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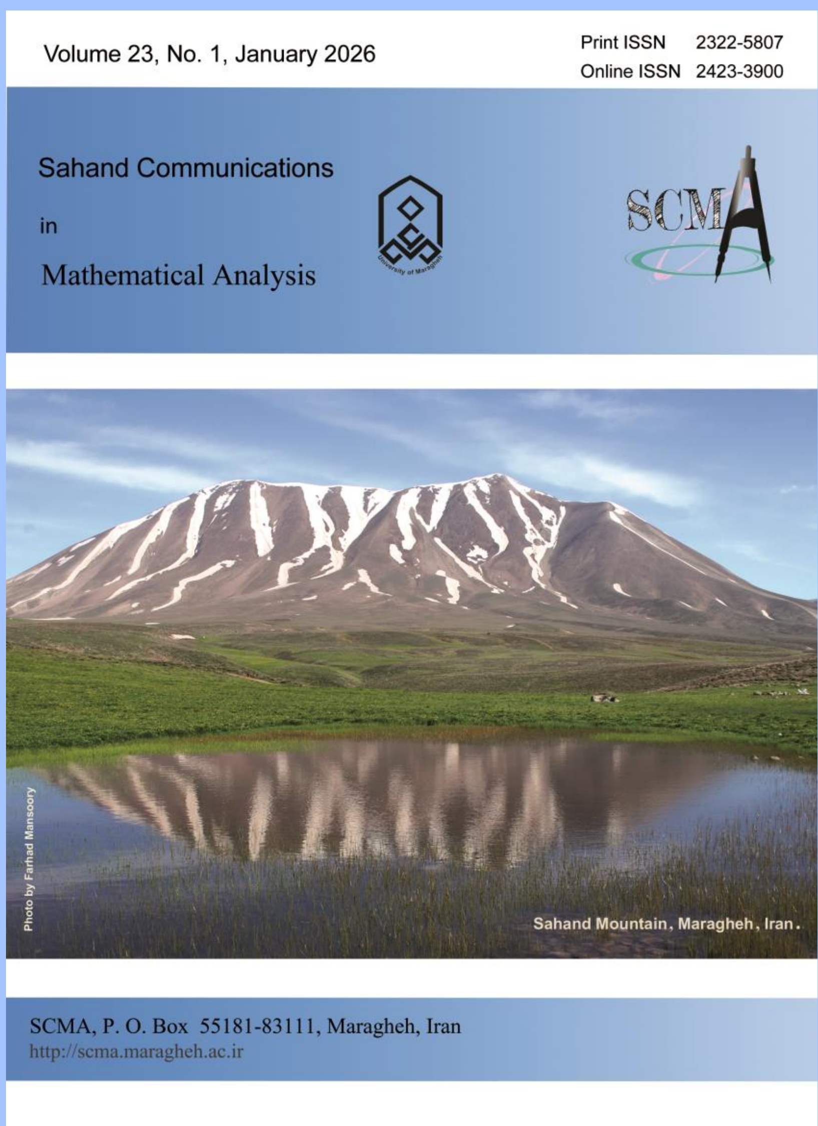

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Scalability of Tensor Products of Multiple Frames in Hilbert Spaces

Asghar Rahimi[1*], Samrand Moayyadzadeh[2] and Bayaz Daraby[3]

Abstract. This paper studies the scalability of tensor products of several frames, which may come from different Hilbert spaces. Although the scalability of single frames and tensor products of two frames has been explored, the case with more than two frames has not been studied and solved yet. We establish sufficient and necessary conditions under which such tensor products are scalable, and we describe these conditions using operator theory, spectral analysis, and numerical methods. To clarify the issue, we provide examples and counterexamples. Finally, we briefly mention how this can be applied in signal processing and sparse representation.

1. Introduction

Frames in Hilbert spaces are now important tools in applied and computational mathematics. They give representations that are redundant but still stable, and they extend the idea of orthonormal bases. Formally, let $\mathcal{H}$ be a real or complex separable Hilbert space. A countable family $\Phi = \{\varphi_j\}_{j\in J} \subset \mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist constants $0 < A \le B < \infty$ such that for all $x \in \mathcal{H}$

$$A\|x\|^2 \le \sum_{j\in J} |\langle x, \varphi_j\rangle|^2 \le B\|x\|^2.$$

The numbers A and B are called the lower and upper frame bounds, respectively. A frame is called tight if $A = B$, and Parseval if $A = B = 1$. Tight frames have optimal numerical stability due to the frame operator which is a scalar multiple of the identity. To convert a general frame

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into a tight one, a natural approach is to scale each frame vector by a nonnegative scalar. If such scalars exist so that the rescaled system becomes a Parseval frame, the original frame is said to be scalable. If all scaling weights are strictly positive, the frame is positively scalable; if the weights are bounded away from zero, it is strictly scalable. The study of scalable frames was initiated and developed by Kutyniok et al. [5], where several geometric and operator-theoretic characterizations were provided.

Zakeri and Ahmadi [9] extended this framework by considering the tensor product of two frames from possibly different Hilbert spaces. Let $\Phi = \{\varphi_i\} \subset \mathcal{H}$ and $\Psi = \{\psi_j\} \subset \mathcal{K}$ be frames. Their tensor product $\Phi \otimes \Psi = \{\varphi_i \otimes \psi_j\} \subset \mathcal{H} \otimes \mathcal{K}$ is itself a frame, but it may not be scalable even when one factor is scalable and the other is tight. Their work gave sufficient conditions for the two-factor case and shows which combinations are not scalable. But the case of tensor products with more than two frames has not been studied in the literature. This gap is important in areas like multi-way graph signal processing on tensors [8], tensor data representations, and multi-dimensional harmonic analysis.

This paper addresses the following main questions:

- Under what conditions on finitely many frames $\Phi^{(1)}, \dots, \Phi^{(n)}$ in the Hilbert spaces $\mathcal{H}_1, \dots, \mathcal{H}_n$, the tensor product

$$\Phi := \Phi^{(1)} \otimes \cdots \otimes \Phi^{(n)} \subset \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$$

  is scalable?
- How do properties like tightness, frame bounds, or individual scalability affect the scalability of the full tensor product?
- What operator-theoretic or numerical criteria can ensure or obstruct scalability?
- Can explicit examples and counterexamples be constructed to illustrate sharpness?

Our contributions include:

- Operator-theoretic characterizations of multi-factor scalability using diagonal operators.
- Spectral bounds relating frame operators of the factors to the tensor product.
- Geometric and numerical conditions for scalable tensor frames.
- Examples in low-dimensional settings.

The results generalize and unify the theories in [5] and [9], and provide new insights for applications in signal processing, numerical linear algebra, and machine learning.

1.1. **Comparison with Related Work:** This subsection explains the new contributions of our work compared to the basic results on scalable frames, focusing on the main generalizations and extensions for tensor products of several frames.

- **Relation to Kutyniok et al. :** While [5] pioneered the theory of scalable frames for *single* Hilbert spaces, giving basic operator-theoretic and geometric characterizations, our work goes beyond this framework and studies tensor products of multiple frames in different Hilbert spaces. Specifically:
  - We generalize the scalability problem to n-fold tensor products $\bigotimes_{i=1}^{n} \mathcal{H}_i$, whereas [5] addressed only the single-space case ($n = 1$).
  - Our spectral condition $\kappa(\Phi) = \prod_{i=1}^{n} \kappa\left(\Phi^{(i)}\right)$ (Theorem 2.10) reduces to $\kappa(\Phi) = \kappa\left(\Phi^{(1)}\right)$ when $n = 1$, recovering their spectral bounds as a special case.
  - The reduced diagram matrix Θ (Theorem 2.3) generalizes their single-frame feasibility criterion to multi-index tensor systems.
- **Relation to Zakeri & Ahmadi :** Though [9] has initiated the study of tensor products for *two* frames, our work resolves the open problem of scalability for *arbitrary finite* tensor factors:
  - We establish necessary *and* sufficient conditions for n-fold tensor products (Theorem 2.3), whereas [9] has provided only sufficient conditions for $n = 2$.
  - Our counterexamples demonstrate that their sufficiency conditions *can not* extend to $n \geq 3$: Scalability of all factors is now shown to be *necessary*, not merely sufficient.
- **Relation to Casazza [1] et al.:** While [2] developed the reduced diagram matrix methodology for single frames, our work significantly extends this framework:
  - We construct multi-index diagram matrices $\Theta \in \mathbb{R}^{D\times M}$ for tensor products (Theorem 2.3), where $D = \sum d_i - n + 1$ and $M = \prod m_i$, overcoming the dimensionality explosion in higher-order tensors.

[1] Professor Peter G. Casazza (June 28, 1945 - October 26, 2025) was a great mathematician who played a huge role in the development of frame theory and the introducing of its applications. Unfortunately, we were shocked by the news of his death while we were preparing the final version of this work. Without a doubt, his scientific achievements and articles will continue to guide researchers and enthusiasts for a long time.

- Our unified criterion $\Theta\mathbf{c} = \mathbf{1}$ subsumes their single-frame condition when $n = 1$ but addresses new algebraic complexities for $n \geq 2$.
- We identify tensor-specific rank deficiencies in Θ that cannot occur in single-frame scenarios, revealing obstructions unique to tensor products.

Synoptic View: Table 1 summarizes how our work unifies and extends the existing literatures:

Property	**[1]**	**[2]**	**Present Work**
Single-frame scalability	✓	×	✓ (generalizes [1])
Two-factor tensor scalability	×	✓	✓ (strengthens [2])
n-factor tensor scalability ($n \geq 3$)	×	×	✓ **(new)**
Necessary/sufficient conditions	Single frame	Sufficient only	**Full characterization**
Diagram matrix method	✓	×	**Multi-factor extension**
Spectral bounds	Single operator	Two operators	**n-operator product**

TABLE 1. Comparison of theoretical contributions

1.2. **Preliminaries and Definitions.** In this paper, we denote by $\mathcal{H}, \mathcal{K}, \mathcal{H}_1, \mathcal{H}_2, \ldots$ separable Hilbert spaces and $\mathcal{H}_L$ stands for a Hilbert space of dimension L. Also, I and J are finite or countable index sets.

Scalable frames are a special type of frames. They are defined by the fact that we can rescale the frame vectors with simple non-negative weights so that the obtained system becomes a Parseval frame. This keeps the nice properties of the original frame, like redundancy and numerical stability, but also gives simple and stable reconstruction formulas. In practice, scalability means, we can precondition a frame in the best way using diagonal operators, which gives high performance in applications such as signal recovery and sparse representation.

Definition 1.1 (Scalable Frame)**.** A frame $\Phi = \{\varphi_j\}_{j\in J} \subset \mathcal{H}$ is called scalable if there exists a sequence of nonnegative scalars $\{c_j\}$ such that the scaled frame $\{c_j\varphi_j\}$ is a Parseval frame. If all $c_j > 0$, Φ is called positively scalable; if in addition $\inf_j c_j > 0$, Φ is strictly scalable (see Kutyniok et al. [5], and Casazza et al. [2]).

For $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$, the rank one operator $x \otimes y : \mathcal{H}_1 \mapsto \mathcal{H}_2$ defined by $(x \otimes y)(z) = \langle x, z\rangle\, y \;\, for \;\, z \in \mathcal{H}$. This operator called the tensor product of x and y. For more study on the properties of the tensor product, see [6].

The tensor product operation constructs a new Hilbert space by combining two given Hilbert spaces in a bilinear, inner-product preserving way. More precisely, given separable Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, one forms their algebraic tensor product and then completes it under the natural inner product

$$\langle \varphi_1 \otimes \psi_1,\ \varphi_2 \otimes \psi_2 \rangle = \langle \varphi_1, \varphi_2 \rangle_{\mathcal{H}_1} \cdot \langle \psi_1, \psi_2 \rangle_{\mathcal{H}_2}$$

for all $\varphi_1, \varphi_2 \in \mathcal{H}_1, \psi_1, \psi_2 \in \mathcal{H}_2$, leading to the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ (see [1, 4]).

When each factor $\Phi^{(1)} \subset \mathcal{H}_1$ and $\Phi^{(2)} \subset \mathcal{H}_2$ is a frame in its respective space, their tensor product

$$\Phi^{(1)} \otimes \Phi^{(2)} = \{\varphi_i \otimes \psi_j\}$$

naturally forms a frame in the tensor-product space. This generalizes the notion of frame constructions into higher dimensions and it is useful especially in multivariate signal or joint representation contexts.

Definition 1.2 (Tensor Product of Frames)**.** Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, and let

$$\Phi^{(1)} = \{\varphi_i\} \subset \mathcal{H}_1, \qquad \Phi^{(2)} = \{\psi_j\} \subset \mathcal{H}_2$$

be frames. Their tensor product frame is defined as

$$\Phi^{(1)} \otimes \Phi^{(2)} = \{\varphi_i \otimes \psi_j : i, j\}.$$

The space $\mathcal{H}_1 \otimes \mathcal{H}_2$ is endowed with the inner product

$$\left\langle \varphi_i \otimes \psi_j, \varphi_{i'} \otimes \psi_{j'} \right\rangle = \langle \varphi_i, \varphi_{i'} \rangle_{\mathcal{H}_1} \cdot \left\langle \psi_j, \psi_{j'} \right\rangle_{\mathcal{H}_2}$$

(see [4] and also standard Hilbert tensor product construction) and $\Phi^{(1)} \otimes \Phi^{(2)}$ is again a frame in the tensor product space [4].

Theorem 1.3 ([4])**.** *Let $\{\varphi_i\} \subset \mathcal{H}$ be a frame for $\mathcal{H}$ with frame bounds A_1 and B_1, and let $\{\psi_j\} \subset \mathcal{K}$ be a frame for $\mathcal{K}$ with frame bounds A_2 and B_2. Then the tensor product system*

$$\{\varphi_i \otimes \psi_j\}_{i,j}$$

is a frame for the tensor product space $\mathcal{H} \otimes \mathcal{K}$, possessing frame bounds

$$A = A_1 A_2, \qquad B = B_1 B_2.$$

Definition 1.4 (Tensor-Product Frame Scalability for Two Frames)**.** Let $\Phi^{(1)} \subset \mathcal{H}_1$ and $\Phi^{(2)} \subset \mathcal{H}_2$ be frames. We say their tensor product $\Phi^{(1)} \otimes \Phi^{(2)}$ is scalable if there exist nonnegative scalars $c_{ij} \geq 0$ such that

$$\{\, c_{ij}(\varphi_i \otimes \psi_j)\}$$

forms a Parseval frame in $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Zakeri and Ahmadi (2020) showed that even if $\Phi^{(1)}$ is scalable and $\Phi^{(2)}$ is tight (but not necessarily Parseval), then the tensor frame need not be scalable, and they provided eigenvalue-based conditions and counterexamples [9].

Theorem 1.5 (Special case: Parseval factor [9]). *Suppose $\{\varphi_i\} \subset \mathcal{H}$ is a scalable frame for $\mathcal{H}$, and $\{\psi_j\} \subset \mathcal{K}$ is a tight frame for $\mathcal{K}$ with frame bound 1 (i.e., a Parseval frame). Then the tensor-product frame*

$$\{\varphi_i \otimes \psi_j\}$$

is scalable in the product space $\mathcal{H} \otimes \mathcal{K}$.

Corollary 1.6 ([9]). *Let $\{\varphi_i\}$ and $\{\psi_j\}$ be frames in $\mathcal{H}$ and $\mathcal{K}$, respectively. The tensor-product frame $\{\varphi_i \otimes \psi_j\}$ is scalable in $\mathcal{H} \otimes \mathcal{K}$ if at least one of the following statements hold:*

(a) *$\{\varphi_i\}$ is Parseval in $\mathcal{H}$ and $\{\psi_j\}$ is scalable in $\mathcal{K}$.*
(b) *Both $\{\varphi_i\}$ and $\{\psi_j\}$ are scalable in their respective spaces.*

Theorem 1.7 ([9]). *If $\{T_i\}$ is a scalable frame for $\mathcal{H} \otimes \mathcal{K}$, then for any nonzero vectors $x_0 \in \mathcal{H}$ and $y_0 \in \mathcal{K}$, the sequences*

$$\left\{ \frac{T_i(y_0)}{\|y_0\|} \right\}_i, \qquad \left\{ \frac{T_i^*(x_0)}{\|x_0\|} \right\}_i$$

are scalable frames in $\mathcal{H}$ and $\mathcal{K}$, respectively.

Theorem 1.8 ([9]). *If $\{T_i\}$ is a scalable frame for $\mathcal{H} \otimes \mathcal{K}$, then its adjoint system $\{T_i^*\}$ forms a scalable frame in the reversed product space $\mathcal{K} \otimes \mathcal{H}$.*

Example 1.9 ([9]). Consider two frames in $\mathbb{R}^2$, each formed by three unit vectors at angles $\theta_1 = 0$, θ_2, θ_3:

$$\varphi_k = \begin{pmatrix} \cos\theta_k \\ \sin\theta_k \end{pmatrix}, \quad k = 1, 2, 3.$$

(i) With $\theta_2 = \frac{\pi}{2}$, $\theta_3 = \frac{2\pi}{3}$,

$$\Phi_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \right\}$$

is scalable - there exist nonnegative weights yielding a Parseval frame.

(ii) With $\theta_2 = \frac{\pi}{6}$, $\theta_3 = \frac{\pi}{3}$,

$$\Phi_2 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \right\}$$

is not scalable - as shown via convex polytope methods, no $c_k \geq 0$ yields a Parseval frame.

Consequently, even though Φ_1 is scalable, the tensor product

$$\Phi_1 \otimes \Phi_2 \subset \mathbb{R}^2 \otimes \mathbb{R}^2$$

is not scalable. This aligns precisely with the counterexample given in [9].

Example 1.10. In $\mathbb{R}^2$, let:

$$\Phi^{(1)} = \{e_1, e_2\}, \qquad \Phi^{(2)} = \{e_1, e_2, u\}, \qquad u = \frac{1}{\sqrt{2}}(e_1 + e_2).$$

Here $\Phi^{(1)}$ is a Parseval frame (tight), and $\Phi^{(2)}$ is scalable but not tight. By Theorem 3.2 of [9], their tensor product

$$\Phi^{(1)} \otimes \Phi^{(2)} \subset \mathbb{R}^4$$

must be scalable, in general failing scalability unless additional spectral conditions apply. This clearly illustrates that:

$$\text{scalable factors} \not\Rightarrow \text{scalable tensor product},$$

aligning precisely with the counterexample given in the original work [9].

1.3. **Eigenvalue and Trace Properties for Tensor-Product Frames.**

Theorem 1.11 ([9])**.** *Let $\{\varphi_i\}_{i=1}^M \subset \mathcal{H}_L$ and $\{\psi_j\}_{j=1}^N \subset \mathcal{H}_K$ be frames with frame operators S_1 and S_2, possessing normalized eigenvectors $\{e_i\}_{i=1}^L$, $\{u_j\}_{j=1}^K$ and corresponding eigenvalues $\{\lambda_i\}_{i=1}^L$, $\{\gamma_j\}_{j=1}^K$. Then*

$$\mathrm{Tr}(S_1 \otimes S_2) = \sum_{i=1}^M \sum_{j=1}^N \|\varphi_i \otimes \psi_j\|^2.$$

Theorem 1.12 ([9])**.** *Let $\{\varphi_i\}_{i=1}^M \subset \mathcal{H}_L$ and $\{\psi_j\}_{j=1}^N \subset \mathcal{H}_K$ be frames with frame operators S_1, S_2, whose eigenvalues satisfy*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_L, \qquad \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_K.$$

Then for the tensor-product frame $\{\varphi_i \otimes \psi_j\}_{i,j}$, the optimal lower and upper bounds are $\lambda_1 \gamma_1$ and $\lambda_L \gamma_K$.

Proposition 1.13 ([9])**.** *Let $\Phi = \{\varphi_i\}_{i=1}^M \subset \mathcal{H}_L$ be a frame, and $\Psi = \{\psi_j\}_{j=1}^N \subset \mathcal{H}_K$ be a* scalable *frame. Then*

$$\min_{s_j \geq 0} \left\| I_L \otimes I_K - \sum_{i=1}^M \sum_{j=1}^N s_j^2 \, \varphi_i \varphi_i^* \otimes \psi_j \psi_j^* \right\|_2 \leq \min_{c_i \geq 0} \left\| I_L - \sum_{i=1}^M c_i^2 \, \varphi_i \varphi_i^* \right\|_2 .$$

2. Main Results

2.1. **Notation.** Let $\mathcal{H}_i$ $(i = 1, \dots, n)$, be separable real or complex Hilbert spaces, and let

$$\Phi^{(i)} = \left\{\varphi^{(i)}_{j_i}\right\}_{j_i \in J_i} \subset \mathcal{H}_i$$

be frames with bounds $A_i, B_i > 0$. We denote the n-fold tensor product

$$\Phi := \Phi^{(1)} \otimes \cdots \otimes \Phi^{(n)} = \left\{\varphi^{(1)}_{j_1} \otimes \cdots \otimes \varphi^{(n)}_{j_n} : j_i \in J_i\right\} \subset \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n.$$

By standard theory [4], Φ is a frame with bounds

$$A = \prod_{i=1}^{n} A_i, \qquad B = \prod_{i=1}^{n} B_i.$$

Lemma 2.1 (Hilbert–Schmidt Identification [3])**.** *There is a canonical isometric isomorphism between the Hilbert space tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ and the space of Hilbert–Schmidt operators $\mathrm{HS}(\mathcal{H}_2, \mathcal{H}_1)$, under which the norm of $x \in \mathcal{H}_1 \otimes \mathcal{H}_2$ equals the Hilbert–Schmidt norm of the associated operator $\widetilde{x}$. Moreover, for any orthonormal basis $\{f_k\}_{k \in K}$ of $\mathcal{H}_2$ one has*

$$\langle x, y\rangle_{\mathcal{H}_1 \otimes \mathcal{H}_2} = \sum_k \langle \widetilde{x}(f_k), \widetilde{y}(f_k)\rangle_{\mathcal{H}_1},$$

and $\|x\|^2 = \|\widetilde{x}\|^2_{\mathrm{HS}} = \mathrm{Tr}(\widetilde{x}^* \widetilde{x})$.

Theorem 2.2 (Tensor Product of Frames)**.** *Let $\mathcal{H}_1, \dots, \mathcal{H}_n$ be separable Hilbert spaces. For each $k = 1, \dots, n$, let $\Phi^{(k)} = \{\varphi^{(k)}_j\}_{j \in J_k}$ be a frame for $\mathcal{H}_k$ with frame bounds $0 < A_k \le B_k < \infty$. Then the tensor product*

$$\Phi = \bigotimes_{k=1}^{n} \Phi^{(k)} = \left\{\varphi^{(1)}_{j_1} \otimes \cdots \otimes \varphi^{(n)}_{j_n}\right\}_{(j_1, \dots, j_n) \in J_1 \times \cdots \times J_n}$$

is a frame for the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ with frame bounds $A = \prod_{k=1}^{n} A_k$ and $B = \prod_{k=1}^{n} B_k$.

Proof. We prove the statement by induction on n. The case $n = 1$ is trivial.

Base case ($n = 2$). Let $\Phi^{(1)} = \{\varphi_i\}_{i \in I} \subset \mathcal{H}_1$ and $\Phi^{(2)} = \{\psi_j\}_{j \in J} \subset \mathcal{H}_2$ be frames with frame bounds (A_1, B_1) and (A_2, B_2) respectively. For $x \in \mathcal{H}_1 \otimes \mathcal{H}_2$ denote by $\widetilde{x} : \mathcal{H}_2 \to \mathcal{H}_1$ the corresponding Hilbert–Schmidt operator (see Lemma 2.1). Using the identification and Fubini-type interchange, we get

$$\sum_{i \in I} \sum_{j \in J} |\langle x, \varphi_i \otimes \psi_j\rangle|^2 = \sum_{j \in J} \sum_{i \in I} |\langle \widetilde{x}\psi_j, \varphi_i\rangle|^2.$$

Applying the frame inequality for $\Phi^{(1)}$ to the vector $\widetilde{x}\psi_j \in \mathcal{H}_1$ we have

$$\sum_{i\in I} |\langle \widetilde{x}\psi_j, \varphi_i\rangle|^2 \le B_1 \|\widetilde{x}\psi_j\|^2, \qquad \sum_{i\in I} |\langle \widetilde{x}\psi_j, \varphi_i\rangle|^2 \ge A_1 \|\widetilde{x}\psi_j\|^2.$$

These follows that

$$A_1 \sum_{j\in J} \|\widetilde{x}\psi_j\|^2 \le \sum_{i,j} |\langle x, \varphi_i \otimes \psi_j\rangle|^2 \le B_1 \sum_{j\in J} \|\widetilde{x}\psi_j\|^2.$$

Now write the sum $\sum_{j\in J} \|\widetilde{x}\psi_j\|^2$ in operator-trace form. Let $S_2 : \mathcal{H}_2 \to \mathcal{H}_2$ be the frame operator of $\Phi^{(2)}$, i.e.

$$S_2 y = \sum_{j\in J} \langle y, \psi_j\rangle \, \psi_j,$$

which is a positive, bounded, and invertible operator satisfying $A_2 I \le S_2 \le B_2 I$. Then

$$\sum_{j\in J} \|\widetilde{x}\psi_j\|^2 = \sum_{j\in J} \langle \widetilde{x}^* \widetilde{x}\psi_j, \psi_j\rangle = \operatorname{Tr}\left(\widetilde{x}^* \widetilde{x}\, S_2\right).$$

Using the operator inequality $A_2 I \le S_2 \le B_2 I$ and positivity of $\widetilde{x}^*\widetilde{x}$ we obtain

$$A_2 \operatorname{Tr}(\widetilde{x}^*\widetilde{x}) \le \operatorname{Tr}(\widetilde{x}^*\widetilde{x}\, S_2) \le B_2 \operatorname{Tr}(\widetilde{x}^*\widetilde{x}).$$

But $\operatorname{Tr}(\widetilde{x}^*\widetilde{x}) = \|\widetilde{x}\|_{\mathrm{HS}}^2 = \|x\|^2$ by Lemma 2.1. Therefore

$$A_1 A_2 \|x\|^2 \le \sum_{i,j} |\langle x, \varphi_i \otimes \psi_j\rangle|^2 \le B_1 B_2 \|x\|^2.$$

This proves that $\Phi^{(1)} \otimes \Phi^{(2)}$ is a frame with bounds $A_1 A_2$ and $B_1 B_2$.

Induction Step ($n \ge 3$): Assume the theorem holds for all tensor products of m frames where $1 \le m \le n-1$. Consider:

(i) **Decomposition:** Define the intermediate space and frame:

$$\widetilde{\mathcal{H}} = \bigotimes_{k=1}^{n-1} \mathcal{H}_k, \qquad \widetilde{\Phi} = \bigotimes_{k=1}^{n-1} \Phi^{(k)}$$

(ii) **Induction Hypothesis:** By the induction hypothesis applied to $n-1$ frames, $\widetilde{\Phi}$ is a frame for $\widetilde{\mathcal{H}}$ with bounds:

$$\widetilde{A} = \prod_{k=1}^{n-1} A_k, \qquad \widetilde{B} = \prod_{k=1}^{n-1} B_k$$

(iii) **Canonical Isomorphism:** Note that $\mathcal{H} = \widetilde{\mathcal{H}} \otimes \mathcal{H}_n$ via the canonical isometric isomorphism:

$$(x_1 \otimes \cdots \otimes x_{n-1}) \otimes x_n \mapsto x_1 \otimes \cdots \otimes x_n$$

(iv) **Base Case Application:** Apply the $n = 2$ case to:

- $\widetilde{\Phi}$ (frame for $\widetilde{\mathcal{H}}$ with bounds $\widetilde{A}, \widetilde{B}$)
- $\Phi^{(n)}$ (frame for $\mathcal{H}_n$ with bounds A_n, B_n)

This shows $\widetilde{\Phi} \otimes \Phi^{(n)}$ is a frame for $\widetilde{\mathcal{H}} \otimes \mathcal{H}_n$ with bounds $\widetilde{A}A_n$ and $\widetilde{B}B_n$.

(v) **Identification:** Under the canonical isomorphism, we have:

$$\widetilde{\Phi} \otimes \Phi^{(n)} \equiv \bigotimes_{k=1}^{n} \Phi^{(k)} = \Phi, \qquad \widetilde{\mathcal{H}} \otimes \mathcal{H}_n \equiv \mathcal{H}$$

Thus Φ is a frame for $\mathcal{H}$ with bounds $\prod_{k=1}^{n} A_k$ and $\prod_{k=1}^{n} B_k$. $\square$

2.2. Scalability Transfer and Tensor Products.

Theorem 2.3 (Scalability Inheritance under Tensor Products)**.** *Let $\left\{\varphi_{j_i}^{(i)}\right\}_{j_i \in J_i}$ be a scalable frame for $\mathcal{H}_i$ with positive scaling coefficients $\left\{c_{j_i}^{(i)}\right\}_{j_i \in J_i}$ for each $1 \leq i \leq n$. Then the tensor product system*

$$\left\{\varphi_{j_1}^{(1)} \otimes \varphi_{j_2}^{(2)} \otimes \cdots \otimes \varphi_{j_n}^{(n)}\right\}_{j_i \in J_i}$$

is a scalable frame for $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$ with positive scaling coefficients

$$\left\{\prod_{i=1}^{n} c_{j_i}^{(i)}\right\}_{j_i \in J_i}.$$

Proof. For each $1 \leq i \leq n$, define the scaled frame:

$$\widetilde{\Phi}^{(i)} = \left\{\widetilde{\varphi}_{j_i}^{(i)}\right\} \quad \text{where} \quad \widetilde{\varphi}_{j_i}^{(i)} = c_{j_i}^{(i)} \varphi_{j_i}^{(i)}.$$

By assumption, each $\widetilde{\Phi}^{(i)}$ is a Parseval frame for $\mathcal{H}_i$. We prove by induction on n that:

$$\widetilde{\Phi}^{(i)} = \left\{\bigotimes_{i=1}^{n} \widetilde{\varphi}_{j_i}^{(i)}\right\}_{(j_1,\ldots,j_n)}$$

is a Parseval frame for $\mathcal{H} = \bigotimes_{i=1}^{n} \mathcal{H}_i$.

Base case ($n = 1$): Trivial since $\widetilde{\Phi} = \widetilde{\Phi}^{(1)}$ is Parseval by assumption.

Induction step ($n \geq 2$): Assume the result holds for $k = n-1$. Define:

$$\mathcal{K} = \bigotimes_{i=1}^{n-1} \mathcal{H}_i, \qquad \widetilde{\mathcal{K}} = \mathcal{H}_n.$$

By the induction hypothesis, the system:

$$\widetilde{\Psi} = \left\{\widetilde{\psi}_{\mathbf{j}'}\right\}_{\mathbf{j}' \in J_1 \times \cdots \times J_{n-1}} \quad \text{with} \quad \widetilde{\psi}_{\mathbf{j}'} = \bigotimes_{i=1}^{n-1} \widetilde{\varphi}_{j_i}^{(i)}$$

is a Parseval frame for $\mathcal{K}$. By assumption, $\widetilde{\Phi}^{(n)}$ is a Parseval frame for $\widetilde{\mathcal{K}}$.

Consider the tensor system:

$$\widetilde{\Phi}^{(i)} = \left\{ \widetilde{\psi}_{\mathbf{j}'} \otimes \widetilde{\varphi}^{(n)}_{j_n} \right\}_{(\mathbf{j}', j_n)}$$

For any $x \in \mathcal{K} \otimes \widetilde{\mathcal{K}}$, we verify the Parseval property:

$$\sum_{(\mathbf{j}', j_n)} \left| \left\langle x, \widetilde{\psi}_{\mathbf{j}'} \otimes \widetilde{\varphi}^{(n)}_{j_n} \right\rangle \right|^2 = \|x\|^2.$$

First, note that $\widetilde{\Phi}^{(i)}$ is a Bessel sequence since:

$$\sum_{(\mathbf{j}', j_n)} \left| \left\langle x, \widetilde{\psi}_{\mathbf{j}'} \otimes \widetilde{\varphi}^{(n)}_{j_n} \right\rangle \right|^2 \leq B_{\widetilde{\Psi}} \cdot B_{\widetilde{\Phi}^{(n)}} \|x\|^2 = 1 \cdot 1 \cdot \|x\|^2,$$

where $B_{\widetilde{\Psi}} = 1$ and $B_{\widetilde{\Phi}^{(n)}} = 1$ are the Bessel bounds. Now, let $\{e_p\}$ be an orthonormal basis for $\widetilde{\mathcal{K}}$. Expand x as:

$$x = \sum_p x_p \otimes e_p, \qquad x_p \in \mathcal{K}, \qquad \sum_p \|x_p\|^2 = \|x\|^2.$$

Then:

$$\left\langle x, \widetilde{\psi}_{\mathbf{j}'} \otimes \widetilde{\varphi}^{(n)}_{j_n} \right\rangle = \sum_p \left\langle x_p, \widetilde{\psi}_{\mathbf{j}'} \right\rangle_{\mathcal{K}} \cdot \left\langle e_p, \widetilde{\varphi}^{(n)}_{j_n} \right\rangle_{\widetilde{\mathcal{K}}}.$$

Using the Parseval identity in $\widetilde{\mathcal{K}}$:

$$\sum_{j_n} \left| \left\langle e_p, \widetilde{\varphi}^{(n)}_{j_n} \right\rangle \right|^2 = \|e_p\|^2 = 1,$$
$$\sum_{j_n} \left\langle e_p, \widetilde{\varphi}^{(n)}_{j_n} \right\rangle \overline{\left\langle e_q, \widetilde{\varphi}^{(n)}_{j_n} \right\rangle} = \langle e_p, e_q \rangle = \delta_{pq}.$$

Now compute:

$$\begin{aligned}
&\sum_{(\mathbf{j}', j_n)} \left| \left\langle x, \widetilde{\psi}_{\mathbf{j}'} \otimes \widetilde{\varphi}^{(n)}_{j_n} \right\rangle \right|^2 \\
&\quad = \sum_{\mathbf{j}', j_n} \left| \sum_p \left\langle x_p, \widetilde{\psi}_{\mathbf{j}'} \right\rangle \left\langle e_p, \widetilde{\varphi}^{(n)}_{j_n} \right\rangle \right|^2 \\
&\quad = \sum_{\mathbf{j}'} \sum_{j_n} \sum_p \sum_q \left\langle x_p, \widetilde{\psi}_{\mathbf{j}'} \right\rangle \overline{\left\langle x_q, \widetilde{\psi}_{\mathbf{j}'} \right\rangle} \left\langle e_p, \widetilde{\varphi}^{(n)}_{j_n} \right\rangle \overline{\left\langle e_q, \widetilde{\varphi}^{(n)}_{j_n} \right\rangle}
\end{aligned}$$

$$= \sum_{\mathbf{j}'} \sum_p \sum_q \left\langle x_p, \widetilde{\psi}_{\mathbf{j}'} \right\rangle \overline{\left\langle x_q, \widetilde{\psi}_{\mathbf{j}'} \right\rangle} \underbrace{\sum_{j_n} \left\langle e_p, \widetilde{\varphi}^{(n)}_{j_n} \right\rangle \overline{\left\langle e_q, \widetilde{\varphi}^{(n)}_{j_n} \right\rangle}}_{\delta_{pq}}$$

$$= \sum_{\mathbf{j}'} \sum_p \left| \left\langle x_p, \widetilde{\psi}_{\mathbf{j}'} \right\rangle \right|^2$$

$$= \sum_p \underbrace{\sum_{\mathbf{j}'} \left| \left\langle x_p, \widetilde{\psi}_{\mathbf{j}'} \right\rangle \right|^2}_{\|x_p\|^2 \text{ (Parseval)}}$$

$$= \sum_p \|x_p\|^2 = \|x\|^2.$$

Thus $\widetilde{\Phi}^{(i)}$ is a Parseval frame. Since:

$$\widetilde{\psi}_{\mathbf{j}'} \otimes \widetilde{\varphi}^{(n)}_{j_n} = \left(\prod_{i=1}^{n} c^{(i)}_{j_i} \right) \left(\bigotimes_{i=1}^{n} \varphi^{(i)}_{j_i} \right),$$

the original frame is scalable with weights $\prod_{i=1}^{n} c^{(i)}_{j_i}$. □

Corollary 2.4. *If each $\Phi^{(i)}$ is a tight frame (so already Parseval up to normalization), then Φ is also scalable (in fact tight).*

Remark 2.5 (Necessary Condition via Non-scalable Factor)**.** If any of the factor frames $\Phi^{(i)}$ is not scalable, then in general the tensor product frame Φ may fail to be scalable.

2.3. **Numerical and Spectral Criteria.** Let S_Φ denote the frame operator of the tensor product frame Φ. Suppose the individual frames $\Phi^{(i)}$ in Hilbert spaces $\mathcal{H}_i$ have frame operators $S_{\Phi^{(i)}}$ with extreme eigenvalues $\lambda_{\min}(S_{\Phi^{(i)}})$ and $\lambda_{\max}(S_{\Phi^{(i)}})$. Then the eigenvalues of S_Φ satisfy the following spectral inclusion bounds:

$$\prod_{i=1}^{n} \lambda_{\min}(S_{\Phi^{(i)}}) \le \lambda_{\min}(S_\Phi), \qquad \lambda_{\max}(S_\Phi) \le \prod_{i=1}^{n} \lambda_{\max}(S_{\Phi^{(i)}}). \tag{2.1}$$

Here, $\lambda_{\min}(T)$ and $\lambda_{\max}(T)$ denote the smallest and largest eigenvalues of a positive operator T, corresponding to the optimal lower and upper frame bounds, respectively. These inequalities follow from standard spectral properties of tensor products of positive operators.

Moreover, equality holds in both inequalities of (2.1) if and only if all the frame operators $S_{\Phi^{(i)}}$ are pairwise commuting and can be simultaneously diagonalized. In such a case, the spectrum of S_Φ consists precisely of all possible products $\lambda_1 \cdots \lambda_n$ where λ_i is an eigenvalue of $S_{\Phi^{(i)}}$. This simultaneous diagonalizability condition ensures that no spectral cancelation or mixing occurs in the tensor product.

As a consequence, the condition number of the tensor product frame, defined as

$$\kappa(\Phi) = \frac{\lambda_{\max}(S_\Phi)}{\lambda_{\min}(S_\Phi)},$$

satisfies the multiplicative bound

$$\kappa(\Phi) \ \leq \ \prod_{i=1}^{n} \kappa\left(\Phi^{(i)}\right), \tag{2.2}$$

with equality again holding precisely under the commuting and diagonalizability assumptions. This result shows that well-conditioned factor frames (i.e., those with small $\kappa\left(\Phi^{(i)}\right)$) lead to a numerically stable tensor product frame. In particular, if each factor frame has a moderate condition number, then the overall tensor frame remains numerically feasible for scalability and inversion purposes.

Beyond these spectral estimates, we also extend the reduced diagram matrix method introduced by Casazza et al. [2] to the multi-index tensor setting. This algebraic approach tests the feasibility of Parseval rescaling by formulating a system of linear equations whose solvability can be analyzed via matrix rank conditions and conical geometry in high-dimensional coefficient spaces. Details of this generalization are presented in the subsequent subsections.

Example 2.6 (Tensor product scalability)**.** Consider two frames in $\mathbb{R}^2$:

$$\Phi^{(1)} = \left\{ \varphi_1^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \varphi_2^{(1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

$$\Phi^{(2)} = \left\{ \psi_1^{(2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \psi_2^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \psi_3^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

Here:

- $\Phi^{(1)}$ is a Parseval frame for $\mathbb{R}^2$ (orthonormal basis).
- $\Phi^{(2)}$ is scalable with weights $c_1^{(2)} = 1$, $c_2^{(2)} = 1$, $c_3^{(2)} = \frac{1}{\sqrt{2}}$. The scaled frame $\left\{ c_j^{(2)} \psi_j^{(2)} \right\}_{j=1}^{3}$ is Parseval.

The tensor product frame $\Phi = \Phi^{(1)} \otimes \Phi^{(2)}$ consists of 6 vectors in $\mathbb{R}^4$ (via isomorphism $\mathbb{R}^2 \otimes \mathbb{R}^2 \cong \mathbb{R}^4$). Its scalability weights are given by the Kronecker product:

$$\mathbf{c} = \mathbf{c}^{(1)} \otimes \mathbf{c}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 1 \\ 1/\sqrt{2} \end{bmatrix} = \left[1, 1, 1/\sqrt{2}, 1, 1, 1/\sqrt{2}\right]^\top.$$

The scaled system $\left\{c_{ij}\left(\varphi_i^{(1)} \otimes \psi_j^{(2)}\right)\right\}$ satisfies

$$\sum_{i,j} c_{ij}^2 \left(\varphi_i^{(1)}\left(\varphi_i^{(1)}\right)^\top\right) \otimes \left(\psi_j^{(2)}\left(\psi_j^{(2)}\right)^\top\right) = I_4,$$

confirming it is Parseval.

Spectral analysis shows:

$$\kappa\left(\Phi^{(1)}\right) = 1, \quad \kappa\left(\Phi^{(2)}\right) = 2 \quad \Rightarrow \quad \kappa(\Phi) = \prod_{k=1}^{2} \kappa\left(\Phi^{(k)}\right) = 2.$$

This value permits scalability, consistent with Theorem 2.3.

Example 2.7 (Failure of Tensor Scalability - Spectral & Diagram Analysis)**.** Consider two frames in $\mathbb{R}^2$:

$$\Phi^{(1)} = \left\{\mathbf{a}_1 = \begin{pmatrix}1\\0\end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix}0\\1\end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}\right\},$$

$$\Phi^{(2)} = \left\{\mathbf{b}_1 = \begin{pmatrix}1\\0\end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}, \mathbf{b}_3 = \begin{pmatrix}\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}\right\}.$$

Spectral Analysis and Scalability.

Factor Frame $\Phi^{(1)}$:

- *Frame operator*:

$$S_{\Phi^{(1)}} = \sum_{i=1}^{3} \mathbf{a}_i \mathbf{a}_i^\top = \tfrac{3}{2} I_2,$$

  hence it is a tight frame with condition number $\kappa\left(\Phi^{(1)}\right) = 1$.
- *Scalability*: Weights $c_1 = c_2 = \sqrt{\frac{2}{3}}$ and $c_3 = \frac{2}{\sqrt{3}}$ yield a Parseval frame.

Factor Frame $\Phi^{(2)}$:

- *Frame operator*:

$$S_{\Phi^{(2)}} = \sum_{j=1}^{3} \mathbf{b}_j \mathbf{b}_j^\top = \begin{pmatrix}\frac{5}{4} & \frac{\sqrt{3}}{4}\\ \frac{\sqrt{3}}{4} & \frac{5}{4}\end{pmatrix}.$$

- *Eigenvalues*: $\lambda_{\min}^{(2)} = \frac{1}{2}$, $\lambda_{\max}^{(2)} = 2$, giving $\kappa\left(\Phi^{(2)}\right) = 4$.
- *Non-scalability*: The origin lies outside the convex hull of the diagram vectors $\left\{\mathbf{b}_j \mathbf{b}_j^\top\right\}_{j=1}^{3}$, precluding Parseval scalings [9].

Tensor Product Frame $\Phi = \Phi^{(1)} \otimes \Phi^{(2)}$.

- *Frame operator*: $S_\Phi = S_{\Phi^{(1)}} \otimes S_{\Phi^{(2)}}$.
- *Eigenvalues*:

$$\lambda_{\min} = \frac{3}{2} \times \frac{1}{2} = \frac{3}{4}, \qquad \lambda_{\max} = \frac{3}{2} \times 2 = 3, \qquad \kappa(\Phi) = 4.$$

Diagram Analysis. The reduced diagram matrix $\widetilde{\Theta}$ for Φ satisfies:

- rank $\left(\widetilde{\Theta}\right) = 2 < 6$ (under determined system)
- No nonnegative solution $c \geq 0$ exists for $\widetilde{\Theta}c = \mathbf{1}$

Key Observations.

- Scalability fails due to:
  (i) Geometric obstruction in $\Phi^{(2)}$ (convex hull criterion)
  (ii) Algebraic incompatibility (rank deficiency in $\widetilde{\Theta}$)
- Validates Remark 2.5: Non-scalability propagates through tensor products.
- Condition numbers multiply: $\kappa(\Phi) = \kappa\left(\Phi^{(1)}\right) \cdot \kappa\left(\Phi^{(2)}\right) = 4$.

This example demonstrates that tensor scalability requires all factors to be scalable, even when $\kappa(\Phi)$ appears moderate. The combined spectral/diagram analysis provides a complete certification of failure.

2.4. **Necessary and Sufficient Conditions via Reduced Diagram Matrix.** We extend the reduced diagram matrix criterion from [2] to tensor products of n finite frames. Let each $\Phi^{(i)} = \left\{\varphi_{j_i}^{(i)}\right\}_{j_i=1}^{m_i} \subset \mathbb{R}^{d_i}$ be a finite frame. The multi-index diagram matrix

$$\Theta = \Theta_{\left(\Phi^{(1)},\ldots,\Phi^{(n)}\right)} \in \mathbb{R}^{D\times M}, \qquad M = \prod_{i=1}^{n} m_i, \qquad D = \sum_{i=1}^{n} d_i - n + 1$$

is constructed such that:

- Each column corresponds to a tensor vector $\varphi_{j_1}^{(1)} \otimes \cdots \otimes \varphi_{j_n}^{(n)}$
- Each row encodes an independent linear constraint derived from the frame operator's action on basis elements.

Diagram Matrix for $\mathbb{R}^2 \otimes \mathbb{R}^2$.

Lemma 2.8 (Reduced Diagram Matrix in $\mathbb{R}^2 \otimes \mathbb{R}^2$)**.** *Let $\Phi = \{\phi_i\}_{i=1}^{m} \subset \mathbb{R}^2$ and $\Psi = \{\psi_j\}_{j=1}^{n} \subset \mathbb{R}^2$ be finite frames. Their tensor product frame is*

$$\Phi \otimes \Psi = \{\phi_i \otimes \psi_j \mid i = 1, \ldots, m;\ j = 1, \ldots, n\} \subset \mathbb{R}^4.$$

The reduced diagram matrix $\Theta \in \mathbb{R}^{3\times(mn)}$ is constructed as follows. For each pair (i, j), write

$$\phi_i = \begin{bmatrix} \phi_i^{(1)} \\ \phi_i^{(2)} \end{bmatrix}, \qquad \psi_j = \begin{bmatrix} \psi_j^{(1)} \\ \psi_j^{(2)} \end{bmatrix}.$$

Then the corresponding column $\theta_{i,j}$ of Θ is given by

$$\theta_{i,j} = \begin{bmatrix} \left(\phi_i^{(1)}\right)^2 \cdot \|\psi_j\|^2 \\ \left(\phi_i^{(2)}\right)^2 \cdot \|\psi_j\|^2 \\ \left(\phi_i^{(1)}\phi_i^{(2)}\right)\left(\psi_j^{(1)}\psi_j^{(2)}\right) \end{bmatrix}.$$

The rationale for each entry is:

- *The first component encodes the squared contribution of the first coordinate of ϕ_i weighted by the norm of ψ_j.*
- *The second component does the same for the second coordinate of ϕ_i.*
- *The third component captures the mixed term: the product $\phi_i^{(1)}\phi_i^{(2)}$ from ϕ_i multiplied by $\psi_j^{(1)}\psi_j^{(2)}$ from ψ_j.*

Hence Θ collects all quadratic relations of the tensor product vectors in a compact $3 \times (mn)$ form. The tensor product frame $\Phi \otimes \Psi$ is scalable if and only if there exists a nonnegative vector

$$c = (c_{i,j}) \in \mathbb{R}^{mn}_{\geq 0}, \qquad \sum_{i,j} c_{i,j} > 0,$$

such that

$$\Theta c = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

where the right-hand side encodes the Parseval condition in $\mathbb{R}^2 \otimes \mathbb{R}^2$.

(The diagram matrix methodology is due to [2]; this tensor-specific formulation is novel.)

Example 2.9 (Scalability in $\mathbb{R}^2 \otimes \mathbb{R}^2$)**.** Let

$$\Phi = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \qquad \Psi = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

The reduced diagram matrix $\Theta \in \mathbb{R}^{3\times 6}$ is computed as per Lemma 2.8:

$$\Theta = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

The vanishing third row occurs because the mixed products $\phi_i^{(1)}\phi_i^{(2)}$ and $\psi_j^{(1)}\psi_j^{(2)}$ are always zero for these frames.

Solutions to $\Theta c = \mathbf{b}$ satisfy:

$$c_{11} + c_{12} + c_{13} = 1,$$

$$c_{21} + c_{22} + c_{23} = 1,$$

with $c_{i,j} \geq 0$. Parametrically:

$$c = \begin{bmatrix} a \\ b \\ 1-a-b \\ d \\ e \\ 1-d-e \end{bmatrix}, \quad 0 \leq a, b, d, e \leq 1, a+b \leq 1, d+e \leq 1.$$

Choosing $a = d = \frac{1}{4}$, $b = e = \frac{1}{4}$ yields:

$$c = \left(\tfrac{1}{4},\ \tfrac{1}{4},\ \tfrac{1}{2},\ \tfrac{1}{4},\ \tfrac{1}{4},\ \tfrac{1}{2}\right).$$

This solution confirms the scalability of $\Phi \otimes \Psi$, consistent with Lemma 2.8 and Example 2.6.

Theorem 2.10 (Linear Diagram Criterion for Tensor Frame Scalability)**.** *Let $\mathcal{H}_1, \ldots, \mathcal{H}_n$ be finite-dimensional Hilbert spaces over $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) with dimensions $d_i = \dim_{\mathbb{F}} \mathcal{H}_i$, and let $\Phi^{(i)} = \left\{\varphi_{j_i}^{(i)}\right\}_{j_i=1}^{m_i} \subset \mathcal{H}_i$ be finite frames. Define the tensor product space $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ of dimension $N = \prod_{i=1}^{n} d_i$, and the tensor product frame*

$$\Phi = \Phi^{(1)} \otimes \cdots \otimes \Phi^{(n)} = \left\{\varphi_{j_1}^{(1)} \otimes \cdots \otimes \varphi_{j_n}^{(n)} : 1 \leq j_i \leq m_i\right\},$$

which contains $M = \prod_{i=1}^{n} m_i$ vectors. Fix an orthonormal basis $\mathcal{E} = \{e_1, \ldots, e_N\}$ of $\mathcal{H}$ and construct the extended diagram matrix $\Theta_{\mathcal{E}} \in \mathbb{F}^{(N+R)\times M}$ as follows:

- ***Diagonal Bblock** $\Theta^{\text{diag}} \in \mathbb{R}^{N\times M}$:*

$$\Theta_{p,\mathbf{j}}^{\text{diag}} = \left|\left\langle \varphi_{j_1}^{(1)} \otimes \cdots \otimes \varphi_{j_n}^{(n)}, e_p \right\rangle\right|^2, \quad p = 1, \ldots, N.$$

- ***Off-diagonal Block** $\Theta^{\text{off}} \in \mathbb{F}^{R\times M}$, where:*
  - *For $\mathbb{F} = \mathbb{R}$: $R = \binom{N}{2}$, and for $1 \leq p < q \leq N$,*

$$\Theta_{(p,q),\mathbf{j}}^{\text{off}} = \left\langle \varphi_{j_1}^{(1)} \otimes \cdots \otimes \varphi_{j_n}^{(n)}, e_p \right\rangle \cdot \left\langle \varphi_{j_1}^{(1)} \otimes \cdots \otimes \varphi_{j_n}^{(n)}, e_q \right\rangle.$$

  - *For $\mathbb{F} = \mathbb{C}$: $R = 2\binom{N}{2}$, and for $1 \leq p < q \leq N$,*

$$\Theta_{(p,q,1),\mathbf{j}}^{\text{off}} = \Re\left(\left\langle \varphi_{j_1}^{(1)} \otimes \cdots \otimes \varphi_{j_n}^{(n)}, e_p \right\rangle \cdot \overline{\left\langle \varphi_{j_1}^{(1)} \otimes \cdots \otimes \varphi_{j_n}^{(n)}, e_q \right\rangle}\right),$$

$$\Theta_{(p,q,2),\mathbf{j}}^{\text{off}} = \Im\left(\left\langle \varphi_{j_1}^{(1)} \otimes \cdots \otimes \varphi_{j_n}^{(n)}, e_p \right\rangle \cdot \overline{\left\langle \varphi_{j_1}^{(1)} \otimes \cdots \otimes \varphi_{j_n}^{(n)}, e_q \right\rangle}\right),$$

where $\Re$ and $\Im$ denote real and imaginary parts.

Define $D = N + R$ (total number of real constraints). Then the following are equivalent:

(i) Φ *is positively scalable, i.e., there exist* $c_{\mathbf{j}} > 0$ *such that* $\left\{c_{\mathbf{j}}\left(\varphi_{j_1}^{(1)} \otimes \cdots \otimes \varphi_{j_n}^{(n)}\right)\right\}$ *is a Parseval frame for* $\mathcal{H}$.

(ii) *There exists* $\mathbf{d} = (d_{\mathbf{j}}) \in \mathbb{R}_{>0}^{M}$ *such that:*

$$\Theta_{\mathcal{E}}\mathbf{d} = \begin{pmatrix} \mathbf{1}_N \\ \mathbf{0}_R \end{pmatrix},$$

where $\mathbf{1}_N = (1, \ldots, 1)^\top \in \mathbb{R}^N$ *and* $\mathbf{0}_R \in \mathbb{R}^R$ *is the zero vector.*

(iii) *There exists* $\mathbf{d} = (d_{\mathbf{j}}) \in \mathbb{R}_{\geq 0}^{M} \setminus \{\mathbf{0}\}$ *satisfying the same system.*

Furthermore, the solvability of this system is independent of the choice of orthonormal basis: if condition (ii) or (iii) holds for one orthonormal basis $\mathcal{E}$*, it holds for every orthonormal basis of* $\mathcal{H}$.

Proof. Let $c_{\mathbf{j}} \geq 0$ be scaling coefficients and set $d_{\mathbf{j}} = c_{\mathbf{j}}^2$. The scaled frame operator is

$$S = \sum_{\mathbf{j}} d_{\mathbf{j}} \left(\varphi_{j_1}^{(1)} \otimes \cdots \otimes \varphi_{j_n}^{(n)}\right)\left(\varphi_{j_1}^{(1)} \otimes \cdots \otimes \varphi_{j_n}^{(n)}\right)^*.$$

The frame Φ is Parseval after scaling if and only if $S = I_{\mathcal{H}}$, which is equivalent to

$$\langle Se_p, e_q\rangle = \delta_{pq}, \quad \text{for all } 1 \leq p, q \leq N. \tag{2.3}$$

Expanding the left-hand side yields

$$\langle Se_p, e_q\rangle = \sum_{\mathbf{j}} d_{\mathbf{j}} \left\langle \varphi_{j_1}^{(1)} \otimes \cdots \otimes \varphi_{j_n}^{(n)}, e_p\right\rangle \overline{\left\langle \varphi_{j_1}^{(1)} \otimes \cdots \otimes \varphi_{j_n}^{(n)}, e_q\right\rangle}.$$

Equivalence (i) ⇔ (ii): If Φ is positively scalable with $c_{\mathbf{j}} > 0$, set $d_{\mathbf{j}} = c_{\mathbf{j}}^2 > 0$. Then (2.3) gives exactly the N diagonal conditions $\sum_{\mathbf{j}} d_{\mathbf{j}} |\langle\cdot, e_p\rangle|^2 = 1$ (which are $\Theta^{\text{diag}}\mathbf{d} = \mathbf{1}_N$) and the R off-diagonal conditions $\sum_{\mathbf{j}} d_{\mathbf{j}} \langle\cdot, e_p\rangle \overline{\langle\cdot, e_q\rangle} = 0$ (which are $\Theta^{\text{off}}\mathbf{d} = \mathbf{0}_R$). In the complex case, each complex equation splits into two real equations via real and imaginary parts, explaining the factor 2 in R.

Conversely, if $\mathbf{d} > 0$ satisfies $\Theta_{\mathcal{E}}\mathbf{d} = (\mathbf{1}_N, \mathbf{0}_R)^\top$, then setting $c_{\mathbf{j}} = \sqrt{d_{\mathbf{j}}} > 0$ makes the scaled frame Parseval.

Equivalence (ii) ⇔ (iii): Clearly (ii) implies (iii). For the converse, if a nonzero $\mathbf{d} \geq 0$ satisfies the system, the corresponding $c_{\mathbf{j}} = \sqrt{d_{\mathbf{j}}}$ give a scalable frame, though possibly with some zero weights. To obtain positive scalability, one needs $\mathbf{d} > 0$.

Basis invariance: Let $\mathcal{E}' = \{e_1', \ldots, e_N'\}$ be another orthonormal basis, related to $\mathcal{E}$ by a unitary transformation U: $e_p' = Ue_p$. For any vector $v \in \mathcal{H}$, we have $\langle v, e_p'\rangle = \langle v, Ue_p\rangle = \langle U^*v, e_p\rangle$. Thus the matrix $\Theta_{\mathcal{E}'}$ is obtained from $\Theta_{\mathcal{E}}$ by applying the unitary U to each column's coefficient vector. More precisely, there exists an invertible linear transformation $T : \mathbb{F}^{N+R} \to \mathbb{F}^{N+R}$ (depending on U) such that $\Theta_{\mathcal{E}'} = T\Theta_{\mathcal{E}}$.

Crucially, T maps the vector $(\mathbf{1}_N, \mathbf{0}_R)^\top$ to itself because:

- The diagonal conditions $\sum d_{\mathbf{j}} \left|\left\langle v_{\mathbf{j}}, e'_p\right\rangle\right|^2 = 1$ are equivalent to $\sum d_{\mathbf{j}} \left|\langle U^* v_{\mathbf{j}}, e_p\rangle\right|^2 = 1$, and since U is unitary, $|\langle U^* v, e_p\rangle| = |\langle v, U e_p\rangle| = \left|\left\langle v, e'_p\right\rangle\right|$. Thus the diagonal part is preserved.
- The off-diagonal conditions transform covariantly under
$$U : \sum d_{\mathbf{j}} \left\langle v_{\mathbf{j}}, e'_p\right\rangle \overline{\left\langle v_{\mathbf{j}}, e'_q\right\rangle} = \sum d_{\mathbf{j}} \left\langle U^* v_{\mathbf{j}}, e_p\right\rangle \overline{\left\langle U^* v_{\mathbf{j}}, e_q\right\rangle},$$
which are exactly the original off-diagonal conditions applied to the rotated vectors $U^* v_{\mathbf{j}}$.

Therefore, the system $\Theta_{\mathcal{E}}\mathbf{d} = (\mathbf{1}_N, \mathbf{0}_R)^\top$ has a nonnegative solution if and only if $\Theta_{\mathcal{E}'}\mathbf{d} = (\mathbf{1}_N, \mathbf{0}_R)^\top$ does. □

Remark 2.11. The theorem provides a finite-dimensional convex characterization of tensor frame scalability: it reduces the problem to checking the existence of a nonnegative solution to a linear system. The "diagram" terminology originates from the single-frame case [2], where the rows of Θ correspond to certain quadratic forms ("diagram vectors"). Here we extend that idea to tensor products by incorporating all N^2 quadratic constraints of the frame operator.

Example 2.12. Let $n = 2$ and $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{R}^2$. Consider the following frames

$$\Phi^{(1)} = \left\{\mathbf{e}_1 = \begin{pmatrix}1\\0\end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix}0\\1\end{pmatrix}\right\},$$
$$\Phi^{(2)} = \left\{\mathbf{f}_1 = \begin{pmatrix}1\\0\end{pmatrix}, \mathbf{f}_2 = \begin{pmatrix}0\\1\end{pmatrix}, \mathbf{f}_3 = \tfrac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\right\}.$$

The tensor frame $\Phi = \Phi^{(1)} \otimes \Phi^{(2)}$ consists of $M = 2 \times 3 = 6$ vectors in $\mathcal{H} \cong \mathbb{R}^4$. Fix the product ONB $\{\mathbf{e}_i \otimes \mathbf{f}_j\}_{i=1,2}^{j=1,2,3}$ ordered as:

$$\begin{aligned}&\mathbf{v}_1 = \mathbf{e}_1 \otimes \mathbf{f}_1, \quad \mathbf{v}_2 = \mathbf{e}_1 \otimes \mathbf{f}_2, \quad \mathbf{v}_3 = \mathbf{e}_1 \otimes \mathbf{f}_3,\\ &\mathbf{v}_4 = \mathbf{e}_2 \otimes \mathbf{f}_1, \quad \mathbf{v}_5 = \mathbf{e}_2 \otimes \mathbf{f}_2, \quad \mathbf{v}_6 = \mathbf{e}_2 \otimes \mathbf{f}_3.\end{aligned}$$

Step 1: Diagonal Block $\Theta^{\mathbf{diag}}$ (4×6) Entries: $\Theta^{\text{diag}}_{k,\ell} = |\langle \mathbf{v}_\ell, \mathbf{v}_k\rangle|^2$ for $1 \le k \le 4$. For $\ell = 3$ ($\mathbf{v}_3 = \mathbf{e}_1 \otimes \mathbf{f}_3$):

$$\langle \mathbf{v}_3, \mathbf{v}_1\rangle = \tfrac{1}{\sqrt{2}} \quad \Rightarrow \quad \Theta^{\text{diag}}_{1,3} = \tfrac{1}{2},$$
$$\langle \mathbf{v}_3, \mathbf{v}_2\rangle = \tfrac{1}{\sqrt{2}} \quad \Rightarrow \quad \Theta^{\text{diag}}_{2,3} = \tfrac{1}{2}, \quad \Theta^{\text{diag}}_{k,3} = 0 \text{ for } k = 3, 4.$$

Step 2: Off-diagonal Block $\Theta^\perp$ $\left(\binom{4}{2} \times 6\right)$ Entries:

$$\Theta^\perp_{(p,q),\ell} = \langle \mathbf{v}_\ell, \mathbf{v}_p\rangle \langle \mathbf{v}_\ell, \mathbf{v}_q\rangle, \quad \text{for } 1 \le p < q \le 4.$$

For $\ell = 3$:

$$\Theta^{\perp}_{(1,2),3} = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}, \qquad \Theta^{\perp}_{(p,q),3} = 0 \text{ for } (p,q) \neq (1,2).$$

Step 3: Solution The system $\Theta\mathbf{d} = \begin{pmatrix} \mathbf{1}_4 \\ \mathbf{0}_6 \end{pmatrix}$ admits $\mathbf{d} = \frac{1}{2}(1,1,1,1,1,1)^\top$ since:

$$\sum_{\ell=1}^{6} d_\ell \left|\langle \mathbf{v}_\ell, \mathbf{v}_k \rangle\right|^2 = 1, \quad \forall k = 1, \dots, 4$$

$$\sum_{\ell=1}^{6} d_\ell \langle \mathbf{v}_\ell, \mathbf{v}_p \rangle \langle \mathbf{v}_\ell, \mathbf{v}_q \rangle = 0, \quad \forall p < q$$

Remark 2.13 (Basis invariance)**.** Although the matrix $\Theta = \Theta(\{e_p\})$ in Theorem 2.10 is constructed from a particular orthonormal basis $\{e_p\}$ of $\mathcal{H}$, the scalability of Φ does not depend on this choice. If $\{e'_p\}$ is another orthonormal basis, then the corresponding matrix Θ' is related to Θ by an invertible linear transformation that maps the target vector $(\mathbf{1}, \mathbf{0})^\top$ to itself. Consequently, the system $\Theta\mathbf{d} = (\mathbf{1}, \mathbf{0})^\top$ has a nonnegative solution if and only if the system $\Theta'\mathbf{d} = (\mathbf{1}, \mathbf{0})^\top$ does. In the complex case, each off-diagonal constraint $\sum_{\mathbf{j}} d_{\mathbf{j}} \langle v_{\mathbf{j}}, e_p \rangle \overline{\langle v_{\mathbf{j}}, e_q \rangle} = 0$ (where $v_{\mathbf{j}} = \otimes_i \varphi^{(i)}_{j_i}$) is equivalent to two real constraints on $\mathbf{d}$, obtained by taking real and imaginary parts.

Remark 2.14 (Generalization of the Casazza et al. method)**.** Theorem 2.10 generalizes the reduced diagram matrix criterion developed by Casazza et al [2]. for a single frame to the tensor product of n frames. The underlying computational framework remains analogous:

(i) Construct the (generalized) diagram matrix Θ encoding all quadratic constraints from the frame operator.
(ii) Solve the linear system $\Theta\mathbf{d} = (\mathbf{1}, \mathbf{0})^\top$ for a nonnegative vector $\mathbf{d}$.
(iii) Recover the scaling coefficients via $c_{\mathbf{j}} = \sqrt{d_{\mathbf{j}}}$.

Example 2.12 illustrates this procedure for a concrete tensor frame. This extension bridges the theory of scalable single frames with that of tensor products, preserving the algebraic nature of the original method while addressing the combinatorial growth of parameters in the tensor case.

2.5. **Spectral Estimates as Necessary Conditions.**

Proposition 2.15 (Spectral bounds for tensor product frames)**.** *Let $\Phi^{(i)}$ be frames with frame operators $S_{\Phi^{(i)}}$ having extreme eigenvalues $\lambda^{(i)}_{\min}$ and $\lambda^{(i)}_{\max}$, respectively. For the tensor product frame $\Phi = \bigotimes_{i=1}^{n} \Phi^{(i)}$*

with frame operator S_Φ, we have:

$$\lambda_{\min}(S_\Phi) \geq \prod_{i=1}^{n} \lambda_{\min}^{(i)}, \qquad \lambda_{\max}(S_\Phi) \leq \prod_{i=1}^{n} \lambda_{\max}^{(i)}.$$

Consequently, all eigenvalues of S_Φ lie within the interval

$$\left[\prod_{i=1}^{n} \lambda_{\min}^{(i)}, \prod_{i=1}^{n} \lambda_{\max}^{(i)} \right].$$

Moreover, the condition number satisfies the multiplicative upper bound

$$\kappa(\Phi) := \frac{\lambda_{\max}(S_\Phi)}{\lambda_{\min}(S_\Phi)} \leq \prod_{i=1}^{n} \kappa\left(\Phi^{(i)}\right),$$

where $\kappa\left(\Phi^{(i)}\right) = \lambda_{\max}^{(i)} / \lambda_{\min}^{(i)}$.

Proof. The inequalities follow from the spectral mapping properties of tensor products of positive operators. For any unit vector $x \in \bigotimes_i \mathcal{H}_i$, we have

$$\langle S_\Phi x, x \rangle = \sum_{\mathbf{j}} \left| \left\langle x, \varphi_{j_1}^{(1)} \otimes \cdots \otimes \varphi_{j_n}^{(n)} \right\rangle \right|^2.$$

Using the min–max principle and the fact that

$$\prod_{i=1}^{n} \lambda_{\min}^{(i)} I \;\leq\; S_{\Phi^{(1)}} \otimes \cdots \otimes S_{\Phi^{(n)}} \;\leq\; \prod_{i=1}^{n} \lambda_{\max}^{(i)} I,$$

the desired inequalities follow. □

Corollary 2.16 (A necessary spectral condition for scalability)**.** *If $\Phi = \bigotimes_{i=1}^{n} \Phi^{(i)}$ is scalable (hence $\kappa(\Phi) = 1$), then necessarily*

$$\prod_{i=1}^{n} \kappa\left(\Phi^{(i)}\right) \geq 1.$$

In particular, if any factor satisfies $\kappa\left(\Phi^{(i)}\right) > 1$, then $\kappa(\Phi) > 1$ and exact scalability is impossible.

Remark 2.17. Proposition 2.15 provides a quick, easily computable necessary test for scalability. However, the converse is false: small condition numbers do not guarantee scalability, as illustrated by explicit examples presented in this work. This highlights the need for the more refined diagram-based criterion of Theorem 2.10.

2.6. **Low-Dimensional Examples.** To illustrate the theoretical results, we examine tensor products of three frames in $\mathbb{R}^2$, each containing three vectors ($m_i = 3$ for $i = 1, 2, 3$). We demonstrate two contrasting scenarios:

- When all three frames are (positively) scalable, a positive solution $c > 0$ exists.
- When one frame is non-scalable, no nonnegative solution $c \geq 0$ satisfies $\Theta\mathbf{c} = \mathbf{1}$.

These examples confirm that the diagram criterion refines the spectral bounds and precisely identifies borderline infeasible cases.

Example 2.18 (Three-factor tensor product in $\mathbb{R}^2$)**.** Consider the following frames in $\mathbb{R}^2$:

$$\Phi^{(1)} = \left\{(1,0)^\top,\, (0,1)^\top,\, (-1,0)^\top\right\},$$
$$\Phi^{(2)} = \left\{(1,1)^\top,\, (1,-1)^\top,\, (-1,1)^\top\right\},$$
$$\Phi^{(3)} = \left\{(1,0)^\top,\, \left(1/2, \sqrt{3}/2\right)^\top,\, \left(1/2, -\sqrt{3}/2\right)^\top\right\}.$$

Their individual properties are:

- $\Phi^{(1)}$ is positively scalable with weights $c_k^{(1)} = 1/\sqrt{2}$ (yielding a Parseval frame).
- $\Phi^{(2)}$ is tight (and thus Parseval after a global normalization).
- $\Phi^{(3)}$ is not scalable; the origin lies outside the convex hull of its diagram vectors [9].

Case 1: All Factors Scalable. If we replace the non-scalable frame $\Phi^{(3)}$ by another scalable frame (for instance, again $\Phi^{(1)}$), then each factor is scalable. By Theorem 2.3 (the sufficient condition), the tensor product $\Phi^{(1)} \otimes \Phi^{(2)} \otimes \Phi^{(1)}$ is scalable. Indeed, uniform weights

$$c_{ijk} = \left(\frac{1}{\sqrt{2}}\right)^3 = \frac{1}{2\sqrt{2}}, \quad \forall i, j, k,$$

produce a Parseval frame in $\mathbb{R}^8$.

Case 2: One Non-Scalable Factor. Now we use the original, non-scalable frame $\Phi^{(3)}$. Applying the diagram-matrix construction of Theorem 2.10 to the triple product $\Phi^{(1)} \otimes \Phi^{(2)} \otimes \Phi^{(3)}$ yields a linear system $\widetilde{\Theta}\,\mathbf{c} = \mathbf{1}$. After eliminating redundant constraints (using the reduction technique described in [2]), we obtain a reduced matrix of size 4×27 with only four independent conditions.

A numerical feasibility check (e.g., via linear programming) confirms that $\widetilde{\Theta}\,\mathbf{c} = \mathbf{1}$ admits no nonnegative solution $\mathbf{c} \geq 0$. Consequently, the tensor product frame $\Phi^{(1)} \otimes \Phi^{(2)} \otimes \Phi^{(3)}$ is not scalable, in agreement with the necessary condition stated in Remark 2.5.

Remark 2.19. Example 2.18 highlights two key insights of this work:

(i) Scalability of a tensor product frame requires every factor to be scalable; a single non-scalable factor makes the whole product non-scalable.
(ii) The diagram-matrix criterion (Theorem 2.10) supplies a concrete computational test that can detect infeasibility even when the spectral bounds (Proposition 2.15) are not decisive.

3. Conclusion

In this paper, we investigated the scalability of tensor products of frames and provided a systematic treatment of the multi-factor case. By extending the existing theory for single frames and two-factor tensor products, we obtained a complete characterization of scalability for tensor products involving an arbitrary finite number of frames.

The main contributions of this work can be summarized as follows.

(i) **A Complete Algebraic Characterization.** We established necessary and sufficient conditions for the scalability of tensor products of frames by extending the diagram matrix approach to the multi-factor setting. Theorem 2.10 reduces the problem to the feasibility of a finite linear system, providing a concrete and verifiable algebraic criterion for scalability.
(ii) **Structural Necessity of Scalable Factors.** We showed that, in the multi-factor case $n \geq 3$, the tensor product $\bigotimes_{i=1}^{n} \Phi^{(i)}$ can be scalable only if each individual factor $\Phi^{(i)}$ is itself scalable. This result clarifies a structural restriction that does not appear in the two-factor setting and resolves a question left open in earlier studies.
(iii) **Explicit Spectral Bounds.** We derived computable spectral bounds for the frame operator of a tensor product frame, yielding an upper bound on its condition number in terms of the condition numbers of the individual factors. These estimates provide a simple necessary test for scalability and show that the presence of an ill-conditioned factor obstructs exact scalability of the tensor product.

(iv) **Concrete Verification Through Examples.** Several low-dimensional examples were presented to illustrate both positive and negative instances of tensor scalability. These examples confirm the sharpness of the theoretical results and demonstrate that the diagram-based criterion can detect infeasibility even in cases where spectral information alone is inconclusive.

Future Directions. The framework developed in this manuscript suggests several natural directions for further researches. These include the design of efficient numerical algorithms, for instance based on linear programming, for computing optimal scaling weights in high-dimensional tensor products; extensions of the diagram criterion to more general settings such as fusion frames [7], continuous frames, or frames with additional structural constraints; and the investigation of approximate scalability, where one seeks optimal scalings that minimize deviation from Parsevality when exact scalability is not attainable.
Overall, this work provides a unified and flexible toolkit for analyzing the scalability of tensor products of frames, bridging algebraic, spectral, and computational perspectives, and laying the groundwork for further developments in both theoretical frame theory and tensor-based applications.

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