rapidly.

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