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# Many-Level Abstract Approximation Spaces

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Abstract. A many-level abstract approximation system on a quantale $\mathcal{L}$ is introduced and studied. Based on this fact, a pair of lower quantic and upper quantic $\mathcal{M}$-approximation operators is specified and discussed. In addition, the concepts of an Alexandrov $\mathcal{M}$-Hutton quasi-fuzzy topology and an Alexandrov $\mathcal{M}$-Hutton fuzzy co-topology on $\mathcal{L}$ are introduced. Moreover, the relationships between them and a lower quantic $\mathcal{M}$-approximation operator (a lower $\mathcal{QM}$-ApprX operator for short) and an upper quantic $\mathcal{M}$-approximation operator (an upper $\mathcal{QM}$-ApprX operator for short) are discussed, respectively. Furthermore, the notion of an $\mathcal{M}$-open structure and its relationships with a lower $\mathcal{QM}$-ApprX operator and the concept of an $\mathcal{M}$-closed structure are established.

## 1. Introduction

Pawlak [21, 22] developed the rough set theory, which is useful for dealing with ambiguous, imprecise, or uncertain data. It's been used successfully in a variety of fields like machine learning, data mining, knowledge discovery, expert systems, granular computing, pattern recognition, algebraic systems, graph theory, and partially ordered sets [4, 12]. After that, as a generalization of rough sets, Dubois and Prade [8] presented fuzzy rough sets. Moreover, Davvaz et al. [5, 6] also have studied various generations of rough approximations. Although it may appear at first glance that the notions of a fuzzy set, a rough set, and a (fuzzy) topological space are fundamentally dissimilar and share nothing in common, this is not the truth. Kortelainen et al., [15, 19] were likely the first to begin investigating the intermediate relationships between fuzzy

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sets, topologies, and rough sets. Moreover, the discussion of various relations between rough sets, fuzzy sets, and some other related notions was done in a lot of papers [9, 16]. Numerous researchers examined the connections between models of the fuzzy rough set and fuzzy topologies in both the unit interval, complete quasi-monoidal lattices, complete distributive lattices, and commutative unital quantales [3, 18]. Extending the axiomatic approaches proposed by Morsi [20] and Yao [32, 33] within a strictly two-sided and commutative (bi-)quantale $\mathcal{L}$ (ref. [14, 24]), Kim [18] established the concept of $\mathcal{L}$-lower and $\mathcal{L}$-upper quasi-approximation spaces.

Further, Šostak [29] presented the concept of an $\mathcal{M}$-approximate system based on a complete lattice. In especially, the demonstration showed that the classifications pertaining to rough sets, along with those linked to fuzzy topology, can be depicted as distinctive subcategories within the broader category of $\mathcal{M}$-approximate systems. Subsequently, building upon Šostak's upper $\mathcal{M}$-approximate operators, J. M. Ko et al. [17] demonstrated the presence of initial upper $\mathcal{M}$-approximate operators and initial $\mathcal{M}$-closed sets within commutative cl-monoids featuring an order reverse involution.

Recent research on rough set theory has advanced beyond the classical models introduced by Pawlak. Contemporary approaches have incorporated various neighborhood-based frameworks to enhance the flexibility and applicability of rough approximations, for example [1, 2].

Our main contributions and motivations of this article are to give an alternative view of the relationships between fuzzy rough sets and $\mathcal{M}$-Hutton fuzzy topological spaces based on a quantale, which is considered a generalization of Šostak [29]. Furthermore, the aim is to create a framework that enables the generalization of these notions and the corresponding theories. The notion of an $\mathcal{M}$-approximation system is the tool that enables us to achieve this goal.

This paper is structured as follows. In Section 2, we provide a brief overview of certain concepts and symbols utilized in this study. In Section 3, we define and discuss a pair of lower $\mathcal{QM}$-ApprX operator and an upper $\mathcal{QM}$-ApprX operator on $\mathcal{L}$. As well, the concept of $\mathcal{M}$-Hutton (resp., an Alexandrov $\mathcal{M}$-Hutton) quasi-fuzzy topology on $\mathcal{L}$ is presented, and the relationship between it and the concept of a lower $\mathcal{QM}$-ApprX operator on $\mathcal{L}$ is discussed. Further, the concept of $\mathcal{M}$-open structure is defined and studied, and its relationships with the lower $\mathcal{QM}$-ApprX operator and the concept of $\mathcal{M}$-closed structure are established, respectively. Also, we define the concept of $\mathcal{M}$-Hutton (resp., an Alexandrov $\mathcal{M}$-Hutton) fuzzy co-topology on $\mathcal{L}$, and the relationship between it and the notion of the upper $\mathcal{QM}$-ApprX operator on $\mathcal{L}$ is studied.

## 2. Preliminaries

In this section, we shall examine a few fundamental notions and symbols employed within this study.

**Definition 2.1.** By a semi-quantale [23] $(\mathcal{L}, \leq, \bigvee, \otimes)$ [$\mathcal{SQ}$unt for short], we mean a $\bigvee$-semilattice $(\mathcal{L}, \leq, \bigvee)$ provided with binary operation $\otimes : \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{L}$, with no additional assumptions. A $\mathcal{SQ}$unt homomorphism is a map preserving arbitrary $\bigvee$ and $\otimes$. Squant is the category of $\mathcal{SQ}$unt and their homomorphisms.

**Definition 2.2.** An $\mathcal{L} \in |\text{Squant}|$ is said to be:

(1) A commutative [28] (resp., a unital [23]) if $\otimes$ is commutative (resp., if $\otimes$ has the unit $e$).
(2) A quantale [27] [$\mathcal{Q}$unt for short] if $\otimes$ is associative and distributive across arbitrary $\bigvee$ from both sides. A $\mathcal{Q}$unt homomorphism is a map preserving arbitrary $\bigvee$ and $\otimes$. quant is the category of $\mathcal{Q}$unt and their homomorphisms.
(3) A coquantale [24] [$\mathcal{C}o\mathcal{Q}$unt for short] if $\otimes$ is associative and distributive across arbitrary $\bigwedge$ from both sides. A $\mathcal{C}o\mathcal{Q}$unt homomorphism is a map preserving arbitrary $\bigwedge$ and $\otimes$. Coquant is the category of $\mathcal{C}o\mathcal{Q}$unt and their homomorphisms.

**Definition 2.3** ([23])**.** Presume that $(\mathcal{L}, \leq, \otimes), (\mathcal{M}, \leq, \odot) \in |Squant|$. A preserving tensor product mapping $\jmath : \mathcal{L} \longrightarrow \mathcal{M}$ is said to be: for $\{a_i : i \in I\} \subseteq \mathcal{L}$.

(*i*) A $(\otimes, \bigvee)$-morphism (or $\mathcal{SQ}$unt morphism) if $\jmath\left(\bigvee_{i\in I} a_i\right) = \bigvee_{i\in I} \jmath(a_i)$;

(*ii*) A $(\otimes, \bigwedge)$-morphism, if $\jmath\left(\bigwedge_{i\in I} a_i\right) = \bigwedge_{i\in I} \jmath(a_i)$;

If a $(\otimes, \bigvee)$-morphism $\jmath : \mathcal{L} \longrightarrow \mathcal{M}$ additionally preserves the top (resp., unit) element, i.e., $\jmath(\top_{\mathcal{L}}) = \top_{\mathcal{M}}$ (resp., $\jmath(e_{\mathcal{L}}) = e_{\mathcal{M}}$), then it is called a strong (resp., unital).

Suppose that $\mathcal{X}$ is a non-empty set and $\mathcal{L} \in |Squant|$. An $\mathcal{L}$-fuzzy subset (or $\mathcal{L}$-subset) of $\mathcal{X}$ is a mapping $\mathcal{F} : \mathcal{X} \longrightarrow \mathcal{L}$. The family of all $\mathcal{L}$-fuzzy subsets on $\mathcal{X}$ is denoted by $\mathcal{L}^{\mathcal{X}}$. The algebraic and lattice-theoretic structures can be extended from the $\mathcal{SQ}$unt $(\mathcal{L}, \leq, \bigvee, \otimes)$ to $\mathcal{L}^{\mathcal{X}}$ pointwisely:

$$\mathcal{F} \leq \mathcal{G} \quad \Leftrightarrow \quad \forall x \in \mathcal{X} : \mathcal{F}(x) \leq \mathcal{G}(x)$$

$$\left(\bigvee_{i\in I}\mathcal{F}_i\right)(x)=\bigvee_{i\in I}\mathcal{F}_i(x)$$
$$(\mathcal{F}\otimes\mathcal{G})(x)=\mathcal{F}(x)\otimes\mathcal{G}(x),\quad x\in\mathcal{X}$$

If $\mathcal{L}$ satisfies the law of double negative [$\mathcal{DN}$egat for short], then we obtain
$$(\mathcal{F}\oplus\mathcal{G})(x)=\mathcal{F}(x)\oplus\mathcal{G}(x),\quad x\in\mathcal{X}$$
The largest and the smallest element in $\mathcal{L}^{\mathcal{X}}$ are indicated by $\underline{\top}$ and $\underline{\bot}$, respectively. The powerset $\mathcal{L}^{\mathcal{X}}$ is again a $\mathcal{SQ}$unt with respect to the tensor product $\otimes$ and arbitrary sups, and when $\mathcal{L}$ is unital, the set $\mathcal{L}^{\mathcal{X}}$ be an unital $\mathcal{SQ}$unt with the unit $\underline{e}$.

Regarding an ordinary mapping $h:\mathcal{X}\longrightarrow\mathcal{Y}$, we define the mappings $h^{\rightarrow}_{\mathcal{L}}:\mathcal{L}^{\mathcal{X}}\longrightarrow\mathcal{L}^{\mathcal{Y}}$ and $h^{\leftarrow}_{\mathcal{L}}:\mathcal{L}^{\mathcal{Y}}\longrightarrow\mathcal{L}^{\mathcal{X}}$ by
$$h^{\rightarrow}_{\mathcal{L}}(\mathcal{F})(y)=\bigvee\{\mathcal{F}(x):x\in\mathcal{X},h(x)=y\}$$
for every $\mathcal{F}\in\mathcal{L}^{\mathcal{X}}$ and every $y\in\mathcal{Y}$, $h^{\leftarrow}_{\mathcal{L}}(\mathcal{G})=\mathcal{G}\circ h$ for every $\mathcal{G}\in\mathcal{L}^{\mathcal{Y}}$, respectively, see [23, 26].

There exists a binary operation $\rightarrow$ (is said to be the residuated or implication operator) [13] is defined by $\lambda\rightarrow\nu=\bigvee\{\eta:\lambda\otimes\eta\le\nu\}$, for any $\lambda,\nu,\eta\in\mathcal{L}$. The residual $\rightarrow:\mathcal{L}\times\mathcal{L}\longrightarrow\mathcal{L}$ on $\mathcal{L}$ can be extended pointwisely to the powerset $\mathcal{L}^{\mathcal{X}}$ as $\rightarrow:\mathcal{L}^{\mathcal{X}}\times\mathcal{L}^{\mathcal{X}}\longrightarrow\mathcal{L}^{\mathcal{X}}$ where
$$(\mathcal{F}\rightarrow\mathcal{G})(x)=\mathcal{F}(x)\rightarrow\mathcal{G}(x).$$
It is claimed that $\mathcal{L}$ satisfies the law of $\mathcal{DN}$egat if
$$(\lambda\rightarrow\bot)\rightarrow\bot=\lambda,\ \text{for all}\ \lambda\in\mathcal{L},$$
For simplicity, we use $\neg\lambda$ to denote $\lambda\rightarrow\bot$. Moreover, for every $\lambda,\nu\in\mathcal{L}$, we define $\lambda\oplus\nu=\neg(\neg\lambda\otimes\neg\nu)$.

**Lemma 2.4** ([10, 34])**.** *If a $\mathcal{Q}$unt $\mathcal{L}$ satisfies the law of $\mathcal{DN}$egat, with $\{\lambda_i:i\in I\}\subseteq\mathcal{L}$ and $\{\nu_j:j\in J\}\subseteq\mathcal{L}$, then $\oplus$ satisfies*

(1) $\bigwedge_{i\in I}\lambda_i\oplus\bigwedge_{j\in J}\nu_j=\bigwedge_{i\in I}\bigwedge_{j\in J}(\lambda_i\oplus\nu_j)$.
(2) $\neg\left(\bigwedge_{i\in I}\lambda_i\right)=\bigvee_{i\in I}(\neg\lambda_i)$.

**Definition 2.5** ([7])**.** Presume that $(\mathcal{L},\le,\otimes),(\mathcal{M},\le,\odot)\in|Squant|$, and $\mathcal{X}$ is a non-empty set.

($i$) A map $\mathcal{T}:\mathcal{L}^{\mathcal{X}}\longrightarrow\mathcal{M}$ is said to be an $\mathcal{LM}$-quasi-fuzzy topology (an $\mathcal{LM}$-q-FTPO for short) on $\mathcal{X}$ iff $\mathcal{T}$ is an $\mathcal{M}$-$F$-$\mathcal{SQ}$unt on $\mathcal{L}^{\mathcal{X}}$, i.e., the next items are fulfilled: $\forall\,\mathcal{F},\mathcal{G}\in\mathcal{L}^{\mathcal{X}}$ and $\{\mathcal{F}_j:j\in J\}\subseteq\mathcal{L}^{\mathcal{X}}$ :
($QT1$) $\mathcal{T}(\mathcal{F})\odot\mathcal{T}(\mathcal{G})\le\mathcal{T}(\mathcal{F}\otimes\mathcal{G})$,
($QT2$) $\bigwedge_{j\in J}\mathcal{T}(\mathcal{F}_j)\le\mathcal{T}\left(\bigvee_{j\in J}\mathcal{F}_j\right)$.

(*ii*) An $\mathcal{LM}$-q-FTPO is said to be strong iff $\mathcal{T}(\underline{\top}) = \top_{\mathcal{M}}$.
(*iii*) If $\mathcal{L}$ is an unital $\mathcal{SQ}$unt with unit $e$. An $\mathcal{LM}$-q-FTPO is said to be an $\mathcal{LM}$-FTPO iff $\mathcal{T}(\underline{e}) = \top_{\mathcal{M}}$.
(*iv*) An $\mathcal{LM}$-FTPO, is said to be Alexandrov if

$$(QT3)\quad \bigwedge_{j\in J} \mathcal{T}(\mathcal{F}_j) \leq \mathcal{T}\left(\bigwedge_{j\in J} \mathcal{F}_j\right).$$

(*vi*) The pair $(\mathcal{X}, \mathcal{T})$ is said to be an $\mathcal{LM}$-q-F (resp., strong $\mathcal{LM}$-q-F, $\mathcal{LM}$-F) topological space if $\mathcal{T}$ is an $\mathcal{LM}$-q- (resp., strong $\mathcal{LM}$-q-, $\mathcal{LM}$-) FTPO on $\mathcal{X}$.

**Definition 2.6** ([11])**.** For $(\mathcal{L}, \leq, \otimes), (\mathcal{M}, \leq, \odot) \in |Squant|$ and a non-empty set $\mathcal{X}$, the mapping $\mathcal{I} : \mathcal{L}^{\mathcal{X}} \times \mathcal{M} \longrightarrow \mathcal{L}^{\mathcal{X}}$ is called:

(*i*) An $\mathcal{LM}$-quasi-fuzzy interior operator (an $\mathcal{LM}$-q-FINTop for short) on $\mathcal{X}$ iff $\mathcal{I}$ fulfills the next axioms: For all $\mathcal{F}, \mathcal{G} \in \mathcal{L}^{\mathcal{X}}, \gamma, \delta \in \mathcal{M}$;
$(I_1)$ $\mathcal{I}(\mathcal{F}, \gamma) \leq \mathcal{I}(\mathcal{G}, \delta)$ whenever $\mathcal{F} \leq \mathcal{G}$, $\delta \leq \gamma$.
$(I_2)$ $\mathcal{I}(\mathcal{F}, \gamma) \leq \mathcal{F}$.
$(I_3)$ $\mathcal{I}(\mathcal{F}, \gamma) \leq \mathcal{I}(\mathcal{I}(\mathcal{F}, \gamma), \gamma)$.
$(I_4)$ $\mathcal{I}(\mathcal{F}, \gamma) \otimes \mathcal{I}(\mathcal{G}, \delta) \leq \mathcal{I}(\mathcal{F} \otimes \mathcal{G}, \gamma \odot \delta)$.
(*ii*) A strong $\mathcal{LM}$-q-FINTop if it satisfies the next item:
$(I_5)$ $\mathcal{I}(\underline{\top}, \gamma) = \underline{\top}$.
(*iii*) An $\mathcal{LM}$-FINTop if $\mathcal{L}$ is a unital $\mathcal{SQ}$unt with unit $e$ and the next item is fulfilled:
$(I_6)$ $\mathcal{I}(\underline{e}, \gamma) = \underline{e}$.

In case $\mathcal{L} = \mathcal{M}$, then a mapping $\mathcal{I} : \mathcal{L}^{\mathcal{X}} \times \mathcal{L} \longrightarrow \mathcal{L}^{\mathcal{X}}$ is called an $\mathcal{L}$-quasi-fuzzy interior on $\mathcal{X}$.

**Definition 2.7** ([10])**.** For $(\mathcal{L}, \leq, \otimes), (\mathcal{M}, \leq, \odot) \in |quant|$ with $\mathcal{L}$ satisfies the law of $\mathcal{DN}$egat and $\mathcal{X}$ be a non-empty set. An $\mathcal{LM}$-fuzzy co-topology ($\mathcal{LM}$-FCoTPO for short) is a map $\mathcal{K} : \mathcal{L}^{\mathcal{X}} \longrightarrow \mathcal{M}$ which fulfills the next items:

$(QCT_1)$ $\mathcal{K}(\underline{\top}) = \mathcal{K}(\underline{\bot}) = \top_{\mathcal{M}}$;
$(QCT_2)$ $\mathcal{K}(\mathcal{F}) \odot \mathcal{K}(\mathcal{G}) \leq \mathcal{K}(\mathcal{F} \oplus \mathcal{G})$ for all $\mathcal{F}, \mathcal{G} \in \mathcal{L}^{\mathcal{X}}$;
$(QCT_3)$ $\bigwedge_{i\in I} \mathcal{K}(\mathcal{F}_i) \leq \mathcal{K}\left(\bigwedge_{i\in I} \mathcal{F}_i\right)$ for every family $\{\mathcal{F}_i : i \in I\} \subseteq \mathcal{L}^{\mathcal{X}}$.

An $\mathcal{LM}$-FCoTPO, is said to be Alexandrov if

$$(QCT_4)\quad \bigwedge_{j\in J} \mathcal{K}(\mathcal{A}_j) \leq \mathcal{K}\left(\bigvee_{j\in J} \mathcal{A}_j\right) \text{ for every family } \{\mathcal{A}_j : j \in J\} \subseteq \mathcal{L}^{\mathcal{X}}.$$

The category of $\mathcal{LM}$-fuzzy co-topological spaces and continuous mappings inducted by $\mathcal{LM}$-FCTop, where whose morphisms from $(\mathcal{X}, \mathcal{K})$ to $(\mathcal{Y}, \theta)$ are all functions $h : \mathcal{X} \longrightarrow \mathcal{Y}$ such that $\theta \leq \mathcal{K} \circ h^{\leftarrow}_{\mathcal{L}}$, where

$h_{\mathcal{L}}^{\leftarrow} : h_{\mathcal{L}^{\mathcal{Y}}} \longrightarrow h_{\mathcal{L}^{\mathcal{X}}}$ is defined by $f_{L}^{\leftarrow}(\mathcal{K}) = \mathcal{K} \circ h$ . Composition and identities in $\mathcal{LM}$-FCTop are the same as in Set.

Naturally, the triple $(\mathcal{X}, \mathcal{T}, \mathcal{K})$ could be referred to as a $\mathcal{LM}$-fuzzy bitopological space.

**Definition 2.8** ([11])**.** The (direct) product of any two $\mathcal{SQ}$unts $(\mathcal{D}, \otimes)$ and $(\mathcal{C}, \odot)$ is again a $\mathcal{SQ}$unt $(\mathcal{D} \times \mathcal{C}, *)$, and the tensor product $*$ on $\mathcal{D} \times \mathcal{C}$ is defined as follows:

$$(x, y) * (z, w) = (x \otimes z, y \odot w).$$

**Example 2.9.** Consider a semi-quantale $\mathcal{L} = (\mathcal{L}, \leq, \otimes)$ and two non-empty sets $\mathcal{X}$ and $\mathcal{Y}$. It is clear that both $\left(\mathcal{L}^{\mathcal{X}}, \leq, \otimes\right)$ and $\left(\mathcal{L}^{\mathcal{Y}}, \leq, \otimes\right)$ are semi-quantales of all $\mathcal{L}$-subsets of a set $\mathcal{X}$ and $\mathcal{Y}$, respectively, then their direct product $\left(\mathcal{L}^{\mathcal{X}} \times \mathcal{L}^{\mathcal{Y}}, \odot\right)$ is the semi-quantale of all pairs of $\mathcal{L}$-subsets $(\alpha, \beta)$, $(\gamma, \delta)$ where $\alpha, \gamma \in \mathcal{L}^{\mathcal{X}}$ and $\beta, \delta \in \mathcal{L}^{\mathcal{Y}}$ , ordered by component-wise inclusion, and the multiplication $\odot$ is defined as

$$(\alpha, \beta) \odot (\gamma, \delta) = (\alpha \otimes \gamma, \beta \otimes \delta).$$

In this paper, if not otherwise specified, $(\mathcal{L}, \leq, \otimes), (\mathcal{M}, \leq, \odot) \in |quant|$.

## 3. A Quantic $\mathcal{M}$-Approximation Systems

The goal of this section is to present and study the concept of a $\mathcal{QM}$-ApprX system as a generalization of Šostak's $\mathcal{M}$-approximate system [29]. Through this section, we define and study the notions of lower and upper $\mathcal{QM}$-ApprX operators, $\mathcal{QM}$-ApprX system, and some related notions.

### 3.1. The Lower $\mathcal{QM}$-Apprx Operator.

**Definition 3.1.** A mapping $\ell : \mathcal{L} \times \mathcal{M} \longrightarrow \mathcal{L}$ is said to be:

- $(i)$ A lower $\mathcal{M}$-ApprX operator on $\mathcal{L}$, if it fulfills the next axioms:
  - $(\ell 1)$ $\ell(c, \gamma) \leq \ell(d, \delta)$, if $c \leq d$ and $\delta \leq \gamma$,
  - $(\ell 2)$ $\ell(c, \gamma) \leq c$,
  - $(\ell 3)$ $\ell(\ell(c, \gamma), \gamma) = \ell(c, \gamma)$,
- $(ii)$ A lower $\mathcal{QM}$-ApprX operator on $\mathcal{L}$, if it is a lower $\mathcal{M}$-ApprX operator and fulfills the next axioms:
  - $(\ell 4)$ $\ell(c, \gamma) \otimes \ell(d, \delta) \leq \ell(c \otimes d, \gamma \odot \delta)$.

Where $c, d \in \mathcal{L}$ and $\gamma, \delta \in \mathcal{M}$. The pair $(\mathcal{L}, \ell)$ is called a lower $\mathcal{QM}$-ApprX system. Let $(\mathcal{L}_1, \ell_1)$ and $(\mathcal{L}_2, \ell_2)$ be a lower $\mathcal{QM}$-ApprX systems. A morphisms $h : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$ is called a lower $\mathcal{QM}$-ApprX map if $\ell_2(h(c), \gamma) \geq h(\ell_1(c, \gamma)), \forall c \in \mathcal{L}_1, \gamma \in \mathcal{M}$.

**Definition 3.2.** A lower $\mathcal{QM}$-ApprX operator $\ell : \mathcal{L} \times \mathcal{M} \longrightarrow \mathcal{L}$ is called:

- Strong when $\ell(\top_{\mathcal{L}},\gamma)=\top_{\mathcal{L}}$.
- Unital when $\mathcal{L}$ is unital and $\ell(e_{\mathcal{L}},\gamma)=e_{\mathcal{L}}$.

**Example 3.3.** (1) Every strong (resp., unital) $\mathcal{M}$-fuzzy quantic conucleus on $\mathcal{L}$ (see Definitions 8, 9 and 10 in [11]) is strong (resp., unital) lower $\mathcal{QM}$-ApprX operator.

(2) In the case where $\mathcal{L}$ is a complete lattice, with a lower $\mathcal{M}$-approximate operator in the sense of Šostak's [29] can be considered as a lower $\mathcal{QM}$-ApprX operator on $\mathcal{L}$ with and $\otimes=\wedge$.

According to Rodabaugh [25], an $\mathcal{M}$-Hutton quasi-fuzzy topological space ($\mathcal{M}$-Hutton q-$FTPO_{SPC}$ for short) is a pair $(\mathcal{L},\mathcal{T})$ where $\mathcal{L}$ is a $\mathcal{SQ}$unt and $\mathcal{T}:\mathcal{L}\longrightarrow\mathcal{M}$ is a mapping satisfying the next axioms: $\forall\lambda,\nu\in\mathcal{L}$ and $\{\lambda_j : j\in J\}\subseteq\mathcal{L}$,

$(HQFT1)$ $\mathcal{T}(\lambda)\odot\mathcal{T}(\nu)\leq\mathcal{T}(\lambda\otimes\nu)$,

$(HQFT2)$ $\bigwedge_{j\in J}\mathcal{T}(\lambda_j)\leq\mathcal{T}\left(\bigvee_{j\in J}\lambda_j\right)$.

- In case $\mathcal{L}=\mathcal{L}^{\mathcal{X}}$, then a mapping $\mathcal{T}:\mathcal{L}^{\mathcal{X}}\longrightarrow\mathcal{M}$ is said to be an $\mathcal{LM}$-q-FTPO on $\mathcal{X}$.
- An $\mathcal{M}$-Hutton q-$FTPO_{SPC}$ is said to be strong if $\mathcal{T}(\top_{\mathcal{L}})=\top_{\mathcal{M}}$.
- If $\mathcal{L}$ is a unital $\mathcal{SQ}$unt with the unit $e_{\mathcal{L}}$, an $\mathcal{M}$-Hutton q-$FTPO_{SPC}$ is an $\mathcal{M}$-Hutton $FTPO_{SPC}$ if $\mathcal{T}(e_{\mathcal{L}})=\top_{\mathcal{M}}$.

A strong $\mathcal{M}$-Hutton q-$FTPO_{SPC}$, is called Alexandrov when $\mathcal{T}$ also satisfies the additional axiom:

$(HQFT3)$ $\bigwedge_{j\in J}\mathcal{T}(c_j)\leq\mathcal{T}\left(\bigwedge_{j\in J}c_j\right)$, for all $\{c_j : j\in J\}\subseteq\mathcal{L}$.

The morphism $f:(\mathcal{L}_1,\mathcal{T}_1)\longrightarrow(\mathcal{L}_2,\mathcal{T}_2)$ of an $\mathcal{M}$-Hutton $FTPO_{SPC}$ are mapping $\mathcal{L}_1\longrightarrow\mathcal{L}_2$ such that $f(\mathcal{T}_2)\leq\mathcal{T}_1$. The category of all $\mathcal{M}$-Hutton q-(resp., $\mathcal{M}$-Hutton ) $FTPO_{SPC}$ and their homomorphisms we denote by is denoted by $\mathcal{M}$-HQFTop (resp., $\mathcal{M}$-HFTop).

In the sequel, we will study the relationship between a lower $\mathcal{QM}$-ApprX operator and an $\mathcal{M}$-Hutton q-$FTPO_{SPC}$.

**Proposition 3.4.** *Presume that $\mathcal{T}:\mathcal{L}\longrightarrow\mathcal{M}$ is an $\mathcal{M}$-Hutton q-(resp., a strong $\mathcal{M}$-Hutton q-) FTPO on $\mathcal{L}$. A map $\ell_{\mathcal{T}}:\mathcal{L}\times\mathcal{M}\longrightarrow\mathcal{L}$ defined by*

$$\ell_{\mathcal{T}}(c,\gamma)=\bigvee\{x\in\mathcal{L}:x\leq c,\mathcal{T}(x)\geq\gamma\},\quad\forall c\in\mathcal{L},\gamma\in\mathcal{M},$$

*is a lower (resp., a strong lower) $\mathcal{QM}$-ApprX on $\mathcal{L}$.*

*Proof.* Suppose that $\mathcal{T}:\mathcal{L}\longrightarrow\mathcal{M}$ is an $\mathcal{M}$-Hutton q-FTPO on $\mathcal{L}$.

($\ell1$) For $c,d\in\mathcal{L}$ and $\gamma,\delta\in\mathcal{M}$ with $c\leq d$ and $\delta\leq\gamma$, we have

$$\ell_{\mathcal{T}}(c,\gamma)=\bigvee\{x\in\mathcal{L}:x\leq c,\mathcal{T}(x)\geq\gamma\}$$

$$\leq \bigvee \{x \in \mathcal{L} : x \leq c \leq d, \mathcal{T}(x) \geq \gamma \geq \delta\}$$
$$= \bigvee \{x \in \mathcal{L} : x \leq d, \mathcal{T}(x) \geq \delta\} = \ell_{\mathcal{T}}(d, \delta)$$

So, $\ell_{\mathcal{T}}(c, \gamma) \leq \ell_{\mathcal{T}}(d, \delta)$.

($\ell$2) According to definition of $\ell_{\mathcal{T}}$, we have

$$\ell_{\mathcal{T}}(c, \gamma) = \bigvee \{x \in \mathcal{L} : x \leq c, \mathcal{T}(x) \geq \gamma\} \leq c.$$

Then, $\ell_{\mathcal{T}}(c, \gamma) \leq c$.

($\ell$3) Since $\ell_{\mathcal{T}}(c, \gamma) \in \mathcal{L}$ and

$$\ell_{\mathcal{T}}(\ell_{\mathcal{T}}(c, \gamma), \gamma) = \bigvee \{x \in \mathcal{L} : x \leq \ell_{\mathcal{T}}(c, \gamma), \mathcal{T}(x) \geq \gamma\},$$

then we get $\mathcal{T}(\ell_{\mathcal{T}}(c, \gamma)) \geq \mathcal{T}(x) \geq \gamma$.

Taking $x = \ell_{\mathcal{T}}(c, \gamma)$, we have

$$\ell_{\mathcal{T}}(\ell_{\mathcal{T}}(c, \gamma), \gamma) = \bigvee \ell_{\mathcal{T}}(c, \gamma),$$

and this implies

$$\ell_{\mathcal{T}}(c, \gamma) \leq \ell_{\mathcal{T}}(\ell_{\mathcal{T}}(c, \gamma), \gamma).$$

Also, from ($\ell$2), we have that

$$\ell_{\mathcal{T}}(c, \gamma) \geq \ell_{\mathcal{T}}(\ell_{\mathcal{T}}(c, \gamma), \gamma).$$

Thus equality is achieved.

($\ell$4) For $c, d \in \mathcal{L}$ and $\gamma, \delta \in \mathcal{M}$, we have

$$\begin{aligned}
&\ell_{\mathcal{T}}(c, \gamma) \otimes \ell_{\mathcal{T}}(d, \delta) \\
&\quad = \bigvee \{x \in \mathcal{L} : x \leq c, \mathcal{T}(x) \geq \gamma\} \otimes \bigvee \{y \in \mathcal{L} : y \leq d, \mathcal{T}(y) \geq \delta\} \\
&\quad = \bigvee \{x \otimes y \in \mathcal{L} : x \otimes y \leq c \otimes d, \mathcal{T}(x) \odot \mathcal{T}(y) \geq \gamma \odot \delta\} \\
&\quad \leq \bigvee \{x \otimes y \in \mathcal{L} : x \otimes y \leq c \otimes d, \mathcal{T}(x \otimes y) \geq \gamma \odot \delta\} \\
&\quad = \bigvee \{w \in \mathcal{L} : w \leq c \otimes d, \mathcal{T}(w) \geq \gamma \odot \delta\} \\
&\quad = \ell_{\mathcal{T}}(c \otimes d, \gamma \odot \delta).
\end{aligned}$$

Then, $\ell_{\mathcal{T}}(c, \gamma) \otimes \ell_{\mathcal{T}}(d, \delta) \leq \ell_{\mathcal{T}}(c \otimes d, \gamma \odot \delta)$. □

If an $\mathcal{M}$-Hutton q-FTPO on $\mathcal{L}$ is strong, then it is obvious that

$$\ell_{\mathcal{T}}(\top_{\mathcal{L}}, \gamma) = \bigvee \{x \in \mathcal{L} : x = \top_{\mathcal{L}}, \mathcal{T}(x) \geq \gamma\} = \top_{\mathcal{L}},$$

and therefore $\ell_{\mathcal{T}}$ is a strong lower $\mathcal{QM}$-ApprX operator on $\mathcal{L}$.

**Remark 3.5.** In case $\mathcal{L}$ is a unital $\mathcal{SQ}$unt, the map $\ell_{\mathcal{T}} : \mathcal{L} \longrightarrow \mathcal{M}$, which defined by $\ell_{\mathcal{T}}(e_{\mathcal{L}}, \gamma) = \bigvee \{x \in \mathcal{L} : x = e_{\mathcal{L}}, \mathcal{T}(x) \geq \gamma\} = e_{\mathcal{L}}$, is a unital lower $\mathcal{QM}$-ApprX on $\mathcal{L}$.

**Proposition 3.6.** *Given a quantic lower (resp., a strong lower) $\mathcal{QM}$-ApprX operator $\ell : \mathcal{L} \times \mathcal{M} \longrightarrow \mathcal{L}$, an $\mathcal{M}$-fuzzy set $\mathcal{T}_\ell : \mathcal{L} \longrightarrow \mathcal{M}$ defined by*

$$\mathcal{T}_\ell(c) = \bigvee \{\gamma \in \mathcal{M} : \ell(c,\gamma) \geq c, c \in \mathcal{L}\},$$

*is an $\mathcal{M}$-Hutton q-(resp., a strong $\mathcal{M}$-Hutton q-) FTPO on $\mathcal{L}$.*

*Proof.* Suppose that $\ell : \mathcal{L} \times \mathcal{M} \to \mathcal{L}$ is a lower $\mathcal{QM}$-ApprX on $\mathcal{L}$. To show that $\mathcal{T}_\ell$ is an $\mathcal{M}$-Hutton q-FTPO on $\mathcal{L}$.

$(HQFT1)$ For a family of $\{c_j : j \in J\} \subset \mathcal{L}$, we fined

$$\begin{aligned}
\mathcal{T}_\ell\left(\bigvee_j c_j\right) &= \bigvee\left\{\gamma \in \mathcal{M} : \ell\left(\bigvee_{j\in J} c_j, \gamma\right) \geq \bigvee_{j\in J} c_j\right\}\\
&= \bigvee\left\{\gamma \in \mathcal{M} : \bigvee_{j\in J} \ell(c_j,\gamma) \geq \bigvee_{j\in J} c_j\right\}\\
&\geq \bigwedge_{j\in J} \bigvee \{\gamma \in \mathcal{M} : \ell(c_j,\gamma) \geq c_j\}\\
&= \bigwedge_{j\in J} \mathcal{T}_\ell(c_j).
\end{aligned}$$

Then, $\mathcal{T}_\ell\left(\bigvee_{j\in J} c_j\right) \geq \bigwedge_{j\in J} \mathcal{T}_\ell(c_j)$.

$(HQFT2)$ For $c, d \in \mathcal{L}$ and $\gamma, \delta \in \mathcal{M}$.

$$\begin{aligned}
\mathcal{T}_\ell(c) \odot \mathcal{T}_\ell(d) &= \bigvee \{\gamma \in \mathcal{M} : \ell(c,\gamma) \geq c\} \odot \bigvee \{\delta \in \mathcal{M} : \ell(d,\delta) \geq d\}\\
&= \bigvee \{\gamma \odot \delta \in \mathcal{M} : \ell(c,\gamma) \otimes \ell(d,\delta) \geq c \otimes d\}\\
&\leq \bigvee \{\gamma \odot \delta \in \mathcal{M} : \ell(c \otimes d, \gamma \odot \delta) \geq c \otimes d\}\\
&= \mathcal{T}_\ell(c \otimes d).
\end{aligned}$$

Then, $\mathcal{T}_\ell(c) \odot \mathcal{T}_\ell(d) \leq \mathcal{T}_\ell(c \otimes d)$.

In the case of a strong lower $\mathcal{QM}$-ApprX, i.e., $\ell(\top_\mathcal{L}, \gamma) = \top_\mathcal{L}$, we get that $\mathcal{T}_\ell(\top_\mathcal{L}) = \top_\mathcal{M}$ so that the proof is complete. □

**Remark 3.7.** When $\mathcal{L}$ is a unital $\mathcal{SQ}$unt, and

$$\mathcal{T}_\ell(e_\mathcal{L}) = \bigvee \{\gamma \in \mathcal{M} : \ell(e_\mathcal{L}, \gamma) = e_\mathcal{L}\} = \top_\mathcal{M},$$

then $\mathcal{T}_\ell$ is an $\mathcal{M}$-Hutton FTPO on $\mathcal{L}$.

**Proposition 3.8.** *Given a strong lower $\mathcal{QM}$-ApprX operator $\ell : \mathcal{L} \times \mathcal{M} \to \mathcal{L}$, an $\mathcal{M}$-fuzzy set $\mathcal{T}_\ell : \mathcal{L} \to \mathcal{M}$ which defined by*

$$\mathcal{T}_\ell(c) = \bigvee \{\gamma \in \mathcal{M} : \ell(c,\gamma) \geq c, c \in \mathcal{L}\},$$

*is an Alexandrov $\mathcal{M}$-Hutton q-FTPO on $\mathcal{L}$.*

*Proof.* We just show the condition $(HQFT3)$. For $c, d \in \mathcal{L}$ and $\gamma, \delta \in \mathcal{M}$, we get

$(HQFT3)$ For a family of $\{c_j : j \in J\} \subset \mathcal{L}$, we get

$$\begin{aligned}\mathcal{T}_\ell\left(\bigwedge_j c_j\right) &= \bigvee\left\{\gamma \in \mathcal{M} : \ell\left(\bigwedge_{j\in J} c_j, \gamma\right) \geq \bigwedge_{j\in J} c_j\right\} \\ &= \bigvee\left\{\gamma \in \mathcal{M} : \bigwedge_{j\in J} \ell\left(c_j, \gamma\right) \geq \bigwedge_{j\in J} c_j\right\} \\ &\geq \bigwedge_{j\in J}\left(\bigvee\left\{\gamma \in \mathcal{M} : \ell\left(c_j, \gamma\right) \geq c_j\right\}\right) \\ &= \bigwedge_{j\in J} \mathcal{T}_\ell\left(c_j\right).\end{aligned}$$

Then, $\mathcal{T}_\ell\left(\bigwedge_{j\in J} c_j\right) \geq \bigwedge_{j\in J} \mathcal{T}_\ell\left(c_j\right)$. □

**Example 3.9.** Every Alexandrov $\mathcal{LM}$-fuzzy topology $\mathcal{T} : \mathcal{L}^{\mathcal{X}} \longrightarrow \mathcal{M}$ (see Definition 2.5) is an Alexandrov $\mathcal{M}$-Hutton q-FTPO on $\mathcal{L}^{\mathcal{X}}$.

**Remark 3.10.** A lower $\mathcal{M}$-approximation operator based on non-increasing, reflexive, unary and transitive $\mathcal{LM}$-fuzzy $\mathcal{G}$ neighborhood system [10] can be consider as an example of a lower $\mathcal{QM}$-ApprX on the power set $\mathcal{Q}$unt $\mathcal{L}^{\mathcal{X}}$.

Next, we will discuss and illustrate the notion of an $\mathcal{M}$-open structure, and discuss the relationship between it and lower $\mathcal{QM}$-ApprX operator.

**Definition 3.11.** A set $O \subset \mathcal{L} \times \mathcal{M}$ is said to be an $\mathcal{M}$-open structure iff it fulfills the next axioms.

- $(O1)$ $(\top, \gamma) \in O$ for each $\gamma \in \mathcal{M}$,
- $(O2)$ If $(c, \gamma), (d, \delta) \in O$, then $(c, \gamma) * (d, \delta) = (c \otimes d, \gamma \odot \delta) \in O$,
- $(O3)$ If $(c_j, \gamma) \in O$ for $j \in J$, then $\left(\bigvee_{j\in J} c_j, \gamma\right) \in O$,
- $(O4)$ If $(c, \gamma) \in O$ and $\delta < \gamma$, $\gamma, \delta \in \mathcal{M}$, then $(c, \delta) \in O$.

The couple $(\mathcal{L}, O)$ is said to be an $\mathcal{M}$-open system. Let $(\mathcal{L}_1, O_1)$ and $(\mathcal{L}_2, O_2)$ be $\mathcal{M}$-open systems. A morphism $\jmath : \mathcal{L}_1 \to \mathcal{L}_2$ is said to be an $\mathcal{M}$-open map if $\left(\jmath^{-1}(c), \gamma\right) \in O_1$ whenever $(c, \gamma) \in O_2$ for all $c \in \mathcal{L}, \gamma \in \mathcal{M}$.

**Example 3.12.** For an infinitely distributive complete lattice $\mathcal{L} = (\mathcal{L}, \leq, \bigwedge, \bigvee)$, and a complete lattice $\mathcal{M}$, with every lower $\mathcal{M}$-approximate operator $\ell : \mathcal{L} \times \mathcal{M} \to \mathcal{L}$ [ [29], Lemma 4.3.], there is a subset

$$O_\ell = \{(c,\alpha) : (c,\alpha) \in \mathcal{L} \times \mathcal{M}, \ell(c,\alpha) = c\} \subseteq \mathcal{L} \times \mathcal{M},$$

with the properties:

($O_\ell 1$) $(1_{\mathcal{L}}, \alpha) \in O_\ell$ for each $\alpha \in \mathcal{M}$,
($O_\ell 2$) $(0_{\mathcal{L}}, 1_{\mathcal{M}}) \in O_\ell$ for each $\alpha \in \mathcal{M}$,
($O_\ell 3$) If $(c,\alpha), (d,\beta) \in O_\ell$, then $(c,\alpha) \wedge (d,\beta) \in O_\ell$,
($O_\ell 4$) If $(c_j,\alpha) \in O_\ell$ for $j \in J$, then $\left(\bigvee_{j\in J} c_j, \alpha\right) \in O_\ell$,
($O_\ell 5$) If $(c,\alpha) \in O_\ell$ and $\beta < \alpha$, $\alpha, \beta \in \mathcal{M}$, then $(c,\beta) \in O_\ell$.

**Proposition 3.13.** *Given a strong lower $\mathcal{QM}$-ApprX operator $\ell : \mathcal{L} \times \mathcal{M} \to \mathcal{L}$, then $O_\ell \subset \mathcal{L} \times \mathcal{M}$ is defined by*

$$O_\ell = \{(c,\gamma) \in L \times \mathcal{M} : \ell(c,\gamma) = c\},$$

*is an $\mathcal{M}$-open structure with $(\ell(c,\gamma), \gamma) \in O_\ell$.*

*Proof.* ($O1$) In the case of a strong lower $\mathcal{QM}$-ApprX, i.e., $\ell(\top_{\mathcal{L}}, \gamma) = \top_L$, we have that $(\top_{\mathcal{L}}, \gamma) \in O_\ell \forall \gamma \in \mathcal{M}$.

($O2$) Let $(c,\gamma), (d,\delta) \in O_\ell$, since $\ell(c,\gamma) = c$ and $\ell(d,\delta) = d$. From ($\ell 4$), we have that

$$\ell(c \otimes d, \gamma \odot \delta) \geq \ell(c,\gamma) \otimes \ell(d,\delta) = c \otimes d.$$

Also, by ($\ell 2$), we have

$$\ell(c \otimes d, \gamma \odot \delta) \leq c \otimes d.$$

Thus equality is achieved. So $(c \otimes d, \gamma \odot \delta) \in O_\ell$.

($O3$) Let $(c_j, \gamma) \in O_\ell$ for $j \in J$. Then

$$\ell\left(\bigvee_{j\in J} c_j, \gamma\right) \geq \bigvee_{i \in J} \ell(c_j,\gamma) = \bigvee_{j\in J} c_j.$$

From ($\ell 2$), we have that $\ell\left(\bigvee_{j\in J} c_j, \gamma\right) = \bigvee_{j\in J} c_j$.
Hence $\left(\bigvee_{j\in J} c_j, \gamma\right) \in O_\ell$.

($O4$) If $(c,\gamma) \in O_\ell$, since $\ell(c,\gamma) = c$, for $\delta < \gamma$, then

$$c = \ell(c,\gamma) \leq \ell(c,\delta) \leq c.$$

So, $(c,\delta) \in O_\ell$. Moreover, from ($\ell 3$), since $\ell(\ell(c,\gamma), \gamma) = \ell(c,\gamma)$, then $(\ell(c,\gamma), \gamma) \in O_\ell$. The proof is finished with this. □

**Proposition 3.14.** *Given an $\mathcal{M}$-open structure $O \subset \mathcal{L} \times \mathcal{M}$, a mapping $\ell_O : \mathcal{L} \times \mathcal{M} \to \mathcal{L}$ which defined by*

$$\ell_O(c,\gamma) = \bigvee \{d : (d,\gamma) \in O, d \leq c\},$$

*is a strong lower $\mathcal{QM}$-ApprX operator.*

*Proof.* ($\ell 1$) If $c \leq d$ and $\delta \leq \gamma$, we have

$$\begin{aligned}\ell_O(c,\gamma) &= \bigvee \{c_1 : (c_1,\gamma) \in O, c_1 \leq c\}\\ &\leq \bigvee \{c_1 : (c_1,\delta) \in O, c_1 \leq d\}\\ &= \ell_O(d,\delta).\end{aligned}$$

($\ell 2$) It follows from the definition of $\ell_O$.

($\ell 3$) From the definition of $\ell_O$ and ($O3$), since $(\ell_O(c,\gamma),\gamma) \in O$ and $\ell_O(c,\gamma) \leq \ell_O(c,\gamma)$, $\ell_O(\ell_O(c,\gamma),\gamma) \geq \ell_O(c,\gamma)$. By ($\ell 2$), and this implies $\ell_O(\ell_O(c,\gamma),\gamma) = \ell_O(c,\gamma)$.

Since $(\top_{\mathcal{L}},\gamma) \in O$, we have $\ell_O(\top_{\mathcal{L}},\gamma) = \top_{\mathcal{L}}$ and therefore $\ell_O$ is a strong lower $\mathcal{M}$-ApprX operator on $\mathcal{L}$.

($\ell 4$) For $c, d \in \mathcal{L}$ and $\gamma, \delta \in \mathcal{M}$, we have

$$\begin{aligned}\ell_O(c,\gamma) \otimes \ell_O(d,\delta) &= \bigvee \{c_1 : (c_1,\gamma) \in O, c_1 \leq c\}\\ &\quad \otimes \bigvee \{d_1 : (d_1,\delta) \in O, d_1 \leq d\}\\ &= \bigvee \{c_1 \otimes d_1 : (c_1,\gamma) * (d_1,\delta) \in O, c_1 \leq c, d_1 \leq d\}\\ &\leq \bigvee \{c_1 \otimes d_1 : (c_1 \otimes d_1, \gamma \odot \delta) \in O, c_1 \otimes d_1 \leq c \otimes d\}\\ &\leq \ell_O(c \otimes d, \gamma \odot \delta).\end{aligned}$$

Then, $\ell_O(c,\gamma) \otimes \ell_O(d,\delta) \leq \ell_O(c \otimes d, \gamma \odot \delta)$. Thus, $\ell_O$ is a strong lower $\mathcal{QM}$-ApprX operator on $\mathcal{L}$. □

**Proposition 3.15.** *Let $(\mathcal{L}_1,\ell_1)$ and $(\mathcal{L}_2,\ell_2)$ be lower $\mathcal{QM}$-ApprX systems. Then $h : (\mathcal{L}_1,\ell_1) \to (\mathcal{L}_2,\ell_2)$ is a lower $\mathcal{QM}$-ApprX map iff the map $h : (\mathcal{L}_1, O_{\ell_1}) \to (\mathcal{L}_2, O_{\ell_2})$ is an $\mathcal{M}$-open.*

*Proof.* ($\Rightarrow$) Let $(c,\gamma) \in O_{\ell_2}$. Then $\ell_2(c,\gamma) = c$.
Since $h(\ell_1(d,\gamma)) \leq \ell_2(h(d),\gamma)$, we put $d = h^{-1}(c)$.
Then

$$\begin{aligned}h\left(\ell_1\left(h^{-1}(c),\gamma\right)\right) &\leq \ell_2\left(h\left(h^{-1}(c)\right),\gamma\right)\\ &\leq \ell_2(c,\gamma)\\ &= c.\end{aligned}$$

Hence $\ell_1\left(h^{-1}(c),\gamma\right) \leq h^{-1}(c)$. Thus $\left(h^{-1}(c),\gamma\right) \in O_{\ell_1}$.

$$\begin{aligned}(\Leftarrow)\quad \ell_2(h(d),\gamma) &= \ell_{O_{\ell_2}}(h(d),\gamma)\\ &= \bigvee \{c : (c,\gamma) \in O_{\ell_2}, c \leq h(d)\}\\ &\geq \bigvee \left\{c : \left(h^{-1}(c),\gamma\right) \in O_{\ell_1}, h^{-1}(c) \leq d\right\}\\ &\geq \bigvee \left\{h\left(h^{-1}(c)\right) : \left(h^{-1}(c),\gamma\right) \in O_{\ell_1}, h^{-1}(c) \leq d\right\}\end{aligned}$$

$$\geq h\left(\bigvee\left\{h^{-1}(c):\left(h^{-1}(c),\gamma\right)\in O_{\ell_1}, h^{-1}(c)\leq d\right\}\right)$$
$$\geq h\left(\ell_1(d,\gamma)\right). \qquad \square$$

**Proposition 3.16.** *Let $(\mathcal{L}_1, O_1)$ and $(\mathcal{L}_2, O_2)$ be $\mathcal{M}$-open systems. Then $h:(\mathcal{L}_1, O_1)\to(\mathcal{L}_2, O_2)$ is an $\mathcal{M}$-open map iff $h:\left(\mathcal{L}_1, \ell_{O_1}\right)\to\left(\mathcal{L}_2, \ell_{O_2}\right)$ is a lower $\mathcal{QM}$-approximate map.*

*Proof.* The proof is identical to that for proposition 3.15. $\square$

### 3.2. **The Upper $\mathcal{QM}$-ApprX Operator.**

**Definition 3.17.** A mapping $\mathfrak{u}:\mathcal{L}\times\mathcal{M}\to\mathcal{L}$ is said to be:

- $(i)$ An upper $\mathcal{M}$-ApprX operator on $\mathcal{L}$ if it fulfills the next axioms:
  - $(u1)$ $\mathfrak{u}(\bot_{\mathcal{L}},\gamma)=\bot_{\mathcal{L}}$.
  - $(u2)$ $\mathfrak{u}(c,\gamma)\leq\mathfrak{u}(d,\delta)$, if $c\leq d$ and $\gamma\leq\delta$.
  - $(u3)$ $c\leq\mathfrak{u}(c,\gamma)$.
  - $(u4)$ $\mathfrak{u}(\mathfrak{u}(c,\gamma),\gamma)=\mathfrak{u}(c,\gamma)$.
- $(ii)$ An upper $\mathcal{QM}$-ApprX on $\mathcal{L}$ if it is an upper $\mathcal{M}$-ApprX with $\mathcal{L}$ satisfy the law of $\mathcal{DN}$egat and satisfies the next condition:
  - $(u5)$ $\mathfrak{u}(c\oplus d,\gamma\odot\delta)\leq\mathfrak{u}(c,\gamma)\oplus\mathfrak{u}(d,\delta)$.

Where $c,d\in\mathcal{L}$ and $\gamma,\delta\in\mathcal{M}$. The couple $(\mathcal{L},\mathfrak{u})$ is said to be an upper $\mathcal{QM}$-ApprX system. Let $(\mathcal{L}_1,\mathfrak{u}_1)$ and $(\mathcal{L}_2,\mathfrak{u}_2)$ be an upper $\mathcal{QM}$-ApprX systems. A morphisms $h:\mathcal{L}_1\to\mathcal{L}_2$ is called an upper $\mathcal{QM}$-ApprX map if $h(\mathfrak{u}_1(c,\gamma))\geq\mathfrak{u}_2(h(c),\gamma), \forall c\in\mathcal{L}_1,\gamma\in\mathcal{M}$.

**Example 3.18.**
(1) As we know from Definitions 6 and 7 in [11], every $\mathcal{M}$-fuzzy quantic nucleus operator on $\mathcal{L}$ is an upper $\mathcal{QM}$-ApprX operator.
(2) In the case where $\mathcal{L}$ is a complete lattice, with an upper $\mathcal{M}$-approximate operator in the sense of Šostak's [29] can be considered as an upper $\mathcal{QM}$-ApprX operator on $\mathcal{L}$ with and $\oplus=\vee$.

**Remark 3.19.**
(1) In case $\mathcal{M}=\{0,1\}$ is a two-point lattice, the upper (resp., lower) $\mathcal{M}$-approximative operator is essentially equivalent to the concept of upper (resp., lower) $\{0,1\}$-approximative operator first defined in [30] and later studied in [31].
(2) In case $\mathcal{M}$ is a one-point lattice $\mathcal{M}=\bullet=\{\cdot\}$, upper (resp., lower) $\mathcal{M}$-approximative operator is essentially equivalent to the upper (resp., lower) $\bullet$-approximate operator [29].

**Remark 3.20.** Every $\mathcal{LM}$-q-(resp., a strong $\mathcal{LM}$-q-, $\mathcal{LM}$-) FINTop on $\mathcal{X}$ is a lower (resp., a strong lower, an unital lower) $\mathcal{QM}$-ApprX on $\mathcal{L}^{\mathcal{X}}$. Also, every an $\mathcal{LM}$-fuzzy closure operator on $\mathcal{X}$ is an upper $\mathcal{QM}$-ApprX on $\mathcal{L}^{\mathcal{X}}$. Thus, the triple $\left(\mathcal{L}^{\mathcal{X}}, int, cl\right)$ is the corresponding $\mathcal{QM}$-ApprX system on $\mathcal{L}^{\mathcal{X}}$.

**Definition 3.21.** Let $\mathcal{L}$ fulfills the law of $\mathcal{DN}$egat, an $\mathcal{M}$-Hutton fuzzy co-topological space ($\mathcal{M}$-Hutton $FCoTPO_{SPC}$ for short) is a pair $(\mathcal{L}, \vartheta)$ where $\mathcal{L}$ is a $\mathcal{SQ}$unt and $\vartheta : \mathcal{L} \longrightarrow \mathcal{M}$ is a mapping satisfying the next conditions: For all $c, d \in \mathcal{L}$ and $\{c_i : i \in I\} \subseteq \mathcal{L}$,

$(HFCT1)$ $\vartheta(\underline{\top}) = \vartheta(\underline{\perp}) = \underline{\top}$,
$(HFCT2)$ $\bigwedge_{i\in I} \vartheta(c_i) \leq \vartheta\left(\bigwedge_{i\in I} c_i\right)$,
$(HFCT3)$ $\vartheta(c) \odot \vartheta(d) \leq \vartheta(c \oplus d)$.

An $\mathcal{M}$-Hutton $FCoTPO_{SPC}$, is said to be Alexandrov if $\vartheta$ also satisfies the additional axiom:

$(HFCT4)$ $\bigwedge_{j\in J} \vartheta(c_j) \leq \vartheta\left(\bigvee_{j\in J} c_j\right)$, for all $\{c_j : j \in J\} \subseteq \mathcal{L}$.

The morphism $f : (\mathcal{L}_1, \vartheta_1) \longrightarrow (\mathcal{L}_2, \vartheta_2)$ of an $\mathcal{M}$-Hutton $FCoTPO_{SPC}$ are mapping $\mathcal{L}_1 \longrightarrow \mathcal{L}_2$ such that $f(\vartheta_2) \leq \vartheta_1$. The category of all $\mathcal{M}$-Hutton $FCoTPO_{SPC}$ and their homomorphisms we denote by is denoted by $\mathcal{M} - HFCTop$.

Clearly, the triple $(\mathcal{L}, \mathcal{T}, \vartheta)$ could be referred to as a $\mathcal{M}$-Hutton fuzzy ditopological space.

**Example 3.22.** Every Alexandrov $\mathcal{LM}$-f Co-TPO $\mathcal{K} : \mathcal{L}^{\mathcal{X}} \to \mathcal{M}$ is an Alexandrov $\mathcal{M}$-Hutton FCoTPO on $\mathcal{L}^{\mathcal{X}}$.

**Remark 3.23.** In case when $\mathcal{L} = \mathcal{L}^{\mathcal{X}}$, and $(\mathcal{L}, \mathcal{T}, \vartheta)$ is an $\mathcal{M}$-Hutton fuzzy ditopological space, the triple $\left(\mathcal{L}^{\mathcal{X}}, \mathcal{T}, \mathcal{K}\right)$ is said to be an $\mathcal{LM}$-fuzzy bitopological space.

In the sequel, we will introduce the relationship between an $\mathcal{M}$-Hutton FCoTPO and an upper $\mathcal{QM}$-ApprX operator on $\mathcal{L}$.

**Proposition 3.24.** *Let $\vartheta : \mathcal{L} \to \mathcal{M}$ be an $\mathcal{M}$-Hutton FCoTPO on $\mathcal{L}$. A map $\mathfrak{u}_\vartheta : \mathcal{L} \times \mathcal{M} \to \mathcal{L}$ defined by*

$$\mathfrak{u}_\vartheta(c, \gamma) = \bigwedge \{x \in \mathcal{L} : x \geq c, \vartheta(x) \geq \gamma\}, \quad \forall c \in L, \gamma \in \mathcal{M},$$

*is an upper $\mathcal{QM}$-ApprX operator on $\mathcal{L}$.*

*Proof.* Suppose that $\vartheta : \mathcal{L} \to \mathcal{M}$ is an $\mathcal{M}$-Hutton FCoTPO on $\mathcal{L}$.

$(u1)$ $\mathfrak{u}(\perp_{\mathcal{L}}, \gamma) = \bigwedge \{x \in \mathcal{L} : x \geq \perp_{\mathcal{L}}, \vartheta(x) \geq \gamma\} = \perp_{\mathcal{L}}$.
$(u2)$ If $c \leq d$ and $\gamma \leq \delta$, then

$$\begin{aligned}\mathfrak{u}_\vartheta(d, \delta) &= \bigwedge \{x \in \mathcal{L} : x \geq d, \vartheta(x) \geq \delta\} \\ &\geq \bigwedge \{x \in \mathcal{L} : x \geq d \geq c, \vartheta(x) \geq \delta \geq \gamma\} \\ &= \bigwedge \{x \in \mathcal{L} : x \geq c, \vartheta(x) \geq \gamma\}\end{aligned}$$

$$= \mathfrak{u}_{\vartheta}(c, \gamma).$$

$(u3)$ From definition of $\mathfrak{u}_{\vartheta}$, we have

$$\mathfrak{u}_{\vartheta}(c, \gamma) = \bigwedge \{x \in \mathcal{L} : x \geq c, \vartheta(x) \geq \gamma\} \geq c.$$

So, $\mathfrak{u}_{\vartheta}(c, \gamma) \geq c$.

$(u4)$ Since $\mathfrak{u}_{\vartheta}(c, \gamma) \in \mathcal{L}$ and

$$\mathfrak{u}_{\vartheta}(\mathfrak{u}_{\vartheta}(c, \gamma), \gamma) = \bigwedge \{x \in \mathcal{L} : x \geq \mathfrak{u}_{\vartheta}(c, \gamma), \vartheta(x) \geq \gamma\},$$

we get $\vartheta(x) \geq \vartheta(\mathfrak{u}_{\vartheta}(c, \gamma)) \geq \gamma$. Then we taking $x = \mathfrak{u}_{\vartheta}(c, \gamma)$, we have $\mathfrak{u}_{\vartheta}(\mathfrak{u}_{\vartheta}(c, \gamma), \gamma) = \bigwedge \mathfrak{u}_{\vartheta}(c, \gamma)$ and this implies

$$\mathfrak{u}_{\vartheta}(c, \gamma) \geq \mathfrak{u}_{\vartheta}(\mathfrak{u}_{\vartheta}(c, \gamma), \gamma).$$

Also, by $(u2)$, we have that

$$\mathfrak{u}_{\vartheta}(c, \gamma) \leq \mathfrak{u}_{\vartheta}(\mathfrak{u}_{\vartheta}(c, \gamma), \gamma).$$

Then equality is achieved. □

**Proposition 3.25.** *Presume $(\mathcal{M}, \leq, \odot) \in |Squant|$ with $\mathcal{L}$ fulfills the law of $\mathcal{DN}egat$ and $\vartheta : \mathcal{L} \to \mathcal{M}$ is an $\mathcal{M}$-Hutton FCoTPO on $\mathcal{L}$. The mapping $\mathfrak{u}_{\vartheta} : \mathcal{L} \times \mathcal{M} \to \mathcal{L}$ defined by*

$$\mathfrak{u}_{\vartheta}(c, \gamma) = \bigwedge \{x \in \mathcal{L} : x \geq c, \vartheta(x) \geq \gamma\}, \quad \forall c \in \mathcal{L}, \gamma \in \mathcal{M},$$

*is an upper $\mathcal{QM}$-ApprX on $\mathcal{L}$.*

*Proof.* We only show the itemize $(u5)$. For $c, d \in \mathcal{L}$ and $\gamma, \delta \in \mathcal{M}$, we get

$$
\begin{aligned}
(u5)\ & \mathfrak{u}_{\vartheta}(c, \gamma) \oplus \mathfrak{u}_{\vartheta}(d, \delta) \\
&= \bigwedge \{x \in \mathcal{L} : x \geq c, \vartheta(x) \geq \gamma\} \oplus \bigwedge \{y \in \mathcal{L} : y \geq d, \vartheta(y) \geq \delta\} \\
&= \bigwedge \{x \oplus y \in \mathcal{L} : x \geq c, y \geq d, \vartheta(x) \geq \gamma, \vartheta(y) \geq \delta\} \\
&= \bigwedge \{x \oplus y \in \mathcal{L} : x \oplus y \geq c \oplus d, \vartheta(x) \odot \vartheta(y) \geq \gamma \odot \delta\} \\
&\geq \bigwedge \{x \oplus y \in \mathcal{L} : x \oplus y \geq c \oplus d, \vartheta(x \oplus y) \geq \gamma \odot \delta\} \\
&= \bigwedge \{z \in \mathcal{L} : z \geq c \oplus d, \vartheta(z) \geq \gamma \odot \delta\} \\
&= \mathfrak{u}_{\vartheta}(c \oplus d, \gamma \odot \delta).
\end{aligned}
$$

Then, $\mathfrak{u}_{\vartheta}(c, \gamma) \oplus \mathfrak{u}_{\vartheta}(d, \delta) \geq \mathfrak{u}_{\vartheta}(c \oplus d, \gamma \odot \delta)$. □

**Proposition 3.26.** *Let $\mathcal{L}$ satisfies the law of $\mathcal{DN}egat$ and given an upper $\mathcal{QM}$-ApprX operator $\mathfrak{u} : \mathcal{L} \times \mathcal{M} \to \mathcal{L}$, then an $\mathcal{M}$-fuzzy set $\vartheta_{\mathfrak{u}} : \mathcal{L} \to \mathcal{M}$ which defined by*

$$\vartheta_{\mathfrak{u}}(c) = \bigvee \{\gamma \in \mathcal{M} : \mathfrak{u}(c, \gamma) \leq c, c \in \mathcal{L}\},$$

*is an $\mathcal{M}$-Hutton FCoTPO on $\mathcal{L}$.*

*Proof.* Assume that $\mathfrak{u} : \mathcal{L} \times \mathcal{M} \to \mathcal{L}$ is an upper $\mathcal{QM}$-ApprX on $\mathcal{L}$. we examine that $\vartheta_{\mathfrak{u}}$ is an $\mathcal{M}$-Hutton FCoTPO on $\mathcal{L}$, we will prove that the conditions $((HFCT1)$-$(HFCT3))$ of the above definition hold.
$(HFCT1)$ is straightforward.
$(HFCT2)$ For $c, d \in \mathcal{L}$ and $\gamma, \delta \in \mathcal{M}$.

$$\begin{aligned}
\vartheta_{\mathfrak{u}}(c) \odot \vartheta_{\mathfrak{u}}(d) &= \bigvee \{\gamma \in \mathcal{M} : \mathfrak{u}(c, \gamma) \leq c\} \odot \bigvee \{\delta \in \mathcal{M} : \mathfrak{u}(d, \delta) \leq d\} \\
&= \bigvee \{\gamma \odot \delta \in \mathcal{M} : \mathfrak{u}(c, \gamma) \oplus \mathfrak{u}(d, \delta) \leq c \oplus d\} \\
&\leq \bigvee \{\gamma \odot \delta \in \mathcal{M} : \mathfrak{u}(c \oplus d, \gamma \odot \delta) \leq c \oplus d\} \\
&= \vartheta_{\mathfrak{u}}(c \oplus d).
\end{aligned}$$

. Then, $\vartheta_{\mathfrak{u}}(c) \odot \vartheta_{\mathfrak{u}}(d) \leq \vartheta_{\mathfrak{u}}(c \oplus d)$.
$(HFCT3)$ For a family of $\{c_i : i \in I\} \subset \mathcal{L}$, we have

$$\begin{aligned}
\vartheta_{\mathfrak{u}}\left(\bigwedge_i c_i\right) &= \bigvee \left\{\gamma \in \mathcal{M} : \mathfrak{u}\left(\bigwedge_{i\in I} c_i, \gamma\right) \leq \bigwedge_{i\in I} c_i\right\} \\
&= \bigvee \left\{\gamma \in \mathcal{M} : \bigwedge_{i\in I} \mathfrak{u}(c_i, \gamma) \leq \bigwedge_{i\in I} c_i\right\} \\
&\geq \bigwedge_{i\in I} \left(\bigvee \{\gamma \in \mathcal{M} : \mathfrak{u}(c_i, \gamma) \leq c_i\}\right) \\
&= \bigwedge_{i\in I} \vartheta_{\mathfrak{u}}(c_i).
\end{aligned}$$

Then, $\vartheta_{\mathfrak{u}}\left(\bigwedge_{i\in I} c_i\right) \geq \bigwedge_{i\in I} \vartheta_{\mathfrak{u}}(c_i)$. □

**Proposition 3.27.** *Let $\mathcal{L}$ satisfies the law of $\mathcal{DN}$egat and given an upper $\mathcal{QM}$-ApprX operator $\mathfrak{u} : \mathcal{L} \times \mathcal{M} \to \mathcal{L}$, then an $\mathcal{M}$-fuzzy set $\vartheta_{\mathfrak{u}} : \mathcal{L} \to \mathcal{M}$ which defined by*

$$\vartheta_{\mathfrak{u}}(c) = \bigvee \{\gamma \in \mathcal{M} : \mathfrak{u}(c, \gamma) \leq c, c \in \mathcal{L}\},$$

*is Alexandrov $\mathcal{M}$-Hutton FCoTPO on $\mathcal{L}$.*

*Proof.* We just check the axiom $(HFCT4)$. For $c, d \in \mathcal{L}$ and $\gamma, \delta \in \mathcal{M}$, we get

$(HFCT4)$ For a family of $\{c_j : j \in J\} \subset \mathcal{L}$, we have

$$\vartheta_{\mathfrak{u}}\left(\bigvee_j c_j\right) = \bigvee \left\{\gamma \in \mathcal{M} : \mathfrak{u}\left(\bigvee_{j\in J} c_j, \gamma\right) \leq \bigvee_{j\in J} c_j\right\}$$

$$= \bigvee \left\{ \gamma \in \mathcal{M} : \bigvee_{j \in J} \mathfrak{u}\left(c_j, \gamma\right) \leq \bigvee_{j \in J} c_j \right\}$$
$$\geq \bigwedge_{j \in J} \bigvee \left\{ \gamma \in \mathcal{M} : \mathfrak{u}\left(c_j, \gamma\right) \leq c_j \right\}$$
$$= \bigwedge_{j \in J} \vartheta_{\mathfrak{u}}\left(c_j\right).$$

Then, $\vartheta_{\mathfrak{u}}\left(\bigvee_{j \in J} c_j\right) \geq \bigwedge_{j \in J} \vartheta_{\mathfrak{u}}\left(c_j\right)$. □

**Remark 3.28.** An upper $\mathcal{M}$-approximation operator based on non-increasing, reflexive, unary and transitive $\mathcal{LM}$-fuzzy $\mathcal{G}$ neighborhood system [10] can be consider as an example of an upper $\mathcal{QM}$-ApprX on the power set $\mathcal{Q}$unt $\mathcal{L}^{\mathcal{X}}$.

**Definition 3.29.** A triple $(\mathcal{L}, \ell, \mathfrak{u})$ is said to be:

- An $\mathcal{M}$-ApprX system on $\mathcal{L}$ if $\ell$ and $\mathfrak{u}$ are lower and upper $\mathcal{M}$-ApprX operators on $\mathcal{L}$.
- The $\mathcal{QM}$-ApprX system on $\mathcal{L}$ if $\ell$ and $\mathfrak{u}$ are lower and upper $\mathcal{QM}$-ApprX operators on $\mathcal{L}$.

**Example 3.30.** For a non-empty set $\mathcal{X}$, $\mathcal{L} = \mathcal{L}^{\mathcal{X}}$, and $(\mathcal{L}, \ell, \mathfrak{u})$ is a $\mathcal{QM}$-ApprX systems, the triple $\left(\mathcal{L}^{\mathcal{X}}, \mathcal{L}, \mathcal{U}\right)$ is called an $\mathcal{M}$-level $\mathcal{L}$-rough approximation space [10].

**Definition 3.31** ([17])**.** A set $\mathcal{C} \subset \mathcal{L} \times \mathcal{M}$ is said to be an $\mathcal{M}$-closed structure iff it satisfies the next axioms.

($C1$) $(\bot, \gamma), (\top, \gamma) \in \mathcal{C}$ for each $\gamma \in M$,
($C2$) If $(c, \gamma), (d, \delta) \in \mathcal{C}$, then $(c, \gamma) * (d, \delta) = (c \oplus d, \gamma \odot \delta) \in \mathcal{C}$,
($C3$) If $(c_j, \gamma) \in \mathcal{C}$ for $j \in J$, then $\left(\bigwedge_{j \in J} c_j, \gamma\right) \in \mathcal{C}$,
($C4$) If $(c, \gamma) \in \mathcal{C}$ and $\delta < \gamma$, $\gamma, \delta \in \mathcal{M}$, then $(c, \delta) \in \mathcal{C}$.

The pair $(\mathcal{L}, \mathcal{C})$ is said to be an $\mathcal{M}$-closed system. Let $(\mathcal{L}_1, \mathcal{C}_1)$ and $(\mathcal{L}_2, \mathcal{C}_2)$ be $\mathcal{M}$-closed systems. A morphism $\jmath : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$ is said to be an $\mathcal{M}$-closed map if $\left(\jmath^{-1}(c), \gamma\right) \in \mathcal{C}_1$ if $(c, \gamma) \in \mathcal{C}_2$ for all $c \in \mathcal{L}, \gamma \in \mathcal{M}$.

**Theorem 3.32** ([17])**.** *Presume $\mathfrak{u}$ is an upper $\mathcal{M}$-approximate operator on $\mathcal{L}$. Define*

$$\mathcal{C}_{\mathfrak{u}} = \{(c, \gamma) \in \mathcal{L} \times \mathcal{M} : \mathfrak{u}(c, \gamma) = c\}$$

*Then $\mathcal{C}_{\mathfrak{u}}$ is an $\mathcal{M}$-closed structure with $(\mathfrak{u}(c, \gamma), \gamma) \in \mathcal{C}_{\mathfrak{u}}$.*

**Theorem 3.33** ([17])**.** *Presume $\mathcal{C} \subset \mathcal{L} \times \mathcal{M}$ is an $\mathcal{M}$-closed structure. We define $\mathfrak{u} : \mathcal{L} \times \mathcal{M} \longrightarrow \mathcal{L}$ as*

$$\mathfrak{u}_{\mathcal{C}}(c, \gamma) = \bigwedge \{d : (d, \gamma) \in C, d \geq c\}.$$

*Then, we possess the following properties.*

(1) $\mathfrak{u}_{\mathcal{C}}$ *is an upper* $\mathcal{M}$*-approximate operator with* $\mathcal{C}_{\mathfrak{u}_{\mathcal{C}}} = \mathcal{C}$.
(2) *When* $\mathfrak{u}$ *is an upper* $\mathcal{M}$*-approximate operator, then* $\mathfrak{u}_{\mathcal{C}_{\mathfrak{u}}} = \mathfrak{u}$.

Now, we will introduce the relationship between an $\mathcal{M}$-closed structure and an $\mathcal{M}$-open structure.

**Proposition 3.34.** *Let* $\mathcal{L}$ *fulfills the law of* $\mathcal{DN}$*egat and given an* $\mathcal{M}$*-closed structure* $\mathcal{C}_{\mathfrak{u}} \subset \mathcal{L} \times \mathcal{M}$. *A subset* $O_\ell \subset \mathcal{L} \times \mathcal{M}$ *defined by*

$$O_\ell = \{(d, \gamma) : (\neg d, \gamma) \in \mathcal{C}_{\mathfrak{u}}\},$$

*is an* $\mathcal{M}$*-open structure.*

*Proof.* (O1) Let $(\bot, \gamma) \in \mathcal{C}_{\mathfrak{u}}$, then $(\top, \gamma) \in O_\ell$.

(O2) Let $(c, \gamma), (d, \delta) \in O_\ell$. Then $(\neg c, \gamma), (\neg d, \delta) \in \mathcal{C}_{\mathfrak{u}}$, we have that $(\neg c, \gamma) * (\neg d, \delta) = (\neg c \oplus \neg d, \gamma \odot \delta) = (\neg (c \otimes d), \gamma \odot \delta) \in \mathcal{C}_{\mathfrak{u}}$. Thus $(c \otimes d, \gamma \odot \delta) \in O_\ell$.

(O3) Let $(c_j, \gamma) \in O_\ell$ for $j \in J$. Then $(\neg c_j, \gamma) \in \mathcal{C}_{\mathfrak{u}}$, we have that $\left(\bigwedge_{j \in J} (\neg c_j), \gamma\right) = \left(\neg \left(\bigvee_{j \in J} c_j\right), \gamma\right) \in \mathcal{C}_{\mathfrak{u}}$. So $\left(\bigvee_{j \in J} c_j, \gamma\right) \in O_\ell$.

(O4) Let $(c, \gamma) \in O_\ell$ . Then $(\neg c, \gamma) \in \mathcal{C}_{\mathfrak{u}}$ for each $\delta < \gamma$, we have that $(\neg c, \delta) \in \mathcal{C}_{\mathfrak{u}}$. Hence $(c, \delta) \in O_\ell$. ☐

**Proposition 3.35.** *Let* $\mathcal{L}$ *fulfills the law of* $\mathcal{DN}$*egat and given an* $\mathcal{M}$*-open structure* $O_\ell \subset \mathcal{L} \times \mathcal{M}$. *A subset* $\mathcal{C}_{\mathfrak{u}} \subset \mathcal{L} \times \mathcal{M}$ *defined by*

$$\mathcal{C}_{\mathfrak{u}} = \{(d, \gamma) : (\neg d, \gamma) \in O_\ell\},$$

*is an* $\mathcal{M}$*-closed structure.*

*Proof.* The proof is the same of Proposition 3.34. ☐

**Proposition 3.36.** *If* $\mathcal{L}$ *satisfy the law of* $\mathcal{DN}$*egat, then an upper* $\mathcal{QM}$*-ApprX operator* $\mathfrak{u}$ *satisfies the next axiom: For all* $c \in \mathcal{L}$, $\gamma \in \mathcal{M}$,

$$\ell(c, \gamma) = \neg \mathfrak{u}(\neg c, \gamma), \qquad \mathfrak{u}(c, \gamma) = \neg \ell(\neg c, \gamma).$$

*Proof.* For any $c, d \in \mathcal{L}$, and $\gamma \in \mathcal{M}$, we have

$$\begin{aligned} \neg \mathfrak{u}(\neg c, \gamma) &= \neg \left[\bigwedge \{d : (d, \gamma) \in C, d \geq \neg c\}\right] \\ &= \bigvee (\neg \{d : (d, \gamma) \in C, d \geq \neg c\}) \\ &= \bigvee \{\neg d : (\neg d, \gamma) \in O, \neg d \leq c\} \\ &= \ell(c, \gamma). \end{aligned}$$

Hence, $\ell(c, \gamma) = \neg\, \mathfrak{u}(\neg c, \gamma)$.

Similarly, one can prove that $\mathfrak{u}(c, \gamma) = \neg \ell(\neg c, \gamma)$. ☐

## 4. Conclusions

In this paper, we have introduced a many-level abstract approximation system on a $\mathcal{Q}$unt $\mathcal{L}$. Furthermore, a pair of lower $\mathcal{QM}$-ApprX and upper $\mathcal{QM}$-ApprX operators based on it have been defined and studied. Also, we have presented and studied the concept of $\mathcal{M}$-Hutton (resp., an Alexandrov $\mathcal{M}$-Hutton) q-FTPO on $\mathcal{L}$ and the relationship between it and the notion of a the lower $\mathcal{QM}$-ApprX operator. As well as the concept of $\mathcal{M}$-open structures has been proposed. Some properties of this concept and its relationships with a lower $\mathcal{QM}$-ApprX operator and the concept of $\mathcal{M}$-closed structures have been discussed, respectively. Moreover, the concept of $\mathcal{M}$-Hutton (resp., an Alexandrov $\mathcal{M}$-Hutton) FCoTPO on $\mathcal{L}$ has been presented and studied, and the relationship between such a concept and an upper $\mathcal{QM}$-ApprX operator has been investigated. This study is considered a new alternative view of the relationships between fuzzy rough sets and $\mathcal{M}$-Hutton fuzzy topological spaces based on a quantale and a generalization of Šostak [29].

As a future work, we explore how the previous notions can be used in fuzzy neural networks, explainable AI, or pattern recognition.

**Data Availability.** No data were used to support the findings of this study.

**Conflicts of Interest.** The author declares no that there are conflicts of interest.

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