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Korovkin-type Theorems via Statistical Derivatives of Deferred Nörlund Summability

Suresh Chandra Mahapatra[1], Bidu Bhusan Jena[2*], Susanta Kumar Paikray[3] and M. Mursaleen[4,5,6]

Abstract. This paper introduces and explores the concept of statistical derivatives within the framework of deferred Nörlund summability, complemented by illustrative examples. Leveraging this approach, we establish a new Korovkin-type theorem for a specific class of algebraic test functions, namely 1, x and x^2, within the Banach space $\mathfrak{C}[0,1]$. Our findings serve as a significant generalization of several classical and statistical Korovkin-type results in approximation theory. Furthermore, we examine the rate of convergence associated with statistical derivatives under deferred Nörlund summability, providing insights into the effectiveness of this summability method. To validate our theoretical results, we present numerical examples alongside graphical visualizations created using MATLAB, offering a clearer perspective on the convergence behavior of the proposed operators.

1. Introduction

In mathematical analysis, fundamental concepts such as limits, continuity and derivatives have been extensively studied for centuries, forming the core principles of calculus and advanced analysis [31]. Over time, these foundational ideas have been expanded and refined to address more complex mathematical problems, leading to the development of new tools and techniques. One such advancement is the study of Cesàro

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limits, a generalization in summability theory that has contributed significantly to understanding the convergence behavior of sequences and series [13]. Building upon these developments, researchers have explored various summability methods, deepening our comprehension of function behavior and extending classical results in analysis [24]. In particular, the work of Connor and Grosse-Erdmann [9] has provided significant contributions by investigating Cesàro continuity, along with statistical limits and statistical continuity. These studies have played a crucial role in broadening the theoretical framework of summability, offering new perspectives on function and sequence convergence. The reader may also consult the recent monographs [4, 5], along with the references cited therein, which focus on spaces of double sequences generated through the domain of specific four-dimensional triangular matrices and related areas. Additionally, further insights into newly introduced double sequence spaces can be found in the research articles [45–47].

Despite these advancements, certain aspects of summability theory remain underdeveloped. In particular, the notions of the deferred Nörlund derivative and the statistical deferred Nörlund derivative have not been extensively examined. These concepts provide a refined approach to summability by introducing a more intricate mechanism for analyzing function convergence, especially in the context of statistical and Nörlund summability. Exploring these ideas can lead to a deeper understanding of the approximation properties of functions and operators.

Approximation theory plays a fundamental role in mathematical analysis, particularly in understanding how well certain operators can approximate functions within a given function space. One of the most significant results in this field is Korovkin's theorem, which provides conditions under which a sequence of positive linear operators converges uniformly to a function. Over the years, many researchers have extended Korovkin-type theorems using different types of summability methods to improve convergence properties. In this direction, statistical summability methods have gained considerable attention due to their effectiveness in dealing with non-traditional convergence sequences.

This paper aims to bridge this gap by formally introducing the deferred Nörlund derivative and the statistical deferred Nörlund derivative within the framework of deferred Nörlund summability technique. We present illustrative examples that demonstrate their effectiveness in approximating functions. Using this methodology, we establish a new Korovkin-type theorem applicable to a specific class of algebraic test functions, namely 1, x and x^2 within the Banach space $[0, 1]$. Our results significantly generalize existing Korovkin-type theorems, providing a broader perspective on function approximation. Additionally, we analyze the convergence rate of statistical derivatives within the framework of deferred Nörlund summability technique. To support our theoretical findings, we include numerical examples along with MATLAB-generated graphical illustrations, offering a visual representation of the convergence behavior of the proposed operators.

2. Preliminaries

Let $\sum_{k=0}^{\infty} a_k$ be a given series and (s_k) be the sequence of its partial sums. Let (p_k) be a sequence of real numbers and $P_k = \sum_{i=0}^{k} p_i$. We suppose that $P_k \neq 0$ for all k. The series $\sum a_k$ is said to be summable (N, p_k) to s when $\lim_{k\to\infty} t_k$ exists and its is equal to s, where

$$t_k = \frac{1}{P_k} \sum_{i=0}^{k} p_{k-i} s_i.$$

Here, the sequence (t_k) is called the k^{th} (N, p_k) mean or k th Nörlund mean.

Let us recall the deferred Nörlund mean (see, Agnew [1], Srivastava *et al.* [42]) of a sequence. Let (α_k) and (β_k) be the sequences in $\mathbb{Z}^{0+}$ with

$$\alpha_k < \beta_k,\ k \in \mathbb{N}, \qquad \lim_{k\to\infty} \beta_k = \infty,$$

leading to the deferred Nörlund $DN(\alpha_k, \beta_k)$-mean of the following form

$$DN(\alpha_k, \beta_k) = \frac{1}{P_k} \sum_{i=\alpha_k+1}^{\beta_k} p_{\beta_k - i} s_i.$$

In the special case with $p_k = \binom{k+\alpha-1}{k} = A_k^{\alpha-1}, \alpha > 0$, the deferred Nörlund mean reduces to the deferred Cesàro mean of order α, (C, α) (see [13]).

The condition for the regularity of summability (N, p_k) is $\frac{p_k}{P_k} \to 0$ and $\sum_{i=0}^{k} |p_i| = O(|P_k|)$ as $k \to \infty$.

If (p_k) is a positive sequence and $p_k = \frac{1}{k+1}$, then it is also easy to see that, the deferred Nörlund mean (see [42]) reduces to the deferred Cesàro mean of order 1 (see [17].

In recent years, statistical convergence has gained significant prominence within modern mathematical discussions, often being favored over traditional point-wise convergence. This shift is largely due to the foundational work of two eminent mathematicians, Fast [11] and Steinhaus [44], whose research has notably expanded the framework and applicability of convergence theory. Today, statistical convergence is extensively utilized across a wide spectrum of disciplines in both pure and applied mathematics, as well as in analytical statistics. Its relevance is particularly evident in fields such as Number Theory, Probability Theory, Machine Learning, Soft Computing and Measure Theory. For a deeper understanding, readers are encouraged to refer to [14, 15].

Next, the natural density of a subset S of natural numbers measures how "large" the set is in terms of its proportion among the first k natural numbers as k grows indefinitely. Formally, the natural density of S is defined as the limit (if it exists)

$$d(S) = \lim_{k\to\infty} \frac{|\{i \leqq k : k \in S\}|}{k},$$

where $|\{i \leqq k : k \in S\}|$ represents the number of elements of S that are less than or equal to k.

In the context of statistical convergence (see [11, 44]), a sequence (a_k) is said to be statistically convergent to ℓ if the set of indices

$$S_\epsilon = \{k \in \mathbb{N} : |a_k - \ell| \geqq \epsilon\}$$

has natural density zero for every $\epsilon > 0$. This means that the proportion of terms in the sequence that deviate significantly from ℓ becomes negligible in the long run, even if such deviations occur infinitely often. Thus, statistical convergence provides a more flexible notion of convergence, allowing occasional large deviations as long as they form an asymptotically small fraction of the sequence.

We now recall the definition of deferred Nörlund statistically convergence criterion for real sequences.

Definition 2.1 ([42])**.** Let (α_k) and (β_k) be the sequences in $\mathbb{Z}^{0+}$. A sequence (a_k) is said to be deferred Nörlund statistically convergent to a finite number ℓ if, for every $\epsilon > 0$

$$\mathcal{F}_\epsilon = \{i : i \leqq P_k \text{ and } p_{\beta_k - i}\,|a_i - \ell| \geqq \epsilon\}$$

ensures zero asymptotic density (see [17]), that is, for every $\epsilon > 0$

$$\delta\big(\mathcal{F}_\epsilon\big) = \frac{|\mathcal{F}_\epsilon|}{P_k} = 0, \quad \text{as}\, k \to \infty.$$

We write

$$\mathrm{stat_{DN}} \lim_{k\to\infty} a_k = \ell.$$

During the latter half of the twentieth century, statistical convergence became a topic of significant interest among mathematicians. One of the early contributors, Šalát [34], examined the boundedness criteria related to statistical convergence and analyzed the behavior of real-number sequences under this framework. Around the same time, Fast [11] introduced a precise definition of statistical Cauchy sequences and derived fundamental results using summability techniques. Building upon these foundational studies, Maddox [23] extended the concept to locally convex spaces, providing crucial insights into statistical convergence in broader mathematical structures. As research in this domain continued to evolve, further advancements were made by Mursaleen and Belen [12], who explored lacunary statistical summability for real sequences. Their work introduced new methodologies that refined existing summability techniques, contributing to a deeper understanding of statistical approximation. These investigations collectively laid the groundwork for modern summability theory, inspiring further research into statistical limits, function approximation and their applications in mathematical analysis. The concept of statistical Cesàro summability was first introduced by Móricz [27], who laid the foundation for its theoretical development and practical applications. Building upon this, Mohiuddine et al. [26] made significant contributions by establishing Korovkin-type theorems within the framework of statistical Cesàro summability, further enhancing its role in approximation theory. The study of statistical convergence via weighted summability means gained prominence through the work of Karakaya and Chishti [20], who played a key role in formalizing and popularizing this approach. Subsequently, Mursaleen et al. [28] refined these ideas by proving essential limit theorems, strengthening the theoretical underpinnings of statistical summability. In more recent research, Baliarsingh [3] introduced and explored a statistically deferred A-convergence framework for uncertain sequences. His work contributed various inclusion theorems, offering new perspectives on convergence analysis and expanding the applicability of statistical summability in mathematical analysis. In 2021, Saini et al. [33] explored equi-statistical convergence in relation to deferred Cesàro and deferred Euler summability product means, establishing new Korovkin-type theorems in this framework. Later, they revisited the topic in Saini et al. [32], extending their findings to applications involving deferred Riesz statistical convergence for complex uncertain sequences. Further advancements in the field were made by Sharma et al. [35], who demonstrated statistical deferred Cesàro convergence for fuzzy-number-valued sequences of order (ξ, ω) showcasing its practical applications. Around the same time, Srivastava et al. [42] investigated equi-statistical convergence using the deferred Nörlund means, broadening the scope of statistical summability techniques. Subsequently, Parida et al. [30] established Korovkin-type theorems for sequences exhibiting equi-statistical convergence through deferred Cesàro means, reinforcing the connection between statistical summability and function approximation. More recently, Srivastava et al. [36] introduced an innovative approach by analyzing equi-statistical convergence using the power-series technique, proving specific approximation results that further expand the theoretical landscape of statistical summability.

3. Deferred Nörlund Derivative Mean

For a function $g : \mathbb{R} \to \mathbb{R}$ and a positive sequence (a_k) satisfying $\lim_{k\to\infty} a_k = 0$, there exists two primary formulas to define the derivative of g at a point a_0 as follows:

$$g'(a_0) = \lim_{k\to\infty} \frac{g(a_0 + a_k) - g(a_0)}{a_k}$$
$$g'(a_0) = \lim_{k\to\infty} \frac{g(a_0 + a_k) - g(a_0 - a_k)}{2a_k}.$$

The first formula represents Newton's difference quotient, which measures the slope of a secant line passing through the function's graph. This approach forms the foundation of classical differentiation by approximating the instantaneous rate of change. The second formula, known as the symmetric difference quotient, calculates the slope of a chord connecting two points on the graph of g symmetrically around a_0. This method often provides a more accurate numerical approximation of the derivative, especially in computational applications.

Adopting a similar approach, we proceed to define the deferred Nörlund derivative.

Definition 3.1. Let (α_k) and (β_k) be the sequences in $\mathbb{Z}^{0+}$ and let (p_k) be a sequence of non-negative numbers with $P_k = \sum_{i=\alpha_k+1}^{\beta_k} p_i$. A function $g : \mathbb{R} \to \mathbb{R}$ possesses a deferred Nörlund derivative (DND) $\ell \in \mathbb{R}$ at a point $a_0 \in \mathbb{R}$ if, for $a_0 > 0$ and $\lim_{k\to\infty} a_k = 0$, the following condition holds:

$$\lim_{k\to\infty} \frac{1}{P_k} \sum_{i=\alpha_k+1}^{\beta_k} p_{\beta_k - i} \left(\frac{g(a_0 + a_i) - g(a_0)}{a_i} \right) = \ell.$$

To illustrate the concept of a deferred Nörlund derivative (DND), we proceed with the following example.

Example 3.2. Let's consider the function $g(x) = x^2$ and analyze its DND at $a_0 = 1$. We choose $a_k = \frac{1}{k}$ so that $a_k = 0$. Define the sequence (α_k) and (β_k) as $\alpha_k = k$, $\beta_k = 2k$. Let $p_i = 1$ for all i, which results in

$P_k = \sum_{i=k+1}^{2k} p_i = k$. Next, the difference quotient is given by

$$\frac{g(a_0 + a_i) - g(a_0)}{a_i} = \frac{(1 + \frac{1}{i})^2 - 1^2}{\frac{1}{i}} = 2 + \frac{1}{i}.$$

The weighted sum over the range $i = k + 1$ to $2k$ is:

$$\begin{aligned}
\frac{1}{P_k} \sum_{i=k+1}^{2k} p_{\beta_k - i} \left(\frac{g(a_0 + a_i) - g(a_0)}{a_i} \right) &= \frac{1}{k} \sum_{i=k+1}^{2k} \left(2 + \frac{1}{i} \right) \\
&= \frac{1}{k} \sum_{i=k+1}^{2k} 2 + \frac{1}{k} \sum_{i=k+1}^{2k} \frac{1}{i} \\
&= \frac{1}{k} 2k + \frac{1}{k} ln(2) \\
&= 2 + 0 \\
&= 2.
\end{aligned}$$

Thus, the deferred Nörlund derivative at $a_0 = 1$ is 2.

We now visualize the behaviour of the Deferred Nörlund Derivative of the Function in Figure 1.

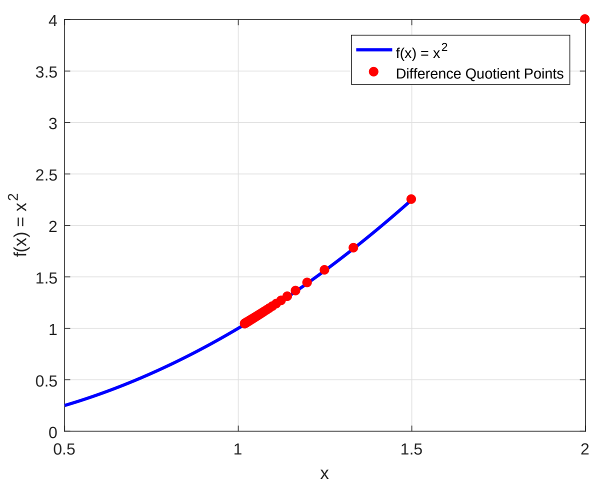

FIGURE 1. Visualization of Deferred Nörlund Derivative of $g(x) = x^2$

Figure 1 illustrates key aspects of the deferred Nörlund derivative (DND) and its computational behavior. The blue curve represents the function $g(x) = x^2$, highlighting its behavior in the vicinity of the point $a_0 = 1$. This visualization is useful for understanding local variations of the function, which are central to the concept of differentiation. The red scatter points in the figure correspond to $(a_0+a_k, g(a_0+a_k))$,indicating the specific locations where the difference quotients are calculated. As the index k increases, these points move closer to a_0, demonstrating the refinement of the approximation. Consequently, the values of the difference quotients converge to the actual derivative $\ell = 2$ as $k \to \infty$.

This behavior reveals that, although the DND employs an averaging mechanism via Nörlund means, it still converges to the classical derivative under suitable conditions. Therefore, the DND framework introduces a more flexible approach to differentiation, especially useful in scenarios where the classical derivative may not exist. Figure 1 effectively visualizes this idea by showcasing the progression of the computed slopes and their convergence behavior.

An equivalent definition in line with the above Definition 3.1 is presented below.

Definition 3.3. Let (α_k) and (β_k) be the sequences in $\mathbb{Z}^{0+}$ and let (p_k) be a sequence of non-negative numbers with $P_k = \sum_{i=\alpha_k+1}^{\beta_k} p_i$. A function $g : \mathbb{R} \to \mathbb{R}$ possesses a deferred Nörlund derivative (DND) $\ell \in \mathbb{R}$ at a point $a_0 \in \mathbb{R}$ if, for $a_0 > 0$ and $\lim_{k\to\infty} a_k = 0$, the following condition holds:

$$\lim_{k\to\infty} \frac{1}{P_k} \sum_{i=\alpha_k+1}^{\beta_k} p_{\beta_k - i} \left(\frac{g(a_0+a_i) - g(a_0-a_i)}{2a_i} \right) = \ell.$$

Next, we propose the definition of statistical deferred Nörlund derivative as follows.

Definition 3.4. Let (α_k) and (β_k) be the sequences in $\mathbb{Z}^{0+}$ and let (p_k) be a sequence of non-negative numbers with $P_k = \sum_{i=\alpha_k+1}^{\beta_k} p_i$. A function $g : \mathbb{R} \to \mathbb{R}$ possesses a statistical deferred Nörlund derivative (SDND) $\ell \in \mathbb{R}$ at a point $a_0 \in \mathbb{R}$ if, for $a_0 > 0$ and $\lim_{k\to\infty} a_k = 0$, the following condition holds:

$$\lim_{k\to\infty} \frac{1}{P_k} \left| \left\{ i : i \leqq P_k \ \text{ and } \ p_{\beta_k - i} \left| \frac{g(a_0+a_k) - g(a_0)}{a_k} - \ell \right| \geqq \epsilon \right\} \right| = 0.$$

We may write

$$\mathrm{SDND}_{\mathrm{stat}} g'(a_k) = \ell.$$

To construct such an example, we consider a function that has oscillatory behavior near a_0, where the usual statistical derivative fails due to irregular fluctuations, but the statistical deferred Nörlund derivative (SDND) can still exist due to its smoothing effect.

Example 3.5. Consider the function $g(x) = x^2 + \sin(1/x)$ for $x > 0$. At $a_0 = 1$, we analyze whether the statistical derivative and the statistical deferred Nörlund derivative exist. The statistical derivative requires the difference quotient,

$$\frac{g(a_0 + a_k) - g(a_0)}{a_k} = \frac{(1 + a_k)^2 + \sin(1/(1 + a_k)) - 1^2 - \sin(1)}{a_k} = \frac{2a_k + a_k^2 + \sin(1/(1 + a_k)) - \sin(1)}{a_k}.$$

For large k, the term $\sin(1/(1 + a_k))$ oscillates between -1 and 1, which prevents the statistical derivative from existing, as the density of deviations away from any limit is nonzero. However, the statistical deferred Nörlund derivative (SDND) exists. Using deferred summation, we define $a_k = 1/k$ so that $\lim a_k = 0$ as $k \to \infty$ and choose $\alpha_k = k$ and $\beta_k = 2k$. Assign weights p_i, leading to $P_k = \sum_{i=k+1}^{2k} p_i = k$. Applying the SDND condition, we have

$$\lim_{k\to\infty} \frac{1}{P_k} \left| \left\{ i : i \leqq P_k \text{ and } p_{\beta_k - i} \left| \frac{g(a_0 + a_k) - g(a_0)}{a_k} - \ell \right| \geqq \epsilon \right\} \right| = 0.$$

Since averaging smooths out the oscillations of $\sin(1/(1 + a_k))$, the fluctuations diminish in density, allowing the SDND to converge to $\ell = 2$.

We now visualize the behaviour of the Deferred Nörlund Derivative of the Function in Figure 2.

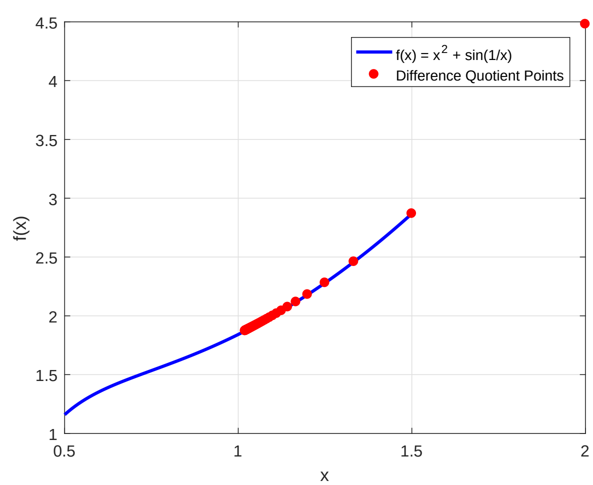

FIGURE 2. Visualization of Deferred Nörlund Derivative of $g(x) = x^2 + sin(1/x)$

Figure 2 presents important insights into the function $g(x) = x^2 + \sin(1/x)$ and demonstrates how the statistical deferred Nörlund derivative (SDND) yields a stable derivative where the standard statistical derivative fails. The blue curve represents the graph of $g(x)$, which features rapid oscillations near the origin due to the $\sin(1/x)$ term. These high-frequency oscillations lead to erratic behavior in the difference quotient, making the classical statistical derivative undefined. The red scatter points indicate the locations at which the difference quotients are computed. Because of the oscillatory nature of $\sin(1/x)$, the computed slopes fluctuate unpredictably at increasingly smaller scales. As a result, no limiting value emerges for the difference quotient and the statistical derivative fails to exist in the usual sense. This failure arises from the fact that standard statistical differentiation depends on density-based convergence, which breaks down in the presence of persistent oscillations.

In contrast, the deferred Nörlund method utilizes Nörlund averaging, which effectively smooths the fluctuations by weighting the difference quotients over a sequence. Although the individual slopes remain highly variable, the averaged sum stabilizes, enabling the SDND to converge to a finite value. Thus, Figure 2 highlights the strength of the SDND approach in handling functions with severe local oscillations, offering a generalized and robust notion of differentiation where traditional methods fall short.

An equivalent definition in line with the above Definition 3.4 is presented below.

Definition 3.6. Let (α_k) and (β_k) be the sequences in $\mathbb{Z}^{0+}$ and let (p_k) be a sequence of non-negative numbers with $P_k = \sum_{i=\alpha_k+1}^{\beta_k} p_i$. A function $g : \mathbb{R} \to \mathbb{R}$ possesses a statistical deferred Nörlund derivative (SDND) $\ell \in \mathbb{R}$ at a point $a_0 \in \mathbb{R}$ if, for $a_0 > 0$ and $\lim_{k\to\infty} a_k = 0$, the following condition holds:

$$\lim_{k\to\infty} \frac{1}{P_k} \left| \left\{ i : i \leqq P_k \text{ and } p_{\beta_k - i} \left| \frac{g(a_0 + a_k) - g(a_0 - a_k)}{2a_k} - \ell \right| \geqq \epsilon \right\} \right| = 0.$$

We may write $\mathrm{stat}_{\mathrm{SDND}} g'(a_k) = \ell$.

The motivation of the study of function approximation plays a crucial role in mathematical analysis, particularly in the development of summability techniques that enhance convergence properties. Korovkin-type theorems provide powerful tools for verifying the convergence of operators, making them essential in approximation theory. However, existing results often rely on classical summability methods, which may not fully capture the behavior of certain sequences. To address this limitation, we introduce statistical derivatives within the deferred Nörlund summability framework, offering a more refined approach. By establishing a new Korovkin-type theorem and analyzing convergence rates, this study aims to extend classical results while providing practical insights through numerical and graphical validation.

4. A Korovkin-type Approximation Theorem

Over the years, mathematicians have devoted significant effort to extending and generalizing Korovkin-type theorems across diverse mathematical settings. These generalizations have been applied to a wide spectrum of contexts, including function spaces, Banach algebras, abstract Banach lattices and Banach spaces. The versatility of these results has made them instrumental in numerous fields, such as partial differential equations, measure theory, optimization, machine learning, data science and functional analysis.

Recent studies have further explored Korovkin-type approximations within the framework of various statistical convergence techniques, as discussed in works such as [10, 14, 16, 37–41]. A notable advancement was made by Srivastava et al. [43], who introduced a more refined result by employing equi-statistical convergence instead of uniform statistical convergence. Additionally, numerous researchers have utilized the concept of equi-statistical convergence to derive distinct results in various mathematical frameworks, as documented in [7, 8, 19, 25, 29, 30, 42].

Building on these developments, this study focuses on investigating the proposed statistical deferred Nörlund derivative means to establish a Korovkin-type theorem in this specific direction of research.

The space $\mathfrak{C}[0,1]$ consists of all continuous functions (real-valued) defined in the interval $[0,1]$, appareled with the norm $\|.\|_\infty$. This norm, defined for $g \in \mathfrak{C}[0,1]$ is the supremum (or maximum) of the absolute values of $g(x)$ over all $x \in [0,1]$ is given by

$$\|g\|_\infty = \sup_{x\in[0,1]} \{|g(x)|\}.$$

Next, for a function $g \in \mathfrak{C}[0,1]$, the modulus of continuity symbolized by $\omega(\delta, g)$ is defined as the supremum of the norm of the difference of g evaluated at $x+y$ and x for all x and y within the range $0 \leqq |y| \leqq \delta$ such that

$$\omega(\delta, g) = \sup_{0\leqq|y|\leqq\delta} \|g(x+y) - g(x)\|_{[0,1]}, \quad g \in \mathfrak{C}[0,1].$$

Here, the norm and modulus of continuity are essential tools for analyzing and measuring the behavior and continuity of functions within the Banach space $\mathfrak{C}[0,1]$.

Let $\mathfrak{R} : \mathfrak{C}[0,1] \to \mathfrak{C}[0,1]$ be a linear operator. In the standard terminology, we say $\mathfrak{R}$ is a positive linear operator under the condition $x \geq 0$ implies $\mathfrak{R}(g) \geqq 0$. The value of $\mathfrak{R}(g)$ at a point $x \in [0,1]$ is denoted by $\mathfrak{R}(g(x); x)$ or simply $\mathfrak{R}(g; x)$.

The classical Korovkin-type theorem stated in [22] is formulated as follows.

Let $\mathfrak{R}_k : \mathfrak{C}[a,b] \to \mathfrak{C}[a,b]$ be a positive linear operators sequence. Then $\lim_{k\to\infty} \|\mathfrak{R}_k(g;x) - g(x)\|_\infty = 0, (\forall g \in \mathfrak{C}[a,b])$ if and only if $\lim_{k\to\infty} \|\mathfrak{R}_k(g_j;x) - g_j(x)\|_\infty = 0, \quad (j = 0,1,2)$ where

$$g_0(x) = 1, \qquad g_1(x) = x, \qquad g_2(x) = x^2.$$

The statistical deferred Nörlund summability version of the theorem settled by Srivastava et al. in [42] is stated as follows.

Let $\mathfrak{R}_k : \mathfrak{C}[0,1] \to \mathfrak{C}[0,1]$ be a sequence of positive linear operators. The statement

$$\text{DN(stat)} \lim_{k\to\infty} \|\mathfrak{R}_k(g;x) - g(x)\|_{[0,1]} = 0, \quad (\forall\, g \in \mathfrak{C}[0,1])$$

holds if and only if

$$\text{DN(stat)} \lim_{k\to\infty} \|\mathfrak{R}_k(g_j;x) - g_j(x)\|_{[0,1]} = 0, \quad (j = 0,1,2)$$

where

$$g_0(x) = 1, \qquad g_1(x) = x, \qquad g_2(x) = x^2.$$

We establish the following theorem by employing the concept of the statistical deferred Nörlund derivative technique.

Theorem 4.1. *Let $\mathfrak{R}_k : \mathfrak{C}[0,1] \to \mathfrak{C}[0,1]$ be a positive linear operators sequence. For all $g \in \mathfrak{C}[0,\infty)$,*

$$\lim_{k\to\infty} \text{SDWD}_{\text{stat}} \; \|\mathfrak{R}_k(g;x) - g(x)\|_{[0,1]} = 0, \tag{4.1}$$

if and only if

$$\lim_{k\to\infty} \text{SDWD}_{\text{stat}} \; \|\mathfrak{R}_k(1;x) - 1\|_{[0,1]} = 0, \tag{4.2}$$

$$\lim_{k\to\infty} \text{SDWD}_{\text{stat}} \; \|\mathfrak{R}_k(t;x) - x\|_{[0,1]} = 0, \tag{4.3}$$

$$\lim_{k\to\infty} \text{SDWD}_{\text{stat}} \; \|\mathfrak{R}_k\left(t^2;x\right) - x^2\|_{[0,1]} = 0. \tag{4.4}$$

Proof. Given that each function $g_j(x) = 1,\ x,\ x^2$, where $j = 0,1,2$, belongs to $\mathfrak{C}(X = [0,1])$, thus the implications from (4.1) to (4.4) are evidently true.

To finalize the proof of the theorem, we initially assume the validity of (4.2) to (4.4). Consider $g \in \mathfrak{C}[X]$, where there exists a positive constant $\mathfrak{K}$ such that

$$|g(x)| \leq \mathfrak{K}, \quad \forall x \in X = [0,1].$$

Thus, for $t, x \in X$, $|g(t) - g(x)| \leq 2\mathfrak{K}$.

Consequently, for given $\epsilon > 0$, there may exists $\delta > 0$ such that

$$|g(t) - g(x)| < \epsilon \tag{4.5}$$

whenever $|t - x| < \delta$, $(\forall\, t, x \in X)$.

Let us write $\chi = \chi(t, x) = (t - x)^2$. For $|t - x| \leqq \delta$, we have

$$|g(t) - g(x)| < \frac{2\mathfrak{K}}{\delta^2}\chi(t, x). \tag{4.6}$$

From equation (4.5) and (4.6), we get

$$|g(t) - g(x)| < \epsilon + \frac{2\mathfrak{K}}{\delta^2}\chi(t, x),$$

this implies that

$$-\epsilon - \frac{2\mathfrak{K}}{\delta^2}\chi(t, x) \leqq g(t) - g(x) \leqq \epsilon + \frac{2\mathfrak{K}}{\delta^2}\chi(t, x). \tag{4.7}$$

Since $\mathfrak{R}_k(1, x)$ is monotone and linear, on application of the operator $\mathfrak{R}_k(1, x)$ to this inequality yields

$$\begin{aligned} \mathfrak{R}_k(1, x)\left(-\epsilon - \frac{2\mathfrak{K}}{\delta^2}\chi(t, x)\right) &\leqq \mathfrak{R}_k(1, x)(g(t) - g(x)) \\ &\leqq \mathfrak{R}_k(1, x)\left(\epsilon + \frac{2\mathfrak{K}}{\delta^2}\chi(t, x)\right). \end{aligned} \tag{4.8}$$

Since x is fixed, $g(x)$ is a constant number. Therefore,

$$-\epsilon\mathfrak{R}_k(g, \xi) - g(\xi)\mathfrak{R}_k(1, x) \leqq \epsilon\mathfrak{R}_k(1, x) + \frac{2\mathfrak{K}}{\delta^2}\mathfrak{R}_k(\chi, x). \tag{4.9}$$

But

$$\mathfrak{R}_k(g, x) - g(x) = [\mathfrak{R}_k(g, x) - g(x)\mathfrak{R}_k(1, x)] + g(x)[\mathfrak{R}_k(1, x) - 1]. \tag{4.10}$$

Using (4.9) and (4.10), we have

$$\mathfrak{R}_k(g, x) - g(x) < \epsilon\mathfrak{R}_k(1, x) + \frac{2\mathfrak{K}}{\delta^2}\mathfrak{R}_k(\chi, x) + g(x)[\mathfrak{R}_k(1, x) - 1]. \tag{4.11}$$

We now estimate $\mathfrak{R}_k(\chi, x)$ as,

$$\begin{aligned} \mathfrak{R}_k(\chi; x) &= \mathfrak{R}_k((t - x)^2; x) = \mathfrak{R}_k\left(t^2 - 2tx + x^2; x\right) \\ &= \mathfrak{R}_k\left(t^2; x\right) - 2x\mathfrak{R}_k(t; x) + x^2\mathfrak{R}_k(1; x) \\ &= \left[\mathfrak{R}_k\left(t^2; x\right) - x^2\right] - 2x\left[\mathfrak{R}_k(t; x) - x\right] + x^2\left[\mathfrak{R}_k(1; x) - x^2\right]. \end{aligned}$$

Using (4.11), we obtain

$$\begin{aligned} \mathfrak{R}_k(g, x) - g(x) < \epsilon\mathfrak{R}_k(1, x) + \frac{2\mathfrak{K}}{\delta^2}\Big\{ &\left[\mathfrak{R}_k\left(t^2; x\right) - x^2\right] - 2x\left[\mathfrak{R}_k(t; x) - x\right] \\ &+ x^2\left[\mathfrak{R}_k(1; x) - x^2\right] \Big\} + g(x)[\mathfrak{R}_k(1; x) - 1]. \end{aligned}$$

$$= \epsilon[\mathfrak{R}_k(1;x)-1]+\epsilon+\frac{2\mathfrak{K}}{\delta^2}\Big\{\left[\mathfrak{R}_k\left(t^2;x\right)-x^2\right]$$
$$-2x\left[\mathfrak{R}_k(t;x)-x\right]+x^2[\mathfrak{R}_k(1,x)-1]\Big\}$$
$$+g(x)[\mathfrak{R}_k(1,x)-1].$$

Since ϵ is arbitrary, we write

(4.12)

$$|\mathfrak{R}_k(g,x)-g(x)| \leqq \epsilon+\left(\epsilon+\frac{2\mathfrak{K}}{\delta^2}+\mathfrak{K}\right)|\mathfrak{R}_k(1,x)-1|$$
$$+\frac{4\mathfrak{K}}{\delta^2}\left|\mathfrak{R}_k(t;x)-x\right|+\frac{2\mathfrak{K}}{\delta^2}\left|\mathfrak{R}_k\left(t^2;x\right)-x^2\right|$$
$$\leqq \mathfrak{B}\left(|\mathfrak{R}_k(1;x)-1|+|\mathfrak{R}_k(t;x)-x|+\left|\mathfrak{R}_k\left(t^2;x\right)-x^2\right|\right),$$

where

$$\mathfrak{B}=\max\left(\epsilon+\frac{2\mathfrak{K}}{\delta^2}+\mathfrak{K}, \frac{4\mathfrak{K}}{\delta^2}, \frac{2\mathfrak{K}}{\delta^2}\right).$$

Next, for a given $r>0$, there exists $\epsilon>0$, such that $\epsilon<r$. Then, by setting

$$\Gamma_k(x,r)=\left\{i : i \leqq P_k \text{ and } p_{\beta_k-i}\left|\frac{\mathfrak{R}_k(g,x)-g(x)}{a_k}-\ell\right| \geqq r\right\},$$

and

$$\Gamma_{j,k}(x,r)=\left\{i : i \leqq P_k \text{ and } p_{\beta_k-i}\left|\frac{\mathfrak{R}_k(g_j,x)-g_j(x)}{a_k}-\ell\right| \geqq \frac{r-\epsilon}{3\mathfrak{K}}\right\}$$

with $a_0>0$ and $a_k \to 0$ as $k\to\infty$, we obtain

$$\Gamma_k(x,r) \leqq \sum_{j=0}^{2}\Gamma_{j,k}(x,r).$$

Clearly,

$$\frac{\|\Gamma_k(x,r)\|_{\mathfrak{C}(X)}}{P_k} \leqq \sum_{j=0}^{2}\frac{\|\Gamma_{j,k}(x,r)\|_{\mathfrak{C}(X)}}{P_k}. \tag{4.13}$$

Applying the assumption mentioned earlier regarding the implications from (4.2) to (4.4) and utilizing Definition 3.4, it becomes evident that the right-hand side of (4.13) tends to zero as k approaches infinity. Consequently, we get

$$\lim_{k\to\infty}\frac{\|\Gamma_k(x,r)\|_{\mathfrak{C}(X)}}{P_k}=0, \quad (r>0).$$

Hence, the assertion in (4.1) is valid. This concludes the proof of Theorem 4.1. □

Taking into account Theorem 4.1, we established a Corollary (below) by using the Definition 3.6.

Corollary 4.2. *Let* $\mathfrak{R}_k : \mathfrak{C}[0,1] \to \mathfrak{C}[0,1]$ *be a positive linear operators sequence. For all* $g \in \mathfrak{C}[0,\infty)$,

$$\lim_{k\to\infty} \text{stat}_{\text{SDND}} \ \|\mathfrak{R}_k(g;x) - g(x)\|_{[0,1]} = 0,$$

if and only if

$$\lim_{k\to\infty} \text{stat}_{\text{SDND}} \ \|\mathfrak{R}_k(1;x) - 1\|_{[0,1]} = 0,$$
$$\lim_{k\to\infty} \text{stat}_{\text{SDND}} \ \|\mathfrak{R}_k(t;x) - x\|_{[0,1]} = 0,$$
$$\lim_{k\to\infty} \text{stat}_{\text{SDND}} \ \|\mathfrak{R}_k\left(t^2;x\right) - x^2\|_{[0,1]} = 0.$$

Proof. The proof for Corollary 4.2 being closely resembles to that of Theorem 4.1. Thus, we opt to omit the intricate details. □

5. Numerical and Geometrical Analysis of Theorem 4.1

We provide an example below under a particular class of positive linear operators known as the Bernstein polynomials and the differential operators, in light of our Theorem 4.1.

Example 5.1. Consider the *Bernstein polynomials* $\mathfrak{G}_k(g;x)$ on $\mathfrak{C}[0,1]$ given by

(5.1)
$$\mathfrak{G}_k(g;x) = \sum_{i=0}^{k} g\left(\frac{i}{k}\right)\binom{k}{i} x^i (1-x)^{k-i}, \quad (x \in [0,1]; k = 0,1,\ldots).$$

Here, in this example, we introduce the positive linear operators on $\mathfrak{C}[0,1]$ under the composition of the Bernstein polynomials and the differential operators

$$x\left(1 + x\frac{d}{dx}\right), \tag{5.2}$$

which was used by Al-Salam [2] as follows:

$$\mathfrak{R}_k(g;x) = [1+g(x)]x\left(1 + x\frac{d}{dx}\right)\mathfrak{G}_k(g;x), \quad (\forall g \in \mathfrak{C}[0,1]), \tag{5.3}$$

where (g) is the same as mentioned in Example 5.1.

We now estimate the values of each of the testing functions 1, x and x^2 by using our proposed operators (5.3) as follows:

$$\mathfrak{R}_k(1;x) = [1+g(x)]x\left(1 + x\frac{d}{dx}\right)1 = [1+g(x)]x,$$

$$\mathfrak{R}_k(x;x) = [1+g]x\left(1+x\frac{d}{dx}\right)x = [1+g]x(1+x)$$

and

$$\begin{aligned}\mathfrak{R}_k\left(x^2;x\right) &= [1+g]x\left(1+x\frac{d}{dx}\right)\left\{x^2+\frac{x(1-x)}{k}\right\}\\ &= [1+g]\left\{x^2\left(2-\frac{3x}{k}\right)\right\}.\end{aligned}$$

So that we obtain,

$$\begin{aligned}&\lim_{k\to\infty}\mathrm{SDND}_{\mathrm{stat}}\ \ \|\mathfrak{R}_k(1;x)-1\|_{[0,1]}=0,\\ &\lim_{k\to\infty}\mathrm{SDND}_{\mathrm{stat}}\ \ \|\mathfrak{R}_k(t;x)-x\|_{[0,1]}=0,\\ &\lim_{k\to\infty}\mathrm{SDND}_{\mathrm{stat}}\ \ \left\|\mathfrak{R}_k\left(t^2;x\right)-x^2\right\|_{[0,1]}=0,\end{aligned}$$

that is, $\mathfrak{R}_k(g;x)$ exhilarates the conditions (4.2) to (4.4). Therefore by Theorem 4.1, we have

$$\lim_{k\to\infty}\mathrm{SDND}_{\mathrm{stat}}\ \ \|\mathfrak{R}_k(g;x)-g(x)\|_{[0,1]}=0. \tag{5.4}$$

Plotting separate figures for the functions 1, x and x^2 as k increases helps to visualize the uniform convergence of the operator $\mathfrak{R}_k(g;x)$ to $g(x)$. Each plot offers distinct insights into the approximation behavior for these basic functions:

Figure 3 illustrates the convergence of the operator $\mathfrak{R}_k(1;x)$ to the constant function 1 as $k\to\infty$. This figure helps in understanding how the operator maintains the fundamental structure of the function while applying its approximation scheme. As k increases, the modifications introduced by the operator diminish and the approximation steadily approaches the original constant function. This behavior confirms that the operator accurately reproduces constant functions in the limit, showcasing its consistency.

Figure 4 illustrates how effectively $\mathfrak{R}_k(x;x)$ approximates the identity function x. For small values of k, noticeable deviations from the true function may occur. However, as k increases, the approximation becomes increasingly accurate, closely aligning with the original function. This behavior confirms that the operator preserves linearity in the limiting process, reinforcing its validity in approximating linear functions.

The quadratic function serves as a benchmark for testing the approximation capabilities of the operator on higher-degree polynomials. Figure 5 illustrates the convergence of $\mathfrak{R}_k\left(x^2;x\right)$ to the function x^2, with the approximation error diminishing as k increases. Any initial discrepancies observed for small k reflect the level of smoothness preserved by the operator. As k grows, the improved alignment with x^2 confirms the operator's ability to accurately approximate nonlinear functions in the limiting process.

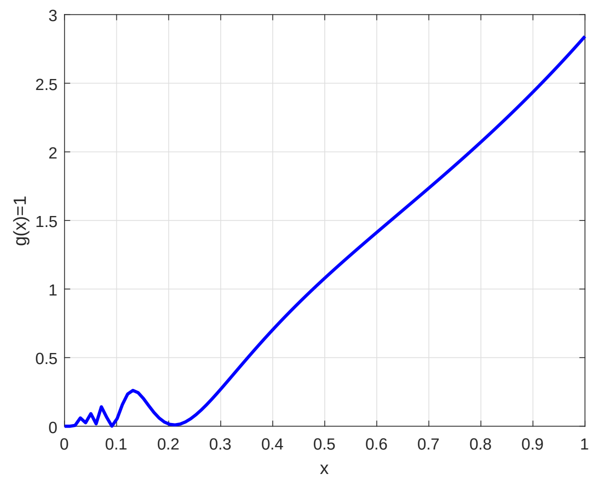

FIGURE 3. Convergence of $R_k(1, x)$

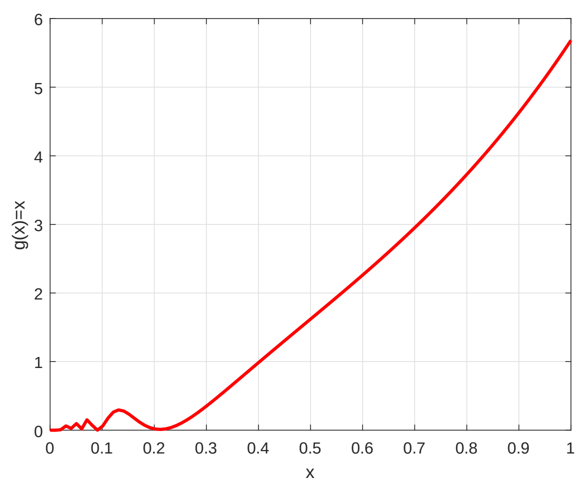

FIGURE 4. Convergence of $R_k(x, x)$

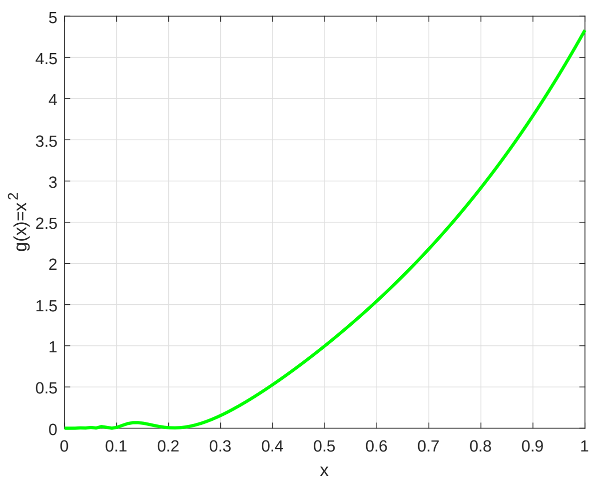

FIGURE 5. Convergence of $R_k(x^2, x)$

Overall, by observing these separate figures, we can confirm that the operator $\mathfrak{R}_k(g; x)$ satisfies the necessary conditions for uniform convergence. The decreasing error across all cases as k increases supports the theoretical result,

$$\lim_{k\to\infty} \text{SDND}_{\text{stat}} \ \|\mathfrak{R}_k(g; x) - g(x)\|_{[0,1]} = 0.$$

These figures provide valuable insight into how effectively $\mathfrak{R}_k(g; x)$ approximates different types of functions and how rapidly convergence is achieved as k increases.

6. Rate of Deferred Nörlund Derivative

In this section, we explore the rates of statistical deferred Nörlund derivatives for a sequence of positive linear operators $\mathfrak{R}(g; x)$ defined on $\mathfrak{C}[0, 1]$. We employ the modulus of continuity in our analysis.

We now present the following definition.

Definition 6.1. Let (α_k) and (β_k) be the sequences in $\mathbb{Z}^{0+}$ and (a_k) be a positive non-increasing sequence. Also, let (p_i) be a non-negative numbers sequence with $P_k = \sum_{i=\alpha_k+1}^{\beta_k} p_i$. A given function $g : \mathbb{R} \to \mathbb{R}$ has statistically deferred Nörlund derivatives ℓ with rate $o(u_k)$, if for every $\epsilon > 0$

$$\lim_{k\to\infty} \frac{1}{a_k P_k} \left| \left\{ i : i \leqq P_k \ \text{ and } \ p_{\beta_k - i} \left| \frac{g(a_0 + a_k) - g(a_0)}{a_k} - \ell \right| \geqq \epsilon \right\} \right| = 0,$$

holds whenever $a_0 > 0$ and $a_k \to 0$ as $k \to \infty$. In this case, we can express

$$g'(a_k) - \ell = \mathrm{SDND}_{\mathrm{stat}} - o(u_k).$$

Let us now proceed to establish the following crucial lemma followed by Theorem 6.3.

Lemma 6.2. *Let* (α_k) and (β_k) *be the sequences in* $\mathbb{Z}^{0+}$. *Also let* (u_k) *and* (v_k) *be positive non-increasing sequences and let* $g(a_k)$ *and* $g(b_k)$ *be two sequences such that*

$$g'(a_k) - \ell = \mathrm{SDND}_{\mathrm{stat}} - o(u_k),$$
$$g'(b_k) - \ell = \mathrm{SDND}_{\mathrm{stat}} - o(v_k).$$

Then the following conditions hold

(i) $(g'(a_k) + g'(b_k)) - (\ell_1 + \ell_2) = \mathrm{SDND}_{\mathrm{stat}} - o(w_k)$,

(ii) $(g'(a_k) - \ell_1)(g'(b_k) - \ell_2) = \mathrm{SDND}_{\mathrm{stat}} - o(u_k v_k)$,

(iii) $\lambda(g'(a_k) - \ell_1) = \mathrm{SDND}_{\mathrm{stat}} - o(u_k)$,

(iv) $\sqrt{|g'(a_k) - \ell_1|} = \mathrm{SDND}_{\mathrm{stat}} - o(u_k)$,

where

$$w_k = \max\{u_k, v_k\}.$$

Proof. In order to prove condition (i) for $\epsilon > 0$ and $x \in [0, 1]$, we define the following sets:

$$\begin{aligned}
&\mathcal{A}_i(a, b; \epsilon)\\
&\quad = \left|\left\{ i : i \leqq P_k \text{ and } p_{\beta_k - i}\left| \left(\frac{g(a_0 + a_k) - g(a_0)}{a_k} + \frac{g(b_0 + b_k) - g(b_0)}{b_k} \right)\right.\right.\right.\\
&\qquad \left.\left.\left. - (\ell_1 + \ell_2)\right| \geqq \epsilon \right\}\right|,
\end{aligned}$$

$$\mathcal{A}_{1,i}(a, \epsilon) = \left|\left\{ i : i \leqq P_k \text{ and } p_{\beta_k - i}\left| \frac{g(a_0 + a_k) - g(a_0)}{a_k} - \ell_1 \right| \geqq \frac{\epsilon}{2} \right\}\right|,$$

$$\mathcal{A}_{2,i}(b, \epsilon) = \left|\left\{ i : i \leqq P_k \text{ and } p_{\beta_k - i}\left| \frac{g(b_0 + b_k) - g(b_0)}{b_k} - \ell_2 \right| \geqq \frac{\epsilon}{2} \right\}\right|.$$

Clearly, we have

$$\mathcal{A}_i(a, b; \epsilon) \subseteq \mathcal{A}_{1,i}(a, \epsilon) \cup \mathcal{A}_{2,i}(b, \epsilon).$$

Since $w_k = \max\{u_k, v_k\}$, by condition (4.1) of Theorem 4.1, we obtain

$$\frac{\|\mathcal{A}_i(a,b;\epsilon)\|_{[0,1]}}{w_k P_k} \leq \frac{\|\mathcal{A}_{1,i}(a,\epsilon)\|_{[0,1]}}{a_k P_k} + \frac{\|\mathcal{A}_{2,i}(b,\epsilon)\|_{[0,1]}}{b_k P_k}.$$

Now, by conditions (4.2) to (4.4) of Theorem 4.1, we obtain

$$\frac{\|\mathcal{A}_i(a,b;\epsilon)\|_{[0,1]}}{w_k P_k} = 0,$$

which establishes (i).

As the proofs of the remaining conditions (ii) to (iv) follow a similar pattern, we choose to omit them. □

Moreover, it is important to note that the modulus of continuity for a function $g \in \mathfrak{C}[0,1]$ is defined as follows:

$$\omega(g,\delta) = \sup_{|t-x|\leq\delta: x,t\in X} |g(t) - g(x)| \quad (\delta > 0).$$

This definition yields

$$|g(t) - g(x)| \leq \omega(g,\delta)\left(\frac{|x-t|}{\delta} + 1\right). \tag{6.1}$$

Theorem 6.3. *Let* $[0,1] \subset \mathbb{R}$ *and let* $\mathfrak{R}_k : \mathfrak{C}[0,1] \to \mathfrak{C}[0,1]$ *be a positive linear operators sequence. Assume that the following conditions hold:*

(i) $\|\mathfrak{R}_k(1,x) - 1\|_{[0,1]} = \mathrm{SDND}_{\mathrm{stat}} - o(u_k)$,

(ii) $\omega(g,\lambda_k) = \mathrm{SDND}_{\mathrm{stat}} - o(v_k)$,

where

$$\lambda_k = \sqrt{\mathfrak{R}_k(\chi^2;x)}, \qquad \chi(t,x) = (x-t)^2.$$

Then, for all $g \in \mathfrak{C}[0,1]$*, the following statement holds*

$$\|\mathfrak{R}_k(g,x) - g\|_{[0,1]} = \mathrm{SDND}_{\mathrm{stat}} - o(w_k), \tag{6.2}$$

$w_k = \max\{u_k, v_k\}$.

Proof. Let $g \in \mathfrak{C}[0,1]$ and $x \in [0,1]$. Using (6.1), we have

$$\begin{aligned}
|\mathfrak{R}_k(g;x) - g(x)| &\leqq \mathfrak{R}_k(|g(t) - g(x)|;x) + |g(x)||\mathfrak{R}_k(1;x) - 1| \\
&\leqq \mathfrak{R}_k\left(\frac{|x-t|}{\lambda_k} + 1; x\right)\omega(g,\lambda_k) + |g(x)||\mathfrak{R}_k(1;x) - 1| \\
&\leqq \mathfrak{R}_k\left(1 + \frac{1}{\lambda_k^2}(x-t)^2; x\right)\omega(g,\lambda_k) + |g(x)||\mathfrak{R}_k(1;x) - 1| \\
&\leqq \left(\mathfrak{R}_k(1;x) + \frac{1}{\lambda_k^2}\mathfrak{R}_k(\chi^2;x)\right)\omega(g,\lambda_k) \\
&\quad + |g(x)||\mathfrak{R}_k(1;x) - 1|.
\end{aligned}$$

Putting $\lambda_k = \sqrt{\mathfrak{R}_k(\chi^2; x)}$, we get

$$\begin{aligned}\|\mathfrak{R}_k(g;x) - g(x)\|_\infty &\leqq 2\omega(g,\lambda_k) + \omega(g,\lambda_k)\|\mathfrak{R}_k(1;x) - 1\|_\infty \\ &\quad + \|g(x)\|\|\mathfrak{R}_k(1;x) - 1\|_{[0,1]} \\ &\leqq \mathfrak{K}\{\omega(g,\lambda_k) + \omega(g,\lambda_k)\|\mathfrak{R}_k(1;x) - 1\|_\infty \\ &\quad + \|\mathfrak{R}_k(1;x) - 1\|_{[0,1]}\},\end{aligned}$$

where $\mathfrak{K} = \{\|g\|_\infty, 2\}$. Thus,

$$\begin{aligned}&\left\|\frac{1}{P_k}\sum_{i=\alpha_k+1}^{\beta_k} p_{\beta_k - i}\left(\mathfrak{R}_k(g;x) - g(x)\right)\right\|_{[0,1]} \\ &\leqq \mathfrak{K}\left\{\omega(g,\lambda_k)\frac{1}{P_k} + \omega(g,\lambda_k)\left\|\frac{1}{P_k}\sum_{i=\alpha_k+1}^{\beta_k} p_{\beta_k - i}\left(\mathfrak{R}_k(g;x) - g(x)\right)\right\|_{[0,1]}\right\} \\ &\quad + \mathfrak{K}\left\{\left\|\frac{1}{P_k}\sum_{i=\alpha_k+1}^{\beta_k} p_{\beta_k - i}\left(\mathfrak{R}_k(g;x) - g(x)\right)\right\|_{[0,1]}\right\}.\end{aligned}$$

Utilizing conditions (i) and (ii) of Theorem 6.3 in conjunction with Lemma 6.2, we establish the correctness of statement (6.2) within Theorem 6.3. Thus, the proof of Theorem 6.3 is completed. □

7. Concluding Remarks and Observations

To conclude our investigation, we present a few final remarks and insights related to the key findings and results discussed throughout this study.

Remark 7.1. Let $g(x)$ be a function given in Example 3.5. As

$$\mathrm{SDND}_{\mathrm{stat}} \lim_{k\to\infty} g'(a_k) \to 2 \text{ on } [0,1]$$

implies

$$\mathrm{SDND}_{\mathrm{stat}} \lim_{k\to\infty} \|\mathfrak{R}_k(g_j, x) - g_j(x)\|_{[0,1]} = 0 \ \ (j = 0,1,2), \tag{7.1}$$

Thus, in view of Theorem 4.1 we write

$$\mathrm{SDND}_{\mathrm{stat}} \lim_{k\to\infty} \|\mathfrak{R}_k(g, x) - g(x)\|_{[0,1]} = 0, \tag{7.2}$$

where

$$g_0(x) = 1, \qquad g_1(x) = x, \qquad g_2(x) = x^2.$$

However, since $g'(a_k)$ does not converge in the classical sense, it also fails to satisfy the conditions for statistical convergence. As a result, the standard Korovkin theorem cannot be employed for the operators defined in (5.3). This scenario clearly demonstrates that Theorem 4.1 offers a meaningful and non-trivial extension of the classical Korovkin-type theorem (see [22]).

Remark 7.2. Let us consider the function $g(x)$, as presented in Example 3.5. Given that

$$\text{SDWDstat} \lim_{k\to\infty} g'(a_k) = 2 \text{ on } [0,1],$$

it follows that equation (7.1) holds true. By applying this result along with Theorem 4.1, we confirm the validity of condition (7.2). Nevertheless, since the sequence $g'(a_k)$ fails to satisfy the criteria for statistical deferred Nörlund convergence, Theorem 1 from the work of Srivastava et al. [42] is not applicable to the operator defined in (5.3). This observation reinforces the fact that Theorem 4.1 constitutes a significant and non-trivial generalization of Theorem 1 by Srivastava et al. [42] (see also [6, 22, 26]). Consequently, these findings substantiate the effectiveness of our proposed approach for the operators in (5.3), thereby illustrating its advantage over earlier Korovkin-type approximation results found in both classical and statistical frameworks, as discussed in [6, 21, 22, 42].

We analyze the deferred Nörlund derivative (DND) for linear, quadratic and trigonometric functions using the given definition:

Remark 7.3. Linear Function: $g(x) = mx + c$

$$\begin{aligned}\frac{g(a_0+a_k)-g(a_0)}{a_k} &= \frac{m(a_0+a_k)+c-(ma_0+c)}{a_k}\\ &= \frac{ma_0+ma_k+c-ma_0-c}{a_k}\\ &= m.\end{aligned}$$

Since m is constant, choosing $\ell = m$ results in:

$$\left|\frac{g(a_0+a_k)-f(a_0)}{a_k} - m\right| = 0.$$

Thus, the SDND is: $\ell = m$.

Here the Figure 6 compute and visualize the statistical deferred Nörlund derivative (SDND) for a linear function $g(x) = mx + c$.

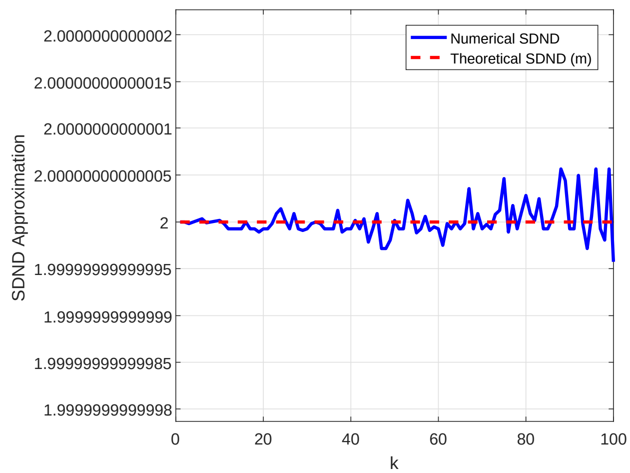

FIGURE 6. SDND of linear function $f(x) = mx + c$

Remark 7.4. Quadratic Function: $g(x) = ax^2 + bx + c$

$$\begin{aligned}
\frac{g(a_0 + a_k) - g(a_0)}{a_k} &= \frac{a(a_0 + a_k)^2 + b(a_0 + a_k) + c - (aa_0^2 + ba_0 + c)}{a_k} \\
&= \frac{aa_0^2 + 2aa_0a_k + aa_k^2 + ba_0 + ba_k + c - aa_0^2 - ba_0 - c}{a_k} \\
&= \frac{2aa_0a_k + aa_k^2 + ba_k}{a_k} \\
&= 2aa_0 + aa_k + b.
\end{aligned}$$

Taking the limit as $k \to \infty$:

$$\ell = 2aa_0 + b.$$

Since $a_k \to 0$, the deviation $|aa_k|$ vanishes, making the SDND:

$$\ell = 2aa_0 + b.$$

Here the Figure 7 compute and visualize the statistical deferred Nörlund derivative (SDND) for a quadratic function $g(x) = ax^2 + bx + c$.

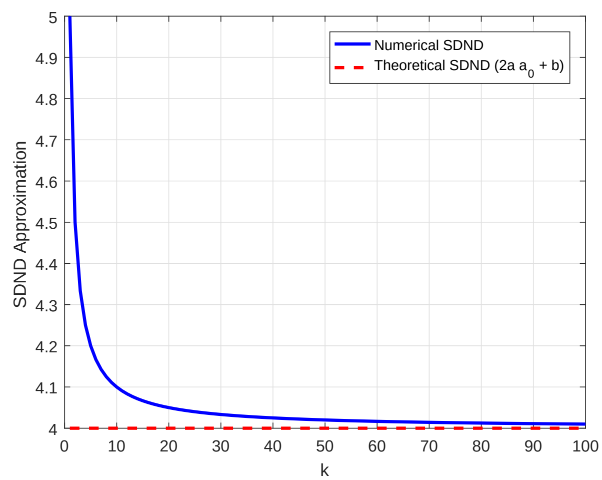

FIGURE 7. SDND of quadratic function: $g(x) = ax^2 + bx + c$

Remark 7.5. Here the Figure 8 compute and visualize the statistical deferred Nörlund derivative (SDND) for the trigonometric function $g(x) = \sin x$.

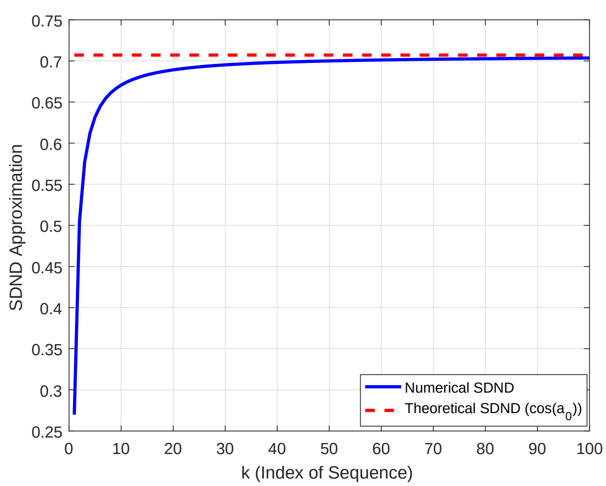

FIGURE 8. SDND of trigonometric function: $g(x) = \sin x$

Trigonometric Function: $g(x) = \sin x$

$$\sin(A + B) - \sin A = 2\cos\left(\frac{2A + B}{2}\right)\sin\left(\frac{B}{2}\right),$$

we get:

$$\frac{\sin(a_0 + a_k) - \sin a_0}{a_k} = \cos(a_0) \frac{\sin a_k}{a_k}.$$

Since $\lim_{a_k \to 0} \frac{\sin a_k}{a_k} = 1$, we obtain:

$$\ell = \cos a_0.$$

The deviation:

$$\left| \frac{\sin(a_0 + a_k) - \sin a_0}{a_k} - \cos a_0 \right| = \left| \cos a_0 \left(\frac{\sin a_k}{a_k} - 1 \right) \right|,$$

vanishes as $a_k \to 0$, proving that the SDND is:

$$\ell = \cos a_0.$$

Remark 7.6. The statistical deferred Nörlund derivative (SDND) has potential applications across various fields. In machine learning and data science, SDND can enhance gradient estimation under noise or uncertainty, improving optimization in stochastic environments. In cryptography, its ability to handle irregular or statistically converging sequences could help model or analyze pseudorandom number generators and side-channel attack resistance. In analytic number theory, SDND may support the study of convergence of arithmetic functions or series with statistical behaviors, offering tools for exploring average-case analysis. Overall, SDND provides a robust framework for approximating derivatives where traditional methods struggle, especially in noisy or probabilistic settings.

Remark 7.7. The present work opens several promising avenues for further investigation. One potential line of research would be to explore alternative formulations of the deferred Nörlund derivative beyond Definition 3.1, by incorporating recently introduced generalized differential operators. In particular, the local generalized derivative proposed by Juan et al. [18] offers an interesting framework that could enrich the analytical properties and applicability of the deferred Nörlund calculus. The combination of these approaches may lead to new generalized models capable of capturing more complex dynamic behaviors, extending the current results and establishing deeper connections with fractional and nonlocal operators. Further studies could also investigate the corresponding integral formulations, numerical schemes and potential applications in mathematical physics and engineering problems where generalized derivatives play a key role.

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References

1. R.P. Agnew, *On deferred Cesàro means*, Ann. Math., 33 (1932), pp. 413-421.
2. W.A. Al-Salam, *Operational representations for the Laguerre and other polynomials*, Duke Math. J., 31 (1964), pp. 127-142.
3. P. Baliarsingh, *On statistical deferred A-convergence of uncertain sequences*, Internat. J. Uncertain. Fuzziness Knowledge-Based Systems, 29 (2021), pp. 499-515.
4. F. Başar, *Summability Theory and its Applications*, 2^{nd} ed., CRC Press/Taylor & Francis Group, Boca Raton · London · New York, 2022.
5. F. Başar and M. Yeşilkayagil Savaşcı, *Double Sequence Spaces and Four Dimensional Matrices*, CRC Press/Taylor & Francis Group, Monographs and Research Notes in Mathematics, Boca Raton · London · New York, 2022.
6. B.D. Boyanov and V. M. Veselinov, *A note on the approximation of functions in an infinite interval by linear positive operators*, Bull. Math. Soc. Sci. Math. Roum., 14 (1970), pp. 9-13.
7. N.L. Braha, V. Loku and H.M. Srivastava, *Λ^2-weighted statistical convergence and Korovkin and Voronovskaya type theorems*, Appl. Math. Comput., 266 (2015), pp. 675-686.
8. N.L. Braha, H.M. Srivastava and S.A. Mohiuddine, *A Korovkin-type approximation theorem for periodic functions via the statistical summability of the generalized de la Vallée Poussin mean*, Appl. Math. Comput., 228 (2014), pp. 162-169.
9. J. Connor and K.G. Grosse-Erdmann, *Sequential definitions of continuity for real functions*, Rocky Mountain J. Math., 33 (2003), pp. 93-121.
10. C. Belen and S.A. Mohiuddine, *Generalized statistical convergence and application*, Appl. Math. Comput., 219 (2013), pp. 9821-9826.
11. H. Fast, *Sur la convergence statistique*, Colloq. Math., 2 (1951), pp. 241-244.
12. M. Mursaleen and C. Belen, *On statistical lacunary summability of double sequences*, Filomat 28 (2014), pp. 231-239.
13. G.H. Hardy, *Divergent Series*, Oxford University Press (1st ed.), Oxford, (1949).
14. B.B. Jena and S.K. Paikray, *Product of deferred Cesàro and deferred weighted statistical probability convergence and its applications to Korovkin-type theorems*, Univ. Sci., 25 (2020), pp. 409-433.

15. B.B. Jena, S.K. Paikray and H. Dutta, *A new approach to Korovkin-type approximation via deferred Cesàro statistical measurable convergence*, Chaos Solitons Fractals, 148 (2021), pp. 1-9.
16. B.B. Jena and S.K. Paikray, *Product of statistical probability convergence and its applications to Korovkin-type theorem*, Miskolc Math. Notes, 20 (2019), pp. 969-984.
17. B.B. Jena, S.K. Paikray and UK. Misra, *Statistical deferred Cesàro summability and its applications to approximation theorems*, Filomat, 32 (2018), pp. 2307-2319.
18. E.N.V. Juan, M.G. Paulo and M.L. Luciano and K. Artion, *The local generalized derivative and Mittag-Leffler function*, Sigma J. Eng. Nat. Sci. 38, 2020, pp. 1007-1017.
19. U. Kadak, N.L. Braha and H.M. Srivastava, *Statistical weighted $\mathcal{B}$-summability and its applications to approximation theorems*, Appl. Math. Comput., 302 (2017), pp. 80-96.
20. V. Karakaya and T.A. Chishti, *Weighted statistical convergence*, Iran. J. Sci. Technol. Trans. A: Sci., 33 (A3) (2009), pp. 219-223.
21. Ö Kişi, M. Gürdal and B. Çakal, *Certain aspects of Nörlund I-statistical convergence of sequences in neutrosophic normed spaces*, Demonstr. Math., 56 (2023), pp. 1-20.
22. P.P. Korovkin, *Convergence of linear positive operators in the spaces of continuous functions* (in Russian), Doklady Akad. Nauk. SSSR (New Ser.), 90 (1953), pp. 961-964.
23. I.J. Maddox, *Statistical convergence in a locally convex space*, Math. Proc. Camb. Phil. Soc., 104 (1988), pp. 141-145.
24. H. Dutta and B.E. Rhoades, *Current Topics in Summability Theory and Applications*, Springer, Singapore Pte Ltd., (2016).
25. S.A. Mohiuddine and B.A. Alamri, *Generalization of equi-statistical convergence via weighted lacunary sequence with associated Korovkin and Voronovskaya type approximation theorems*, Rev. Real Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. (RACSAM), 113 (2019), pp. 1955-1973.
26. S.A. Mohiuddine, A. Alotaibi and M. Mursaleen, *Statistical summability $(C, 1)$ and a Korovkin-type approximation theorem*, J. Inequal. Appl., 2012 (2012), pp. 1-8.
27. F. Móricz, *Tauberian conditions under which statistical convergence follows from statistical summability* $(C, 1)$, J. Math. Anal. Appl., 275 (2002), pp. 277-287.
28. M. Mursaleen, V. Karakaya, M. Ertürk and F. Gürsoy, *Weighted statistical convergence and its application to Korovkin-type approximation theorem*, Appl. Math. Comput., 218 (2012), pp. 9132-9137.

29. S.K. Paikray, P. Parida and S.A. Mohiuddine, *A certain class of relatively equi-statistical fuzzy approximation theorems*, Eur. J. Pure Appl. Math., 13 (2020), pp. 1212-1230.
30. P. Parida, S.K. Paikray and B.B. Jena, *Generalized deferred Cesàro equi-statistical convergence and analogous approximation theorems*, Proyecciones J. Math., 39 (2020), pp. 307-331.
31. W. Ruddine, *Principles of Mathematical Analysis*, McGraw-Hill, Inc. (3rd ed.), London, (2023).
32. K. Saini and K. Raj, *Applications of statistical convergence in complex uncertain sequences via deferred Riesz mean*, Internat. J. Uncertain. Fuzziness Knowledge-Based Systems, 29 (2021), pp. 337-351.
33. K. Saini, K. Raj and M. Mursaleen, *Deferred Cesàro and deferred Euler equi-statistical convergence and its applications to Korovkin-type approximation theorem*, Int. J. Gen. Syst., 50 (2021), pp. 567-579.
34. T. Šalát, *On statistically convergent sequences of real numbers*, Math Slovaca, 30 (1980), pp. 139-150.
35. S. Sharma, U.P. Singh and K. Raj, *Applications of deferred Cesàro statistical convergence of sequences of fuzzy numbers of order* (ξ, ω), J. Intell. Fuzzy Syst., 41 (2021), pp. 1-10.
36. H.M. Srivastava, B.B. Jena and S.K. Paikray, *A certain class of equi-statistical convergence in the sense of the deferred power-series method*, Axioms, 12 (2023).
37. H.M. Srivastava, B.B. Jena and S.K. Paikray, *Deferred Cesàro statistical convergence of martingale sequence and Korovkin-type approximation theorems*, Miskolc Math. Notes, 23 (2022), pp. 443-456.
38. H.M. Srivastava, B.B. Jena and S.K. Paikray, *Statistical product convergence of martingale sequences and its applications to Korovkin-type approximation theorems*, Math. Methods Appl. Sci., 44 (2021), pp. 9600-9610.
39. H.M. Srivastava, B.B. Jena and S.K. Paikray, *Statistical Riemann and Lebesgue integrable sequence of functions with Korovkin-type approximation theorems*, Axioms, 10 (2021), pp. 1-16.
40. H.M. Srivastava, B.B. Jena and S.K. Paikray, *Statistical deferred Nörlund summability and Korovkin-type approximation theorem*, Mathematics, 8 (2020), pp. 1-11.
41. H.M. Srivastava, B.B. Jena and S.K. Paikray, *Deferred Cesàro statistical probability convergence and its applications to approximation theorems*, J. Nonlinear Convex Anal., 20 (2019), pp. 1777-1792.

42. H.M. Srivastava, B.B. Jena, S.K. Paikray and U. K. Misra, *Generalized equi-statistical convergence of the deferred Nörlund summability and its applications to associated approximation theorems*, Rev. Real Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. (RACSAM), 112 (2018), pp. 1487-1501.
43. H.M. Srivastava, M. Mursaleen and A. Khan, *Generalized equi-statistical convergence of positive linear operators and associated approximation theorems*, Math. Comput. Model., 55 (2012), pp. 2040-2051.
44. H. Steinhaus, *Sur la convergence ordinaire et la convergence asymtotique*, Colloq. Math. 2 (1951), 73-74.
45. M. Yeşilkayagil and F. Başar, *$AK(\vartheta)$-property of the spaces of double series*, Bull. Malays. Math. Sci. Soc., 44 (2021), pp. 881-889.
46. M. Yeşilkayagil and F. Başar, *On the paranormed space of bounded variation double sequences*, Bull. Malays. Math. Sci. Soc., 43 (2020), pp. 2701-2712.
47. M. Yeşilkayagil and F. Başar, *Domain of Riesz mean in some spaces of double sequences*, Indag. Math. (N.S.), 29 (2018), pp. 1009-1029.

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