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Dense Fuzzy Sets in Fuzzy Topological Spaces and Separability

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Abstract. Dense fuzzy sets play an inevitable role in exploring various aspects of fuzzy topological spaces. The present paper aims at investigating the properties of dense fuzzy sets in different fuzzy topological spaces. Several characterization theorems and other novel results are obtained by means of dense fuzzy sets in specific contexts as well as in general settings. The relationship between quasi-coincidence of fuzzy sets and denseness property is also analyzed. Additionally, the concepts of separability, Q-separability, β-separability and Q-β-separability are examined.

1. Introduction

In 1965, L. A. Zadeh [23] established the theory of fuzzy sets and later, in 1968, C. L. Chang [4] introduced fuzzy topology containing fuzzy subsets of a set X as open sets. In order to extend topological concepts like neighbourhood, convergence etc. to a fuzzy context, C. K. Wong [22] proposed the concept of fuzzy points. Later, in 1980, P. P. Ming and L. Y. Ming [7] modified the definition of fuzzy points so that crisp point could be presented as a special case of fuzzy points. They also established Moore-Smith convergence of fuzzy nets using quasi-coincidence and Q-neighbourhoods. In [8], the authors defined denseness and Q-denseness of a collection of fuzzy points instead of denseness of subsets of X as in crisp case. The denseness and semi-denseness of fuzzy subsets of X were studied by G. Thangaraj and G. Balasubramanian [15, 16]. G. Thangaraj and S. Anjalmose proposed the concept of fuzzy Baire

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spaces using fuzzy nowhere dense sets and presented several properties and characterizations of fuzzy Baire spaces [12–14].

In 2014, S. Anjalmose and G. Thangaraj [2] defined fuzzy semi-nowhere dense sets and fuzzy semi-Baireness in fuzzy topological spaces. In 2018, G. Thangaraj and R. Palani [18] introduced the concept of fuzzy β-nowhere dense sets and defined fuzzy β-Baire spaces. In [20], G. Thangaraj and A. Vinothkumar introduced the notion of fuzzy fraction dense spaces and established several characterizations of fuzzy fraction dense spaces. Some other generalizations of dense fuzzy sets such as fuzzy baire dense sets, fuzzy neutrosophic dense sets etc. have also been studied by many authors [1, 5, 19, 21].

The notions of quasi-coincidence and Q-neighborhoods in fuzzy topology are defined parallel to belongingness and neighbourhoods in crisp topology. They are highly instrumental in generalizing many topological properties such as convergence and separation axioms to fuzzy topology [9, 10]. Extending these concepts to fuzzy metric spaces is also worthwhile. Analogous to classical topology, using dense fuzzy sets, complex fuzzy topological spaces can be approximated using a smaller, often countable subset of the space. By enabling approximations and connections between various regions of the space, dense fuzzy sets offer a potent tool for comprehending fuzzy topological spaces and associated mathematical structures. Though fuzzy topological spaces have been studied extensively, dense fuzzy sets and separability are not adequately explored.

In the present paper, some properties of dense fuzzy sets and fuzzy separability in fuzzy topological spaces are presented. In Section 2, some basic definitions and results in fuzzy topology that act as pre-requisites for the subsequent sections are provided. In Section 3, the nature of dense fuzzy sets in various fuzzy topological spaces are investigated. In Section 4, characterizations of dense fuzzy sets are obtained and a relation between dense fuzzy sets and Q-dense families of fuzzy points is derived. In Section 5, the concepts of fuzzy separability and β-separability are introduced and studied.

2. Preliminaries

This section encompasses a few basic definitions in fuzzy topology, that serve as a base for the sequel. For the sake of uniformity, we have changed certain notations in some of the definitions. The fundamental concepts in fuzzy set theory can be found in [6].

A fuzzy topology is a family δ of fuzzy sets in X such that $\phi, X \in \delta$; if $A, B \in \delta$, then $A \cap B \in \delta$ and if $A_i \in \delta$ for each $i \in I, then \cup_I A_i \in \delta$. The pair (X, δ) is called a fuzzy topological space. Each member of δ is called an open fuzzy set. A fuzzy set is closed if and only if its complement is open [4].

The fuzzy topology that contains only the fuzzy sets X and ϕ is called the indiscrete fuzzy topology on X and that contains all the fuzzy subsets of X is called the discrete fuzzy topology on X [4]. (X, δ) is said to be fuzzy co-finite if $\delta = \phi \cup \{A \in I^X : Supp(A^c) \text{ is finite}\}$ and is said to be fuzzy co-countable if $\delta = \phi \cup \{A \in I^X : Supp(A^c) \text{ is countable}\}$.

A fuzzy set in X is called a fuzzy point if and only if it takes the value 0 for all $y \in X$ except for one point, say $x \in X$. If its value at x is $\lambda, 0 < \lambda \leq 1$, we denote this fuzzy point by x_λ, where the point x is called its support. The fuzzy point x_λ belongs to a fuzzy set A, denoted by $x_\lambda \in A$, if and only if $\lambda \leq A(x)$. Evidently, each fuzzy set A can be expressed as the union of all the fuzzy points which belong to A [7]. Two fuzzy points in X are said to be distinct if and only if their supports are distinct [11].

A fuzzy point x_λ is said to be quasi-coincident with the fuzzy set A, denoted by $x_\lambda q A$, if and only if $\lambda > A^c(x)$, or $\lambda + A(x) > 1$. A fuzzy set A in the fuzzy topological space (X, δ) is said to be quasi-coincident with a fuzzy set B, denoted by AqB, if and only if there exists $x \in X$ such that $A(x) > B^c(x)$, or $A(x) + B(x) > 1$. If this is true, we also say that A and B are quasi-coincident (with each other) at x [7].

A fuzzy set A in (X, δ) is called a Q-neighbourhood (Q-nbhd) of x_λ if and only if there exists a $B \in \delta$ such that $x_\lambda q B \leq A$.The family consisting of all the Q-neighbourhoods of x_λ is called the system of Q-neighbourhoods of x_λ. A Q-neighbourhood of a fuzzy point may not contain the point itself. A fuzzy topological space (X, δ) is fuzzy T_2 (Hausdorff) space if and only if for any two fuzzy points x_λ and y_γ satisfying $x \neq y$, there exist Q-neighbourhoods B and C of x_λ and y_γ, respectively, such that $B \cap C = \phi$. Let A be a fuzzy set in the fuzzy topological space (X, δ). The intersection of all closed sets containing A is called the closure of A, denoted by $\bar{A}$. Obviously, $\bar{A}$ is the smallest closed set containing A [7].

Two fuzzy sets A_1 and A_2 in a fuzzy topological space (X, δ) are said to be Q-separated if and only if there exist closed fuzzy sets H_i $(i = 1, 2)$ such that $H_i \supseteq A_i$ $(i = 1, 2)$ and $H_1 \cap A_2 = \phi = H_2 \cap A_1$. It is obvious that A_1 and A_2 are Q-separated if and only if $\bar{A}_1 \cap A_2 = \phi = \bar{A}_2 \cap A_1$. A fuzzy set D in a fuzzy topological space (X, δ) is called disconnected if and only if there exist two non-empty fuzzy sets A and B in the subspace $D_0 = Supp(D)$ such that A and B are Q-separated and $D = A \cup B$. A fuzzy set is called connected if and only if it is not disconnected [7].

Let $f : X \to Y$ be a function. For a fuzzy set A in X, $f(A)$ is defined as follows:

$$f(A)(y) = \begin{cases} \sup\{A(x) : x \in X \text{ and } f(x) = y\}, & \text{when } f^{-1}(y) \neq \phi \\ 0, & \text{when } f^{-1}(y) = \phi. \end{cases}$$

A family $\Omega = \{e_\alpha\}$ of fuzzy points e_α in X is said to be dense in (X, δ) if and only if every non-empty open set contains some member of Ω. Ω is said to be Q-dense if and only if every non-empty open set is quasi-coincident with some member of Ω. A fuzzy topological space (X, δ) is said to be Q-separable if and only if there exists a countable family of fuzzy points in X which is Q-dense in (X, δ) [8].

Let (X, δ) be a fuzzy topological space. A fuzzy set A of X is called fuzzy β-open if $A \leq cl(\text{int}(cl(A)))$. The complement of a fuzzy β-open set is called fuzzy β-closed. In other words, a fuzzy set A of X is called fuzzy β-closed if $\text{int}(cl(\text{int}(A))) \leq A$ [3]. Let (X, δ) be a fuzzy topological space and let A be a fuzzy subset of X. Then, the fuzzy β-closure of A is defined by the intersection of all fuzzy β-closed sets containing A and is symbolized by $\beta cl(A)$. That is,

$$\beta cl(A) = \inf\{K : A \leq K, K \text{ is fuzzy } \beta - \text{closed}\}$$

A fuzzy set A in a fuzzy topological space (X, δ) is called fuzzy β-dense if there exists no fuzzy β-closed set B in (X, δ) such that $A \leq B < X$ [17].

Theorem 2.1 ([7]). *In a fuzzy topological space (X, δ), a fuzzy point $e \in \bar{A}$ if and only if there is a fuzzy net S in A such that S converges to e.*

Lemma 2.2 ([8]). *Let $f : X \to Y$ be a function and let A and B be fuzzy sets in X and Y, respectively. Then, for a fuzzy point x_λ in X, $f(x_\lambda)$ is a fuzzy point in Y and $f(x_\lambda) = (f(x))_\lambda$.*

Theorem 2.3 ([8]). *If $f : (X, \delta) \to (Y, \tau)$ is a fuzzy continuous function, then for each fuzzy net $S = \{S_n : n \in D\}$, if S fuzzy converges to e, then $f \circ S = \{f(S_n) : n \in D\}$ is a fuzzy net in Y and converges to $f(e)$.*

Theorem 2.4 ([7]). *In a fuzzy topological space (X, δ) every fuzzy net does not converge to two fuzzy points with different supports if and only if (X, δ) is a fuzzy Hausdorff space.*

3. Dense Fuzzy Sets in Fuzzy Topological Spaces

The notion of dense fuzzy sets is very useful in exploring the properties of fuzzy topological spaces. There is even a connection between the number of dense fuzzy sets in a fuzzy topological space and the fineness of the fuzzy topology. Let A and B be fuzzy subsets of X. Throughout the paper, (X, δ) denotes a fuzzy topological space. Also, $A \subset B$ indicates that A is a proper fuzzy subset of B. That is, $A(x) < B(x)$, for some $x \in X$.

In [15], the authors defined fuzzy dense sets as follows: A fuzzy set A in (X, δ) is called fuzzy dense if there exists no fuzzy closed set B in (X, δ) such that $A \subseteq B \subset X$ [15]. This definition can be reformulated as follows. A fuzzy subset A of X is said to be dense in (X, δ), if $\bar{A} = X$. In other words, A is dense in (X, δ) if and only if X is the only closed fuzzy set containing A.

Example 3.1. Let X be any non-empty set and let A be a non-empty proper fuzzy subset of X. Consider, $\delta = \left\{B \in I^X : B \supseteq A\right\} \cup \{\phi\}$, a fuzzy topology on X. Since, A^c is the largest closed fuzzy subset of X, any proper fuzzy superset of A^c is dense in (X, δ). In particular, let $X = N$, the set of all natural numbers, and let A be a fuzzy subset of N such that $A(x) = \frac{1}{x}$, $\forall x \in N$. Then, $A^c(x) = \frac{x-1}{x}, \forall x \in N$. Therefore, in (N, δ), the fuzzy set D of N defined by $D(x) = \frac{x}{x+1}, \forall x \in N$ is a dense fuzzy set as $D \supset A^c$.

Now, we investigate the nature of dense fuzzy sets in various fuzzy topological spaces.

Proposition 3.2. *If (X, δ) is a discrete fuzzy topological space, then X is the only dense fuzzy subset of X.*

Remark 3.3. Unlike in classical topology, the converse of Proposition 3.2 may not hold, in general, in fuzzy topology, as is evident from the following counterexample:

Let X be a given set and let δ be the fuzzy topology on X generated by the union of the sets $\left\{A \in I^X : Supp(A) = X \text{ and } A(x) \leq \frac{1}{2}, \forall x \in X\right\}$ and $\left\{A \in I^X : A \text{ is a crisp subset of } X\right\}$. Then, the closed fuzzy sets of X are precisely the crisp subsets of X and the fuzzy sets contained in $\left\{A \in I^X : A(x) = 0 \text{ or } A(x) \geq \frac{1}{2}, \forall x \in X\right\}$. We claim that no proper fuzzy subset of X is dense in (X, δ). For, let A be a non-empty proper fuzzy subset of X. First, assume that $Supp(A) \neq X$. Then, $\exists x \in X$ such that $x \notin Supp(A)$. Now, the fuzzy set B such that $B(x) = 0$ and $B(y) = 1, \forall y \neq x$ is a proper closed fuzzy subset of X containing A. Therefore, $\bar{A} \neq X$. So, no fuzzy set A with $Supp(A) \neq X$ is dense in (X, δ).

Now, let A be a proper fuzzy subset of X with $Supp(A) = X$. Then, by the construction of δ, there exists a closed fuzzy set B such that $A \subseteq B \subset X$. Therefore, no proper fuzzy set with $Supp(A) = X$ is dense in (X, δ). So, X is the only dense fuzzy subset of X. But, by the construction of δ, for $x \in X$, the fuzzy point $x_{2/3} \notin \delta$. Hence, (X, δ) is not fuzzy discrete. Thus, there exists a non-discrete fuzzy topological space (X, δ) in which X is the only dense fuzzy subset of X.

However, in the case of indiscrete, co-finite and co-countable fuzzy topological spaces, we obtain exact generalizations of corresponding results in classical topology.

Proposition 3.4. *If (X, δ) is an indiscrete fuzzy topological space, then every non-empty fuzzy subset of X is dense in (X, δ).*

Proposition 3.5. *If (X, δ) is an infinite co-finite fuzzy topological space, then a fuzzy subset A of X is dense in (X, δ) if and only if $Supp(A)$ is infinite.*

Proposition 3.6. *If (X, δ) is an uncountable co-countable fuzzy topological space, then a fuzzy subset A of X is dense in (X, δ) if and only if $Supp(A)$ is uncountable.*

By Propositions 3.2 and 3.4, we get an intuitive idea that as the fuzzy topology on X gets finer the number of dense fuzzy sets decreases. This is because, as the fuzzy topology gets finer, the number of closed fuzzy sets increases and so the number of dense fuzzy sets decreases.

The nature of dense fuzzy sets in various fuzzy topological spaces is summarized in Table 1.

TABLE 1. Dense fuzzy sets in different fuzzy topological spaces

Type of Fuzzy Topological Space	**Nature of Dense Fuzzy Sets**
Indiscrete fuzzy topological space	Every non-empty fuzzy subset of X is dense.
Co-finite fuzzy topological space	If X is infinite, then dense fuzzy sets are precisely those with infinite support.
Co-countable fuzzy topological space	If X is uncountable, then dense fuzzy sets are precisely those with uncountable support.
Discrete fuzzy topological space	X is the only dense fuzzy subset.

In the indiscrete case, we obtain the following characterization of the spaces in terms of dense fuzzy sets:

Theorem 3.7. *A fuzzy topological space (X, δ) is fuzzy indiscrete if and only if every non-empty fuzzy subset of X is dense in (X, δ).*

Proof. If (X, δ) is fuzzy indiscrete, then the only non-empty closed fuzzy subset of X is X itself. Therefore, $\bar{A} = X$, for any non-empty fuzzy subset A of X. Hence, every non-empty fuzzy subset of X is dense in (X, δ).

Conversely, suppose that every non-empty fuzzy subset of X is dense in (X, δ). If possible, let A be a non-empty proper open fuzzy subset of X. Then, A^c is a non-empty proper closed fuzzy subset of X. So, $\overline{A^c} = A^c \neq X$, which is a contradiction. Therefore, X is the only non-empty open fuzzy subset of X. Hence, (X, δ) is fuzzy indiscrete. □

Next, we investigate the properties of fuzzy topological spaces based on certain characteristics of dense fuzzy sets that possess. Before that, we need the following proposition and definition.

Proposition 3.8. *If δ is a fuzzy topology on X, then the collection of supports of elements in δ is a crisp topology on X.*

Proof. Let δ be a fuzzy topology on X and τ be the family of supports of all fuzzy sets in δ. Then, $Supp(X) = X \in \tau$ and $Supp(\phi) = \phi \in \tau$. Now, let $\{A_i\}$ be a subfamily of elements in τ. For each i, choose $A_i' \in \delta$ such that $Supp\,(A_i') = A_i$. Since, δ is a fuzzy topology on X, $\bigcup_i A_i' \in \delta$. Moreover, $Supp\left(\bigcup_i A_i'\right) = \bigcup_i A_i$. So, $\bigcup_i A_i \in \tau$. Let $A_1, A_2, \ldots, A_n$ be elements in τ. For $i = 1, 2, \ldots, n$, choose $A_i' \in \delta$ such that $Supp\,(A_i') = A_i$. Evidently, $\bigcap_{i=1}^{n} A_i' \in \delta$ and $Supp\left(\bigcap_{i=1}^{n} A_i'\right) = \bigcap_{i=1}^{n} A_i$. Hence, $\bigcap_{i=1}^{n} A_i \in \tau$. Thus, τ is a crisp topology on X. □

In light of Proposition 3.8, we formulate the following definition of support topology of a fuzzy topology.

Definition 3.9. The support topology of a fuzzy topology δ on X is the crisp topology of the supports of elements of δ. In other words, the support topology of a fuzzy topology δ on X is the smallest crisp topology on X that contains the support of each open fuzzy set in (X, δ).

Example 3.10. Consider the fuzzy topology δ on X defined in Example 3.1. Then, the crisp topology, $\tau = \{B \subseteq X : B \supseteq Supp(A)\} \cup \{\phi\}$ is the support topology of δ on X. In particular, if $X = N$ and A is such that $A(x) = \frac{1}{x}, \forall x \in N$, then the support topology of δ is the indiscrete fuzzy topology $\{N, \phi\}$.

Now, we explore some properties of the fuzzy topological spaces that do not have proper dense fuzzy sets. Firstly, we prove that such a space (X, δ) has infinitely many open fuzzy points with support x, for each $x \in X$.

Theorem 3.11. *If $X \neq \phi$ is the only dense fuzzy set in (X, δ), then for each $x \in X$, there exist infinitely many open fuzzy points in (X, δ) with support x.*

Proof. Suppose that X is the only dense fuzzy set in (X, δ). Then, $\bar{A} \neq X$, for any proper fuzzy subset A of X. Let $x \in X$. For each $n \in N$, consider the fuzzy set A_n defined by

$$A_n(y) = \begin{cases} 1 - \frac{1}{n}, & \text{when } y = x \\ 1, & \text{when } y \neq x. \end{cases}$$

Since, $A_n(x) \neq 1, \forall n \in N$, A_n is a proper fuzzy subset of X, $\forall n \in N$. So, $\bar{A_n} \neq X, \forall n$.

Let $\bar{A_1} = B_0$. Then, evidently, $B_0(y) = 1, \forall y \neq x$. Moreover, since, $B_0 \neq X$, $B_0(x) \in [0, 1)$. Now, by the Archimedean property, $\exists\ n_1 \in N$ such that $\frac{1}{n_1} < 1 - B_0(x)$ or $B_0(x) < 1 - \frac{1}{n_1}$. So, $B_0(x) < A_{n_1}(x)$. Therefore, $\overline{A_{n_1}} \neq B_0$. Let $\overline{A_{n_1}} = B_1$. Then, $B_1(x) \in (B_0(x), 1)$. Obviously, B_0 and B_1 are distinct proper closed fuzzy subsets of X.

Again, by Archimedean property, $\exists n_2 \in N$ such that $\frac{1}{n_2} < 1 - B_1(x)$ or $B_1(x) < 1 - \frac{1}{n_2}$ and therefore, $\overline{A_{n_2}} \neq B_1$. Let $\overline{A_{n_2}} = B_2$. It is clear that $B_2(x) \in (B_1(x), 1)$. It is apparent that B_0, B_1 and B_2 are distinct proper closed fuzzy subsets of X.

Continuing like this, we get infinitely many distinct proper closed fuzzy sets $B_0, B_1, B_2, \ldots$ in (X, δ) with core $X - \{x\}$ because the process will never terminate as $\bar{A_n} \neq X, \forall n \in N$. Hence, $\{B_0^c, B_1^c, B_2^c, \ldots\}$ form an infinite collection of open fuzzy points in (X, δ) with support x. Since, $x \in X$ was arbitrary, for each $x \in X$, there exist infinitely many open fuzzy points in (X, δ) with support x. □

The closed fuzzy sets $B_1, B_2, \ldots$ obtained in the proof of Theorem 3.11 have support X. Furthermore, $B_0 \subset B_1 \subset B_2 \subset \cdots$.

Now, we have the following corollaries of Theorem 3.11.

Corollary 3.12. *If $X \neq \phi$ is the only dense fuzzy set in (X, δ), then δ is infinite.*

Corollary 3.13. *If $X \neq \phi$ is the only dense fuzzy set in (X, δ), then there exist infinitely many closed fuzzy sets with support X and core $X - \{x\}$, for each $x \in X$.*

Corollary 3.14. *If $X \neq \phi$ is the only dense fuzzy set in (X, δ), then the support topology of δ is discrete.*

Proof. Suppose that $X \neq \phi$ is the only dense fuzzy set in (X, δ). Let τ be the support topology of δ and let A be a non-empty crisp subset of X. By Theorem 3.11, for each $x \in X$, there exist infinitely many open fuzzy points in (X, δ) with support x. So, corresponding to each $x \in A \subseteq X$, choose an open fuzzy point in (X, δ) with support x. Then, the union of all these chosen fuzzy points is an open fuzzy set in (X, δ) with support A. Therefore, $A \in \tau$. Since, A was arbitrary, each non-empty crisp subset of X is contained in τ. So, τ contains all the crisp subsets of X. Hence, τ is the discrete topology on X. □

The following theorem characterizes fuzzy topological spaces having X as the only dense fuzzy subset.

Theorem 3.15. *(X, δ) is a fuzzy topological space having $X \neq \phi$ as the only dense fuzzy set if and only if for each $x \in X$, X can be expressed as the infinite union of a strictly increasing sequence of proper closed fuzzy sets in (X, δ) each having core $X - \{x\}$.*

Proof. Suppose $X \neq \phi$ is the only dense fuzzy set in (X, δ). Let $x \in X$. Consider the fuzzy sets $A_n, n \in N$ defined in the proof of Theorem 3.11. Then, by continuing the process demonstrated in the proof of Theorem 3.11, we get a sequence of natural numbers, $n_1 < n_2 < ...$, such that $\overline{A_{n_i}} = B_i$. Clearly, each B_i is a proper closed fuzzy subset of X with core $X - \{x\}$ and $B_1 \subset B_2 \subset \cdots$. Now, $A_{n_i} \subseteq B_i, \forall i$.

$$\Rightarrow A_{n_i}(x) \leq B_i(x), \forall i$$
$$\Rightarrow 1 - \frac{1}{n_i} \leq B_i(x), \forall i$$
$$\Rightarrow \sup_i \left\{1 - \frac{1}{n_i}\right\} \leq \sup_i \{B_i(x)\}$$

But, $\sup_i \left\{1 - \frac{1}{n_i}\right\} = 1$. Therefore,

$$1 \leq \sup_i \{B_i(x)\} \leq 1.$$

So, $\sup_i \{B_i(x)\} = 1$. That is, $\left(\bigcup_{i=1}^{\infty} B_i\right)(x) = 1$. Moreover, we have, $B_i(y) = 1, \forall y \neq x$ and $\forall i$. Hence, $\bigcup_{i=1}^{\infty} B_i = X$. So, since, x was arbitrary, for each $x \in X$, X can be expressed as the infinite union of a strictly increasing sequence of proper closed fuzzy sets in (X, δ) each having core $X - \{x\}$.

Conversely, assume that for each $x \in X$, X can be expressed as the infinite union of a strictly increasing sequence of proper closed fuzzy sets in (X, δ) each having core $X - \{x\}$. Let A be a proper fuzzy subset of X. Then, $\exists x \in X$ such that $A(x) \neq 1$. By our assumption, corresponding to this x, there exists a strictly increasing sequence of proper closed fuzzy sets in (X, δ) with core $X - \{x\}$, say $\{B_1, B_2, \ldots\}$ such that $\bigcup_{i=1}^{\infty} B_i = X$. Evidently, $A(x) < \sup_i \{B_i(x)\} = 1$. So, by the property of supremum, $\exists j \in N$ such that $B_j(x) > A(x)$. Now, since B_j has core $X - \{x\}$, $A \subset B_j$. That is, B_j is a proper closed fuzzy set in (X, δ) containing A. Therefore, $\bar{A} \subseteq B_j \neq X$. Thus, A is not dense in (X, δ). That is, no proper fuzzy subset of X is dense in (X, δ). Hence, X is the only dense fuzzy subset of X. □

Next, we examine the properties of dense fuzzy points and those of spaces having dense fuzzy points.

Theorem 3.16. *If x_λ is a dense fuzzy point in (X, δ), then its dual has empty interior.*

Proof. Suppose that x_λ is a dense fuzzy point in (X, δ).
$\Rightarrow \exists$ no proper closed fuzzy set A in (X, δ) such that $\lambda \leq A(x)$.
$\Rightarrow \exists$ no proper closed fuzzy set A in (X, δ) such that $1 - \lambda \geq 1 - A(x)$.
$\Rightarrow \exists$ no non-empty open fuzzy set B in (X, δ) such that $B(x) \leq 1 - \lambda$.
$\Rightarrow x_{1-\lambda}$ is not contained in any non-empty open fuzzy set in (X, δ).
$\Rightarrow \text{int}(x_{1-\lambda}) = \phi$.
Thus, the dual of x_λ has empty interior. □

Theorem 3.17. *If x_λ is a dense fuzzy point in (X, δ), then the support of each non-empty open fuzzy subset of X contains x.*

Proof. Let x_λ be a dense fuzzy point in (X, δ). If possible, suppose that there exists a non-empty open fuzzy set, say A, in (X, δ) such that $x \notin Supp(A)$. That is, $A(x) = 0$. Then, evidently, A^c is a proper closed fuzzy set in (X, δ) such that $A^c(x) = 1$. Therefore, A^c is a proper closed fuzzy set in (X, δ) containing x_λ. So, $\overline{x_\lambda} \subseteq A^c \neq X$, which is a contradiction. Hence, the support of each non-empty open fuzzy set in (X, δ) contains x. □

Corollary 3.18. *If (X, δ) has dense fuzzy points with support x for each $x \in X$, then the support topology of δ is indiscrete.*

4. Further Properties of Dense Fuzzy Sets

Now, we study dense fuzzy sets in a more general setting. The next theorem provides a characterization of dense fuzzy sets in an arbitrary fuzzy topological space.

Theorem 4.1. *A fuzzy subset A of X is dense in (X, δ) if and only if A is quasi-coincident with each non-empty open fuzzy subset of X.*

Proof. Let (X, δ) be a fuzzy topological space. The case when $X = \phi$ is trivial. Let $X \neq \phi$ and let A be a dense fuzzy set in (X, δ). Then, $\bar{A} = X$ and $A \neq \phi$. Therefore, $A(x) \neq 0$, for some $x \in X$. Let B be a non-empty open fuzzy subset of X. If $B = X$, then $B(x) = 1, \forall x \in X$. Therefore, $A(x) + B(x) > 1$, for some $x \in X$. If $B \neq X$, then, B^c is a non-empty proper closed fuzzy subset of X. Since, $\bar{A} = X$, $A \nsubseteq B^c$.

$\Rightarrow$ There exists $x \in X$ such that $A(x) > B^c(x)$.
$\Rightarrow A(x) > 1 - B(x)$.
$\Rightarrow A(x) + B(x) > 1$.

So, in both cases, A is quasi-coincident with B. Since, B was arbitrary, A is quasi-coincident with each non-empty open fuzzy subset of X.

Conversely, suppose that A is quasi-coincident with each non-empty open fuzzy subset of X. Then, obviously, $A \neq \phi$. If $\bar{A} \neq X$, then there exists a closed fuzzy set B such that $A \subseteq B$ and $B \neq X$. Then, B^c is a proper non-empty open fuzzy subset of X. Then, by assumption, AqB^c.

$\Rightarrow$ There exists $x \in X$ such that $A(x) + B^c(x) > 1$.
$\Rightarrow A(x) + 1 - B(x) > 1$.
$\Rightarrow A(x) > B(x)$.
$\Rightarrow A \nsubseteq B$, which is a contradiction.

So, $\bar{A} = X$ and hence A is dense in (X, δ). □

The forthcoming theorem guarantees that for a fuzzy subset A to be dense in (X, δ) it is enough that A is quasi-coincident with each member of a base for δ.

Theorem 4.2. *A fuzzy subset A of X is dense in (X, δ) if and only if A is quasi-coincident with every non-empty member of a base for δ.*

Proof. Let (X, δ) be a fuzzy topological space and let $\mathcal{B}$ be a base for δ. Suppose that A is a dense fuzzy subset of X. Then, by Theorem 4.1, A is quasi-coincident with every non-empty open fuzzy subset of X. So, since $\mathcal{B} \subseteq \delta$, A is quasi-coincident with every non-empty member of $\mathcal{B}$.

Conversely, suppose that A is quasi-coincident with every non-empty member of a base, say $\mathcal{B}$, for δ. We claim that A is quasi-coincident with each non-empty open fuzzy subset of X. For, let O be a non-empty open fuzzy subset of X. Then, since $\mathcal{B}$ is a base for δ, we have $O = \bigcup_i B_i$, for some $B_i \in \mathcal{B}$. Therefore, there exists $B \in \mathcal{B}$ such that $B \subseteq O$ and $B \neq \phi$. Then, by our assumption AqB and hence AqO. Thus, A is quasi-coincident with each non-empty open fuzzy subset of X. Hence, by Theorem 4.1, A is dense in (X, δ). □

The following theorem proves that the existence of a connected dense fuzzy set in a fuzzy topological space guarantees the connectedness of the space.

Theorem 4.3. *A fuzzy topological space* (X, δ) *has a connected dense fuzzy set if and only if* X *is connected.*

Proof. Suppose that (X, δ) has a connected dense fuzzy set, say D. If possible, assume that X is disconnected. Then, there exist two non-empty fuzzy sets A and B in X such that A and B are Q-separated and $X = A \cup B$. That is, $\bar{A} \cap B = \phi = A \cap \bar{B}$ and $X = A \cup B$.

Let $G = A \cap D$ and $H = B \cap D$. Then, G and H are fuzzy sets in $D_0 = Supp(D)$. Also,

$$G \cup H = (A \cap D) \cup (B \cap D) = (A \cup B) \cap D = X \cap D = D.$$

Let $\widetilde{C}$ be denote the relative closure of the fuzzy set C in D. Now,

$$\begin{aligned} \widetilde{G} \cap H &= (\widetilde{A \cap D}) \cap (B \cap D) \\ &= (\overline{A \cap D}) \cap D \cap (B \cap D) \\ &\subseteq (\bar{A} \cap \bar{D}) \cap (B \cap D) \\ &= (\bar{A} \cap B) \cap D \\ &= \phi. \end{aligned}$$

Also,

$$\begin{aligned} G \cap \widetilde{H} &= (A \cap D) \cap (\widetilde{B \cap D}) \\ &= (A \cap D) \cap (\overline{B \cap D}) \cap D \\ &\subseteq (A \cap D) \cap (\bar{B} \cap \bar{D}) \\ &= (A \cap \bar{B}) \cap D \\ &= \phi. \end{aligned}$$

Therefore, $\widetilde{G} \cap H = \phi = G \cap \widetilde{H}$ and so G and H are Q-separated in the subspace D_0.

If $G = \phi$, then, since, $G \cup H = D$, $H = B \cap D = D$. That is, $B = D$. Therefore, $\bar{B} = \bar{D} = X$. Then, $A \cap \bar{B} = A \cap X = A \neq \phi$, which is a contradiction. Hence, $G \neq \phi$. Similarly, if $H = \phi$, then $G = A \cap D = D$ and so $A = D$. Therefore, $\bar{A} = X$ and $\bar{A} \cap B = X \cap B = B \neq \phi$, which is a contradiction. So, $H \neq \phi$. Hence, G and H are two non-empty Q-separated fuzzy sets in $D_0 = Supp(D)$ such that $D = G \cup H$. This implies that D is disconnected, which is a contradiction. Hence, X is connected.

The converse is obvious because if X is connected, then X itself is a connected dense fuzzy set in (X, δ). □

As $\bar{A} = X$ for any dense fuzzy subset A of X in (X, δ), by Theorem 2.1, we have the following characterization of dense fuzzy sets in terms of convergence of fuzzy nets.

Theorem 4.4. *A fuzzy subset A of X is dense in (X, δ) if and only if for each fuzzy point x_λ in X there is a fuzzy net S in A such that S converges to x_λ.*

The ensuing theorem, which is an application of Theorem 4.4, asserts that if two continuous functions from a fuzzy topological space (X, δ) into a Hausdorff space agree on the collection of all fuzzy points contained in a dense fuzzy subset of X, then they agree on all of X.

Theorem 4.5. *If two continuous functions $f, g : (X, \delta) \to (Y, \tau)$, where (Y, τ) is fuzzy Hausdorff, agree on the collection of all fuzzy points contained in a dense fuzzy subset of X, then $f = g$ on X.*

Proof. Let f and g be continuous functions from a fuzzy topological space (X, δ) to a Hausdorff fuzzy topological space (Y, τ). Let D be a dense fuzzy subset of X and suppose that $f(e) = g(e)$ for each fuzzy point $e \in D$.

Let $x \in X$ and consider the fuzzy point x_λ. Since, D is dense in (X, δ), by Theorem 4.4, there exists a fuzzy net $S = \{S_n : n \in E\}$ in D converging to x_λ, where E is a directed set. Since, f is fuzzy continuous, by Theorem 2.3, $\{f(S_n) : n \in E\}$ is a fuzzy net in Y converging to $f(x_\lambda) = (f(x))_\lambda$. Similarly, since, g is fuzzy continuous, $\{g(S_n) : n \in E\}$ is a fuzzy net in Y converging to $g(x_\lambda) = (g(x))_\lambda$. Since, f and g agree on each fuzzy point contained in D, $f(S_n) = g(S_n), \forall n \in E$. Therefore, $\{f(S_n) : n \in E\} = \{g(S_n) : n \in E\}$. So, $(f(x))_\lambda$ and $(g(x))_\lambda$ are the limits of the fuzzy sequence $\{f(S_n) : n \in E\}$ in Y. Since, (Y, τ) is fuzzy Hausdorff, by Theorem 2.4, $f(x) = g(x)$. Since, $x \in X$ was arbitrary, $f(x) = g(x), \forall x \in X$. □

Next, we obtain a relation between fuzzy dense sets and Q-dense families of fuzzy points in a fuzzy topological space by using the following definition and lemmas.

Definition 4.6. A fuzzy point x_λ is called a maximal fuzzy point contained in a fuzzy set A if $x_\lambda \in A$ and $x_\mu \notin A$ for any $\mu > \lambda$. Obviously, x_λ is a maximal fuzzy point contained in A if and only if $\lambda = A(x)$.

Example 4.7. Consider the fuzzy set A defined on N, the set of all natural numbers, in Example 3.1. Then, the maximal fuzzy points contained in A are $x_{(\frac{1}{x})}, \forall x \in N$.

Lemma 4.8. *If A and B are fuzzy subsets of X such that AqB, then there exists at least one maximal fuzzy point contained in B which is quasi-coincident with A.*

Proof. If AqB, then there exists $x \in X$ such that $A(x) + B(x) > 1$. Consider the fuzzy point x_λ, where $\lambda = B(x)$. Then, evidently, x_λ is a maximal fuzzy point contained in B and is quasi-coincident with A. □

Lemma 4.9. *A fuzzy set A is the union of all maximal fuzzy points contained in A.*

In the following theorem, we prove that corresponding to each dense fuzzy subset in (X, δ), there exists a Q-dense family of fuzzy points and vice versa.

Theorem 4.10. *A fuzzy subset A of X is dense in (X, δ) if and only if the set of all maximal fuzzy points contained in A is Q-dense in (X, δ).*

Proof. Suppose A is dense in (X, δ) and let M be the collection of all maximal fuzzy points contained in A. Then, by Theorem 4.1, each non-empty open fuzzy subset of X is quasi-coincident with A. Therefore, by Lemma 4.8, every non-empty open fuzzy subset of X is quasi-coincident with some member of M. Hence, M is Q-dense in (X, δ).

Conversely, suppose that M is Q-dense in (X, δ). Then, by definition, every non-empty open fuzzy subset of X is quasi-coincident with some member of M. By Lemma 4.9, the union of all the fuzzy points contained in M is A. So, A is quasi-coincident with every non-empty open fuzzy subset of X. Therefore, by Theorem 4.1, A is dense in (X, δ). □

5. Fuzzy Separability and β-separability in Fuzzy Topology

In this section, we define fuzzy separability and β-separability in fuzzy topological spaces and explore their properties. We begin with the definition of a separable fuzzy topological space.

Definition 5.1. A fuzzy topological space (X, δ) is said to be fuzzy separable if it has a dense fuzzy set A with countable support.

Now, we give some examples of separable fuzzy topological spaces.

Example 5.2. If X is a countable set, then in any fuzzy topological space (X, δ), X is a dense fuzzy set such that $Supp(X)$ is countable. Hence if X is countable, then (X, δ) is a separable fuzzy topological space.

Example 5.3. By Theorem 3.7, it is obvious that an indiscrete fuzzy topological space (X, δ) is always separable.

Theorem 5.4. *A discrete fuzzy topological space (X, δ) is fuzzy separable if and only if X is countable.*

Proof. Let (X, δ) be a discrete fuzzy topological space. Suppose that (X, δ) is fuzzy separable. That is, there exists a dense fuzzy set in (X, δ) with countable support. Since, (X, δ) is fuzzy discrete, X is the only dense fuzzy subset of X. Therefore, X has countable support. Hence, X is countable.

Conversely, suppose that X is countable. Then, evidently, X itself is a dense fuzzy set in (X, δ) with countable support. Hence, (X, δ) is fuzzy separable. □

The next theorem states that every C_{11} fuzzy topological space is separable.

Theorem 5.5. *If (X, δ) is a C_{11} fuzzy topological space, then (X, δ) is fuzzy separable.*

Proof. Let (X, δ) be a C_{11} fuzzy topological space. Let $\mathcal{B} = \{B_1, B_2, \ldots\}$ be a countable base for δ. Without loss of generality, assume that each B_i is non-empty. For each $B_i \in \mathcal{B}$, choose an element $x_i \in X$ such that x_i has non-zero membership value in B_i. So, corresponding to each $B_i \in \mathcal{B}$, we get an element $x_i \in X$. Now, form the fuzzy set A such that for each i, x_i has membership value 1 in A and all other points in X has membership value zero in A. Then, A has countable support and A is quasi-coincident with each element in $\mathcal{B}$. So, by Theorem 4.2, A is a dense fuzzy subset of X. Thus, A is a dense fuzzy subset of X with countable support and hence the fuzzy topological space (X, δ) is fuzzy separable. □

The upcoming theorem shows that fuzzy separability and Q-separability are equivalent.

Theorem 5.6. *A fuzzy topological space (X, δ) is fuzzy separable if and only if it is Q-separable.*

Proof. Suppose that (X, δ) is fuzzy separable. Then, there exists a dense fuzzy set, say A, in (X, δ) such that $Supp(A)$ is countable. Let M be the collection of all maximal fuzzy points contained in A. Then, by Theorem 4.10, M is Q-dense in (X, δ). Since, A has countable support, M is also countable. Therefore, there exists a countable family of fuzzy points M in X which is Q-dense in (X, δ). Hence, the fuzzy topological space (X, δ) is Q-separable.

Conversely suppose that (X, δ) is Q-separable. Then, there exists a countable family of fuzzy points in X, say M, which is Q-dense in (X, δ). Let A be the fuzzy set obtained by taking the union of all the fuzzy points contained in M. Since, M is countable, A has countable support. Also, M contains all the maximal fuzzy points contained in A. Therefore, by Theorem 4.10, A is a dense fuzzy set in (X, δ). Hence (X, δ) is fuzzy separable. □

Next, we look into the properties of β-dense fuzzy sets and introduce β-separability in fuzzy topology. The forthcoming definition elucidates Q-β-denseness of a collection of fuzzy points.

Definition 5.7. A family, $P = \{e_\alpha\}$, of fuzzy points e_α in X is said to be Q-β-dense in (X, δ) if every non-empty β-open fuzzy set is quasi-coincident with some member of P.

Now, we define β-dense fuzzy sets in fuzzy topological spaces.

Definition 5.8. A fuzzy set A of X in (X, δ) is called β-dense in (X, δ) if $\beta cl(A) = X$.

The following are some examples of β-dense fuzzy sets in fuzzy topological spaces.

Example 5.9. Let (X, δ) be fuzzy indiscrete and suppose that $X \neq \phi$. Evidently, X and ϕ are β-open fuzzy sets in (X, δ). Now, let A be a proper non-empty fuzzy subset of X. Then,

$$\text{int}(cl(\text{int}(A))) = \text{int}(cl(\phi)) = \text{int}(\phi) = \phi \leq A,$$
$$cl(\text{int}(cl(A))) = cl(\text{int}(X)) = cl(X) = X \geq A.$$

Therefore, A is both β-open and β-closed. So, each fuzzy subset of X is β-closed in X. Hence, in an indiscrete fuzzy topological space (X, δ), X is the only β-dense fuzzy subset of X.

Example 5.10. Let (X, δ) be fuzzy discrete. Let A be a fuzzy subset of X. Then,

$$cl(\text{int}(cl(A))) = cl(\text{int}(A)) = cl(A) = A,$$
$$\text{int}(cl(\text{int}(A))) = \text{int}(cl(A)) = \text{int}(A) = A.$$

Therefore, A is both β-open and β-closed. Hence, X is the only β-dense fuzzy set in (X, δ).

Remark 5.11. Examples 5.9 and 5.10 show that unlike in the case of dense fuzzy sets in a fuzzy topological space, the fineness of the fuzzy topology has no influence on the number of β-dense fuzzy sets in the fuzzy topological space.

Next, we characterize β-dense fuzzy sets in (X, δ) by means of quasi-coincidence.

Theorem 5.12. *A fuzzy subset A of X is β-dense in (X, δ) if and only if A is quasi-coincident with each non-empty β-open fuzzy subset of X.*

Proof. If A is β-dense in (X, δ), then $\beta cl(A) = X$. Evidently, A is quasi-coincident with X. Let $B \neq X$ be a non-empty β-open fuzzy subset of X. Then, B^c is a non-empty proper β-closed fuzzy subset of X. Since, $\beta cl(A) = X$, $A \nsubseteq B^c$.

$\Rightarrow \exists x \in X$ such that $A(x) > B^c(x)$.

$\Rightarrow A(x) > 1 - B(x)$.

$\Rightarrow A(x) + B(x) > 1.$
$\Rightarrow AqB.$

Therefore, A is quasi-coincident with each non-empty β-open fuzzy subset of X.

Conversely, assume that A is quasi-coincident with each non-empty β-open fuzzy subset of X. Let $B \neq X$ be a non-empty β-closed fuzzy set in X. Then, B^c is a non-empty β-open fuzzy subset of X. Then, by assumption , AqB^c.

$\Rightarrow \exists x \in X$ such that $A(x) + B^c(x) > 1.$
$\Rightarrow A(x) + 1 - B(x) > 1.$
$\Rightarrow A(x) > B(x).$
$\Rightarrow A \nsubseteq B.$

Therefore, A is not contained in any non-empty proper β-closed fuzzy subset of X. So, $\beta cl(A) = X$ and hence A is β-dense in (X, δ). □

Now, we establish a relation between β-dense fuzzy sets in a fuzzy topological space (X, δ) and Q-β-dense collection of fuzzy points in (X, δ).

Theorem 5.13. *A fuzzy subset A of X is β-dense in (X, δ) if and only if the set of all maximal fuzzy points contained in A is Q-β-dense in (X, δ).*

Proof. Let A be a β-dense fuzzy subset of X in (X, δ). Let P be the collection of all maximal fuzzy points contained in A. Since, A is β-dense in (X, δ), by Theorem 5.12, A is quasi-coincident with each non-emptyβ-open fuzzy subset of X. So, every non-empty β-open fuzzy subset of X is quasi-coincident with some member of P. Hence, P is Q-β-dense in (X, δ).

Conversely, suppose that the set of all maximal fuzzy points contained in the fuzzy set A is Q-β-dense in (X, δ). Then, by definition, each non-empty β-open fuzzy subset of X in (X, δ) is quasi-coincident with some member of P. The union of all the fuzzy points in P is the fuzzy set A. Therefore, A is quasi-coincident with every non-empty β-open fuzzy subset of X. So, by Theorem 5.12, A is β-dense in (X, δ). □

Next, we define Q-β-separability of a fuzzy topological space.

Definition 5.14. A fuzzy topological space (X, δ) is said to be Q-β-separable if it has a countable Q-β-dense family of fuzzy points.

The following is the definition of β-separability of a fuzzy topological space.

Definition 5.15. A fuzzy topological space (X, δ) is said to be β-separable if it has a β-dense fuzzy set with countable support.

Example 5.16. By Examples 5.9 and 5.10, an indiscrete fuzzy topological space or a discrete fuzzy topological space is β-separable if and only if X is countable.

Now, we establish the equivalence of the concepts of β-separability and Q-β-separability.

Theorem 5.17. *A fuzzy topological space (X, δ) is β-separable if and only if it is Q-β-separable.*

Proof. Suppose that (X, δ) is β-separable. Then, there exists a β-dense fuzzy set, say A, in (X, δ) with countable support. Let P be the collection of all maximal fuzzy points in A. Then, by Theorem 5.13, P is Q-β-dense in (X, δ). Since, A has countable support, P is countable. That is, (X, δ) has a countable Q-β-dense set of fuzzy points. Hence, (X, δ) is Q-β-separable.

Conversely, suppose that (X, δ) is Q-β-separable. Then, there exists a countable collection of fuzzy points, say P, which is Q-β-dense in (X, δ). Let A be the fuzzy set obtained by taking the union of all the fuzzy points contained in P. Then, by Theorem 5.13, A is β-dense in (X, δ). Hence, (X, δ) is β-separable. □

6. Conclusion

The concepts of dense fuzzy sets and separability are effective tools in fuzzy topology for analyzing the properties of fuzzy topological spaces. An extensive study of denseness in fuzzy topology is carried out in the present paper. The features of dense fuzzy sets in various fuzzy topological spaces are explored. Also, dense fuzzy sets are characterized in different ways within a general framework. Moreover, the concepts of fuzzy separability, Q-β-separability, and β-separability are proposed and studied. It is proved that every C_{11} fuzzy topological space is fuzzy separable and the equivalence of fuzzy separability and Q-separability of a fuzzy topological space is established. The features of dense fuzzy sets can be used to identify the nature of the fuzzy topology in many fuzzy topological spaces. Dense fuzzy sets enhance the applications of fuzzy sets across various domains, including medical diagnosis, control systems, image processing and more. Future work includes studying further topological properties such as convergence, compactness, Lindeloffness etc. by way of dense fuzzy sets. Also, the study on dense fuzzy sets can be extended to fuzzy metric spaces and other fuzzy topological spaces such as bipolar fuzzy topological space, intuitionistic fuzzy topological space etc.

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