

## Some Novelty Quantum Hermite–Hadamard Type Inequalities Pertaining To Coordinated $(s, t)$ -Convex Functions

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ABSTRACT. Convexity assumptions play an important role in mathematical analysis. In this work, we establish a midpoint-type inequality for the average value of  $(s, t)$ -convex functions of the second kind within the framework of quantum calculus. We begin by presenting a new identity for functions that are twice partially quantum differentiable. Using this identity, we derive two inequalities that provide error estimates for a generalized Hadamard inequality. We also demonstrate how these results apply to continuous functions whose second-order quantum partial derivatives are  $(s, t)$ -convex of the second kind in absolute value. Graphical examples are included to illustrate and compare the results. Finally, we present applications involving special means of positive real numbers.

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### 1. INTRODUCTION

Convexity is one of the most fundamental and important concepts in mathematics. Since Minkowski's pioneering work in the early 20th century, the theory of convexity has been steadily and systematically developing. Today, the field encompasses several areas, which can be broadly classified into three main branches: convex geometry, convex analysis, and discrete or combinatorial convexity. This classification reflects the different concepts, techniques, and tools employed in each area.

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In convex geometry, measures of symmetry are frequently studied, highlighting the rich and diverse nature of convexity. The literature on convex bodies of constant width provides numerous concrete examples of such symmetry measures, and this area continues to grow (see, for example, [30]). This motivates the researchers to explore more new and general features via different generalized convex functions and fractional and classical integrals.

The most celebrated Hermite–Hadamard inequality [13] holds:

$$\mathcal{F}\left(\frac{\hat{a} + \hat{b}}{2}\right) \leq \frac{1}{\hat{b} - \hat{a}} \int_{\hat{a}}^{\hat{b}} \mathcal{F}(\nu) d\nu \leq \frac{\mathcal{F}(\hat{a}) + \mathcal{F}(\hat{b})}{2},$$

if  $\mathcal{F} : [\hat{a}, \hat{b}] \rightarrow \mathbb{R}$  is a convex function with  $\hat{a} < \hat{b}$ . Recall that a function  $\mathcal{F} : \mathcal{J} \rightarrow \mathbb{R}$  is said to be convex if

$$\mathcal{F}(\varrho x + (1 - \varrho)y) \leq \varrho \mathcal{F}(x) + (1 - \varrho) \mathcal{F}(y)$$

holds for all  $x, y \in \mathcal{J}$  and  $\varrho \in [0, 1]$ .

Let  $\Xi := [\hat{a}, \hat{b}] \times [\hat{c}, \hat{d}] \subset \mathbb{R}^2$ ,  $0 \leq \hat{a} < \hat{b} < \infty$ ,  $0 \leq \hat{c} < \hat{d} < \infty$ .

In 2001, Dragomir [10], presented the concept of coordinated convex function utilizing the notion of partial mappings as follows:

A function  $\mathcal{F} : \Xi \rightarrow \mathbb{R}$  is said to be convex on the co-ordinates on  $\Xi$  if the partial mappings  $\mathcal{F}_y : [\hat{a}, \hat{b}] \rightarrow \mathbb{R}$ ,  $\mathcal{F}_y(u) = \mathcal{F}(u, y)$  and  $\mathcal{F}_x : [\hat{c}, \hat{d}] \rightarrow \mathbb{R}$ ,  $\mathcal{F}_x(v) = \mathcal{F}(x, v)$  are convex for all  $x \in [\hat{a}, \hat{b}]$ ,  $y \in [\hat{c}, \hat{d}]$ .

The formal definition is stated as follows:

**Definition 1.1.** A function  $\mathcal{F} : \Xi \rightarrow \mathbb{R}$  is called coordinated convex on  $\Xi$ , if the following inequality holds:

$$\begin{aligned} \mathcal{F}\left(\varrho \hat{a} + (1 - \varrho)\hat{b}, \varsigma \hat{c} + (1 - \varsigma)\hat{d}\right) &\leq \varrho \varsigma \mathcal{F}(\hat{a}, \hat{c}) + \varrho(1 - \varsigma) \mathcal{F}(\hat{a}, \hat{d}) \\ &\quad + (1 - \varrho) \varsigma \mathcal{F}(\hat{b}, \hat{c}) \\ &\quad + (1 - \varrho)(1 - \varsigma) \mathcal{F}(\hat{b}, \hat{d}), \end{aligned}$$

for all  $\varrho, \varsigma \in [0, 1]$  and  $(\hat{a}, \hat{c}), (\hat{a}, \hat{d}), (\hat{b}, \hat{c}), (\hat{b}, \hat{d}) \in \Xi$ .

The following well known Hermite–Hadamard type inequality for coordinated convex functions was also proved in [10]:

**Theorem 1.2.** *Let  $\mathcal{F} : \Xi \rightarrow \mathbb{R}$  be a coordinated convex function on  $\Xi$  and  $\mathcal{F} \in L_1(\Xi)$ . Then the following inequalities are satisfied:*

(1.1)

$$\begin{aligned}
 & \mathcal{F} \left( \frac{\hat{a} + \hat{b}}{2}, \frac{\hat{c} + \hat{d}}{2} \right) \\
 & \leq \frac{1}{2} \left[ \frac{1}{\hat{b} - \hat{a}} \int_{\hat{a}}^{\hat{b}} \mathcal{F} \left( \nu, \frac{\hat{c} + \hat{d}}{2} \right) d\nu + \frac{1}{\hat{d} - \hat{c}} \int_{\hat{c}}^{\hat{d}} \mathcal{F} \left( \frac{\hat{c} + \hat{d}}{2}, \eta \right) d\eta \right] \\
 & \leq \frac{1}{(\hat{b} - \hat{a})(\hat{d} - \hat{c})} \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} \mathcal{F}(\nu, \eta) d\eta d\nu \\
 & \leq \frac{1}{4(\hat{b} - \hat{a})} \int_{\hat{a}}^{\hat{b}} [\mathcal{F}(\nu, \hat{c}) + \mathcal{F}(\nu, \hat{d})] d\nu \\
 & \quad + \frac{1}{4(\hat{d} - \hat{c})} \int_{\hat{a}}^{\hat{b}} [\mathcal{F}(\hat{a}, \eta) + \mathcal{F}(\hat{b}, \eta)] d\eta \\
 & \leq \frac{\mathcal{F}(\hat{a}, \hat{c}) + \mathcal{F}(\hat{a}, \hat{d}) + \mathcal{F}(\hat{b}, \hat{c}) + \mathcal{F}(\hat{b}, \hat{d})}{4}.
 \end{aligned}$$

In 2012, Latif and Dragomir [21] developed error estimates for the first and second inequality in (1.1).

**Theorem 1.3.** *Let  $\mathcal{F} : \Xi \rightarrow \mathbb{R}$  be a partial differentiable mapping on  $\Xi$ .*

*If  $\left| \frac{\partial^2 \mathcal{F}}{\partial u \partial v} \right|$  is convex on the coordinates on  $\Xi$ , then the following inequality holds:*

(1.2)

$$\begin{aligned}
 & \left| \frac{1}{(\hat{b} - \hat{a})(\hat{d} - \hat{c})} \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} \mathcal{F}(u, v) dv du + \mathcal{F} \left( \frac{\hat{a} + \hat{b}}{2}, \frac{\hat{c} + \hat{d}}{2} \right) - \Psi \right| \\
 & \leq \frac{(\hat{b} - \hat{a})(\hat{d} - \hat{c})}{16} \left[ \frac{\left| \frac{\partial^2 \mathcal{F}(\hat{a}, \hat{b})}{\partial u \partial v} \right| + \left| \frac{\partial^2 \mathcal{F}(\hat{a}, \hat{d})}{\partial u \partial v} \right| + \left| \frac{\partial^2 \mathcal{F}(\hat{b}, \hat{c})}{\partial u \partial v} \right| + \left| \frac{\partial^2 \mathcal{F}(\hat{b}, \hat{d})}{\partial u \partial v} \right|}{4} \right],
 \end{aligned}$$

where

$$\Psi = \frac{1}{\hat{b} - \hat{a}} \int_{\hat{a}}^{\hat{b}} \mathcal{F} \left( x, \frac{\hat{c} + \hat{d}}{2} \right) dx + \frac{1}{\hat{b} - \hat{a}} \int_{\hat{c}}^{\hat{d}} \mathcal{F} \left( \frac{\hat{a} + \hat{b}}{2}, y \right) dy.$$

In 1994, Hudzik and Maligranda [15] considered, among others, the class of functions which are  $s$ -convex in the second sense. This family of functions is defined in the following way:

A function  $\mathcal{F} : [0, \infty) \rightarrow \mathbb{R}$  is said to be  $s$ -convex in the second sense if  $\mathcal{F}(\varrho x + (1 - \varrho)y) \leq \varrho^s \mathcal{F}(x) + (1 - \varrho)^s \mathcal{F}(y)$  holds for all  $x, y \in [0, \infty)$ ,  $\varrho \in [0, 1]$  and for some fixed  $s \in (0, 1]$ .

It can be easily seen that for  $s = 1$ ,  $s$ -convexity reduces to ordinary convexity of functions defined on  $[0, \infty)$ .

In 2008, Alomari and Darus [2] extended  $s$ -convexity of second kind to the case of two variables as follows:

**Definition 1.4.** Let  $s \in [0, 1]$ . The function  $\mathcal{F} : \Xi \rightarrow \mathbb{R}$  is coordinated  $s$ -convex function of second kind on  $\Xi$ , if the following inequality holds:

$$\begin{aligned} \mathcal{F}\left(\varrho\hat{a} + (1 - \varrho)\hat{b}, \varsigma\hat{c} + (1 - \varsigma)\hat{d}\right) &\leq \varrho^s \varsigma^s \mathcal{F}(\hat{a}, \hat{c}) + \varrho^s (1 - \varsigma)^s \mathcal{F}(\hat{a}, \hat{d}) \\ &\quad + (1 - \varrho)^s \varsigma^s \mathcal{F}(\hat{b}, \hat{c}) \\ &\quad + (1 - \varrho)^s (1 - \varsigma)^s \mathcal{F}(\hat{b}, \hat{d}), \end{aligned}$$

for all  $\varrho, \varsigma \in [0, 1]$  and  $(\hat{a}, \hat{c}), (\hat{a}, \hat{d}), (\hat{b}, \hat{c}), (\hat{b}, \hat{d}) \in \Xi$ .

In 2019, Raees and Anwar [24] recapture the notion of coordinated  $(s, t)$ -convex functions from coordinated  $(p_1, h_1)$ - $(p_2, h_2)$ -convex functions as follows:

**Definition 1.5.** Let  $(s, t) \in (0, 1]^2$ . The function  $\mathcal{F} : \Xi \rightarrow \mathbb{R}$  is coordinated  $(s, t)$ -convex function of second kind on  $\Xi$ , if the following inequality holds:

$$\begin{aligned} \mathcal{F}\left(\varrho\hat{a} + (1 - \varrho)\hat{b}, \varsigma\hat{c} + (1 - \varsigma)\hat{d}\right) &\leq \varrho^s \varsigma^t \mathcal{F}(\hat{a}, \hat{c}) + \varrho^s (1 - \varsigma)^t \mathcal{F}(\hat{a}, \hat{d}) \\ &\quad + (1 - \varrho)^s \varsigma^t \mathcal{F}(\hat{b}, \hat{c}) \\ &\quad + (1 - \varrho)^s (1 - \varsigma)^t \mathcal{F}(\hat{b}, \hat{d}), \end{aligned}$$

for all  $\varrho, \varsigma \in [0, 1]$  and  $(\hat{a}, \hat{c}), (\hat{a}, \hat{d}), (\hat{b}, \hat{c}), (\hat{b}, \hat{d}) \in \Xi$ .

They considered  $h_1 = s$  and  $h_2 = t$ . This is an extension of the function given in Definition 1.4.

Following this commencement, the researchers put a lot of effort into extending and improving the inequality (1.1) by using both classical and fractional integrals. We believe interested readers should additionally take into account [1, 23, 24, 27] and the references therein for the results that alter, refine, and generalize the inequality (1.1).

Over the past decade, there has been a growing interest in quantum calculus among both mathematicians and physicists. Quantum calculus focuses on calculus in the absence of limits and is often used in the study of non-smooth structures. It has significant applications in various areas, including the theory of relativity, orthogonal polynomials, combinatorics, number theory, and basic hyper-geometric functions (see, e.g., [5, 6, 9, 11, 12, 14, 17, 18] and the references therein).

Tariboon and Ntouyas [28, 29] introduced the  $q$ -derivative and  $q$ -integral for continuous functions on finite intervals and established several of their basic properties. Since then, many classical inequalities have been extended to the setting of  $q$ -calculus, such as the Hölder, Hermite–Hadamard, trapezoid, Ostrowski, and Cauchy–Bunyakovsky–Schwarz inequalities, leading to a number of useful results.

In particular, Alp and Sarikaya [3] obtained a corrected version of the  $q$ -Hermite–Hadamard inequality for co-ordinated convex functions, while Kalsoom et al. [19] derived Simpson-type inequalities using co-ordinated convexity. Latif and Dragomir [21] established new Hermite–Hadamard-type inequalities for functions of two variables via quantum integrals. Bermudo et al. [7] later introduced the  $q^b$ -calculus, and Budak and Ali [8] proved several Hermite–Hadamard inequalities in both the  $q_a$ - and  $q^b$ -frameworks. Moreover, Alqudah et al. [4] refined mixed-type quantum Hermite–Hadamard inequalities, Raees and Anwar [25] provided a general error estimate, and Wannalookkhee et al. [31] presented further refinements in this direction.

Motivated by the developments mentioned above, and in particular by the results in [21], our goal is to establish a new inequality within the framework of quantum calculus that gives the error estimate of the inequalities stated in Theorems 2.11 very similar to (1.2). For the purpose we use the coordinated  $(s, t)$ -convex functions which is an extension of coordinated convexity.

The paper’s structure is organized as follows: In Section 2, we provide a concise overview of the fundamental principles of  $q$ -calculus, encompassing both single variable and bivariate functions. In Section 3, we discuss the main findings. We first introduce a generic identity that facilitates the derivation of error estimates, followed by a critical examination of the inequality’s limitations and a proposed reformulation to address these shortcomings. In Section 4, we present a comparative analysis of the obtained results, utilizing graphical visualizations to elucidate the relationships between the proposed inequalities. In Section 5, we give some applications about the special means of different positive real numbers. In Section 6, we conclude the key findings and future plans.

## 2. PRELIMINARIES

In this section we give some definitions and properties of convexity hypothesis and basic theory of quantum calculus. This foundational knowledge serves as a precursor to the subsequent sections.

Here and onward  $\bar{q} \in (0, 1)$ .

**Definition 2.1** ([18]). The difference operator expressed by the quotient

$$(2.1) \quad D_{\bar{q}}\mathcal{F}(\omega) = \frac{\mathcal{F}(\omega) - \mathcal{F}(\bar{q}\omega)}{\omega - \bar{q}\omega},$$

is called quantum derivative or  $\bar{q}$ -derivative.

Note that,  $\lim_{\bar{q} \rightarrow 1^-} D_{\bar{q}}\mathcal{F}(\omega) = \frac{d\mathcal{F}(\omega)}{d\omega}$ .

**Definition 2.2** ([16, 18]). Suppose  $0 < \hat{r} < \hat{s}$ . The definite  $\bar{q}$ -integral (Jackson  $\bar{q}$ -integral) is defined as:

$$(2.2) \quad \int_0^{\hat{s}} \mathcal{F}(\omega) d_{\bar{q}}\omega = (1 - \bar{q}) \hat{s} \sum_{\nu=0}^{\infty} \bar{q}^{\nu} \mathcal{F}(\bar{q}^{\nu} \hat{s}),$$

and

$$(2.3) \quad \int_{\hat{r}}^{\hat{s}} \mathcal{F}(\omega) d_{\bar{q}}\omega = \int_0^{\hat{s}} \mathcal{F}(\omega) d_{\bar{q}}\omega - \int_0^{\hat{r}} \mathcal{F}(\omega) d_{\bar{q}}\omega.$$

In [28, 29], Tariboon et al. extended the notions of quantum derivative given by the quotient (2.1) and Jackson integral characterized by the series (2.2) over a finite interval as follows:

**Definition 2.3.** Let  $\mathcal{F} : [\hat{a}, \hat{b}] \rightarrow \mathbb{R}$  be a continuous function and  $\bar{q} \in (0, 1)$ . Then the  $\bar{q}_{\hat{a}}$ -derivative of  $\mathcal{F}$  at  $\omega \in [\hat{a}, \hat{b}]$  is characterized by the quotient:

$$(2.4) \quad {}_{\hat{a}}D_{\bar{q}}\mathcal{F}(\omega) = \frac{\mathcal{F}(\omega) - \mathcal{F}(\bar{q}\omega + (1 - \bar{q})\hat{a})}{(1 - \bar{q})(\omega - \hat{a})}, \quad \omega \neq \hat{a}.$$

The function  $\mathcal{F}$  is called  $\bar{q}_{\hat{a}}$ -differentiable on  $[\hat{a}, \hat{b}]$ , if  ${}_{\hat{a}}D_{\bar{q}}\mathcal{F}(\omega)$  exists for all  $\omega \in [\hat{a}, \hat{b}]$ . Evidently,

$${}_{\hat{a}}D_{\bar{q}}\mathcal{F}(\hat{a}) = \lim_{\omega \rightarrow \hat{a}} {}_{\hat{a}}D_{\bar{q}}\mathcal{F}(\omega).$$

Moreover, if  $\hat{a} = 0$ , then the  $\bar{q}_{\hat{a}}$ -derivative reduces to Jackson  $\bar{q}$ -difference operator given by (2.1).

**Example 2.4.** Let  $\mathcal{F}(\omega) = \omega^{\frac{3}{2}}$ , then

$${}_{\hat{a}}D_{\bar{q}}\mathcal{F}(\omega) = \frac{\omega^{\frac{3}{2}} - (\bar{q}\omega + (1 - \bar{q})\hat{a})^{\frac{3}{2}}}{(1 - \bar{q})(\omega - \hat{a})}$$

$$= \frac{\omega + \omega^{\frac{1}{2}} (\bar{q}\omega + (1 - \bar{q}) \hat{a})^{\frac{1}{2}} + (\bar{q}\omega + (1 - \bar{q}) \hat{a})}{\omega^{\frac{1}{2}} + (\bar{q}\omega + (1 - \bar{q}) \hat{a})^{\frac{1}{2}}}.$$

If  $\hat{a} = 0$ , then we have the Jackson derivative

$${}_0D_{\bar{q}}\mathcal{F}(\omega) = \frac{(1 + \bar{q})\omega + \omega(\bar{q})^{\frac{1}{2}}}{\omega^{\frac{1}{2}}(1 + \bar{q}^{\frac{1}{2}})},$$

and if  $\bar{q} \rightarrow 1^-$ , then  $\mathcal{F}'(\omega) = \frac{3\sqrt{\omega}}{2}$ .

**Definition 2.5.** Let  $\mathcal{F} : [\hat{a}, \hat{b}] \rightarrow \mathbb{R}$  be a continuous function and  $\bar{q} \in (0, 1)$ . Then the  $\bar{q}_{\hat{a}}$ -integral of the function  $\mathcal{F}$  is defined by the series expression

$$(2.5) \quad \int_{\hat{a}}^{\hat{z}} \mathcal{F}(\omega) {}_{\hat{a}}d_{\bar{q}}\omega = (1 - \bar{q})(\hat{z} - \hat{a}) \sum_{\nu=0}^{\infty} \bar{q}^{\nu} \mathcal{F}(\bar{q}^{\nu} \hat{z} + (1 - \bar{q}^{\nu}) \hat{a}), \quad \hat{z} \in [\hat{a}, \hat{b}].$$

Clearly, if  $\hat{a} = 0$ , then the  $\bar{q}_{\hat{a}}$ -integral reduces to Jackson integral given by (2.2).

**Example 2.6.** Let  $0 \leq \hat{a} < \hat{b}$ ,  $\bar{q} \in (0, 1)$  and  $\hat{m}$  be a non-negative real number, then

$$\int_{\hat{a}}^{\hat{b}} \sqrt[\hat{m}]{\omega} {}_{\hat{a}}d_{\bar{q}}\omega = (1 - \bar{q})(\hat{b} - \hat{a}) \sum_{\nu=0}^{\infty} \bar{q}^{\nu} \sqrt[\hat{m}]{(\bar{q}^{\nu}(\hat{b} - \hat{a}) + \hat{a})}.$$

If  $\hat{a} = 0$ ,  $\hat{b} = 1$ ,  $\bar{q} = \frac{1}{2}$  and  $\hat{m} = 2$ , then the numerical sum of the above infinite series:

$$\int_0^1 \sqrt{\omega} {}_0d_{\frac{1}{2}}\omega = \left(1 - \frac{1}{2}\right) \sum_{\nu=0}^{\infty} \left(\frac{1}{2}\right)^{\nu} \sqrt{\left(\left(\frac{1}{2}\right)^{\nu} + 1\right)} = 1.2831.$$

Similarly, for  $\hat{a} = 0$ ,  $\hat{b} = 1$ ,  $\bar{q} = \frac{1}{2}$  and  $\hat{m} = 1$ , then the numerical sum converges to:

$$\int_0^1 \sqrt{\omega} {}_0d_{\frac{1}{2}}\omega = \frac{1}{2} \sum_{\nu=0}^{\infty} \left(\frac{1}{2}\right)^{\nu} \left(\left(\frac{1}{2}\right)^{\nu} + 1\right) = 1.6667.$$

In the same paper, the following  $\hat{q}$ -Hölder inequality is proved.

**Theorem 2.7.** Let  $\mathcal{F}_1, \mathcal{F}_2 : [\hat{a}, \hat{b}] \rightarrow \mathbb{R}$  be two continuous functions, then the inequality holds:

$$(2.6) \quad \int_{\hat{a}}^{\hat{z}} |\mathcal{F}_1(\omega)\mathcal{F}_2(\omega)| {}_{\hat{a}}d_{\bar{q}}\omega$$

$$\leq \left( \int_{\hat{a}}^{\hat{z}} |\mathcal{F}_1(\omega)|^{\gamma_1} {}_{\hat{a}}d_{\hat{q}}\omega \right)^{\frac{1}{\gamma_1}} \left( \int_{\hat{a}}^{\hat{z}} |\mathcal{F}_2(\omega)|^{\gamma_2} {}_{\hat{a}}d_{\hat{q}}\omega \right)^{\frac{1}{\gamma_2}}$$

for all  $z \in [\hat{a}, \hat{b}]$ , where  $\gamma_1^{-1} + \gamma_2^{-1} = 1$ .

In [26], Raees et al. gave a counter example to establish the limiting feature of the inequality (2.6).

In 2017, Latif et al. [22] presented the concept of partial quantum derivative of a function  $\mathcal{F}(u, v)$  of two variables by considering two constants  $\bar{q}, \check{q} \in (0, 1)$ . They introduced the notion of partial  $\bar{q}_{\hat{a}}, \check{q}_{\hat{c}}, \bar{q}_{\hat{a}}\check{q}_{\hat{c}}$ -derivative and associated quantum integrals. For brevity, we will use the notations partial  $\bar{q}_{\hat{a}}$ -derivatives of a function  $\mathcal{F}(u, v)$  with respect to  $u$  as  $\frac{\partial \mathcal{F}}{{}_{\hat{a}}\partial_{\bar{q}}u}$  instead of  $\frac{{}_{\hat{a}}\partial_{\bar{q}}\mathcal{F}}{\partial_{\bar{q}}u}$ . Analogously, we will use  $\frac{\partial \mathcal{F}}{{}_{\hat{c}}\partial_{\check{q}}v}$  instead of  $\frac{{}_{\hat{c}}\partial_{\check{q}}\mathcal{F}}{\partial_{\check{q}}v}$  for partial  $\check{q}_{\hat{c}}$ -derivative and  $\frac{\partial^2 \mathcal{F}}{{}_{\hat{a}}\partial_{\bar{q}}u {}_{\hat{c}}\partial_{\check{q}}v}$  instead of  $\frac{{}_{\hat{a}, \hat{c}}\partial_{\bar{q}\check{q}}^2 \mathcal{F}}{\partial_{\bar{q}}u \partial_{\check{q}}v}$  for twice partial  $\bar{q}_{\hat{a}}\check{q}_{\hat{c}}$ -derivative:

**Definition 2.8.** Let  $\mathcal{F}$  be a function of two variables and  $(\bar{q}, \check{q}) \in (0, 1)^2$ , then the partial  $\bar{q}_{\hat{a}}$ -derivative,  $\check{q}_{\hat{c}}$ -derivative and  $\bar{q}_{\hat{a}}\check{q}_{\hat{c}}$ -derivative are defined at  $(u, v) \in \Xi$  respectively, as follows:

$$\begin{aligned} \frac{\partial \mathcal{F}(u, v)}{{}_{\hat{a}}\partial_{\bar{q}}u} &= \frac{\mathcal{F}(u, v) - \mathcal{F}(\bar{q}u + (1 - \bar{q})\hat{a}, v)}{(1 - \bar{q})(u - \hat{a})}, \quad u \neq \hat{a}, \\ \frac{\partial \mathcal{F}(u, v)}{{}_{\hat{c}}\partial_{\check{q}}v} &= \frac{\mathcal{F}(u, v) - \mathcal{F}(u, \check{q}v + (1 - \check{q})\hat{c})}{(1 - \check{q})(v - \hat{c})}, \quad v \neq \hat{c} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 \mathcal{F}(u, v)}{{}_{\hat{a}}\partial_{\bar{q}}u {}_{\hat{c}}\partial_{\check{q}}v} &= \frac{(1 - \bar{q})^{-1}(u - \hat{a})^{-1}}{(1 - \check{q})(v - \hat{c})} \left[ \mathcal{F}(u, v) - \mathcal{F}(\bar{q}u + (1 - \bar{q})\hat{a}, v) \right. \\ &\quad \left. - \mathcal{F}(u, \check{q}v + (1 - \check{q})\hat{c}) + \mathcal{F}(\bar{q}u + (1 - \bar{q})\hat{a}, \check{q}v + (1 - \check{q})\hat{c}) \right], \end{aligned}$$

$u \neq \hat{a}, v \neq \hat{c}$ .

**Definition 2.9.** Let  $\mathcal{F}$  be a continuous function of two variables and  $(\bar{q}, \check{q}) \in (0, 1)^2$ , then the  $\bar{q}_{\hat{a}}\check{q}_{\hat{c}}$ -integral is expressed by:

$$\begin{aligned} &\int_{\hat{a}}^x \int_{\hat{c}}^y \mathcal{F}(\theta, \omega) {}_{\hat{c}}d_{\check{q}}\omega {}_{\hat{a}}d_{\bar{q}}\theta \\ &= (1 - \bar{q})(1 - \check{q})(x - \hat{a})(y - \hat{c}) \\ &\quad \times \sum_{\nu=0}^{\infty} \sum_{\vartheta=0}^{\infty} \bar{q}^{\nu} \check{q}^{\vartheta} \mathcal{F} \left( \bar{q}^{\nu}x + (1 - \bar{q}^{\nu})\hat{a}, \check{q}^{\vartheta}y + (1 - \check{q}^{\vartheta})\hat{c} \right), \end{aligned}$$

for all  $(x, y) \in \Xi$ .

In 2019, Kunt et al. [20] gave the following lemma:

**Lemma 2.10.** *If the conditions of the Definition 2.9 are satisfied, then*

$$\begin{aligned}
 (2.7) \quad & \int_r^x \int_s^y \mathcal{F}(u, v) {}_c d_{\tilde{q}} v {}_{\hat{a}} d_{\tilde{q}} u \\
 &= \int_{\hat{a}}^x \int_{\hat{c}}^y \mathcal{F}(u, v) {}_c d_{\tilde{q}} v {}_{\hat{a}} d_{\tilde{q}} u - \int_{\hat{a}}^x \int_{\hat{c}}^s \mathcal{F}(u, v) {}_c d_{\tilde{q}} v {}_{\hat{a}} d_{\tilde{q}} u \\
 &\quad - \int_{\hat{a}}^r \int_{\hat{c}}^y \mathcal{F}(u, v) {}_c d_{\tilde{q}} v {}_{\hat{a}} d_{\tilde{q}} u + \int_{\hat{a}}^r \int_{\hat{c}}^s \mathcal{F}(u, v) {}_c d_{\tilde{q}} v {}_{\hat{a}} d_{\tilde{q}} u,
 \end{aligned}$$

for all  $(u, v) \in \Xi$ .

In 2020, Alp and Sarikaya [3] established the following correct inequality involving quantum integrals:

**Theorem 2.11.** *Let  $\mathcal{F} : \Xi \rightarrow \mathbb{R}$  be a coordinated convex and partially differentiable function on  $\Xi$ . Then one has the inequalities:*

$$\begin{aligned}
 (2.8) \quad & \mathcal{F} \left( \frac{\tilde{q}\hat{a} + \hat{b}}{1 + \tilde{q}}, \frac{\check{q}\hat{c} + \hat{d}}{1 + \check{q}} \right) \\
 & \leq \frac{1}{2} \left[ \frac{1}{\hat{b} - \hat{a}} \int_{\hat{a}}^{\hat{b}} \mathcal{F} \left( x, \frac{\check{q}\hat{c} + \hat{d}}{1 + \check{q}} \right) {}_{\hat{a}} d_{\tilde{q}} x + \frac{1}{\hat{d} - \hat{c}} \int_{\hat{c}}^{\hat{d}} \mathcal{F} \left( \frac{\tilde{q}\hat{a} + \hat{b}}{1 + \tilde{q}}, y \right) {}_c d_{\check{q}} y \right] \\
 & \leq \frac{1}{(\hat{b} - \hat{a})(\hat{d} - \hat{c})} \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} \mathcal{F}(x, y) {}_c d_{\check{q}} y {}_{\hat{a}} d_{\tilde{q}} x \\
 & \leq \frac{\check{q}}{2(1 + \check{q})(\hat{b} - \hat{a})} \int_{\hat{a}}^{\hat{b}} \mathcal{F}(x, \hat{c}) {}_{\hat{a}} d_{\tilde{q}} x + \frac{1}{2(1 + \check{q})(\hat{b} - \hat{a})} \int_{\hat{a}}^{\hat{b}} \mathcal{F}(x, \hat{d}) {}_{\hat{a}} d_{\tilde{q}} x \\
 & \quad + \frac{\tilde{q}}{2(1 + \tilde{q})(\hat{d} - \hat{c})} \int_{\hat{c}}^{\hat{d}} \mathcal{F}(\hat{a}, y) {}_c d_{\check{q}} y + \frac{1}{2(1 + \tilde{q})(\hat{d} - \hat{c})} \int_{\hat{c}}^{\hat{d}} \mathcal{F}(\hat{b}, y) {}_c d_{\check{q}} y \\
 & \leq \frac{\tilde{q}\check{q}\mathcal{F}(\hat{a}, \hat{c}) + \tilde{q}\mathcal{F}(\hat{a}, \hat{d}) + \check{q}\mathcal{F}(\hat{b}, \hat{c}) + \mathcal{F}(\hat{b}, \hat{d})}{(1 + \tilde{q})(1 + \check{q})}.
 \end{aligned}$$

### 3. MAIN RESULTS

We devote this section to our main findings. We divide this section into three subsections. In the first subsection, we prove a generic new identity for twice partially  $\tilde{q}_{\hat{a}}\check{q}_{\hat{c}}$ -differentiable function defined over the bi-dimensional open interval  $\Xi$ . In the second subsection, we give a new inequality which gives the error in the second and third inequality of inequality (2.8). In the third and final subsection, we establish another inequality which gives a new variant of the above inequality.

**3.1. General Novel Quantum Integral Identity.** Now, we present the novel identity which is a vital part of our main findings.

**Lemma 3.1.** *Let  $\mathcal{F} : \Xi \subseteq [0, \infty)^2 \rightarrow \mathbb{R}$  be a twice partial  $\bar{q}_a \check{q}_c$ -differentiable function defined on  $\Xi$ . If the partial  $\bar{q}_a \check{q}_c$ -derivative  $\frac{\partial^2 \mathcal{F}(u,v)}{\bar{a} \partial_{\bar{q}} u \check{c} \partial_{\check{q}} v}$  is continuous and  $\bar{q}_a \check{q}_c$ -integrable over  $\Xi$ , then the following identity holds:*

$$(3.1) \quad \begin{aligned} & \frac{1}{(\hat{b} - \hat{a})(\hat{d} - \hat{c})} \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} \mathcal{K}_1(\kappa, \eta) \frac{\partial^2 \mathcal{F}(\kappa, \eta)}{\bar{a} \partial_{\bar{q}} \kappa \check{c} \partial_{\check{q}} \eta} \check{c} d_{\check{q}} \eta \bar{a} d_{\bar{q}} \kappa \\ &= \frac{1}{(\hat{b} - \hat{a})(\hat{d} - \hat{c})} \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} \mathcal{F}(x, y) \check{c} d_{\check{q}} y \bar{a} d_{\bar{q}} x + \mathcal{F} \left( \frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}, \frac{\check{q}\hat{c} + \hat{d}}{1 + \check{q}} \right) - \mathcal{V}_*, \end{aligned}$$

where

$$(3.2) \quad \begin{aligned} \mathcal{V}_* := & \frac{1}{\hat{b} - \hat{a}} \int_{\hat{a}}^{\hat{b}} \mathcal{F} \left( u, \frac{\check{q}\hat{c} + \hat{d}}{1 + \check{q}} \right) \bar{a} d_{\bar{q}} u + \frac{1}{\hat{d} - \hat{c}} \int_{\hat{c}}^{\hat{d}} \mathcal{F} \left( \frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}, v \right) \check{c} d_{\check{q}} v \\ & + \frac{\check{q}\hat{c} - \hat{c}}{(\hat{b} - \hat{a})(\hat{d} - \hat{c})} \int_{\hat{a}}^{\hat{b}} \left[ \mathcal{F}(u, \hat{d}) - \mathcal{F}(u, \hat{c}) \right] \bar{a} d_{\bar{q}} u \\ & + \frac{\bar{q}\hat{a} - \hat{a}}{(\hat{b} - \hat{a})(\hat{d} - \hat{c})} \int_{\hat{c}}^{\hat{d}} \left[ \mathcal{F}(\hat{b}, v) - \mathcal{F}(\hat{a}, v) \right] \check{c} d_{\check{q}} v \\ & + \frac{(\bar{q}\hat{a} - \hat{a}) \left[ \mathcal{F} \left( \hat{a}, \frac{\check{q}\hat{c} + \hat{d}}{1 + \check{q}} \right) - \mathcal{F} \left( \hat{b}, \frac{\check{q}\hat{c} + \hat{d}}{1 + \check{q}} \right) \right]}{(\hat{b} - \hat{a})} \\ & + \frac{(\check{q}\hat{c} - \hat{c}) \left[ \mathcal{F} \left( \frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}, \hat{c} \right) - \mathcal{F} \left( \frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}, \hat{d} \right) \right]}{(\hat{d} - \hat{c})} \\ & + \frac{(\bar{q}\hat{a} - \hat{a})(\check{q}\hat{c} - \hat{c}) \left[ \mathcal{F}(\hat{b}, \hat{c}) - \mathcal{F}(\hat{a}, \hat{c}) - \mathcal{F}(\hat{b}, \hat{d}) + \mathcal{F}(\hat{a}, \hat{d}) \right]}{(\hat{b} - \hat{a})(\hat{d} - \hat{c})} \end{aligned}$$

and

$$\mathcal{K}_1(u, v) := \begin{cases} (\bar{q}u - \hat{a})(\check{q}v - \hat{c}), & \text{if } (u, v) \in \left[\hat{a}, \frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}\right] \times \left[\hat{c}, \frac{\check{q}\hat{c} + \hat{d}}{1 + \check{q}}\right]; \\ (\bar{q}u - \hat{a})(\check{q}v - \hat{d}), & \text{if } (u, v) \in \left[\hat{a}, \frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}\right] \times \left[\frac{\check{q}\hat{c} + \hat{d}}{1 + \check{q}}, \hat{d}\right]; \\ (\bar{q}u - \hat{b})(\check{q}v - \hat{c}), & \text{if } (u, v) \in \left[\frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}, \hat{b}\right] \times \left[\hat{c}, \frac{\check{q}\hat{c} + \hat{d}}{1 + \check{q}}\right]; \\ (\bar{q}u - \hat{b})(\check{q}v - \hat{d}), & \text{if } (u, v) \in \left[\frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}, \hat{b}\right] \times \left[\frac{\check{q}\hat{c} + \hat{d}}{1 + \check{q}}, \hat{d}\right]. \end{cases}$$

*Proof.* At start we notice by utilizing the Lemma 2.10, that

$$\begin{aligned} (3.3) \quad & \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} \mathcal{K}_1(\varrho, \eta) \frac{\partial^2 \mathcal{F}(\varrho, \eta)}{{}_a \partial_{\bar{q}} \varrho \, {}_c \partial_{\check{q}} \eta} \, {}_c d_{\check{q}} \eta \, {}_a d_{\bar{q}} \\ &= \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} (\bar{q}\varrho - \hat{b})(\check{q}\eta - \hat{d}) \frac{\partial^2 \mathcal{F}(\varrho, \eta)}{{}_a \partial_{\bar{q}} \varrho \, {}_c \partial_{\check{q}} \eta} \, {}_c d_{\check{q}} \eta \, {}_a d_{\bar{q}} \\ &+ (\hat{b} - \hat{a})(\hat{d} - \hat{c}) \int_{\hat{a}}^{\frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}} \int_{\hat{c}}^{\frac{\check{q}\hat{c} + \hat{d}}{1 + \check{q}}} \frac{\partial^2 \mathcal{F}(\varrho, \eta)}{{}_a \partial_{\bar{q}} \varrho \, {}_c \partial_{\check{q}} \eta} \, {}_c d_{\check{q}} \eta \, {}_a d_{\bar{q}} \\ &+ (\hat{b} - \hat{a}) \int_{\hat{a}}^{\frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}} \int_{\hat{c}}^{\hat{d}} (\check{q}\eta - \hat{d}) \frac{\partial^2 \mathcal{F}(\varrho, \eta)}{{}_a \partial_{\bar{q}} \varrho \, {}_c \partial_{\check{q}} \eta} \, {}_c d_{\check{q}} \eta \, {}_a d_{\bar{q}} \\ &+ (\hat{d} - \hat{c}) \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\frac{\check{q}\hat{c} + \hat{d}}{1 + \check{q}}} (\bar{q}\varrho - \hat{b}) \frac{\partial^2 \mathcal{F}(\varrho, \eta)}{{}_a \partial_{\bar{q}} \varrho \, {}_c \partial_{\check{q}} \eta} \, {}_c d_{\check{q}} \eta \, {}_a d_{\bar{q}} \\ &= R_1 + R_2 + R_3 + R_4. \end{aligned}$$

Now by making use of the partial quantum derivatives and integrals expressed in the Definition 2.8 and 2.9, respectively, we have

$$\begin{aligned} (3.4) \quad R_1 &= (\hat{b} - \hat{a})(\hat{d} - \hat{c}) \sum_{\nu=0}^{\infty} \sum_{\kappa=0}^{\infty} \bar{q}^{\nu+1} \check{q}^{\kappa+1} \\ &\times \left[ \begin{array}{l} \mathcal{F} \left( \bar{q}^{\nu} (\hat{b} - \hat{a}) + \hat{a}, \check{q}^{\kappa} (\hat{d} - \hat{c}) + \hat{c} \right) \\ - \mathcal{F} \left( \bar{q}^{\nu+1} (\hat{b} - \hat{a}) + \hat{a}, \check{q}^{\kappa} (\hat{d} - \hat{c}) + \hat{c} \right) \\ - \mathcal{F} \left( \bar{q}^{\nu} (\hat{b} - \hat{a}) + \hat{a}, \check{q}^{\kappa+1} (\hat{d} - \hat{c}) + \hat{c} \right) \\ + \mathcal{F} \left( \bar{q}^{\nu+1} (\hat{b} - \hat{a}) + \hat{a}, \check{q}^{\kappa+1} (\hat{d} - \hat{c}) + \hat{c} \right) \end{array} \right] \\ &+ (\hat{b} - \hat{a})(\check{q}\hat{c} - \hat{d}) \sum_{\nu=0}^{\infty} \sum_{\kappa=0}^{\infty} \bar{q}^{\nu+1} \end{aligned}$$

$$\begin{aligned}
& \times \left[ \begin{array}{c} \mathcal{F} \left( \bar{q}^\nu (\hat{b} - \hat{a}) + \hat{a}, \check{q}^\kappa (\hat{d} - \hat{c}) + \hat{c} \right) \\ -\mathcal{F} \left( \bar{q}^{\nu+1} (\hat{b} - \hat{a}) + \hat{a}, \check{q}^\kappa (\hat{d} - \hat{c}) + \hat{c} \right) \\ -\mathcal{F} \left( \bar{q}^\nu (\hat{b} - \hat{a}) + \hat{a}, \check{q}^{\kappa+1} (\hat{d} - \hat{c}) + \hat{c} \right) \\ +\mathcal{F} \left( \bar{q}^{\nu+1} (\hat{b} - \hat{a}) + \hat{a}, \check{q}^{\kappa+1} (\hat{d} - \hat{c}) + \hat{c} \right) \end{array} \right] \\
& + \left( \bar{q}\hat{a} - \hat{b} \right) (\hat{d} - \hat{c}) \sum_{\nu=0}^{\infty} \sum_{\kappa=0}^{\infty} \check{q}^{\kappa+1} \\
& \times \left[ \begin{array}{c} \mathcal{F} \left( \bar{q}^\nu (\hat{b} - \hat{a}) + \hat{a}, \check{q}^\kappa (\hat{d} - \hat{c}) + \hat{c} \right) \\ -\mathcal{F} \left( \bar{q}^{\nu+1} (\hat{b} - \hat{a}) + \hat{a}, \check{q}^\kappa (\hat{d} - \hat{c}) + \hat{c} \right) \\ -\mathcal{F} \left( \bar{q}^\nu (\hat{b} - \hat{a}) + \hat{a}, \check{q}^{\kappa+1} (\hat{d} - \hat{c}) + \hat{c} \right) \\ +\mathcal{F} \left( \bar{q}^{\nu+1} (\hat{b} - \hat{a}) + \hat{a}, \check{q}^{\kappa+1} (\hat{d} - \hat{c}) + \hat{c} \right) \end{array} \right] \\
& + \left( \bar{q}\hat{a} - \hat{b} \right) (\check{q}\hat{c} - \hat{d}) \\
& \times \sum_{\nu=0}^{\infty} \sum_{\kappa=0}^{\infty} \left[ \begin{array}{c} \mathcal{F} \left( \bar{q}^\nu (\hat{b} - \hat{a}) + \hat{a}, \check{q}^\kappa (\hat{d} - \hat{c}) + \hat{c} \right) \\ -\mathcal{F} \left( \bar{q}^{\nu+1} (\hat{b} - \hat{a}) + \hat{a}, \check{q}^\kappa (\hat{d} - \hat{c}) + \hat{c} \right) \\ -\mathcal{F} \left( \bar{q}^\nu (\hat{b} - \hat{a}) + \hat{a}, \check{q}^{\kappa+1} (\hat{d} - \hat{c}) + \hat{c} \right) \\ +\mathcal{F} \left( \bar{q}^{\nu+1} (\hat{b} - \hat{a}) + \hat{a}, \check{q}^{\kappa+1} (\hat{d} - \hat{c}) + \hat{c} \right) \end{array} \right] \\
& = S_1 + S_2 + S_3 + S_4.
\end{aligned}$$

Applying the integral expressed in the Definition 2.9, we have

$$\begin{aligned}
S_1 &= (\hat{b} - \hat{a}) (\hat{d} - \hat{c}) \left[ \bar{q}\check{q} \sum_{\nu=0}^{\infty} \sum_{\kappa=0}^{\infty} \bar{q}^\nu \check{q}^\kappa \mathcal{F} \left( \bar{q}^\nu (\hat{b} - \hat{a}) + \hat{a}, \check{q}^\kappa (\hat{d} - \hat{c}) + \hat{c} \right) \right. \\
& - \check{q} \sum_{\nu=1}^{\infty} \sum_{\kappa=0}^{\infty} \bar{q}^\nu \check{q}^\kappa \mathcal{F} \left( \bar{q}^\nu (\hat{b} - \hat{a}) + \hat{a}, \check{q}^\kappa (\hat{d} - \hat{c}) + \hat{c} \right) \\
& - \bar{q} \sum_{\nu=0}^{\infty} \sum_{\kappa=1}^{\infty} \bar{q}^\nu \check{q}^\kappa \mathcal{F} \left( \bar{q}^\nu (\hat{b} - \hat{a}) + \hat{a}, \check{q}^\kappa (\hat{d} - \hat{c}) + \hat{c} \right) \\
& \left. + \sum_{\nu=1}^{\infty} \sum_{\kappa=1}^{\infty} \bar{q}^\nu \check{q}^\kappa \mathcal{F} \left( \bar{q}^\nu (\hat{b} - \hat{a}) + \hat{a}, \check{q}^\kappa (\hat{d} - \hat{c}) + \hat{c} \right) \right],
\end{aligned}$$

which leads to:

$$S_1 = \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} \mathcal{F}(x, y) {}_c d_{\check{q}} y {}_{\hat{a}} d_{\bar{q}} x - (\hat{d} - \hat{c}) \int_{\hat{a}}^{\hat{b}} \mathcal{F}(x, \hat{d}) {}_{\hat{a}} d_{\bar{q}} x$$

$$- (\hat{b} - \hat{a}) \int_{\hat{c}}^{\hat{d}} \mathcal{F}(\hat{b}, y) {}_c d_{\check{q}} y + (\hat{b} - \hat{a}) (\hat{d} - \hat{c}) \mathcal{F}(\hat{b}, \hat{d}).$$

Similarly,

$$\begin{aligned} S_2 = & - (\check{q}\hat{c} - \hat{d}) \int_{\hat{a}}^{\hat{b}} [\mathcal{F}(x, \hat{d}) - \mathcal{F}(x, \hat{c})] {}_a d_{\check{q}} x + (\hat{b} - \hat{a}) (\check{q}\hat{c} - \hat{d}) \mathcal{F}(\hat{b}, \hat{d}) \\ & - (\hat{b} - \hat{a}) (\check{q}\hat{c} - \hat{d}) \mathcal{F}(\hat{b}, \hat{c}), \end{aligned}$$

$$\begin{aligned} S_3 = & - (\bar{q}\hat{a} - \hat{b}) \int_{\hat{c}}^{\hat{d}} [\mathcal{F}(\hat{b}, y) - \mathcal{F}(\hat{a}, y)] {}_c d_{\check{q}} y + (\hat{d} - \hat{c}) (\bar{q}\hat{a} - \hat{b}) \mathcal{F}(\hat{b}, \hat{d}) \\ & - (\hat{d} - \hat{c}) (\bar{q}\hat{a} - \hat{b}) \mathcal{F}(\hat{a}, \hat{d}) \end{aligned}$$

$$S_4 = (\bar{q}\hat{a} - \hat{b}) (\check{q}\hat{c} - \hat{d}) [\mathcal{F}(\hat{b}, \hat{d}) - \mathcal{F}(\hat{a}, \hat{d}) - \mathcal{F}(\hat{b}, \hat{c}) + \mathcal{F}(\hat{a}, \hat{c})].$$

Substituting  $S_1, S_2, S_3$  and  $S_4$  into (3.4) yields

$$\begin{aligned} (3.5) \quad R_1 = & \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} \mathcal{F}(x, y) {}_c d_{\check{q}} y {}_a d_{\check{q}} x \\ & - \int_{\hat{c}}^{\hat{d}} [(\bar{q}\hat{a} - \hat{a}) \mathcal{F}(\hat{b}, y) - (\bar{q}\hat{a} - \hat{b}) \mathcal{F}(\hat{a}, y)] {}_c d_{\check{q}} y \\ & - \int_{\hat{a}}^{\hat{b}} [(\check{q}\hat{c} - \hat{c}) \mathcal{F}(x, \hat{d}) - (\check{q}\hat{c} - \hat{d}) \mathcal{F}(x, \hat{c})] {}_a d_{\check{q}} x \\ & + (\bar{q}\hat{a} - \hat{a}) (\check{q}\hat{c} - \hat{c}) \mathcal{F}(\hat{b}, \hat{d}) + (\bar{q}\hat{a} - \hat{b}) (\check{q}\hat{c} - \hat{d}) \mathcal{F}(\hat{a}, \hat{c}) \\ & - (\bar{q}\hat{a} - \hat{a}) (\check{q}\hat{c} - \hat{d}) \mathcal{F}(\hat{b}, \hat{c}) - (\bar{q}\hat{a} - \hat{b}) (\check{q}\hat{c} - \hat{c}) \mathcal{F}(\hat{a}, \hat{d}). \end{aligned}$$

Proceeding in a manner analogous to  $R_1$ , we obtain

$$\begin{aligned} (3.6) \quad R_2 = & (\hat{b} - \hat{a}) (\hat{d} - \hat{c}) \left[ \mathcal{F} \left( \frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}, \frac{\check{q}\hat{c} + \hat{d}}{1 + \check{q}} \right) \right. \\ & \left. - \mathcal{F} \left( \frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}, \hat{c} \right) - \mathcal{F} \left( \hat{a}, \frac{\check{q}\hat{c} + \hat{d}}{1 + \check{q}} \right) + \mathcal{F}(\hat{a}, \hat{c}) \right], \end{aligned}$$

$$\begin{aligned} (3.7) \quad R_3 = & (\hat{b} - \hat{a}) \int_{\hat{c}}^{\hat{d}} \left[ \mathcal{F}(\hat{a}, y) - \mathcal{F} \left( \frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}, y \right) \right] {}_c d_{\check{q}} y \\ & + \left[ \mathcal{F} \left( \frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}, \hat{d} \right) - \mathcal{F}(\hat{a}, \hat{d}) \right] (\check{q}\hat{c} - \hat{c}) (\hat{b} - \hat{a}) \end{aligned}$$

$$\begin{aligned}
& + \left[ \mathcal{F}(\hat{a}, \hat{c}) - \mathcal{F}\left(\frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}, \hat{c}\right) \right] (\hat{b} - \hat{a}) (\check{q}\hat{c} - \hat{d}), \\
(3.8) \quad R_4 &= (\hat{d} - \hat{c}) \int_{\hat{a}}^{\hat{b}} \left[ \mathcal{F}(x, \hat{c}) - \mathcal{F}\left(x, \frac{\check{q}\hat{c} + \hat{d}}{1 + \check{q}}\right) \right] {}_c d_{\bar{q}} y \\
& + \left[ \mathcal{F}\left(\hat{b}, \frac{\check{q}\hat{c} + \hat{d}}{1 + \check{q}}\right) - \mathcal{F}(\hat{b}, \hat{c}) \right] (\bar{q}\hat{a} - \hat{a}) (\hat{d} - \hat{c}) \\
& + \left[ \mathcal{F}(\hat{a}, \hat{c}) - \mathcal{F}\left(\hat{a}, \frac{\check{q}\hat{c} + \hat{d}}{1 + \check{q}}\right) \right] (\bar{q}\hat{a} - \hat{b}) (\hat{d} - \hat{c}).
\end{aligned}$$

Now utilizing equations (3.5)–(3.8) in the equation (3.3) and multiplying the outcome of with  $\frac{1}{(\hat{b}-\hat{a})(\hat{d}-\hat{c})}$ , we get the desired identity (3.1).  $\square$

**3.2. New Theorem for Case Study.** Before proceeding to the main results, we give some quantum integrals which are vital for further investigation.

Let  $\hat{a}, \hat{b} \in \mathbb{R}$  such that  $\hat{a} \leq \hat{b}$  and  $\bar{q} \in (0, 1)$ , then

(1)

$$\begin{aligned}
& {}_{\hat{a}}\mathcal{R}_{\frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}}(s, \bar{q}) \\
& := \int_{\hat{a}}^{\frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}} |\bar{q}u - \hat{a}| \left( \frac{\frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}} - u}{\frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}} - \hat{a}} \right)^s {}_{\hat{a}} d_{\bar{q}} u \\
& = \begin{cases} \left( \frac{(\hat{b} - \hat{a})(1 - \bar{q})}{(\bar{q} + 1)^2} \right. \\ \quad \times \sum_{\beta=0}^{\infty} \bar{q}^{\beta} \left( \hat{a} (1 - \bar{q}^2) - (\hat{b} - \hat{a}) \bar{q}^{\beta+1} \right) (1 - \bar{q}^{\beta})^s, & \text{if } \frac{\hat{a}}{\bar{q}} \geq \frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}; \\ \\ \left. \frac{2\hat{a}^2(1 - \bar{q})^3}{\bar{q}(\hat{b} - \hat{a})^s} \sum_{\beta=0}^{\infty} \bar{q}^{\beta} (1 - \bar{q}^{\beta}) \left( \hat{b} - \hat{a} - \hat{a}\bar{q}^{\beta-1} (1 - \bar{q}^2) \right)^s \right. \\ \quad - \frac{(\hat{b} - \hat{a})(1 - \bar{q})}{(\bar{q} + 1)^2} \\ \quad \times \sum_{\beta=0}^{\infty} \bar{q}^{\beta} \left( \hat{a} (1 - \bar{q}^2) - (\hat{b} - \hat{a}) \bar{q}^{\beta+1} \right) (1 - \bar{q}^{\beta})^s, & \text{if } \frac{\hat{a}}{\bar{q}} < \frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}. \end{cases}
\end{aligned}$$

(2)

$$\begin{aligned}
 & \hat{a} \mathcal{S}_{1+\bar{q}}^{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}}(s, \bar{q}) \\
 & := \int_{\hat{a}}^{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}} |\bar{q}u - \hat{a}| \left( \frac{u - \hat{a}}{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}} - \hat{a}} \right)^s \hat{a} d_{\bar{q}}u \\
 & = \begin{cases} \frac{(1-\bar{q})(\hat{b}-\hat{a})}{(\bar{q}+1)^2} \sum_{\beta=0}^{\infty} \bar{q}^{(1+s)\beta} \left( \hat{a} (1 - \bar{q}^2) - (\hat{b} - \hat{a}) \bar{q}^{1+\beta} \right), & \text{if } \frac{\hat{a}}{\bar{q}} \geq \frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}; \\ \frac{2\hat{a}^{2+s}(1+\bar{q})^s(1-\bar{q})^{3+s}}{\bar{q}^{1+s}(\hat{b}-\hat{a})^s} \sum_{\beta=0}^{\infty} \bar{q}^{(1+s)\beta} (1 - \bar{q}^\beta) \\ - \frac{(1-\bar{q})(\hat{b}-\hat{a})}{(\bar{q}+1)^2} \sum_{\beta=0}^{\infty} \bar{q}^{(1+s)\beta} \left( \hat{a} (1 - \bar{q}^2) - (\hat{b} - \hat{a}) \bar{q}^{1+\beta} \right), & \text{if } \frac{\hat{a}}{\bar{q}} < \frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}. \end{cases}
 \end{aligned}$$

(3)

$$\begin{aligned}
 & \frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}} \mathcal{T}^{\hat{b}}(s, \bar{q}) \\
 & := \int_{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}}^{\hat{b}} \left| (\hat{q}u - \hat{b}) \right| \left( \frac{\hat{b} - u}{\hat{b} - \frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}} \right)^s \hat{a} d_{\bar{q}}u \\
 & = \frac{(1-\bar{q})(\hat{b}-\hat{a})(\bar{q}+1)^s}{\bar{q}^s} \\
 & \quad \times \sum_{\beta=0}^{\infty} \bar{q}^\beta (1 - \bar{q}^\beta)^s \left( \hat{b} - \hat{a}\bar{q} - (\hat{b} - \hat{a}) \bar{q}^{1+\beta} \right) - \frac{(1-\bar{q})(\hat{b}-\hat{a})}{(\bar{q}+1)^2 \bar{q}^s} \\
 & \quad \times \sum_{\beta=0}^{\infty} \bar{q}^\beta (\bar{q} - \bar{q}^\beta + 1)^s \left( (\hat{b} - \hat{a}\bar{q})(1 + \bar{q}) - (\hat{b} - \hat{a}) \bar{q}^{1+\beta} \right),
 \end{aligned}$$

(4)

$$\begin{aligned}
 & \frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}} \mathcal{U}^{\hat{b}}(s, \bar{q}) \\
 & := \int_{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}}^{\hat{b}} |\hat{q}u - \hat{b}| \left( \frac{u - \frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}}{\hat{b} - \frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}} \right)^s \hat{a} d_{\bar{q}}u \\
 & = \frac{(\hat{b} - \hat{a})(1 - \bar{q})}{\bar{q}^s}
 \end{aligned}$$

$$\begin{aligned}
& \times \sum_{\beta=0}^{\infty} \bar{q}^{\beta} \left( \bar{q}^{\beta} + \bar{q}^{\beta+1} - 1 \right)^s \left( \hat{b} - \hat{a}\bar{q} - \left( \hat{b} - \hat{a} \right) \bar{q}^{1+\beta} \right) \\
& - \frac{(1 - \bar{q}) \left( \hat{b} - \hat{a} \right)}{\bar{q}^s (\bar{q} + 1)^2} \\
& \times \sum_{\beta=0}^{\infty} \bar{q}^{\beta} \left( \bar{q}^{\beta} - 1 \right)^s \left( \left( \hat{b} - \hat{a}\bar{q} \right) (1 + \bar{q}) - \left( \hat{b} - \hat{a} \right) \bar{q}^{1+\beta} \right).
\end{aligned}$$

**Remark 3.2.** Here we deduce some integrals using different values of parameters involved. For the purpose we use software Mathematica and Scientific work place to calculate infinite geometric series. For instance, if  $\hat{a} = 0, \hat{b} = 1$ , then we obtain following integrals with specific value of  $\bar{q}$ :

(1)

$$\begin{aligned}
{}_0\mathcal{R}^{\frac{2}{3}} \left( \frac{1}{2}, \frac{1}{2} \right) &= \frac{\sqrt{3}}{2\sqrt{2}} \int_0^{\frac{2}{3}} \frac{1}{2} u \left( \frac{2}{3} - u \right)^{\frac{1}{2}} {}_0d_{\frac{1}{2}}u \\
&= \frac{\frac{1}{2} \left( 1 - \frac{1}{2} \right)}{\left( \frac{1}{2} + 1 \right)^2} \sum_{\beta=0}^{\infty} \left( \frac{1}{2} \right)^{2\beta} \left( 1 - \left( \frac{1}{2} \right)^{\beta} \right)^{\frac{1}{2}} \\
&\approx 2.7843 \times 10^{-2}.
\end{aligned}$$

(2)

$$\begin{aligned}
{}_0\mathcal{S}^{\frac{2}{3}} \left( \frac{1}{2}, \frac{1}{2} \right) &= \frac{\sqrt{3}}{2\sqrt{2}} \int_0^{\frac{2}{3}} \frac{1}{2} u^{\frac{3}{2}} {}_0d_{\frac{1}{2}}u \\
&= \frac{\frac{1}{2} \left( 1 - \frac{1}{2} \right)}{\left( \frac{1}{2} + 1 \right)^2} \sum_{\beta=0}^{\infty} \left( \frac{1}{2} \right)^{\left( \frac{5}{2} \right)\beta} \\
&\approx 0.13497.
\end{aligned}$$

(3)

$$\begin{aligned}
{}_0\mathcal{T}^{\frac{2}{3}} \left( \frac{1}{2}, \frac{3}{5} \right) &= \sqrt{\frac{8}{3}} \int_{\frac{3}{5}}^1 \left| \left( \frac{3}{5}u - 1 \right) \right| (1 - u)^{\frac{1}{2}} {}_0d_{\frac{3}{5}}u \\
&= \frac{(1 + 0.6)^{\frac{1}{2}} (1 - 0.6)}{(0.6)^{\frac{1}{2}}} \sum_{k=0}^{\infty} (0.6)^k \left( 1 - (0.6)^k \right)^{\frac{1}{2}} \left( 1 - (0.6)^{1+k} \right)^{\frac{1}{2}} \\
&- \frac{(1 - 0.6)}{(1 + 0.6)^2 (0.6)^{\frac{1}{2}}}
\end{aligned}$$

$$\begin{aligned} & \times \sum_{k=0}^{\infty} (0.6)^k \left(0.6 + 1 - (0.6)^k\right)^{\frac{1}{2}} \left(1 + 0.6 - (0.6)^{k+1}\right) \\ & \approx 6.0699 \times 10^{-2}. \end{aligned}$$

(4)

$$\begin{aligned} & {}_{\frac{2}{3}}\mathcal{U}^1\left(1, \frac{3}{5}\right) \\ & = \frac{8}{3} \int_{\frac{5}{8}}^1 \left|\frac{3}{5}u - 1\right| \left(u - \frac{5}{8}\right) {}_0d_{\frac{3}{8}}u \\ & = \frac{(1 - (0.6))}{(0.6)} \sum_{k=0}^{\infty} (0.6)^k \left((0.6)^k + (0.6)^{k+1} - 1\right) \left(1 - (0.6)^{1+k}\right) \\ & \quad - \frac{(1 - (0.6))}{(1 + (0.6))^2 (0.6)} \sum_{k=0}^{\infty} (0.6)^k \left((0.6)^k - 1\right) \left(1 + (0.6) - (0.6)^{1+k}\right) \\ & \approx 0.15446. \end{aligned}$$

**Remark 3.3.** Now we printout that when  $\bar{q} < 1$ , then  $\bar{q}^\beta \rightarrow 0$ . Therefore  $(\bar{q}^\beta + \bar{q}^{\beta+1} - 1)^s$  leads to an imaginary number. The integral  $\frac{\bar{q}^{\hat{a}+\hat{b}}}{1+\bar{q}} \mathcal{U}^{\hat{b}}(s, \bar{q})$  is real and finite only, if either  $s = 1$  or  $\bar{q} \rightarrow 1^-$ .

**Theorem 3.4.** Let  $(s, t) \in (0, 1]^2$ . Assume that the function  $\mathcal{F} : \Xi \rightarrow \mathbb{R}$  is a twice partial  $\bar{q}_a \bar{q}_c$ -differentiable on  $\Xi^\circ$  such that the partial  $\bar{q}_a \bar{q}_c$ -derivative  $\frac{\partial^2 \mathcal{F}(u, v)}{{}_a \partial_{\bar{q}} u {}_c \partial_{\bar{q}} v}$  is continuous and  $\bar{q}_a \bar{q}_c$ -integrable over  $\Xi$ . If  $\left| \frac{\partial^2 \mathcal{F}(u, v)}{{}_a \partial_{\bar{q}} u {}_c \partial_{\bar{q}} v} \right|$  is coordinated  $(s, t)$ -convex of second kind on  $\Xi$ , then the following inequality holds:

$$\begin{aligned} (3.9) \quad & \left| \frac{1}{(\hat{b} - \hat{a})(\hat{d} - \hat{c})} \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} \mathcal{F}(x, y) {}_c d_{\bar{q}} y {}_a d_{\bar{q}} x + \mathcal{F}\left(\frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}, \frac{\bar{q}\hat{c} + \hat{d}}{1 + \bar{q}}\right) - \mathcal{V}_* \right| \\ & \leq \frac{1}{(\hat{b} - \hat{a})(\hat{d} - \hat{c})} [\Theta + \Phi + \Psi + \Upsilon], \end{aligned}$$

where

$$\begin{aligned} \Theta & := \left| \frac{\partial^2 \mathcal{F}(\hat{a}, \hat{c})}{{}_a \partial_{\bar{q}} u {}_c \partial_{\bar{q}} v} \right| {}_a \mathcal{R}^{\frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}}(s, \bar{q}) {}_c \mathcal{R}^{\frac{\bar{q}\hat{c} + \hat{d}}{1 + \bar{q}}}(t, \bar{q}) \\ & \quad + \left| \frac{\partial^2 \mathcal{F}\left(\hat{a}, \frac{\bar{q}\hat{c} + \hat{d}}{1 + \bar{q}}\right)}{{}_a \partial_{\bar{q}} u {}_c \partial_{\bar{q}} v} \right| {}_a \mathcal{R}^{\frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}}(s, \bar{q}) {}_c \mathcal{S}^{\frac{\bar{q}\hat{c} + \hat{d}}{1 + \bar{q}}}(t, \bar{q}) \end{aligned}$$

$$\begin{aligned}
& + \left| \frac{\partial^2 \mathcal{F} \left( \frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}, \hat{c} \right)}{\hat{a}\partial_{\bar{q}}u \hat{c}\partial_{\bar{q}}v} \right| \hat{a}\mathcal{S}^{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}}(s, \bar{q}) \hat{c}\mathcal{R}^{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}}(t, \check{q}) \\
& + \left| \frac{\partial^2 \mathcal{F} \left( \frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}, \frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}} \right)}{\hat{a}\partial_{\bar{q}}u \hat{c}\partial_{\bar{q}}v} \right| \hat{a}\mathcal{S}^{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}}(s, \bar{q}) \hat{c}\mathcal{S}^{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}}(t, \check{q}); \\
\Phi := & \left| \frac{\partial^2 \mathcal{F} \left( \hat{a}, \frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}} \right)}{\hat{a}\partial_{\bar{q}}u \hat{c}\partial_{\bar{q}}v} \right| \hat{a}\mathcal{R}^{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}}(s, \bar{q}) \frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}\mathcal{T}^{\hat{d}}(t, \check{q}) \\
& + \left| \frac{\partial^2 \mathcal{F} \left( \hat{a}, \hat{d} \right)}{\hat{a}\partial_{\bar{q}}u \hat{c}\partial_{\bar{q}}v} \right| \hat{a}\mathcal{R}^{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}}(s, \bar{q}) \frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}\mathcal{U}^{\hat{d}}(t, \check{q}) \\
& + \left| \frac{\partial^2 \mathcal{F} \left( \frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}, \frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}} \right)}{\hat{a}\partial_{\bar{q}}u \hat{c}\partial_{\bar{q}}v} \right| \hat{a}\mathcal{S}^{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}}(s, \bar{q}) \frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}\mathcal{T}^{\hat{d}}(t, \check{q}) \\
& + \left| \frac{\partial^2 \mathcal{F} \left( \frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}, \hat{d} \right)}{\hat{a}\partial_{\bar{q}}u \hat{c}\partial_{\bar{q}}v} \right| \hat{a}\mathcal{S}^{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}}(s, \bar{q}) \frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}\mathcal{U}^{\hat{d}}(t, \check{q}); \\
\Psi := & \left| \frac{\partial^2 \mathcal{F} \left( \frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}, \hat{c} \right)}{\hat{a}\partial_{\bar{q}}u \hat{c}\partial_{\bar{q}}v} \right| \frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}\mathcal{T}^{\hat{b}}(s, \bar{q}) \hat{c}\mathcal{R}^{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}}(t, \check{q}) \\
& + \left| \frac{\partial^2 \mathcal{F} \left( \frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}, \frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}} \right)}{\hat{a}\partial_{\bar{q}}u \hat{c}\partial_{\bar{q}}v} \right| \frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}\mathcal{T}^{\hat{b}}(s, \bar{q}) \hat{c}\mathcal{S}^{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}}(t, \check{q}) \\
& + \left| \frac{\partial^2 \mathcal{F} \left( \hat{b}, \hat{c} \right)}{\hat{a}\partial_{\bar{q}}u \hat{c}\partial_{\bar{q}}v} \right| \frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}\mathcal{U}^{\hat{b}}(s, \bar{q}) \hat{c}\mathcal{R}^{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}}(t, \check{q}) \\
& + \left| \frac{\partial^2 \mathcal{F} \left( \hat{b}, \frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}} \right)}{\hat{a}\partial_{\bar{q}}u \hat{c}\partial_{\bar{q}}v} \right| \frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}\mathcal{U}^{\hat{b}}(s, \bar{q}) \hat{c}\mathcal{S}^{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}}(t, \check{q}); \\
\Upsilon := & \left| \frac{\partial^2 \mathcal{F} \left( \frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}, \frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}} \right)}{\hat{a}\partial_{\bar{q}}u \hat{c}\partial_{\bar{q}}v} \right| \frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}\mathcal{T}^{\hat{b}}(s, \bar{q}) \frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}\mathcal{T}^{\hat{d}}(t, \check{q})
\end{aligned}$$

$$\begin{aligned}
 & + \left| \frac{\partial^2 \mathcal{F} \left( \frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}, \hat{d} \right)}{\hat{a}\partial_{\bar{q}}u \; \hat{c}\partial_{\bar{q}}v} \right| \frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}} \mathcal{T}^{\hat{b}}(s, \bar{q}) \frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}} \mathcal{U}^{\hat{d}}(t, \bar{q}) \\
 & + \left| \frac{\partial^2 \mathcal{F} \left( \hat{b}, \frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}} \right)}{\hat{a}\partial_{\bar{q}}u \; \hat{c}\partial_{\bar{q}}v} \right| \frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}} \mathcal{U}^{\hat{b}}(s, \bar{q}) \frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}} \mathcal{T}^{\hat{d}}(t, \bar{q}) \\
 & + \left| \frac{\partial^2 \mathcal{F} \left( \hat{b}, \hat{d} \right)}{\hat{a}\partial_{\bar{q}}u \; \hat{c}\partial_{\bar{q}}v} \right| \frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}} \mathcal{U}^{\hat{b}}(s, \bar{q}) \frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}} \mathcal{U}^{\hat{d}}(t, \bar{q}).
 \end{aligned}$$

*Proof.* Let us first consider the identity mapping with some convenient notation:

$$(3.10) \quad \mathcal{I}_{\mathcal{A}}^{\mathcal{B}}(\lambda) := \frac{\mathcal{B} - \lambda}{\mathcal{B} - \mathcal{A}} \mathcal{A} + \frac{\lambda - \mathcal{A}}{\mathcal{B} - \mathcal{A}} \mathcal{B}.$$

Taking modulus on both sides of equation (3.1), we have

$$\begin{aligned}
 (3.11) \quad & \left| \frac{1}{(\hat{b} - \hat{a})(\hat{d} - \hat{c})} \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} \mathcal{F}(x, y) \; \hat{c}d_{\bar{q}}y \; \hat{a}d_{\bar{q}}x + \mathcal{U} - \mathcal{V} \right| \\
 & \leq \frac{1}{(\hat{b} - \hat{a})(\hat{d} - \hat{c})} \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} |\mathcal{K}_1(u, v)| \left| \frac{\partial^2 \mathcal{F}(u, v)}{\hat{a}\partial_{\bar{q}}u \; \hat{c}\partial_{\bar{q}}v} \right|^{\omega} \; \hat{c}d_{\bar{q}}v \; \hat{a}d_{\bar{q}}u \\
 & = \frac{1}{(\hat{b} - \hat{a})(\hat{d} - \hat{c})} \\
 & \quad \times \left[ \int_{\hat{a}}^{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}} \int_{\hat{c}}^{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}} |(\bar{q}u - \hat{a})(\bar{q}v - \hat{c})| \left| \frac{\partial^2 \mathcal{F}(u, v)}{\hat{a}\partial_{\bar{q}}u \; \hat{c}\partial_{\bar{q}}v} \right| \; \hat{c}d_{\bar{q}}v \; \hat{a}d_{\bar{q}}u \right. \\
 & \quad + \int_{\hat{a}}^{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}} \int_{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}}^{\hat{d}} |(\bar{q}u - \hat{a})(\bar{q}v - \hat{d})| \left| \frac{\partial^2 \mathcal{F}(u, v)}{\hat{a}\partial_{\bar{q}}u \; \hat{c}\partial_{\bar{q}}v} \right| \; \hat{c}d_{\bar{q}}v \; \hat{a}d_{\bar{q}}u \\
 & \quad + \int_{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}}^{\hat{b}} \int_{\hat{c}}^{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}} |(\bar{q}u - \hat{b})(\bar{q}v - \hat{c})| \left| \frac{\partial^2 \mathcal{F}(u, v)}{\hat{a}\partial_{\bar{q}}u \; \hat{c}\partial_{\bar{q}}v} \right| \; \hat{c}d_{\bar{q}}v \; \hat{a}d_{\bar{q}}u \\
 & \quad \left. + \int_{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}}^{\hat{b}} \int_{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}}^{\hat{d}} |(\bar{q}u - \hat{b})(\bar{q}v - \hat{d})| \left| \frac{\partial^2 \mathcal{F}(u, v)}{\hat{a}\partial_{\bar{q}}u \; \hat{c}\partial_{\bar{q}}v} \right| \; \hat{c}d_{\bar{q}}v \; \hat{a}d_{\bar{q}}u \right] \\
 & = \frac{1}{(\hat{b} - \hat{a})(\hat{d} - \hat{c})} [\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3 + \mathcal{W}_4].
 \end{aligned}$$

By the given coordinated convexity of  $\left| \frac{\partial^2 \mathcal{F}}{\hat{a} \partial_{\bar{q}} u \hat{c} \partial_{\bar{q}} v} \right|$  and utilizing the identity function (3.10), we have

$$\begin{aligned}
\mathcal{W}_1 &= \int_{\hat{a}}^{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}} \int_{\hat{c}}^{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}} \frac{|\bar{q}u - \hat{a}|}{|\bar{q}v - \hat{c}|^{-1}} \left| \frac{\partial^2 \mathcal{F}}{\hat{a} \partial_{\bar{q}} u \hat{c} \partial_{\bar{q}} v} \left( \mathcal{I}_{\hat{a}}^{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}}(u), \mathcal{I}_{\hat{c}}^{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}}(v) \right) \right| \hat{c} d_{\bar{q}} v \hat{a} d_{\bar{q}} u \\
&\leq \left| \frac{\partial^2 \mathcal{F}(\hat{a}, \hat{c})}{\hat{a} \partial_{\bar{q}} u \hat{c} \partial_{\bar{q}} v} \right| \\
&\quad \times \int_{\hat{a}}^{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}} \int_{\hat{c}}^{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}} \frac{|\bar{q}u - \hat{a}|}{|\bar{q}v - \hat{c}|^{-1}} \left( \frac{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}} - u}{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}} - \hat{a}} \right)^s \left( \frac{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}} - v}{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}} - \hat{c}} \right)^t \hat{c} d_{\bar{q}} v \hat{a} d_{\bar{q}} u \\
&\quad + \left| \frac{\partial^2 \mathcal{F} \left( \hat{a}, \frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}} \right)}{\hat{a} \partial_{\bar{q}} u \hat{c} \partial_{\bar{q}} v} \right| \\
&\quad \times \int_{\hat{a}}^{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}} \int_{\hat{c}}^{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}} \frac{|\bar{q}u - \hat{a}|}{|\bar{q}v - \hat{c}|^{-1}} \left( \frac{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}} - u}{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}} - \hat{a}} \right)^s \left( \frac{v - \hat{c}}{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}} - \hat{c}} \right)^t \hat{c} d_{\bar{q}} v \hat{a} d_{\bar{q}} u \\
&\quad + \left| \frac{\partial^2 \mathcal{F} \left( \frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}, \hat{c} \right)}{\hat{a} \partial_{\bar{q}} u \hat{c} \partial_{\bar{q}} v} \right| \\
&\quad \times \int_{\hat{a}}^{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}} \int_{\hat{c}}^{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}} \frac{|\bar{q}u - \hat{a}|}{|\bar{q}v - \hat{c}|^{-1}} \left( \frac{u - \hat{a}}{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}} - \hat{a}} \right)^s \left( \frac{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}} - v}{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}} - \hat{c}} \right)^t \hat{c} d_{\bar{q}} v \hat{a} d_{\bar{q}} u \\
&\quad + \left| \frac{\partial^2 \mathcal{F} \left( \frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}, \frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}} \right)}{\hat{a} \partial_{\bar{q}} u \hat{c} \partial_{\bar{q}} v} \right| \\
&\quad \times \int_{\hat{a}}^{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}} \int_{\hat{c}}^{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}} \frac{|\bar{q}u - \hat{a}|}{|\bar{q}v - \hat{c}|^{-1}} \left( \frac{u - \hat{a}}{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}} - \hat{a}} \right)^s \left( \frac{v - \hat{c}}{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}} - \hat{c}} \right)^t \hat{c} d_{\bar{q}} v \hat{a} d_{\bar{q}} u.
\end{aligned}$$

Using the integrals given in the subsection 3.2, we obtain

$$\mathcal{W}_1 \leq \Theta.$$

Similarly, for the rectangles  $\left[ \hat{a}, \frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}} \right] \times \left[ \frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}, \hat{d} \right]$ ,  $\left[ \frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}, \hat{b} \right] \times \left[ \hat{c}, \frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}} \right]$  and  $\left[ \frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}, \hat{b} \right] \times \left[ \frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}, \hat{d} \right]$ , we have respectively,

$$\mathcal{W}_2 \leq \Phi, \quad \mathcal{W}_3 \leq \Psi, \quad \mathcal{W}_4 \leq \Upsilon.$$

Using all the inequalities in (3.11), we have the desired inequality.  $\square$

**Remark 3.5.** Now, we see that the outcome of Theorem 3.4 involves integral  $\frac{\hat{q}\hat{a}+\hat{b}}{1+\hat{q}}\mathcal{U}^{\hat{b}}(s)$ . The integral integral  $\frac{\check{q}\hat{a}+\hat{b}}{1+\check{q}}\mathcal{U}^{\hat{b}}(s)$  contains a function which is discontinuous near  $x = a$ . This fact reduces the range of situations where the Theorem 3.4 remains valid.

Now, we present some results that are deduced from Theorem 3.4.

**Corollary 3.6.** *In addition to the conditions of Theorem 3.4, if  $s = t = 1$ , then*

$$\begin{aligned}
 (3.12) \quad & \left| \frac{1}{(\hat{b} - \hat{a})(\hat{d} - \hat{c})} \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} \mathcal{F}(x, y) {}_c d_{\check{q}} y {}_a d_{\check{q}} x + \mathcal{F} \left( \frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}, \frac{\check{q}\hat{c} + \hat{d}}{1 + \check{q}} \right) - \mathcal{V} \right| \\
 & \leq \frac{1}{(\hat{b} - \hat{a})(\hat{d} - \hat{c})} \\
 & \times \left\{ \left[ \left| \frac{\partial^2 F(\hat{a}, \hat{c})}{a \partial_{\bar{q}} u_c \partial_{\check{q}} v} \right|_{\hat{c}} \mathcal{R}^{\frac{\bar{q}\hat{c} + \hat{d}}{1 + \bar{q}}}(1, \check{q}) + \left| \frac{\partial^2 F(\hat{a}, \hat{d})}{a \partial_{\bar{q}} u_c \partial_{\check{q}} v} \right|_{\frac{\check{q}\hat{c} + \hat{d}}{1 + \check{q}}} \mathcal{U}^{\hat{d}}(1, \check{q}) \right] \hat{a} \mathcal{R}^{\frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}}(1, \bar{q}) \right. \\
 & + \left[ \left| \frac{\partial^2 F(\hat{b}, \hat{c})}{a \partial_{\bar{q}} u_c \partial_{\check{q}} v} \right|_{\hat{c}} \mathcal{R}^{\frac{\bar{q}\hat{c} + \hat{d}}{1 + \bar{q}}}(1, \check{q}) + \left| \frac{\partial^2 F(\hat{b}, \hat{d})}{a \partial_{\bar{q}} u_c \partial_{\check{q}} v} \right|_{\frac{\check{q}\hat{c} + \hat{d}}{1 + \check{q}}} \mathcal{U}^{\hat{d}}(1, \check{q}) \right] \frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}} \mathcal{U}^{\hat{b}}(1, \bar{q}) \\
 & + \left[ \left| \frac{\partial^2 F \left( a, \frac{\check{q}\hat{c} + \hat{d}}{1 + \check{q}} \right)}{a \partial_{\bar{q}} u_c \partial_{\check{q}} v} \right|_{\hat{a}} \mathcal{R}^{\frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}}(1, \bar{q}) + \left| \frac{\partial^2 F \left( b, \frac{\check{q}\hat{c} + \hat{d}}{1 + \check{q}} \right)}{a \partial_{\bar{q}} u_c \partial_{\check{q}} v} \right|_{\frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}} \mathcal{U}^{\hat{b}}(1, \bar{q}) \right] \\
 & \times \left[ \hat{c} \mathcal{S}^{\frac{\bar{q}\hat{c} + \hat{d}}{1 + \bar{q}}}(1, \check{q}) + \frac{\bar{q}\hat{c} + \hat{d}}{1 + \bar{q}} \mathcal{T}^{\hat{d}}(1, \check{q}) \right] \\
 & + \left[ \left| \frac{\partial^2 F \left( \frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}, c \right)}{a \partial_{\bar{q}} u_c \partial_{\check{q}} v} \right|_{c} \mathcal{R}^{\frac{c + \hat{d}}{2}}(1, \check{q}) + \left| \frac{\partial^2 F \left( \frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}, d \right)}{a \partial_{\bar{q}} u_c \partial_{\check{q}} v} \right|_{\frac{c + \hat{d}}{2}} \mathcal{U}^{\hat{d}}(1, \check{q}) \right] \\
 & \times \left[ \hat{a} \mathcal{S}^{\frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}}(1, \bar{q}) + \frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}} \mathcal{T}^{\hat{b}}(1, \bar{q}) \right] \\
 & + \left| \frac{\partial^2 F \left( \frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}, \frac{\check{q}\hat{c} + \hat{d}}{1 + \check{q}} \right)}{a \partial_{\bar{q}} u_c \partial_{\check{q}} v} \right| \left[ \frac{\bar{q}\hat{c} + \hat{d}}{1 + \bar{q}} \mathcal{T}^{\hat{d}}(1, \check{q}) + \hat{c} \mathcal{R}^{\frac{\bar{q}\hat{c} + \hat{d}}{1 + \bar{q}}}(1, \check{q}) \right] \\
 & \times \left. \left[ \frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}} \mathcal{T}^{\hat{b}}(1, \bar{q}) + \hat{c} \mathcal{S}^{\frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}}(1, \bar{q}) \right] \right\}.
 \end{aligned}$$

**Corollary 3.7.** *Assuming the conditions of Theorem 3.4 hold, and further if  $\bar{q}, \tilde{q} \rightarrow 1^-$ , then*

(3.13)

$$\begin{aligned}
& \left| \frac{1}{(\hat{b} - \hat{a})(\hat{d} - \hat{c})} \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} \mathcal{F}(x, y) dy dx + \mathcal{F}\left(\frac{\hat{a} + \hat{b}}{2}, \frac{\hat{c} + \hat{d}}{2}\right) - \Psi_* \right| \\
& \leq \frac{(\hat{b} - \hat{a})(\hat{d} - \hat{c})}{16(2+s)(2+t)} \left[ \frac{\left| \frac{\partial^2 \mathcal{F}(\hat{a}, \hat{c})}{\partial u \partial v} \right| + \left| \frac{\partial^2 \mathcal{F}(\hat{a}, \hat{d})}{\partial u \partial v} \right| + \left| \frac{\partial^2 \mathcal{F}(\hat{b}, \hat{c})}{\partial u \partial v} \right| + \left| \frac{\partial^2 \mathcal{F}(\hat{b}, \hat{d})}{\partial u \partial v} \right|}{(1+s)(1+t)} \right] \\
& \quad + \frac{(\hat{b} - \hat{a})(\hat{d} - \hat{c})}{16(2+s)(2+t)} \left[ 2 \frac{\left| \frac{\partial^2 \mathcal{F}(\hat{a}, \frac{\hat{c} + \hat{d}}{2})}{\partial u \partial v} \right| + \left| \frac{\partial^2 \mathcal{F}(\hat{b}, \frac{\hat{c} + \hat{d}}{2})}{\partial u \partial v} \right|}{1+s} \right] \\
& \quad + 2 \left[ \frac{\left| \frac{\partial^2 \mathcal{F}(\frac{\hat{a} + \hat{b}}{2}, \hat{c})}{\partial u \partial v} \right| + \left| \frac{\partial^2 \mathcal{F}(\frac{\hat{a} + \hat{b}}{2}, \hat{d})}{\partial u \partial v} \right|}{1+t} + 4 \left| \frac{\partial^2 \mathcal{F}\left(\frac{\hat{a} + \hat{b}}{2}, \frac{\hat{c} + \hat{d}}{2}\right)}{\partial u \partial v} \right| \right] \\
& \leq \left( \frac{1}{(1+s)(1+t)} + \frac{1}{(1+s)2^{s+t-2}} + \frac{1}{(1+t)2^{s+t-2}} + \frac{1}{2^{s+t-2}} \right) \\
& \quad \times \frac{(\hat{b} - \hat{a})(\hat{d} - \hat{c})}{16(2+s)(2+t)} \\
& \quad \times \left[ \left| \frac{\partial^2 \mathcal{F}(\hat{a}, \hat{c})}{\partial u \partial v} \right| + \left| \frac{\partial^2 \mathcal{F}(\hat{a}, \hat{d})}{\partial u \partial v} \right| + \left| \frac{\partial^2 \mathcal{F}(\hat{b}, \hat{c})}{\partial u \partial v} \right| + \left| \frac{\partial^2 \mathcal{F}(\hat{b}, \hat{d})}{\partial u \partial v} \right| \right],
\end{aligned}$$

where

$$\Psi_* := \frac{1}{2} \left[ \frac{1}{\hat{b} - \hat{a}} \int_{\hat{a}}^{\hat{b}} \mathcal{F}\left(x, \frac{\hat{c} + \hat{d}}{2}\right) dx + \frac{1}{\hat{d} - \hat{c}} \int_{\hat{c}}^{\hat{d}} \mathcal{F}\left(\frac{\hat{a} + \hat{b}}{2}, y\right) dy \right].$$

**3.3. General Quantum Integrals.** In this subsection, we give very general quantum integrals which are of vital importance for a new error estimate that rectifies the limiting feature of inequality.

Let  $\hat{a}, \hat{b} \in \mathbb{R}$ . If  $\bar{q}, \tilde{q}$  are two fixed real numbers such that  $\bar{q}, \tilde{q} \in (0, 1)$ , then we have following  $\bar{q}$ -integrals which are denoted and defined by:

(i)

$$\begin{aligned}
 \mathcal{C}_1 &:= \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} |\bar{q}u - \hat{b}| |\check{q}v - \hat{d}| \left( \frac{\hat{b} - u}{\hat{b} - \hat{a}} \right)^s \left( \frac{\hat{d} - v}{\hat{d} - \hat{c}} \right)^t {}_{\hat{c}}d_{\check{q}}v {}_{\hat{a}}d_{\bar{q}}u \\
 &= (1 - \bar{q}) (\hat{b} - \hat{a}) \sum_{\alpha=0}^{\infty} \bar{q}^{\alpha} \left[ (\hat{b} - \hat{a}\bar{q}) - \bar{q}^{\alpha+1} (\hat{b} - \hat{a}) \right] (1 - \bar{q}^{\alpha})^s \\
 &\quad \times (1 - \check{q}) (\hat{d} - \hat{c}) \sum_{\beta=0}^{\infty} \check{q}^{\beta} \left[ (\hat{d} - \hat{c}\check{q}) - \check{q}^{\beta+1} (\hat{d} - \hat{c}) \right] (1 - \check{q}^{\beta})^t;
 \end{aligned}$$

(ii)

$$\begin{aligned}
 \mathcal{C}_2 &:= \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} |\bar{q}u - \hat{b}| |\check{q}v - \hat{d}| \left( \frac{\hat{b} - u}{\hat{b} - \hat{a}} \right)^s \left( \frac{v - \hat{c}}{\hat{d} - \hat{c}} \right)^t {}_{\hat{c}}d_{\check{q}}v {}_{\hat{a}}d_{\bar{q}}u \\
 &= (1 - \bar{q}) (\hat{b} - \hat{a}) \sum_{\alpha=0}^{\infty} \bar{q}^{\alpha} \left[ (\hat{b} - \hat{a}\bar{q}) - \bar{q}^{\alpha+1} (\hat{b} - \hat{a}) \right] (1 - \bar{q}^{\alpha})^s \\
 &\quad \times (1 - \check{q}) (\hat{d} - \hat{c}) \sum_{\beta=0}^{\infty} \check{q}^{\beta} \left[ (\hat{d} - \hat{c}\check{q}) - \check{q}^{\beta+1} (\hat{d} - \hat{c}) \right] \check{q}^{s_2\beta};
 \end{aligned}$$

(iii)

$$\begin{aligned}
 \mathcal{C}_3 &:= \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} |\bar{q}u - \hat{b}| |\check{q}v - \hat{d}| \left( \frac{u - \hat{a}}{\hat{b} - \hat{a}} \right)^s \left( \frac{\hat{d} - v}{\hat{d} - \hat{c}} \right)^t {}_{\hat{c}}d_{\check{q}}v {}_{\hat{a}}d_{\bar{q}}u \\
 &= (1 - \bar{q}) (\hat{b} - \hat{a}) \sum_{\alpha=0}^{\infty} \bar{q}^{\alpha} \left[ (\hat{b} - \hat{a}\bar{q}) - \bar{q}^{\alpha+1} (\hat{b} - \hat{a}) \right] \bar{q}^{s_1\alpha} \\
 &\quad \times (1 - \check{q}) (\hat{d} - \hat{c}) \sum_{\beta=0}^{\infty} \check{q}^{\beta} \left[ (\hat{d} - \hat{c}\check{q}) - \check{q}^{\beta+1} (\hat{d} - \hat{c}) \right] (1 - \check{q}^{\beta})^t;
 \end{aligned}$$

(iv)

$$\begin{aligned}
 \mathcal{C}_4 &:= \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} |\bar{q}u - \hat{b}| |\check{q}v - \hat{d}| \left( \frac{u - \hat{a}}{\hat{b} - \hat{a}} \right)^s \left( \frac{v - \hat{c}}{\hat{d} - \hat{c}} \right)^t {}_{\hat{c}}d_{\check{q}}v {}_{\hat{a}}d_{\bar{q}}u \\
 &= (1 - \bar{q}) (\hat{b} - \hat{a}) \sum_{\alpha=0}^{\infty} \bar{q}^{\alpha} \left[ (\hat{b} - \hat{a}\bar{q}) - \bar{q}^{\alpha+1} (\hat{b} - \hat{a}) \right] \bar{q}^{s_1\alpha} \\
 &\quad \times (1 - \check{q}) (\hat{d} - \hat{c}) \sum_{\beta=0}^{\infty} \check{q}^{\beta} \left[ (\hat{d} - \hat{c}\check{q}) - \check{q}^{\beta+1} (\hat{d} - \hat{c}) \right] \check{q}^{s_2\beta};
 \end{aligned}$$

(v)

$$\begin{aligned} \mathcal{D}_1 &:= \int_{\hat{a}}^{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}} \int_{\hat{c}}^{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}} \left( \frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}} - u \right)^s \left( \frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}} - v \right)^t {}_c d_{\bar{q}} v {}_a d_{\bar{q}} u \\ &= \left( \frac{\hat{b} - \hat{a}}{1 + \bar{q}} \right) \left( \frac{\hat{d} - \hat{c}}{1 + \bar{q}} \right) (1 - \bar{q}) (1 - \bar{q}) \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} \bar{q}^{\beta} \bar{q}^{\alpha} (1 - \bar{q}^{\alpha})^s (1 - \bar{q}^{\beta})^t; \end{aligned}$$

(vi)

$$\begin{aligned} \mathcal{D}_2 &:= \int_{\hat{a}}^{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}} \int_{\hat{c}}^{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}} \left( \frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}} - u \right)^s \left( \frac{v - \hat{c}}{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}} - \hat{c}} \right)^t {}_c d_{\bar{q}} v {}_a d_{\bar{q}} u \\ &= \left( \frac{\hat{b} - \hat{a}}{1 + \bar{q}} \right) \left( \frac{\hat{d} - \hat{c}}{1 + \bar{q}} \right) (1 - \bar{q}) (1 - \bar{q}) \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} \bar{q}^{\alpha} (1 - \bar{q}^{\alpha})^s \bar{q}^{(1+s_2)\beta}; \end{aligned}$$

(vii)

$$\begin{aligned} \mathcal{D}_3 &:= \int_{\hat{a}}^{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}} \int_{\hat{c}}^{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}} \left( \frac{u - \hat{a}}{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}} - \hat{a}} \right)^s \left( \frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}} - v \right)^t {}_c d_{\bar{q}} v {}_a d_{\bar{q}} u \\ &= \left( \frac{\hat{b} - \hat{a}}{1 + \bar{q}} \right) \left( \frac{\hat{d} - \hat{c}}{1 + \bar{q}} \right) (1 - \bar{q}) (1 - \bar{q}) \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} \bar{q}^{(1+s_1)\alpha} \bar{q}^{\beta} (1 - \bar{q}^{\beta})^t; \end{aligned}$$

(viii)

$$\begin{aligned} \mathcal{D}_4 &:= \int_{\hat{a}}^{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}} \int_{\hat{c}}^{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}} \left( \frac{u - \hat{a}}{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}} - \hat{a}} \right)^s \left( \frac{v - \hat{c}}{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}} - \hat{c}} \right)^t {}_c d_{\bar{q}} v {}_a d_{\bar{q}} u \\ &= \left( \frac{\hat{b} - \hat{a}}{1 + \bar{q}} \right) \left( \frac{\hat{d} - \hat{c}}{1 + \bar{q}} \right) (1 - \bar{q}) (1 - \bar{q}) \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} \bar{q}^{(1+s_1)\alpha} \bar{q}^{(1+s_2)\beta}; \end{aligned}$$

(ix)

$$\begin{aligned} \mathcal{H}_1 &:= \int_{\hat{a}}^{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}} \int_{\hat{c}}^{\hat{d}} |\bar{q}v - \hat{d}| \left( \frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}} - u \right)^s \left( \frac{\hat{d} - v}{\hat{d} - \hat{c}} \right)^t {}_c d_{\bar{q}} v {}_a d_{\bar{q}} u \\ &= \left( \frac{\hat{b} - \hat{a}}{1 + \bar{q}} \right) (\hat{d} - \hat{c}) (1 - \bar{q}) (1 - \bar{q}) \\ &\quad \times \sum_{\beta=0}^{\infty} \sum_{\alpha=0}^{\infty} \bar{q}^{\alpha} \bar{q}^{\beta} (1 - \bar{q}^{\alpha})^s (\hat{d} - \bar{q}\hat{c} - \bar{q}^{\beta+1}(\hat{d} - \hat{c})) (1 - \bar{q}^{\beta})^t; \end{aligned}$$

(x)

$$\begin{aligned}
\mathcal{H}_2 &:= \int_{\hat{a}}^{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}} \int_{\hat{c}}^{\hat{d}} |\check{q}v - \hat{d}| \left( \frac{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}} - u}{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}} - \hat{a}} \right)^s \left( \frac{v - \hat{c}}{\hat{d} - \hat{c}} \right)^t \check{c}d_{\check{q}}v \hat{a}d_{\bar{q}}u \\
&= \left( \frac{\hat{b} - \hat{a}}{1 + \bar{q}} \right) (\hat{d} - \hat{c}) (1 - \check{q}) (1 - \bar{q}) \\
&\quad \times \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} \bar{q}^{\alpha} (1 - \bar{q}^{\alpha})^s \check{q}^{(1+s_2)\beta} (\hat{d} - \check{q}\hat{c} - \check{q}^{\beta+1} (\hat{d} - \hat{c}));
\end{aligned}$$

(xi)

$$\begin{aligned}
\mathcal{H}_3 &:= \int_{\hat{a}}^{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}} \int_{\hat{c}}^{\hat{d}} |\check{q}v - \hat{d}| \left( \frac{u - \hat{a}}{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}} - \hat{a}} \right)^s \left( \frac{\hat{d} - v}{\hat{d} - \hat{c}} \right)^t \check{c}d_{\check{q}}v \hat{a}d_{\bar{q}}u \\
&= \left( \frac{\hat{b} - \hat{a}}{1 + \bar{q}} \right) (\hat{d} - \hat{c}) (1 - \bar{q}) (1 - \check{q}) \\
&\quad \times \sum_{\beta=0}^{\infty} \sum_{\alpha=0}^{\infty} \bar{q}^{(1+s_1)\alpha} \check{q}^{\beta} (\hat{d} - \check{q}\hat{c} - \check{q}^{\beta+1} (\hat{d} - \hat{c})) (1 - \check{q}^{\beta})^t;
\end{aligned}$$

(xii)

$$\begin{aligned}
\mathcal{H}_4 &:= \int_{\hat{a}}^{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}} \int_{\hat{c}}^{\hat{d}} |\check{q}v - \hat{d}| \left( \frac{u - \hat{a}}{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}} - \hat{a}} \right)^s \left( \frac{v - \hat{c}}{\hat{d} - \hat{c}} \right)^t \check{c}d_{\check{q}}v \hat{a}d_{\bar{q}}u \\
&= \left( \frac{\hat{b} - \hat{a}}{1 + \bar{q}} \right) (\hat{d} - \hat{c}) (1 - \bar{q}) (1 - \check{q}) \\
&\quad \times \sum_{\beta=0}^{\infty} \sum_{\alpha=0}^{\infty} \bar{q}^{(1+s_1)\alpha} \check{q}^{(1+s_2)\beta} (\hat{d} - \check{q}\hat{c} - \check{q}^{\beta+1} (\hat{d} - \hat{c}));
\end{aligned}$$

(xiii)

$$\begin{aligned}
\mathcal{J}_1 &:= \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}} |\bar{q}u - \hat{b}| \left( \frac{\hat{b} - u}{\hat{b} - \hat{a}} \right)^s \left( \frac{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}} - v}{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}} - \hat{c}} \right)^t \check{c}d_{\check{q}}v \hat{a}d_{\bar{q}}u \\
&= (\hat{b} - \hat{a}) \left( \frac{\hat{d} - \hat{c}}{1 + \check{q}} \right) (1 - \check{q}) (1 - \bar{q}) \\
&\quad \times \sum_{\beta=0}^{\infty} \sum_{\alpha=0}^{\infty} \bar{q}^{\alpha} \check{q}^{\beta} \left( \hat{b} - \bar{q}\hat{a} - \bar{q}^{\alpha+1} \left( \frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}} - \hat{a} \right) \right) (1 - \bar{q}^{\alpha})^s (1 - \check{q}^{\beta})^t;
\end{aligned}$$

(xiv)

$$\begin{aligned}
\mathcal{J}_2 &:= \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\frac{\check{q}\hat{c}+\hat{d}}{1+\check{q}}} |\bar{q}u - \hat{b}| \left( \frac{\hat{b} - u}{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}} - \hat{a}} \right)^s \left( \frac{v - \hat{c}}{\frac{\check{q}\hat{c}+\hat{d}}{1+\check{q}} - \hat{c}} \right)^t \varepsilon d_{\bar{q}}v \hat{a} d_{\bar{q}}u \\
&= (\hat{b} - \hat{a}) \left( \frac{\hat{d} - \hat{c}}{1 + \check{q}} \right) (1 - \check{q}) (1 - \bar{q}) \\
&\quad \times \sum_{\beta=0}^{\infty} \sum_{\alpha=0}^{\infty} \bar{q}^{\alpha} \check{q}^{(1+s_2)\beta} \left( \hat{b} - \bar{q}\hat{a} - \bar{q}^{\alpha+1} \left( \frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}} - \hat{a} \right) \right) (1 - \bar{q}^{\alpha})^s;
\end{aligned}$$

(xv)

$$\begin{aligned}
\mathcal{J}_3 &:= \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\frac{\check{q}\hat{c}+\hat{d}}{1+\check{q}}} |\bar{q}u - \hat{b}| \left( \frac{u - \hat{a}}{\hat{b} - \hat{a}} \right)^s \left( \frac{\frac{\check{q}\hat{c}+\hat{d}}{1+\check{q}} - v}{\frac{\check{q}\hat{c}+\hat{d}}{1+\check{q}} - \hat{c}} \right)^t \varepsilon d_{\bar{q}}v \hat{a} d_{\bar{q}}u \\
&= (\hat{b} - \hat{a}) \left( \frac{\hat{d} - \hat{c}}{1 + \check{q}} \right) (1 - \check{q}) (1 - \bar{q}) \\
&\quad \times \sum_{\beta=0}^{\infty} \sum_{\alpha=0}^{\infty} \bar{q}^{(1+s_1)\alpha} \check{q}^{\beta} \left( \hat{b} - \bar{q}\hat{a} - \bar{q}^{\alpha+1} \left( \frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}} - \hat{a} \right) \right) (1 - \check{q}^{\beta})^t,
\end{aligned}$$

(xvi)

$$\begin{aligned}
\mathcal{J}_4 &:= \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\frac{\check{q}\hat{c}+\hat{d}}{1+\check{q}}} |\bar{q}u - \hat{b}| \left( \frac{u - \hat{a}}{\hat{b} - \hat{a}} \right)^s \left( \frac{v - \hat{c}}{\frac{\check{q}\hat{c}+\hat{d}}{1+\check{q}} - \hat{c}} \right)^t \varepsilon d_{\bar{q}}v \hat{a} d_{\bar{q}}u \\
&= (\hat{b} - \hat{a}) \left( \frac{\hat{d} - \hat{c}}{1 + \check{q}} \right) (1 - \check{q}) (1 - \bar{q}) \\
&\quad \times \sum_{\beta=0}^{\infty} \sum_{\alpha=0}^{\infty} \bar{q}^{(1+s_1)\alpha} \check{q}^{(1+s_2)\beta} \left( \hat{b} - \bar{q}\hat{a} - \bar{q}^{\alpha+1} \left( \frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}} - \hat{a} \right) \right).
\end{aligned}$$

**Remark 3.8.** It is worth mentioning that all the integrals listed above are real and finite over the specified domain.

The above mentioned integrals can be evaluated for fixed values of  $\hat{a}, \hat{b}, \hat{c}, \hat{d}, s$  and  $t$  as evaluated in Remark 3.2 just by putting values of parameters in the integrals.

Now, we present the following general and useful inequality to give new error estimate.

**Theorem 3.9.** Let  $(s, t) \in (0, 1]^2$ . Assume that the function  $\mathcal{F} : \Xi \rightarrow \mathbb{R}$  is a twice partial  $\bar{q}_a \check{q}_c$ -differentiable on  $\Xi^\circ$  such that the partial  $\bar{q}_a \check{q}_c$ -derivative  $\frac{\partial^2 \mathcal{F}(u, v)}{\bar{a} \partial_{\bar{q}} u \check{c} \partial_{\check{q}} v}$  is continuous and  $\bar{q}_a \check{q}_c$ -integrable over  $\Xi$ . If  $\left| \frac{\partial^2 \mathcal{F}(u, v)}{\bar{a} \partial_{\bar{q}} u \check{c} \partial_{\check{q}} v} \right|^\omega$  is coordinated  $(s, t)$ -convex of second kind on  $\Xi$  for some  $\omega \geq 1$ , then the following inequality holds:

(3.14)

$$\begin{aligned} & \left| \frac{1}{(\hat{b} - \hat{a})(\hat{d} - \hat{c})} \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} \mathcal{F}(x, y) {}_{\check{c}}d_{\check{q}} y {}_{\hat{a}}d_{\bar{q}} x + \mathcal{F}\left(\frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}, \frac{\check{q}\hat{c} + \hat{d}}{1 + \check{q}}\right) - \mathcal{V}_* \right| \\ & \leq \frac{1}{(\hat{b} - \hat{a})(\hat{d} - \hat{c})} \left[ \left( \frac{(\hat{b} - \hat{a})(\hat{b} - \bar{q}^2 \hat{a})(\hat{d} - \hat{c})(\hat{d} - \check{q}^2 \hat{c})}{(1 + \bar{q})(1 + \check{q})} \right)^{1 - \frac{1}{\omega}} (\mathcal{C}^*)^{\frac{1}{\omega}} \right. \\ & \quad + (\hat{b} - \hat{a})(\hat{d} - \hat{c}) \left( \frac{\hat{b} - \hat{a} \hat{d} - \hat{c}}{1 + \bar{q} \ 1 + \check{q}} \right)^{1 - \frac{1}{\omega}} (\mathcal{D}^*)^{\frac{1}{\omega}} \\ & \quad + (\hat{b} - \hat{a}) \left( \frac{(\hat{b} - \hat{a})(\hat{d} - \hat{c})(\hat{d} - \check{q}^2 \hat{c})}{(1 + \bar{q})(1 + \check{q})} \right)^{1 - \frac{1}{\omega}} (\mathcal{H}^*)^{\frac{1}{\omega}} \\ & \quad \left. + (\hat{d} - \hat{c}) \left( \frac{(\hat{d} - \hat{c})(\hat{b} - \hat{a})(\hat{b} - \bar{q}^2 \hat{a})}{(1 + \check{q})(1 + \bar{q})} \right)^{1 - \frac{1}{\omega}} (\mathcal{J}^*)^{\frac{1}{\omega}} \right], \end{aligned}$$

where

$$\begin{aligned} \mathcal{C}^* & := \left| \frac{\partial^2 \mathcal{F}(\hat{a}, c)}{\bar{a} \partial_{\bar{q}} u \check{c} \partial_{\check{q}} v} \right|^\omega \mathcal{C}_1 + \left| \frac{\partial^2 \mathcal{F}(\hat{a}, \hat{d})}{\bar{a} \partial_{\bar{q}} u \check{c} \partial_{\check{q}} v} \right|^\omega \mathcal{C}_2 + \left| \frac{\partial^2 \mathcal{F}(\hat{b}, c)}{\bar{a} \partial_{\bar{q}} u \check{c} \partial_{\check{q}} v} \right|^\omega \mathcal{C}_3 \\ & \quad + \left| \frac{\partial^2 \mathcal{F}(\hat{b}, \hat{d})}{\bar{a} \partial_{\bar{q}} u \check{c} \partial_{\check{q}} v} \right|^\omega \mathcal{C}_4; \\ \mathcal{D}^* & := \left| \frac{\partial^2 \mathcal{F}(\hat{a}, \hat{c})}{\bar{a} \partial_{\bar{q}} u \check{c} \partial_{\check{q}} v} \right|^\omega \mathcal{D}_1 + \left| \frac{\partial^2 \mathcal{F}(\hat{a}, \frac{\check{q}\hat{c} + \hat{d}}{1 + \check{q}})}{\bar{a} \partial_{\bar{q}} u \check{c} \partial_{\check{q}} v} \right|^\omega \mathcal{D}_2 + \left| \frac{\partial^2 \mathcal{F}(\frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}, \hat{c})}{\bar{a} \partial_{\bar{q}} u \check{c} \partial_{\check{q}} v} \right|^\omega \mathcal{D}_3 \\ & \quad + \left| \frac{\partial^2 \mathcal{F}(\frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}, \frac{\check{q}\hat{c} + \hat{d}}{1 + \check{q}})}{\bar{a} \partial_{\bar{q}} u \check{c} \partial_{\check{q}} v} \right|^\omega \mathcal{D}_4; \end{aligned}$$

$$\begin{aligned}
\mathcal{H}^* &:= \left| \frac{\partial^2 \mathcal{F}(\hat{a}, \hat{c})}{\hat{a} \partial_{\bar{q}} u \hat{c} \partial_{\bar{q}} v} \right|^\omega \mathcal{H}_1 + \left| \frac{\partial^2 \mathcal{F}(\hat{a}, \hat{d})}{\hat{a} \partial_{\bar{q}} u \hat{c} \partial_{\bar{q}} v} \right|^\omega \mathcal{H}_2 + \left| \frac{\partial^2 \mathcal{F}\left(\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}, \hat{c}\right)}{\hat{a} \partial_{\bar{q}} u \hat{c} \partial_{\bar{q}} v} \right|^\omega \mathcal{H}_3 \\
&\quad + \left| \frac{\partial^2 \mathcal{F}\left(\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}, \hat{d}\right)}{\hat{a} \partial_{\bar{q}} u \hat{c} \partial_{\bar{q}} v} \right|^\omega \mathcal{H}_4; \\
\mathcal{J}^* &:= \left| \frac{\partial^2 \mathcal{F}(\hat{a}, \hat{c})}{\hat{a} \partial_{\bar{q}} u \hat{c} \partial_{\bar{q}} v} \right|^\omega \mathcal{J}_1 + \left| \frac{\partial^2 \mathcal{F}\left(\hat{a}, \frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}\right)}{\hat{a} \partial_{\bar{q}} u \hat{c} \partial_{\bar{q}} v} \right|^\omega \mathcal{J}_2 + \left| \frac{\partial^2 \mathcal{F}(\hat{b}, \hat{c})}{\hat{a} \partial_{\bar{q}} u \hat{c} \partial_{\bar{q}} v} \right|^\omega \mathcal{J}_3 \\
&\quad + \left| \frac{\partial^2 \mathcal{F}\left(\hat{b}, \frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}\right)}{\hat{a} \partial_{\bar{q}} u \hat{c} \partial_{\bar{q}} v} \right|^\omega \mathcal{J}_4.
\end{aligned}$$

*Proof.* By considering the identity (3.3) and applying the properties of the modulus, we have

$$\begin{aligned}
(3.15) \quad & \left| \frac{1}{(\hat{b}-\hat{a})(\hat{d}-\hat{c})} \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} \mathcal{F}(x, y) \hat{c} d_{\bar{q}} y \hat{a} d_{\bar{q}} x + \mathcal{U} - \mathcal{V} \right| \\
& \leq \frac{1}{(\hat{b}-\hat{a})(\hat{d}-\hat{c})} \left[ \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} \left| (\bar{q}u - \hat{b})(\bar{q}v - \hat{d}) \frac{\partial^2 \mathcal{F}(u, v)}{\hat{a} \partial_{\bar{q}} u \hat{c} \partial_{\bar{q}} v} \right| \hat{c} d_{\bar{q}} v \hat{a} d_{\bar{q}} u \right. \\
& \quad + (\hat{b}-\hat{a})(\hat{d}-\hat{c}) \int_{\hat{a}}^{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}} \int_{\hat{c}}^{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}} \left| \frac{\partial^2 \mathcal{F}(u, v)}{\hat{a} \partial_{\bar{q}} u \hat{c} \partial_{\bar{q}} v} \right| \hat{c} d_{\bar{q}} v \hat{a} d_{\bar{q}} u \\
& \quad + (\hat{b}-\hat{a}) \int_{\hat{a}}^{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}} \int_{\hat{c}}^{\hat{d}} \left| (\bar{q}v - \hat{d}) \frac{\partial^2 \mathcal{F}(u, v)}{\hat{a} \partial_{\bar{q}} u \hat{c} \partial_{\bar{q}} v} \right| \hat{c} d_{\bar{q}} v \hat{a} d_{\bar{q}} u \\
& \quad \left. + (\hat{d}-\hat{c}) \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}} \left| (\bar{q}u - \hat{b}) \frac{\partial^2 \mathcal{F}(u, v)}{\hat{a} \partial_{\bar{q}} u \hat{c} \partial_{\bar{q}} v} \right| \hat{c} d_{\bar{q}} v \hat{a} d_{\bar{q}} u \right].
\end{aligned}$$

Utilizing the identity mappings given in (3.10) and applying the indicated coordinated convexity of  $\left| \frac{\partial^2 \mathcal{F}(u, v)}{\hat{a} \partial_{\bar{q}} u \hat{c} \partial_{\bar{q}} v} \right|^\omega$ , we have

$$\begin{aligned}
 & \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} |\bar{q}u - \hat{b}| |\check{q}v - \hat{d}| \left| \frac{\partial^2 \mathcal{F} \left( m \mathcal{I}_{\hat{a}}^{\hat{b}}(u), m \mathcal{I}_{\hat{c}}^{\hat{d}}(v) \right)}{\hat{a} \partial_{\bar{q}} u \hat{c} \partial_{\check{q}} v} \right|^{\omega} {}_{\hat{c}} d_{\check{q}} v {}_{\hat{a}} d_{\bar{q}} u \\
 & \leq \left| \frac{\partial^2 \mathcal{F}(\hat{a}, c)}{\hat{a} \partial_{\bar{q}} u \hat{c} \partial_{\check{q}} v} \right|^{\omega} \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} |\bar{q}u - \hat{b}| |\check{q}v - \hat{d}| \left( \frac{\hat{b} - u}{\hat{b} - \hat{a}} \right)^s \left( \frac{\hat{d} - v}{\hat{d} - \hat{c}} \right)^t {}_{\hat{c}} d_{\check{q}} v {}_{\hat{a}} d_{\bar{q}} u \\
 & \quad + \left| \frac{\partial^2 \mathcal{F}(\hat{a}, \hat{d})}{\hat{a} \partial_{\bar{q}} u \hat{c} \partial_{\check{q}} v} \right|^{\omega} \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} |\bar{q}u - \hat{b}| |\check{q}v - \hat{d}| \left( \frac{\hat{b} - u}{\hat{b} - \hat{a}} \right)^s \left( \frac{v - \hat{c}}{\hat{d} - \hat{c}} \right)^t {}_{\hat{c}} d_{\check{q}} v {}_{\hat{a}} d_{\bar{q}} u \\
 & \quad + \left| \frac{\partial^2 \mathcal{F}(\hat{b}, c)}{\hat{a} \partial_{\bar{q}} u \hat{c} \partial_{\check{q}} v} \right|^{\omega} \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} |\bar{q}u - \hat{b}| |\check{q}v - \hat{d}| \left( \frac{u - \hat{a}}{\hat{b} - \hat{a}} \right)^s \left( \frac{\hat{d} - v}{\hat{d} - \hat{c}} \right)^t {}_{\hat{c}} d_{\check{q}} v {}_{\hat{a}} d_{\bar{q}} u \\
 & \quad + \left| \frac{\partial^2 \mathcal{F}(\hat{b}, \hat{d})}{\hat{a} \partial_{\bar{q}} u \hat{c} \partial_{\check{q}} v} \right|^{\omega} \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} |\bar{q}u - \hat{b}| |\check{q}v - \hat{d}| \left( \frac{u - \hat{a}}{\hat{b} - \hat{a}} \right)^s \left( \frac{v - \hat{c}}{\hat{d} - \hat{c}} \right)^t {}_{\hat{c}} d_{\check{q}} v {}_{\hat{a}} d_{\bar{q}} u.
 \end{aligned}$$

The above inequality yields,

$$(3.16) \quad \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} |\bar{q}u - \hat{b}| |\check{q}v - \hat{d}| \left| \frac{\partial^2 \mathcal{F}(u, v)}{\hat{a} \partial_{\bar{q}} u \hat{c} \partial_{\check{q}} v} \right|^{\omega} {}_{\hat{c}} d_{\check{q}} v {}_{\hat{a}} d_{\bar{q}} u \leq C^*.$$

By the application of power mean inequality and inequality (3.16), we have

$$\begin{aligned}
 (3.17) \quad & \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} |\bar{q}u - \hat{b}| |\check{q}v - \hat{d}| \left| \frac{\partial^2 \mathcal{F}(u, v)}{\hat{a} \partial_{\bar{q}} u \hat{c} \partial_{\check{q}} v} \right|^{\omega} {}_{\hat{c}} d_{\check{q}} v {}_{\hat{a}} d_{\bar{q}} u \\
 & \leq \left( \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} |\bar{q}u - \hat{b}| |\check{q}v - \hat{d}| {}_{\hat{c}} d_{\check{q}} v {}_{\hat{a}} d_{\bar{q}} u \right)^{1 - \frac{1}{\omega}} \\
 & \quad \times \left( \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} |\bar{q}u - \hat{b}| |\check{q}v - \hat{d}| \left| \frac{\partial^2 \mathcal{F}(u, v)}{\hat{a} \partial_{\bar{q}} u \hat{c} \partial_{\check{q}} v} \right|^{\omega} {}_{\hat{c}} d_{\check{q}} v {}_{\hat{a}} d_{\bar{q}} u \right)^{\frac{1}{\omega}} \\
 & \leq \left( \frac{(\hat{b} - \hat{a})(\hat{b} - \bar{q}^2 \hat{a})(\hat{d} - \hat{c})(\hat{d} - \check{q}^2 \hat{c})}{(1 + \bar{q})(1 + \check{q})} \right)^{1 - \frac{1}{\omega}} (C^*)^{\frac{1}{\omega}}.
 \end{aligned}$$

Considering the rectangles  $[\hat{a}, \frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}] \times [\hat{c}, \frac{\check{q}\hat{c} + \hat{d}}{1 + \check{q}}]$ ,  $[\hat{a}, \frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}] \times [\hat{c}, \hat{d}]$  and  $[\hat{a}, \hat{b}] \times [\hat{c}, \frac{\check{q}\hat{c} + \hat{d}}{1 + \check{q}}]$  and applying the indicated convexity of  $\left| \frac{\partial^2 \mathcal{F}(u, v)}{\hat{a} \partial_{\bar{q}} u \hat{c} \partial_{\check{q}} v} \right|^{\omega}$ ,

we obtain

$$(3.18) \quad \int_{\hat{a}}^{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}} \int_{\hat{c}}^{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}} \left| \frac{\partial^2 \mathcal{F}(u, v)}{\hat{a} \partial_{\bar{q}} u \hat{c} \partial_{\bar{q}} v} \right| \hat{c} d_{\bar{q}} v \hat{a} d_{\bar{q}} u \leq \left( \frac{\hat{b} - \hat{a} \hat{d} - \hat{c}}{1 + \bar{q} \ 1 + \bar{q}} \right)^{1-\frac{1}{\omega}} (\mathcal{D}^*)^{\frac{1}{\omega}},$$

and

$$(3.19) \quad \int_{\hat{a}}^{\frac{\bar{q}\hat{a}+\hat{b}}{1+\bar{q}}} \int_{\hat{c}}^{\hat{d}} \left| \hat{q}v - \hat{d} \right| \left| \frac{\partial^2 \mathcal{F}(u, v)}{\hat{a} \partial_{\bar{q}} u \hat{c} \partial_{\bar{q}} v} \right| \hat{c} d_{\bar{q}} v \hat{a} d_{\bar{q}} u \\ \leq \left( \frac{(\hat{b} - \hat{a})(\hat{d} - \hat{c})(\hat{d} - \bar{q}^2 \hat{c})}{(1 + \bar{q})(1 + \bar{q})} \right)^{1-\frac{1}{\omega}} (\mathcal{H}^*)^{\frac{1}{\omega}}$$

also

$$(3.20) \quad \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\frac{\bar{q}\hat{c}+\hat{d}}{1+\bar{q}}} \left| \bar{q}u - \hat{b} \right| \left| \frac{\partial^2 \mathcal{F}(u, v)}{\hat{a} \partial_{\bar{q}} u \hat{c} \partial_{\bar{q}} v} \right| \hat{c} d_{\bar{q}} v \hat{a} d_{\bar{q}} u \\ \leq \left( \frac{(\hat{d} - \hat{c})(\hat{b} - \hat{a})(\hat{b} - \bar{q}^2 \hat{a})}{(1 + \bar{q})(1 + \bar{q})} \right)^{1-\frac{1}{\omega}} (\mathcal{J}^*)^{\frac{1}{\omega}}.$$

Making use of inequalities (3.17)–(3.20) in (3.15), we get the desired inequality (3.14).  $\square$

**Remark 3.10.** It is pertinent to mention that the inequality (3.14) is a general and better inequality. This inequality involve no kernel where the function is discontinuous.

Now, we present following results that are deduced from Theorem 3.9:

**Corollary 3.11.** *Under the conditions of Theorem 3.9, and with the additional assumption that  $s = t = \omega = 1$ , then*

$$(3.21)$$

$$\left| \frac{1}{(\hat{b} - \hat{a})(\hat{d} - \hat{c})} \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} \mathcal{F}(x, y) \hat{c} d_{\bar{q}} y \hat{a} d_{\bar{q}} x + \mathcal{F} \left( \frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}, \frac{\bar{q}\hat{c} + \hat{d}}{1 + \bar{q}} \right) - \mathcal{V}_* \right| \\ \leq \frac{1}{(\bar{q} + 1)(\bar{q} + 1)(\bar{q}^2 + \bar{q} + 1)(\bar{q}^2 + \bar{q} + 1)}$$

$$\begin{aligned}
 & \times \left[ \frac{\bar{q}\check{q} \left( \hat{a} - 2\hat{b} + 2\hat{a}\bar{q} - 3\hat{b}\bar{q} + 2\hat{a}\bar{q}^2 + \hat{a}\bar{q}^3 + \hat{a}\bar{q}^4 - 2\hat{b}\bar{q}^2 \right) \times \left( c - 2d + 2c\check{q} - 3d\check{q} + 2c\check{q}^2 + c\check{q}^3 + c\check{q}^4 - 2d\check{q}^2 \right)}{(1 + \bar{q})(1 + \check{q})} \right] \\
 & \times \left| \frac{\partial^2 \mathcal{F}(\hat{a}, \hat{c})}{\hat{a}\partial_{\bar{q}}u \ \check{c}\partial_{\check{q}}v} \right| \\
 & + \left[ \frac{-\bar{q}(d - c\check{q}^3) \left( a - 2b + 2a\bar{q} - 3b\bar{q} + 2a\bar{q}^2 + a\bar{q}^3 + a\bar{q}^4 - 2b\bar{q}^2 \right)}{(\bar{q} + 1)} \right] \left| \frac{\partial^2 \mathcal{F}(\hat{a}, \hat{d})}{\hat{a}\partial_{\bar{q}}u \ \check{c}\partial_{\check{q}}v} \right| \\
 & + \left[ \frac{-\check{q}(\hat{b} - \hat{a}\bar{q}^3) \left( \hat{c} - 2\hat{d} + 2\hat{c}\check{q} - 3\hat{d}\check{q} + 2\hat{c}\check{q}^2 + \hat{c}\check{q}^3 + \hat{c}\check{q}^4 - 2\hat{d}\check{q}^2 \right)}{(\check{q} + 1)} \right] \left| \frac{\partial^2 \mathcal{F}(\hat{b}, \hat{c})}{\hat{a}\partial_{\bar{q}}u \ \check{c}\partial_{\check{q}}v} \right| \\
 & + (\hat{b} - \hat{a}\bar{q}^3) (\hat{d} - \hat{c}\check{q}^3) \left| \frac{\partial^2 \mathcal{F}(\hat{b}, \hat{d})}{\hat{a}\partial_{\bar{q}}u \ \check{c}\partial_{\check{q}}v} \right| \\
 & + \left[ \frac{-\bar{q}(\check{q}^2 + \check{q} + 1) (\hat{d} - \hat{c}) \left( \hat{a} - 2\hat{b} + 2\hat{a}\bar{q} - 3\hat{b}\bar{q} + 2\hat{a}\bar{q}^2 + \hat{a}\bar{q}^3 + \hat{a}\bar{q}^4 - 2\hat{b}\bar{q}^2 \right)}{(\bar{q} + 1)(\check{q} + 1)} \right] \left| \frac{\partial^2 \mathcal{F}\left(\hat{a}, \frac{\check{q}\hat{c} + \hat{d}}{1 + \check{q}}\right)}{\hat{a}\partial_{\bar{q}}u \ \check{c}\partial_{\check{q}}v} \right| \\
 & + \left[ \frac{-\check{q}(\bar{q}^2 + \bar{q} + 1) (\hat{b} - \hat{a}) \left( \hat{c} - 2\hat{d} + 2\hat{c}\check{q} - 3\hat{d}\check{q} + 2\hat{c}\check{q}^2 + \hat{c}\check{q}^3 + \hat{c}\check{q}^4 - 2\hat{d}\check{q}^2 \right)}{(\bar{q} + 1)(\check{q} + 1)} \right] \left| \frac{\partial^2 \mathcal{F}\left(\frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}, \hat{c}\right)}{\hat{a}\partial_{\bar{q}}u \ \check{c}\partial_{\check{q}}v} \right| \\
 & + \frac{(\bar{q}^2 + \bar{q} + 1) (\hat{b} - \hat{a}) (\hat{d} - \hat{c}\check{q}^3)}{(\bar{q} + 1)} \left| \frac{\partial^2 \mathcal{F}\left(\frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}, \hat{d}\right)}{\hat{a}\partial_{\bar{q}}u \ \check{c}\partial_{\check{q}}v} \right| \\
 & + \frac{(\check{q}^2 + \check{q} + 1) (\hat{b} - \hat{a}\bar{q}^3) (\hat{d} - \hat{c})}{(\check{q} + 1)} \left| \frac{\partial^2 \mathcal{F}\left(\hat{b}, \frac{\check{q}\hat{c} + \hat{d}}{1 + \check{q}}\right)}{\hat{a}\partial_{\bar{q}}u \ \check{c}\partial_{\check{q}}v} \right|
 \end{aligned}$$

$$\begin{aligned}
& + \frac{(\bar{q}^2 + \bar{q} + 1)(\check{q}^2 + \check{q} + 1)(\hat{b} - \hat{a})(\hat{d} - \hat{c})}{(\bar{q} + 1)(\check{q} + 1)} \left| \frac{\partial^2 \mathcal{F} \left( \frac{\bar{q}\hat{a} + \hat{b}}{1 + \bar{q}}, \frac{\check{q}\hat{c} + \hat{d}}{1 + \check{q}} \right)}{\hat{a}\partial_{\bar{q}}u \ \hat{c}\partial_{\check{q}}v} \right| \Bigg\} \\
\leq & \frac{1}{(\bar{q} + 1)(\check{q} + 1)(\bar{q}^2 + \bar{q} + 1)(\check{q}^2 + \check{q} + 1)} \\
& \times \left[ \frac{\begin{aligned} & \bar{q}\check{q} \left( 2\hat{a} - 3\hat{b} + 4\hat{a}\bar{q} - 6\hat{b}\bar{q} + 5\hat{a}\bar{q}^2 + 3\hat{a}\bar{q}^3 + 2\hat{a}\bar{q}^4 \right. \\ & \left. - 6\hat{b}\bar{q}^2 + \hat{a}\bar{q}^5 - 2\hat{b}\bar{q}^3 \right) \left( 2\hat{c} - 3\hat{d} + 4\hat{c}\check{q} - 6\hat{d}\check{q} \right. \\ & \left. + 5\hat{c}\check{q}^2 + 3\hat{c}\check{q}^3 + 2\hat{c}\check{q}^4 - 6\hat{d}\check{q}^2 + \hat{c}\check{q}^5 - 2\hat{d}\check{q}^3 \right) \end{aligned}}{(\bar{q} + 1)^2 (\check{q} + 1)^2} \left| \frac{\partial^2 \mathcal{F}(\hat{a}, \hat{c})}{\hat{a}\partial_{\bar{q}}u \ \hat{c}\partial_{\check{q}}v} \right| \right] \\
& + \left[ \frac{\begin{aligned} & \bar{q} \left( \hat{c} - 2\hat{d} + \hat{c}\check{q} - 3\hat{d}\check{q} + \hat{c}\check{q}^2 + \hat{c}\check{q}^3 + 2\hat{c}\check{q}^4 \right. \\ & \left. - 2\hat{d}\check{q}^2 + \hat{c}\check{q}^5 \right) \left( 2\hat{a} - 3\hat{b} + 4\hat{a}\bar{q} - 6\hat{b}\bar{q} \right. \\ & \left. + 5\hat{a}\bar{q}^2 + 3\hat{a}\bar{q}^3 + 2\hat{a}\bar{q}^4 - 6\hat{b}\bar{q}^2 + \hat{a}\bar{q}^5 - 2\hat{b}\bar{q}^3 \right) \end{aligned}}{(\bar{q} + 1)^2 (\check{q} + 1)^2} \left| \frac{\partial^2 \mathcal{F}(\hat{a}, \hat{d})}{\hat{a}\partial_{\bar{q}}u \ \hat{c}\partial_{\check{q}}v} \right| \right] \\
& + \left[ \frac{\begin{aligned} & \check{q} \left( \hat{a} - 2\hat{b} + \hat{a}\bar{q} - 3\hat{b}\bar{q} + \hat{a}\bar{q}^2 + \hat{a}\bar{q}^3 + 2\hat{a}\bar{q}^4 \right. \\ & \left. - 2\hat{b}\bar{q}^2 + \hat{a}\bar{q}^5 \right) \left( 2\hat{c} - 3\hat{d} + 4\hat{c}\check{q} - 6\hat{d}\check{q} + 5\hat{c}\check{q}^2 \right. \\ & \left. + 3\hat{c}\check{q}^3 + 2\hat{c}\check{q}^4 - 6\hat{d}\check{q}^2 + \hat{c}\check{q}^5 - 2\hat{d}\check{q}^3 \right) \end{aligned}}{(\bar{q} + 1)^2 (\check{q} + 1)^2} \left| \frac{\partial^2 \mathcal{F}(\hat{b}, \hat{c})}{\hat{a}\partial_{\bar{q}}u \ \hat{c}\partial_{\check{q}}v} \right| \right]
\end{aligned}$$

$$+ \left[ \frac{\begin{aligned} & \left( \hat{a} - 2\hat{b} + \hat{a}\bar{q} - 3\hat{b}\bar{q} + \hat{a}\bar{q}^2 + \hat{a}\bar{q}^3 + 2\hat{a}\bar{q}^4 \right. \\ & \left. - 2\hat{b}\bar{q}^2 + \hat{a}\bar{q}^5 \right) \left( \hat{c} - 2\hat{d} + \hat{c}\check{q} - 3\hat{d}\check{q} + \hat{c}\check{q}^2 \right. \\ & \left. + \hat{c}\check{q}^3 + 2\hat{c}\check{q}^4 - 2\hat{d}\check{q}^2 + \hat{c}\check{q}^5 \right) \\ & \left. \right]}{(\bar{q} + 1)^2 (\check{q} + 1)^2} \left| \frac{\partial^2 \mathcal{F}(\hat{b}, \hat{d})}{\hat{a} \partial_{\bar{q}} u \check{c} \partial_{\check{q}} v} \right| \Bigg\}.$$

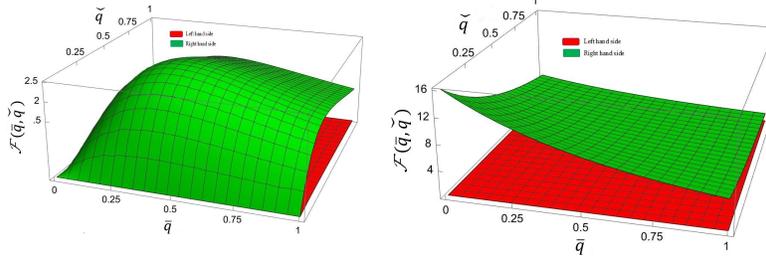
**Corollary 3.12.** *Suppose further that, in addition to the conditions of Theorem 3.9, if  $\bar{q}, \check{q} \rightarrow 1^-$ ,  $\omega = 1$ , then*

(3.22)

$$\begin{aligned} & \left| \frac{1}{(\hat{b} - \hat{a})(\hat{d} - \hat{c})} \int_{\hat{a}}^{\hat{b}} \int_{\hat{c}}^{\hat{d}} \mathcal{F}(x, y) dx dy + \mathcal{F}\left(\frac{\hat{a} + \hat{b}}{2}, \frac{\hat{c} + \hat{d}}{2}\right) - \Psi_* \right| \\ & \leq (\hat{b} - \hat{a})(\hat{d} - \hat{c}) \\ & \quad \times \left\{ \frac{(3s + 4)(3t + 4) |\mathcal{F}_{12}(\hat{a}, \hat{c})|}{4(s^2 + 3s + 2)(t^2 + 3t + 2)} + \frac{(3s + 4) |\mathcal{F}_{12}(\hat{a}, \hat{d})|}{2(s^2 + 3s + 2)(t^2 + 3t + 2)} \right. \\ & \quad + \frac{(3t + 4) |\mathcal{F}_{12}(\hat{b}, \hat{c})|}{2(s^2 + 3s + 2)(t^2 + 3t + 2)} + \frac{|\mathcal{F}_{12}(\hat{b}, \hat{d})|}{(s^2 + 3s + 2)(t^2 + 3t + 2)} \\ & \quad + \frac{(3s + 4) |\mathcal{F}_{12}(\hat{a}, \frac{\hat{c} + \hat{d}}{2})|}{4(t + 1)(s^2 + 3s + 2)} + \frac{(3t + 4) |\mathcal{F}_{12}(\frac{\hat{a} + \hat{b}}{2}, \hat{c})|}{4(s + 1)(t^2 + 3t + 2)} \\ & \quad + \frac{|\mathcal{F}_{12}(\hat{b}, \frac{\hat{c} + \hat{d}}{2})|}{2(s^2 + 3s + 2)(t + 1)} + \frac{|\mathcal{F}_{12}(\frac{\hat{a} + \hat{b}}{2}, \hat{d})|}{2(s + 1)(t^2 + 3t + 2)} + \frac{|\mathcal{F}_{12}(\frac{\hat{a} + \hat{b}}{2}, \frac{\hat{c} + \hat{d}}{2})|}{4(s + 1)(t + 1)} \\ & \leq \frac{(\hat{b} - \hat{a})(\hat{d} - \hat{c})}{(s^2 + 3s + 2)(t^2 + 3t + 2)} \\ & \quad \times \left\{ \frac{2^{s+t}(3s + 4)(3t + 4) + 22s + 22t + 13st + 36}{2^{s+t+2}} |\mathcal{F}_{12}(\hat{a}, \hat{c})| \right. \\ & \quad + \frac{2^{s+t+1}(3s + 4) + 18s + 10t + 7st + 28}{2^{s+t+2}} |\mathcal{F}_{12}(\hat{a}, \hat{d})| \\ & \quad + \frac{2^{s+t+1}(3t + 4) + 10s + 18t + 7st + 28}{2^{s+t+2}} |\mathcal{F}_{12}(\hat{b}, \hat{c})| \\ & \quad \left. + \frac{2^{s+t+2} + 6s + 6t + st + 20}{2^{s+t+2}} |\mathcal{F}_{12}(\hat{b}, \hat{d})| \right\}, \end{aligned}$$

where  $\Psi_*$  is same as in equation (3.14).

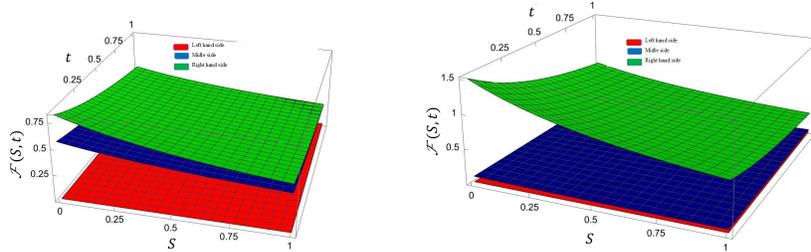
#### 4. DISCUSSION



(A) Graph of inequality (3.12). (B) Graph of inequality (3.21).

FIGURE 1. Graphical illustration of Theorems 3.4 and 3.9

Now, we investigate the behavior of our outcomes given in Theorems 3.4 and 3.9. Consider a function  $\mathcal{F}(x, y) = \frac{1}{4}x^2y^2$  with  $\bar{q} = q_1, \check{q} = q_2 \in (0, 1)$ . Then, the twice quantum partial derivative is  $\left| \frac{\partial^2 \mathcal{F}(x, y)}{\partial \bar{q} \partial \check{q}} \right| = \frac{(1+q_1)(1+q_2)xy}{4}$ , which is coordinated convex. With the aid of this function, we have following plots depicted in Figure 1a and 1b respectively corresponding to the inequalities (3.12) and (3.21). The plots validates the inequalities. It is quite clear that the image 1(A) gives a better estimate and hence the corresponding inequality (3.12) gives least error and hence a better estimate.



(A) Graph of inequality (3.13). (B) Graph of inequality (3.22).

FIGURE 2. Comparison of Theorems 3.4 and 3.9 for the case when  $\bar{q} = \check{q} \rightarrow 1^-$ .

We further investigate the scenario by keeping the function  $\mathcal{F}(x, y) = \frac{x^{1+s}y^{t+1}}{(s+1)(t+1)}$ ,  $(s, t) \in (0, 1]^2$ , where  $q_1, q_2 \rightarrow 1^-$ . The case yielding two surfaces represented by plot Figure 2a for inequality (3.13) and plot Figure

2b for the inequality (3.22). It is easy to figure out that both inequalities are behaving well in certain cases and so are of equal importance. The plot Figure 2b shows that inequality (3.22) provides a better first estimate whereas the inequality (3.13) yields a sharp estimate for the second case, as indicated.

Finally, we note that outcome due to Theorem 3.4 holds true for classical Riemann integrals involving  $(s, t)$ -convex functions or for quantum integrals containing coordinated  $(1, 1)$ -convex functions. However, the outcome exhibits a limiting feature, as it involves functions that are not real-valued in the neighborhood of the point  $a$ . This limitation renders the quantum calculus of Tariboon inapplicable, analogous to the case of Jackson integrals, which require continuity of functions near 0. In contrast, inequality (3.14) due to Theorem 3.9 remains valid for quantum integrals involving coordinated  $(s, t)$ -convex functions. These findings enhance our understanding of quantum calculus and underscore the necessity for a careful approach when employing quantum calculus.

### 5. APPLICATIONS TO SPECIAL MEANS

Let us recall some useful special means of positive real numbers  $\theta_1 < \theta_2$ .

- The arithmetic mean:  $\mathcal{A}(\theta_1, \theta_2) := \frac{\theta_1 + \theta_2}{2}$ .
- The geometric mean:  $\mathcal{G}(\theta_1, \theta_2) = \sqrt{\theta_1 \theta_2}$ .
- The generalized logarithmic mean:  $\mathcal{L}_r(\theta_1, \theta_2) := \left( \frac{\theta_2^{r+1} - \theta_1^{r+1}}{(r+1)(\theta_2 - \theta_1)} \right)^{\frac{1}{r}}$ ,  
 $r \in \mathbb{Z} \setminus \{-1, 0\}$ .

Now, we have the following applications of our results in terms of above special means of different positive real numbers.

**Proposition 5.1.** *Let  $\hat{a}, \hat{b}, \hat{c}, \hat{d} \in \mathbb{R}, 0 < \hat{a} < \hat{b}, 0 < \hat{c} < \hat{d}$  and  $(s, t) \in (0, 1]^2$ . Then, we have*

$$\begin{aligned}
 (5.1) \quad & \left| \left( \frac{\mathcal{L}_{1+s}^{1+s}(a, b) - \mathcal{A}_{1+s}^{1+s}(a, b)}{1+s} \right) \left( \frac{\mathcal{L}_{1+t}^{1+t}(c, d) - \mathcal{A}_{1+t}^{1+t}(\hat{c}, \hat{d})}{1+t} \right) \right| \\
 & \leq \frac{(\hat{b} - \hat{a})(\hat{d} - \hat{c}) \mathcal{A}(a^s, b^s) \mathcal{A}(c^t, d^t)}{4(1+s)(2+s)(1+t)(2+t)} + \frac{(\hat{b} - \hat{a})(\hat{d} - \hat{c})}{4(2+s)(2+t)} \\
 & \quad \times \left[ \frac{\mathcal{A}(a^s, b^s) \mathcal{A}^t(c, d)}{1+s} + \frac{\mathcal{A}^s(a, b) \mathcal{A}(c^t, d^t)}{1+t} + \mathcal{A}^s(a, b) \mathcal{A}^t(c, d) \right]
 \end{aligned}$$

$$\begin{aligned} &\leq \frac{(\hat{b} - \hat{a})(\hat{d} - \hat{c}) \mathcal{A}(a^s, b^s) \mathcal{A}(c^t, d^t)}{4(2+s)(2+t)} \\ &\quad \times \left( \frac{1}{(1+s)(1+t)} + \frac{2^{2-s-t}}{1+s} + \frac{2^{2-s-t}}{1+t} + 2^{2-s-t} \right). \end{aligned}$$

*Proof.* The result is immediate from Corollary 3.7, with

$$\mathcal{F}(x, y) = \frac{x^{1+s}y^{1+t}}{(1+s)(1+t)}. \quad \square$$

**Proposition 5.2.** *Let  $\hat{a}, \hat{b}, \hat{c}, \hat{d} \in \mathbb{R}, 0 < \hat{a} < \hat{b}, 0 < \hat{c} < \hat{d}$  and  $(s, t) \in (0, 1]^2$ . Then, we get*

$$\begin{aligned} (5.2) \quad &\left| \left( \frac{\mathcal{L}_{1+s}^{1+s}(\hat{a}, \hat{b}) - \mathcal{A}_{1+s}^{1+s}(\hat{a}, \hat{b})}{1+s} \right) \left( \frac{\mathcal{L}_{1+t}^{1+t}(\hat{c}, \hat{d}) - \mathcal{A}_{1+t}^{1+t}(\hat{c}, \hat{d})}{1+t} \right) \right| \\ &\leq \frac{(\hat{b} - \hat{a})(\hat{d} - \hat{c})}{(s^2 + 3s + 2)(t^2 + 3t + 2)} \\ &\quad \times \left[ \left( \frac{3}{2}\hat{a}^s s + \hat{a}^s + 2\mathcal{A}(\hat{a}^s, \hat{b}^s) \right) + (s+2)\mathcal{A}^s(\hat{a}, \hat{b}) \right] \\ &\quad \times \left[ \left( \frac{3}{2}\hat{c}^t t + \hat{c}^t + 2\mathcal{A}(\hat{c}^t, \hat{d}^t) \right) + (t+2)\mathcal{A}^t(\hat{c}, \hat{d}) \right] \\ &\leq \frac{(\hat{b} - \hat{a})(\hat{d} - \hat{c})}{(s^2 + 3s + 2)(t^2 + 3t + 2)} \\ &\quad \times \left\{ \frac{2^{s+t}(3s+4)(3t+4) + 22s + 22t + 13st + 36}{2^{s+t+2}} G^2(\hat{a}^s, \hat{c}^t) \right. \\ &\quad + \frac{2^{s+t+1}(3s+4) + 18s + 10t + 7st + 28}{2^{s+t+2}} G^2(\hat{a}^s, \hat{d}^t) \\ &\quad + \frac{2^{s+t+1}(3t+4) + 10s + 18t + 7st + 28}{2^{s+t+2}} G^2(\hat{b}^s, \hat{c}^t) \\ &\quad \left. + \frac{2^{s+t+2} + 6s + 6t + st + 20}{2^{s+t+2}} G^2(\hat{b}^s, \hat{d}^t) \right\}. \end{aligned}$$

*Proof.* Let  $\mathcal{F}(x, y) = \frac{x^{1+s}y^{1+t}}{(1+s)(1+t)}$ . Then desired outcome follows from Corollary 3.12.  $\square$

## 6. CONCLUSION

This investigation has culminated in the establishment of a novel error estimation framework for midpoint type inequalities within the realm of quantum calculus. Our investigation into coordinated  $(s, t)$ -convexity has revealed subtle insights into the intricate relationships between quantum and classical frameworks. We develop inequality (3.9), which interestingly, is found inapplicable in the quantum realm, retains its validity in the context of classical Riemann-integrals. To address this discrepancy, we developed a corrective approach, culminating in an alternative inequality namely (3.14), that harmoniously reconciles the quantum-classical divide. Furthermore, our findings underscore the validity of  $q$ -integrals of Tariboon and Ntouyas [28] for real-valued functions in the neighborhood of a point  $a$ . Through comparative analyses and applications to generalized special means of different positive real numbers, we demonstrate the broad applicability and versatility of our corrected inequality. This study showcases the potential of quantum calculus to offer fresh insights into longstanding problems, and we expect that it will stimulate further investigation and innovation in quantum and post-quantum calculus research. In future, we will try to introduce new quantum derivatives and integrals that remove the requirement of function to be continuous in the entire domain of discourse. We also hope that the results of this study can be addressed for other types of convexity classes. Finally, we feel that researchers working in the field of inequalities should discuss this problem in post-quantum calculus and time scales frameworks.

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