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Application of the Quasilinearization Method to Approximate Non-linear Delay Differential Equations in Physical Problems

Fateme Sheikhi¹ and Bahman Ghazanfari^{2*}

ABSTRACT. This paper investigates the application of the Quasilinearization Method (QLM) for approximating non-linear delay differential equations (DDEs), which are prevalent in fields such as control systems and population dynamics. QLM effectively transforms these complex non-linear problems into a system of linear equations, a key advantage for computational efficiency. Our work provides two main contributions: a rigorous mathematical proof demonstrating the quadratic convergence of the proposed technique and numerical examples that illustrate its practical applicability and reliability. We apply QLM to DDEs with various non-linear forms, including quadratic and exponential types and with fixed, discrete delays. The results confirm that the method is highly accurate, computationally efficient and easy to implement, making it a valuable tool for future research.

1. INTRODUCTION

Differential equations with delays, commonly-known as delay differential equations (DDEs) are extensively utilized in various fields, including control systems, population dynamics and engineering sciences [6, 16, 18, 19, 24]. The initial value problem for the non-linear delay differential equation is presented in Equation (1.1), which consists of the main differential equation and its initial function defined over a specific

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interval as follows:

$$(1.1) \quad \begin{cases} u^{(n)}(r) = F(r, u(r), u'(r), \dots, u^{(n-1)}(r), u(r - \tau_0), \\ \quad u'(r - \tau_1), \dots, u^{(n-1)}(r - \tau_{n-1})), & r \geq r_0, \\ u(r) = \phi(r), & r \leq r_0, \end{cases}$$

where $0 < \tau_0 \leq \tau_1 \leq \dots \leq \tau_{n-1}$ and $r_0 = 0$.

Delay differential equations are characterized by their dependence on prior values of the solution and its derivatives, along with an initial function instead of the conventional initial conditions. This dependence on prior values, defined by the delay terms, introduces a considerable challenge in analyzing and numerically solving these types of equations. Various methods have been documented in the literature for solving these equations. Senu et al. [26] used the two-derivative Runge-Kutta type method with Newton interpolation, while Chatzarakis et al. [8] solved these types of equations using scillation criteria. Bellour et al. in [4] applied the Taylor collocation method. Hoo et al. [13] solved these equations using the direct two-point block method. Kocak et al. in [17] employed the homotopy perturbation and variational iteration methods. Jaaffar et al. [14] proposed a fifth-order direct multistep block method. Also, Aibinu solved the delay differential equations via Sumudu transforms [1] and Rihan utilized the numerical method based on modified ODE formulae [25].

Among these methods, the QLM effectively approximates solutions to equations of the form (1.1), particularly those arising in physical contexts. The QLM has been applied to a variety of non-linear ODEs [20–23, 27, 28].

However, its application to DDEs requires careful consideration of the specific challenges created by the delay terms. We demonstrate that with a sufficient number of iterations, it is possible to achieve efficient numerical results with rapid convergence.

Generally, this article focuses on the application of the QLM to the approximation of delay differential equations. While some articles have applied this method to other types of equations, its application to physical delay differential equations has been limited.

The framework of the paper is as follows:

In Section 2, describe of the technique. Section 3 presents a brief discussion on convergence analysis. In Section 4, we present examples that illustrate the effectiveness of the proposed algorithm. Finally, the last Section provides a conclusion.

2. DESCRIPTION OF THE TECHNIQUE

The QLM was originally developed by Bellman and Kalaba [15]. The main idea of this technique is to replace a non-linear differential equation with a sequence of linear differential equations. Nowadays, this method has been utilized and further expanded by various researchers [20, 21, 23, 27, 28].

Now, we consider the non-linear delay differential equation (1.1) as a general DDE of order n , while F is a non-linear function and $\tau_0, \tau_1, \dots, \tau_{n-1}$ are the delays. To simplify, we use the following symbols:

$$(2.1) \quad \begin{aligned} u_0(r) &= u(r), u_1(r) = u'(r), \dots, u_{n-1}(r) = u^{(n-1)}(r), \\ u_{0\tau_0}(r) &= u(r - \tau_0), u_{1\tau_1}(r) = u(r - \tau_1), \\ &\vdots \\ u_{(n-1)\tau_{n-1}}(r) &= u(r - \tau_{n-1}). \end{aligned}$$

So, by using these expressions, we can rewrite equation (1.1) as follows:

$$(2.2) \quad L^n u(r) = F(r, u_0(r), \dots, u_{n-1}(r), u_{0\tau_0}(r), \dots, u_{(n-1)\tau_{n-1}}(r)) = 0.$$

On the other hand, we choose initial guesses for $u_0(r), u_1(r), \dots, u_{n-1}(r)$ respectively as follows:

$$(2.3) \quad u_{0,0}(r), u_{1,0}(r), \dots, u_{(n-1),0}(r).$$

By using the Quasilinearization method for solving (2.2), we obtain the iterative approximation $u_{n+1}(r)$ as a solution of the following linear delay differential equation:

$$(2.4) \quad \begin{aligned} L^n u_{k+1}(r) &= F(r, u_{0,k}(r), u_{1,k}(r), \dots, u_{n-1\tau_{n-1},k}(r)) \\ &+ \sum_{i=0}^{n-1} (u_{i,(k+1)}(r) - u_{i,k}(r)) \frac{\partial F}{\partial u_i}(r, u_{0,k}(r), u_{1,k}(r), \dots, u_{n-1\tau_{n-1},k}(r)) \\ &+ \sum_{i=0}^{n-1} (u_{i\tau_i,k+1}(r) - u_{i\tau_i,k}(r)) \frac{\partial F}{\partial u_{i\tau_i}}(r, u_{0,k}(r), u_{1,k}(r), \dots, u_{n-1\tau_{n-1},k}(r)). \end{aligned}$$

In the interval $0 \leq r \leq \tau_0$, we obtain:

$$(2.5) \quad \begin{aligned} u(r - \tau_0) &= \phi(r - \tau_0), \\ u'(r - \tau_0) &= \phi'(r - \tau_0), \\ &\vdots \end{aligned}$$

Then, the QLM is applied for $\tau_0 \leq r \leq \tau_j$, where j is the smallest index such that $\tau_0 \neq \tau_j$. This procedure is continued iteratively until an approximate solution of (1.1) is obtained.

3. CONVERGENCE ANALYSIS

The convergence of the method will be briefly demonstrated in this section [3]. This proof will require the Green's function $G(r, \theta)$, which for the homogeneous system:

$$(3.1) \quad y_{k+1}''(r) = 0, \quad y_{k+1}(0) = y_{k+1}(r_f) = 0,$$

can be obtained easily [9, 10, 29]:

$$(3.2) \quad G(r, \theta) = \frac{r(\theta - r_f)}{r_f}, \quad 0 \leq r \leq \theta,$$

$$(3.3) \quad G(r, \theta) = \frac{\theta(r - r_f)}{r_f}, \quad \theta \leq r \leq r_f.$$

To address the issue of convergence, it can be shown that the sequence of functions $u_k(r)$ converges to the solution of the original equation if the value of r_f or the length of the interval $[0, r_f]$ is sufficiently small. For simplicity, the sequence defined by the following system will be considered:

$$(3.4) \quad u_{k+1}''(r) = f(u_k(r)) + (u_{k+1}(r) - u_k(r))f_{u_k(r)}(u_k(r))$$

$$(3.5) \quad u_{k+1}(0) = u_{k+1}(r_f) = 0.$$

Our proof follows that of Kalaba [15]. Equation (3.4) can be written as:

$$(3.6) \quad u_k''(r) = f(u_{k-1}(r)) + (u_k(r) - u_{k-1}(r))f_{u_{k-1}(r)}(u_{k-1}(r)).$$

By subtracting equation (3.4) from equation (3.6), we obtain:

$$(3.7) \quad u_{k+1}''(r) - u_k''(r) = f(u_k(r)) - f(u_{k-1}(r)) \\ - (u_k(r) - u_{k-1}(r))f_{u_{k-1}(r)}(u_{k-1}(r)) \\ + (u_{k+1}(r) - u_k(r))f_{u_k(r)}(u_k(r)),$$

where the term $(u_{k+1}(r) - u_k(r))$ represents the difference between successive approximations and can be considered as the unknown in this context. Using the mean value theorem, we can demonstrate that:

$$(3.8) \quad f(u_k(r)) - f(u_{k-1}(r)) - (u_k(r) - u_{k-1}(r))f_{u_{k-1}(r)}(u_{k-1}(r)) \\ = \frac{1}{2}(u_k(r) - u_{k-1}(r))^2 f_{vv}(v),$$

where v lies between $u_{k-1}(r)$ and $u_k(r)$. By substituting equation (3.8) to the equation (3.7) and converting it into an integral equation, we find

that equation (3.7) becomes:

$$(3.9) \quad u_{k+1}(r) - u_k(r) = \int_0^{r_f} G(r, \theta) \left[\frac{1}{2} (u_k(r) - u_{k-1}(r))^2 f_{vv}(v) + (u_{k+1}(r) - u_k(r)) f_{u_k(r)}(u_k(r)) \right] d\theta.$$

Consider the absolute values and let $m_1 = \max |f_{vv}(v)|$ and the larger value of $\max |f(u(r))|$ and $\max |f(u_k(r))|$ by m , so we get:

$$(3.10) \quad |u_{k+1}(r) - u_k(r)| \leq \frac{r_f}{4} \int_0^{r_f} \left[\frac{m_1}{2} (u_k(r) - u_{k-1}(r))^2 + m |u_{k+1}(r) - u_k(r)| \right] d\theta.$$

Now, the equation (3.10) can be rearranged:

$$(3.11) \quad |u_{k+1}(r) - u_k(r)| \leq \left(\frac{m_1 \frac{r_f^2}{8}}{1 - m \frac{r_f^2}{4}} \right) (u_k(r) - u_{k-1}(r))^2 = \kappa (|u_k(r) - u_{k-1}(r)|)^2.$$

The above equation shows that the convergence is quadratic if there is convergence at all. Rewrite equation (3.11) as follows:

$$(3.12) \quad \begin{aligned} |u_{k+1}(r) - u_k(r)| &\leq \kappa (|u_k(r) - u_{k-1}(r)|)^2, \\ |u_k(r) - u_{k-1}(r)| &\leq \kappa (|u_{k-1}(r) - u_{k-2}(r)|)^2, \\ |u_{k-1}(r) - u_{k-2}(r)| &\leq \kappa (|u_{k-2}(r) - u_{k-3}(r)|)^2, \\ &\vdots \\ |u_2(r) - u_1(r)| &\leq \kappa (|u_1(r) - u_0(r)|)^2. \end{aligned}$$

By the use of simple substitution, we have:

$$(3.13) \quad \begin{aligned} |u_{k+1}(r) - u_k(r)| &\leq \kappa (|u_k(r) - u_{k-1}(r)|)^2 \\ &\leq \kappa \left(\kappa (|u_{k-1}(r) - u_{k-2}(r)|)^2 \right)^2 \\ &\leq \dots \end{aligned}$$

that the following inequality can be obtained:

$$(3.14) \quad \begin{aligned} |u_{k+1}(r) - u_k(r)| &\leq \kappa \left[\kappa^{2^k - 2} (|u_1(r) - u_0(r)|)^{2^k} \right] \\ &= \frac{[\kappa (u_1(r) - u_0(r))]^{2^k}}{\kappa}. \end{aligned}$$

If the quantity $[\kappa(u_1(r) - u_0(r))] < 1$ holds, the right-hand side of (3.14) will approach zero as k increases. Therefore, $u_k(r)$ will approach a function $u(r)$ an equation (3.2) is reduced to:

$$(3.15) \quad u(r) = \int_0^{r_f} G(r, \theta) f(u(\theta)) d\theta.$$

Thus, the function $u(r)$ satisfies the original equation. Observe that if The value of r , or the length of the interval $[0, r]$ is sufficiently small, the quantity

$$\kappa(|u_1(r) - u_0(r)|)$$

will be less than one. It is interesting to further observe that this quantity also depends on the maximum value of

$$|u_1(r) - u_0(r)|.$$

Thus, if the interval $[0, r_f]$ is too large, we can always, at least theoretically, choose a better initial approximation $u_0(r)$ so that the maximum difference between the absolute values of $u_1(r)$ and $u_0(r)$, will be small enough to make

$$[\kappa(|u_1(r) - u_0(r)|)]$$

less than one.

4. NUMERICAL EXAMPLES

To demonstrate the applicability of the presented numerical method, we consider some examples.

Example 4.1. Consider the following delay differential equation:

$$(4.1) \quad u'(r) + u''(r - 5) = -gu^n(r)u'(r - 5), \quad u(0) = 1$$

subject to

$$(4.2) \quad u(r) = 1 + r, \quad r \leq 0.$$

For $0 \leq r < 5$, we can obtain $-5 \leq r - 5 < 0$. So, the following relation can be concluded from relation (4.2):

$$(4.3) \quad u'(r - 5) = 1 \Rightarrow u''(r - 5) = 0.$$

By substituting (4.3) in (4.1), we get the following equation:

$$(4.4) \quad u'(r) = -g \cdot u^n(r), \quad u(0) = 1,$$

which is a non-linear ordinary differential equation suggested in [5].

In this situation, we have:

$$(4.5) \quad u(r) = \frac{1}{(1 + (n - 1)gr)^{-\frac{1}{n-1}}}.$$

Now, for $5 \leq r < 10$, we have $0 \leq r - 5 < 5$. So, by using relation (4.5), we obtain:

$$(4.6) \quad u'(r - 5) = f'(r - 5), \quad \Rightarrow \quad u''(r - 5) = f''(r - 5).$$

Then by substituting equation (4.6) into equation (4.1), we have:

$$(4.7) \quad u'(r) + f''(r - 5) = -gu^n(r)f'(r - 5).$$

We get the initial condition for (4.7) by substituting $r = 5$ in equation (4.5). Then, by applying equation (2.4) to equations (4.4) and (4.7) respectively, we get:

$$(4.8) \quad u'_{k+1}(r) + ngu_k^{n-1}(r)u_{k+1}(r) = (n - 1)gu_k^n(r), \quad u_{k+1}(0) = 1,$$

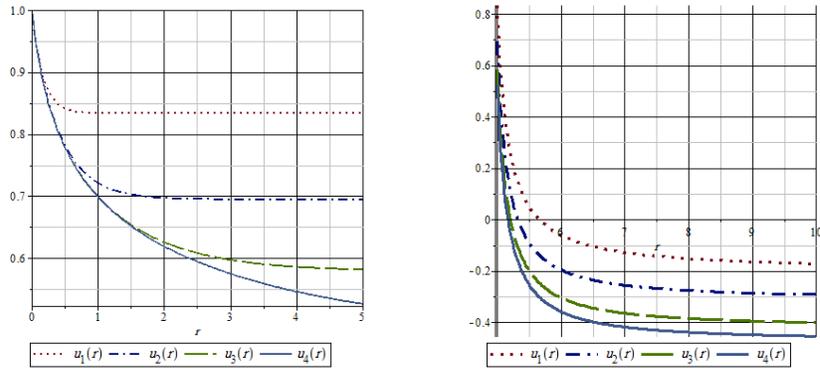
and

$$(4.9) \quad u'_{k+1}(r) = -f''(r - 5) - gu_k^n(r) + (u_{k+1}(r) - u_k(r)) (ngu_k^{n-1}(r)f'(r - 5)),$$

$$u_{k+1}(5) = 0.5212011081.$$

The given boundary value will be used as the initial guess for the relations (4.8) and (4.9) i.e., $u_0(r) = 1$. These equations are solved and the results are plotted in Figure 1 (a) for $0 \leq r < 5$, (b) for $5 \leq r < 10$ and in Figure 2 for $0 \leq r < 10$.

As shown in Figures 1 and 2, the Quasilinearization Method (QLM) demonstrates rapid convergence. The difference between successive iterations approaches zero quickly, with the method achieving a highly accurate solution after only a few iterations. This result confirms the effectiveness of our approach for this specific type of non-linear delay differential equation.



(A) The obtained graphs for $0 \leq r < 5$. (B) The obtained graphs for $5 \leq r < 10$.

FIGURE 1. Numerical solutions of example 4.1.

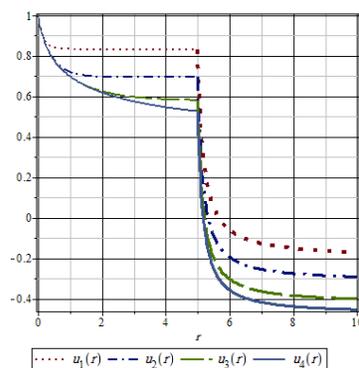


FIGURE 2. The obtained graphs of example 4.1 for $0 \leq r < 10$.

Example 4.2. We analyze the following delay differential equation:

$$(4.10) \quad u''(r) + u(r) + gu^3(r)u(r-3) = 0, \quad u'(0) = 0,$$

subject to

$$(4.11) \quad u(r) = 1; \quad r \leq 0.$$

For $0 \leq r \leq 3$, we have $-3 \leq r-3 < 0$. Then, as a result of relation (4.11), We get:

$$(4.12) \quad u(r-3) = 1.$$

By substituting (4.12) in (4.10), we have:

$$(4.13) \quad u''(r) + u(r) + gu^3(r) = 0, \quad u(0) = 1, \quad u'(0) = 0,$$

which is a form of the classical anharmonic oscillator, a non-linear Duffing equation.

Now, for $3 \leq r < 6$, we have $0 \leq r-3 < 3$, which is outside the range defined in relation (4.11). So, we obtain $u(r)$ using an interpolation method and then we calculate $u(r-3) = q(r)$. Substituting this into (4.10) yields:

$$(4.14) \quad u''(r) + u(r) + gu^3(r)q(r) = 0,$$

with initial conditions $u(3)$ and $u'(3)$ obtained using the interpolation function.

Then, by using equation (2.4) on equations (4.13) and (4.14) Respectively, we have:

$$(4.15) \quad u''_{k+1}(r) + (1 + 3gu_k^2(r))u_{k+1}(r) - 2gu_k^3(r) = 0, \quad u_{k+1}(0) = 1, \quad u'_{k+1}(0) = 0,$$

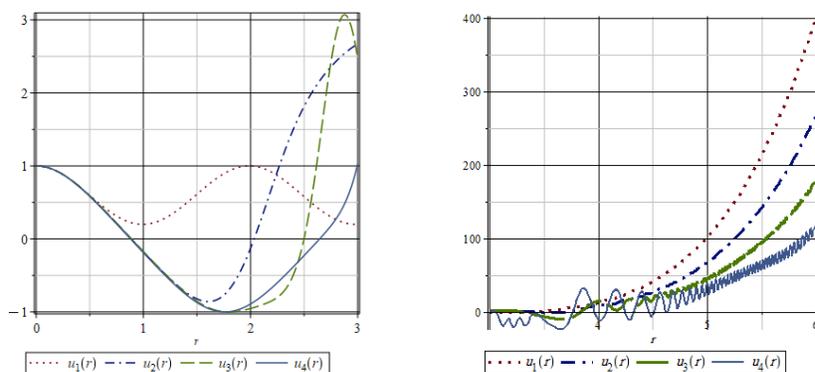
and

$$(4.16) \quad u''_{k+1}(r) + u_{k+1}(r) + 3gu_k^2(r)u_{k+1}(r)q(r) - 2gu_k^3(r)q(r) = 0,$$

$$u_{k+1}(3) = 0.552323766, \quad u'_{k+1}(3) = 1.149835921.$$

The initial guess for the iterative process in (4.15) is $u_0(r) = 1$. The presented equations are solved with five iterations and the results are plotted in Figure 3 (a) for $0 \leq r < 3$, (b) for $3 \leq r < 6$ and in Figure 4 for $0 \leq r < 6$.

As shown in Figures 3 and 4, the Quasilinearization Method (QLM) demonstrates rapid convergence. The difference between successive iterations approaches zero quickly, with the method achieving a highly accurate solution after only a few iterations. This result confirms the effectiveness of our approach for this specific type of non-linear delay differential equation.



(A) The obtained graphs for $0 \leq r < 3$. (B) The obtained graphs for $3 \leq r < 6$.

FIGURE 3. Numerical solutions of Example 4.2.

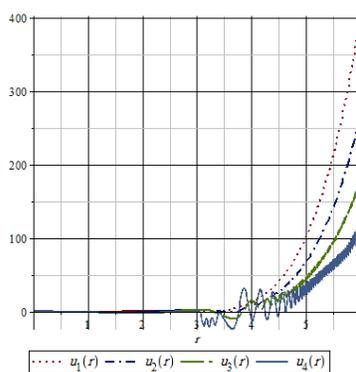


FIGURE 4. The obtained graphs of example 4.2 for $0 \leq r < 6$.

Example 4.3. In this example, consider the following third-order delay differential equation:

$$(4.17) \quad u'''(r) + \frac{1}{2}u''(r)u(r)u'(r-4) = 0,$$

subject to

$$(4.18) \quad u(r) = \frac{1}{16}r^2(r-4), \quad r \leq 0.$$

For $r \leq 0$, from (4.18) we have the initial conditions at $r = 0$:

$$(4.19) \quad u(0) = 0, \quad u'(0) = 0.$$

The value $u'(4) = 1$ can also be calculated.

For $0 \leq r < 4$, we have $-4 \leq r - 4 < 0$. Thus, by relation 4.18, we can obtain:

$$(4.20) \quad u'(r-4) = 1.$$

By substituting (4.20) in equation (4.17), we get:

$$(4.21) \quad u'''(r) + \frac{1}{2}u''(r)u(r) = 0, \quad u(0) = u'(0) = 0, \quad u'(1) = 1,$$

which is a form of the non-linear Blasius equation.

Now, for $4 \leq r < 8$, we have $0 \leq r - 4 < 4$, which is outside the range defined in (4.18). So, we obtain $u(r)$ using an interpolation method and then we calculate $u(r-4) = q(r)$ and $u'(r-4) = q'(r)$. Then, by substituting these quantities in equation (4.17), we obtain:

$$(4.22) \quad u'''(r) + \frac{1}{2}u''(r)u(r)q'(r) = 0, \quad u(4) = u'(4) = 0, \quad u'(8) = 1.$$

Then, by using equation (2.4) in equations (4.21) and (4.22) Respectively, we have:

$$(4.23) \quad \begin{aligned} u'''_{k+1}(r) + u_k(r)u''_{k+1}(r) + u_{k+1}(r)u''_k(r) - u_k(r)u''_k(r) &= 0, \\ u_{k+1}(0) = u'_{k+1}(0) = 0, \quad u'_{k+1}(4) &= 1, \end{aligned}$$

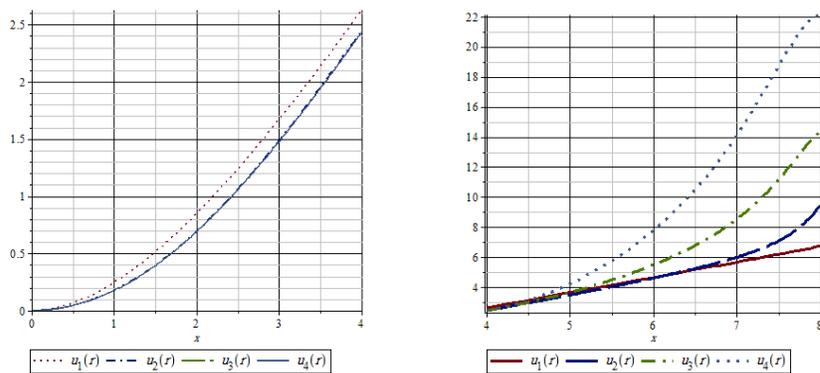
and

$$(4.24) \quad \begin{aligned} u'''_{k+1}(r) &= -\frac{1}{2}u''_k(r)u_k(r)q'(r) + (u_{k+1}(r) - u_k(r)) \left(-\frac{1}{2}u''_k(r)q'(r) \right) \\ &\quad + (u''_{k+1}(r) - u''_k(r)) \left(-\frac{1}{2}u_k(r)q'(r) \right), \\ u_{k+1}(4) = u'_{k+1}(4) &= 0, \quad u'_{k+1}(8) = 1. \end{aligned}$$

In this context, the initial estimation for the equations (4.23) and (4.24) is chosen to be $u_0(r) = 1$. These equations are solved and at last, the

results are plotted in Figure 5 (a) for $0 \leq r < 4$, (b) for $4 \leq r < 8$ and in Figure 6 for $0 \leq r < 8$.

As shown in Figures 5 and 6, the Quasilinearization Method (QLM) demonstrates rapid convergence. The difference between successive iterations approaches zero quickly, with the method achieving a highly accurate solution after only a few iterations. This result confirms the effectiveness of our approach for this specific type of non-linear delay differential equation.



(A) The obtained graphs for $0 \leq r < 4$. (B) The obtained graphs for $4 \leq r < 8$.

FIGURE 5. Numerical solutions of Example 4.3.

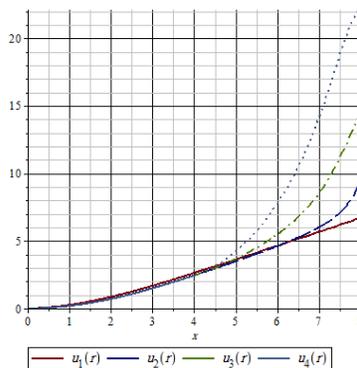


FIGURE 6. The obtained graphs of Example 4.3 for $0 \leq r < 8$.

Example 4.4. The delay Lane-Emden type equation is as follows:

$$(4.25) \quad u''(r) + \frac{u^m(r)}{r^{m-1}} + u(r-1) = 0$$

with initial conditions:

$$(4.26) \quad u(0) = 0, \quad u'(0) = 1,$$

and defined $u(r)$ for $r \leq 0$. In this type of equation, the parameter m is related to the equation of state, with a physical range of $0 \leq m \leq 5$.

In this example, we set $m = 4$. Thus, the equation becomes:

$$(4.27) \quad u''(r) + \frac{u^4(r)}{r^3} + u(r-1) = 0, \quad u(0) = 0, \quad u'(0) = 1,$$

Subject to:

$$(4.28) \quad u(r) = r, \quad r \leq 0.$$

For $0 < r \leq 1$, we have $-1 < r-1 \leq 0$. According to relation (4.28), we have:

$$(4.29) \quad u(r-1) = r.$$

By substituting (4.29) in equation (4.27), we get:

$$(4.30) \quad u''(r) + \frac{u^4(r)}{r^3} + r = 0, \quad u(0) = 0, \quad u'(0) = 1.$$

Now, for $1 < r \leq 2$, we have $0 < r-1 \leq 1$, which is outside the range defined in (4.28). So, we obtain $u(r)$ on this interval. Then, we calculate $u(r-1) = p(r)$ and by substituting this into equation (4.27), we get:

$$(4.31) \quad u''(r) + \frac{u^4(r)}{r^3} + p(r) = 0, \quad u(1) = 0, \quad u'(1) = 1.$$

Applying equation (2.4) to equations (4.30) and (4.31) respectively yields:

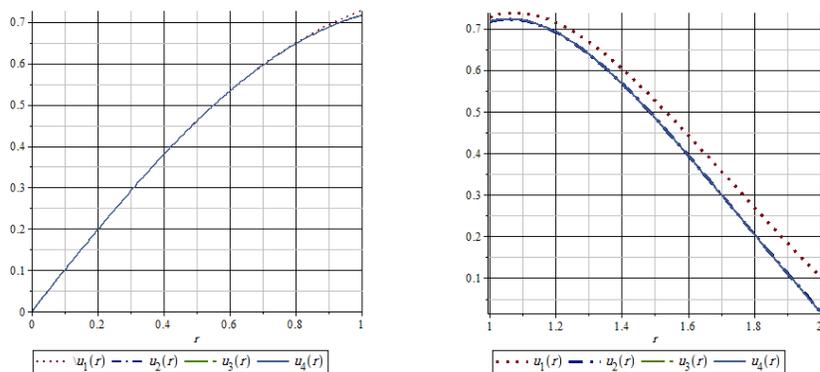
$$(4.32) \quad u''_{k+1}(r) - 3\frac{u_k^4(r)}{r^3} + 4\frac{u_k^3(r)}{r^3}u_{k+1}(r) + r = 0, \\ u_{k+1}(0) = 0, \quad u'_{k+1}(0) = 1,$$

and

$$(4.33) \quad u''_{k+1}(r) - 3\frac{u_k^4(r)}{r^3} + 4\frac{u_k^3(r)}{r^3}u_{k+1}(r) + p(r) = 0, \\ u_{k+1}(1) = 0.7164238617, \quad u'_{k+1}(1) = 0.2241501898.$$

Finally, the results obtained from solving these equations are reported in Figure 7 (a) for $0 \leq r < 1$, (b) for $1 \leq r < 2$ and in Figure 8 for $0 \leq r < 2$.

As shown in Figures 7 and 8, the Quasilinearization Method (QLM) demonstrates rapid convergence. The difference between successive iterations approaches zero quickly, with the method achieving a highly accurate solution after only a few iterations. This result confirms the effectiveness of our approach for this specific type of non-linear delay differential equation.



(A) The obtained graphs for $0 \leq r < 1$. (B) The obtained graphs for $1 \leq r < 2$.

FIGURE 7. Numerical solutions of Example 4.4.

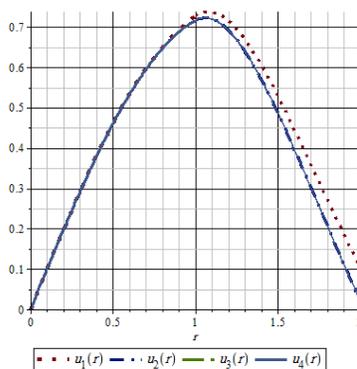


FIGURE 8. The obtained graphs of Example 4.4 for $0 \leq r < 2$.

Example 4.5. We consider $m = 5$ for the Lane-Emden equation presented in previous example. Therefore, we have equations (4.25)-(4.26) as follows:

$$(4.34) \quad u''(r) + \frac{u^5(r)}{r^4} + u(r-1) = 0, \quad u'(0) = 1,$$

subject to

$$(4.35) \quad u(r) = 0, \quad r \leq 0.$$

For $0 < r \leq 1$, we have $-1 < r-1 \leq 0$. Thus, from relation (4.35), we get:

$$(4.36) \quad u(r-1) = 0.$$

Now, by substituting (4.36) into equation (4.34), we get:

$$(4.37) \quad u''(r) + \frac{u^5(r)}{r^4} = 0, \quad u(0) = 0, \quad u'(0) = 1.$$

For $1 < r \leq 2$, we have $0 < r - 1 \leq 1$, which is outside the range defined in (4.35). Therefore, using the solution obtained for equation (4.37), we calculate $u(r - 1) = g(r)$ and substitute it into (4.34) to get:

$$(4.38) \quad u''(r) + \frac{u^5(r)}{r^4} + g(r) = 0, \quad u(0) = 0, \quad u'(0) = 1.$$

Applying equation (2.4) to equations (4.37) and (4.38) yields:

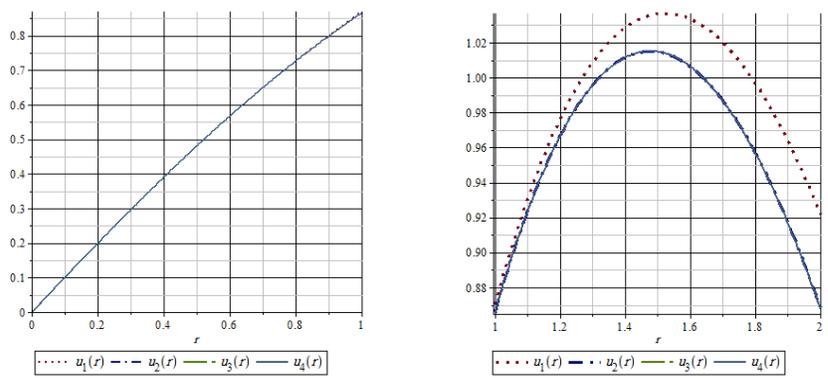
$$(4.39) \quad u''_{k+1}(r) + 5 \frac{u_k^4(r)}{r^4} u_{k+1}(r) - 4 \frac{u_k^5(r)}{r^4} = 0, \quad u_{k+1}(0) = 0, \quad u'_{k+1}(0) = 1,$$

and

$$(4.40) \quad u''_{k+1}(r) + 5 \frac{u_k^4(r)}{r^4} u_{k+1}(r) - 4 \frac{u_k^5(r)}{r^4} + \frac{1}{\sqrt{1 + \frac{(r-1)^2}{3}}} = 0,$$

$$u_{k+1}(1) = 0.866025566, \quad u'_{k+1}(1) = 0.649879038.$$

Finally, the results obtained from solving these equations are reported in Figure 9 (a) for $0 \leq r < 1$, (b) for $1 \leq r < 2$ and for $0 \leq r < 2$ in Figure 10. As shown in Figures 9 and 10, the Quasilinearization Method (QLM) demonstrates rapid convergence. The difference between successive iterations approaches zero quickly, with the method achieving a highly accurate solution after only a few iterations. This result confirms the effectiveness of our approach for this specific type of non-linear delay differential equation.



(A) The obtained graphs for $0 \leq r < 1$. (B) The obtained graphs for $1 \leq r < 2$.

FIGURE 9. Numerical solutions of example 4.5.

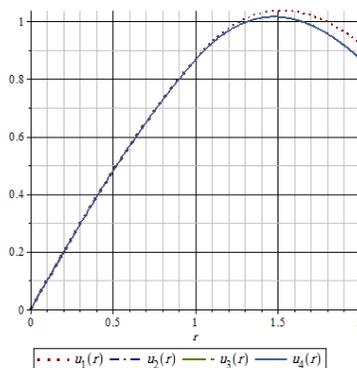


FIGURE 10. The obtained graphs of example 4.5 for $0 \leq r < 2$.

5. CONCLUSION

In this study, we successfully applied the Quasilinearization Method (QLM) to a new class of non-linear delay differential equations (DDEs) prevalent in various physical applications. Our approach effectively transforms these complex non-linear problems into a sequence of linear differential equations, a key process that ensures computational efficiency. We provided a rigorous mathematical proof demonstrating the quadratic convergence of our method and our numerical results consistently confirmed its high accuracy and straightforward implementation. These findings establish QLM as a valuable and robust tool for solving this class of equations.

While our method demonstrates exceptional performance, it is important to acknowledge its limitations. The current approach is primarily suitable for a specific class of non-linearities and constant delays within the scope of our numerical examples. Further research is required to explore its computational performance when dealing with more complex non-linear terms or state-dependent delays over larger time intervals.

Furthermore, our work opens several promising avenues for future research [2, 7, 11, 12, 30, 31]. One significant direction involves extending the application of the QLM to DDEs formulated using other generalized differential operators, such as the local generalized derivative or the conformable derivative. The use of these alternative operators could offer new insights into the behavior of complex systems and broaden the applicability of the QLM approach. In summary, our findings establish QLM as a powerful and innovative tool for analyzing delay differential equations, which enables researchers to address a wider variety of physical problems using more accurate mathematical models.

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