Sahand Communications in Mathematical Analysis (SCMA) Vol. ··· No. ··· (20??), ···-···
http://scma.maragheh.ac.ir
DOI: ...

Erdélyi–Kober Fractional Integral Equations in Applied Sciences: A Fixed Point Perspective

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Abstract. This paper explores a solution method for a nonlinear Erdélyi-Kober type fractional integral equation (NLFIE), leveraging fixed point theory, particularly the Darbo fixed point theory. The equation, with deviating arguments and the Erdélyi-Kober operator, offers insights applicable to diverse scientific domains. Notably, by specializing parameters, it aligns with models describing infectious disease propagation. Additionally, it underscores the utility of Erdélyi-Kober fractional integrals in characterizing media with non-integer mass dimensions, with applications spanning porous media to electrochemistry. This analysis advances our understanding of solving complex nonlinear integral equations, offering interdisciplinary insights with practical implications.

1. Introduction

Integral equations serve as a sophisticated framework for comprehensively characterizing a myriad of physical phenomena, spanning disciplines as diverse as viscoelasticity and electrochemistry. Among the arsenal of available mathematical tools, nonlinear integral equations (NLIEs) emerge as indispensable constructs for effectively modeling many practical problems encountered in these domains. Furthermore, the scope of integral equations broadens substantially with the incorporation of quadratic formulations, which find application in pivotal areas such as radiative transfer, neutron transport, and the kinetic theory of gases.

2020 *Mathematics Subject Classification.* 45G10, 26A33, 47H10.
Key words and phrases. Erdélyi–Kober operator, Nonlinear integral equations, Fixed point theory, Fractional calculus applications
Received: 2025-09-24, Accepted: 2025-10-19.
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A wealth of scholarly literature, exemplified by references [2, 5, 8, 11–22] is dedicated to meticulously examining NLIEs, encompassing various types, including quadratic and fractional integral equations involving Riemann–Liouville fractional integral. This scholarly pursuit delves deeply into the nuanced analysis of solution existence, probing the intricate interplay of mathematical formulations and physical realities.

Earlier studies related to fractional integral equations cited above involve the Riemann–Liouville fractional integral. Few studies related to Erdélyi-Kober operator are witnessed. Erdélyi-Kober fractional integrals are largely used to depict the medium with non-integer mass dimensions. Also, Erdélyi-Kober type fractional integrals are found in porous media, electrochemistry, and viscoelasticity [7, 9, 10]. This leads the findings to be more relevant in terms of physical application and implementation.

In this paper, we have taken the following nonlinear FIE with deviating arguments and containing Erdélyi-Kober operator, in which the indispensability of the Darbo fixed point theory will be highlighted.

(1.1)
$$x(s) = \left(p(s, x(\tilde{a}(s))) + F\left(s, \frac{\alpha}{\Gamma(\beta)} \int_0^s \frac{\zeta^{\alpha-1} u(s, \zeta, x(\tilde{c}(\zeta)))}{(s^\alpha - \zeta^\alpha)^{1-\beta}} d\zeta\right)\right) \times \left(h(s) + g\left(s, x(\tilde{b}(s))\right) + \int_0^s q\left(s, \zeta, x(\tilde{d}(\zeta))\right) d\zeta\right)$$

where Γ denotes the Gamma function, $s \in R_+$, $0 < \alpha < 1$, $\beta > 0$, $h(s)$ is a known function, F, p and g are unknown functions. Also, u, $q : I \times I \times R \to R$ are Lebesgue integrable functions. Moreover $\tilde{a}(s)$, $\tilde{b}(s)$, $\tilde{c}(s)$ and $\tilde{d}(s)$ act continuously from R_+ to R_+.

Particularly for $\alpha = \beta = 1$, if we take

$$\begin{aligned}
&p(s, x(s)) = p(s),\\
&F\left(s, \int_0^s u(s, \zeta, x(\zeta))\, d\zeta\right) = -\int_0^s A(s-\zeta)\, x(\zeta)\, d\zeta,\\
&h(s) + g(s, x(s)) = f(s),\\
&q(s, \zeta, x(\zeta)) = a(s-\zeta)\, x(\zeta).
\end{aligned}$$

We get the following equation of the form

$$x(s) = k\left(p(s) - \int_0^s A(s-\zeta)\, x(\zeta)\, d\zeta\right)\left(f(s) + \int_0^s a(s-\zeta)\, x(\zeta)\, d\zeta\right)$$

that has been studied in [6] which describes the proliferation of an infectious disease where $k > 0$ is a constant and where the functions p and f

take into account the effects of the infection before $s = 0$. Here the quantities $\left(p(s) - \int_0^s A\left(s-\zeta\right) x\left(\zeta\right) d\zeta\right)$ and $\left(f\left(s\right) + \int_0^s a\left(s-\zeta\right) x\left(\zeta\right) d\zeta\right)$ represent number of susceptibles and the total infectivity respectively and $x\left(s\right)$ is the rate at which susceptibles become infected. Also, the equation (1.1) involves the celebrated Chandrasekhar integral equation as a particular case. The article is arranged in the following manner. The second section is dedicated to all preliminary concepts, Section 3 is focused on key outcomes. In Section 4, a few suitable numerical examples are studied for validation purposes, and finally, the article ends in Section 5 with some concluding remarks.

2. Preliminary Concepts

In this section, we elucidate the fundamental ideas that connect solvability criteria with the nature of solutions for the Erdélyi–Kober (EK) type integral-differential equation (IDE). We consider an equation of type (1.1), formulated with this operator to capture the interaction dynamics.

Definition 2.1. The Erdélyi–Kober (EK) integral operator $I_\alpha^{\mu,\beta}$, where $\alpha > 0$, $\beta > 0$, and $\mu \in \mathbb{R}$, for a sufficiently regular function $\varphi(t)$, is defined by

$$I_\alpha^{\mu,\beta}\varphi\left(t\right) = \frac{\alpha}{\Gamma\left(\beta\right)} t^{-\alpha(\beta+\mu)} \int_0^t \frac{\tau^{\alpha(\mu+1)-1}\varphi(\tau)}{\left(t^\alpha - \tau^\alpha\right)^{1-\beta}} d\tau.$$

In particular, when $\mu = 0$,

$$I_\alpha^{0,\beta}\varphi\left(t\right) = \frac{\alpha}{\Gamma\left(\beta\right)} t^{-\alpha\beta} \int_0^t \frac{\tau^{\alpha-1}\varphi(\tau)}{\left(t^\alpha - \tau^\alpha\right)^{1-\beta}} d\tau,$$

or equivalently,

$$t^{\alpha\beta} I_\alpha^{0,\beta}\varphi\left(t\right) = \frac{\alpha}{\Gamma\left(\beta\right)} \int_0^t \frac{\tau^{\alpha-1}\varphi(\tau)}{\left(t^\alpha - \tau^\alpha\right)^{1-\beta}} d\tau.$$

Lemma 2.2. *Suppose $g : \mathbb{R}_+ \to \mathbb{R}_+$ is concave with $g(0) = 0$. Then g is subadditive for any $r_1, r_2 \in \mathbb{R}_+$, i.e.,*

$$g\left(r_1 + r_2\right) \leq g\left(r_1\right) + g(r_2).$$

Lemma 2.3. *Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $g\left(r\right) = r^\alpha$. If $0 < \alpha < 1$ and $r_2 > r_1 \geq 0$, then*

$$r_2^\alpha - r_1^\alpha \leq (r_2 - r_1)^\alpha.$$

Proof. Since $g''\left(r\right) = \alpha(\alpha-1)r^{\alpha-2} < 0$ for $r > 0$, the function g is concave. By Lemma 2.2, g is subadditive, and the stated inequality follows. □

To discuss the existence of a solution to the integral equation (1.1), we use the technique of the measure of noncompactness satisfying condition (m), which ensures solvability of operator equations in Banach algebras. This criterion was first implemented in [4] for the Hausdorff measure of noncompactness in the Banach algebra $C(I)$.

We consider an infinite-dimensional Banach space $(E, \|\cdot\|)$ with zero element denoted by θ'. The closure and convex closure of a subset $Z \subset E$ are denoted by $\overline{Z}$ and Conv$Z$, respectively. Let $B(x,r)$ be the closed ball centered at x with radius r, and write $B_r := B(\theta', r)$. Denote by $\mathcal{S}_E$ the family of all nonempty bounded subsets of E, and by $\mathcal{W}_E \subset \mathcal{S}_E$ the subfamily of all nonempty relatively compact subsets.

Definition 2.4. A mapping $\nu : \mathcal{S}_E \to \mathbb{R}_+$ is called a *measure of noncompactness* in E if the following hold:

(i) $\ker \nu := \{Z \in \mathcal{S}_E : \nu(Z) = 0\}$ is nonempty and $\ker \nu \subset \mathcal{W}_E$.
(ii) If $Z \subset N$, then $\nu(Z) \le \nu(N)$.
(iii) $\nu\left(\overline{Z}\right) = \nu(Z)$.
(iv) $\nu(\text{Conv}Z) = \nu(Z)$.
(v) $\nu(\lambda Z + (1-\lambda) Z) \le \lambda \nu(Z) + (1-\lambda)\nu(Z)$ for all $\lambda \in [0,1]$.
(vi) If $Z_{n+1} \subset Z_n$ for $n \in \mathbb{N}$, each Z_n is closed and $\lim_{n\to\infty} \nu(Z_n) = 0$, then $Z_\infty := \cap_{n=1}^{\infty} Z_n$ is nonempty.

We work in the Banach algebra $BC(\mathbb{R}_+)$ with the norm $\|y\| = \sup\{|y(s)| : s \in \mathbb{R}_+\}$. The measure of noncompactness in $BC(\mathbb{R}_+)$ described in [3] is used here. For a nonempty bounded set $Y \subset BC(\mathbb{R}_+)$ and $T > 0$, define the (truncated) modulus of continuity

$$w^T(\overline{y}, \varepsilon) = \sup\{|\overline{y}(s_2) - \overline{y}(s_1)| : s_1, s_2 \in [0,T],\ |s_2 - s_1| \le \varepsilon\}, \quad \overline{y} \in Y,$$

and

$$\begin{aligned} w^T(Y, \varepsilon) &= \sup\{w^T(\overline{y}, \varepsilon) : \overline{y} \in Y\}, \\ w_0^T(Y) &= \lim_{\varepsilon \to 0^+} w^T(Y, \varepsilon), \\ w_0^\infty(Y) &= \lim_{T\to\infty} w_0^T(Y). \end{aligned}$$

For each fixed $s \in \mathbb{R}_+$, set

$$Y(s) = \{\overline{y}(s) : \overline{y} \in Y\}, \qquad \text{diam}Y(s) = \sup\{|\overline{x}(s) - \overline{y}(s)| : \overline{x}, \overline{y} \in Y\},$$

and define

$$c(Y) = \lim_{s\to\infty} \sup \text{diam}Y(s).$$

Then

$$\nu_c(Y) = w_0^\infty(Y) + c(Y) \tag{2.1}$$

gives the degree of noncompactness in $BC(\mathbb{R}_+)$ [3].

Definition 2.5. In the Banach algebra E, the measure ν_c satisfies condition (m) if for all arbitrary sets $H, D \in \mathcal{S}_E$ the inequality

$$\nu_c\left(HD\right) \leq \|H\| \nu_c\left(D\right) + \|D\| \nu_c\left(H\right)$$

holds.

Theorem 2.6 ([1])**.** *Let ν_c be the measure given by (2.1). Then ν_c satisfies condition (m) on $\mathcal{S}_E$ where $E = BC(\mathbb{R}_+)$ (a Banach algebra), provided the functions from each such collection of subsets are nonnegative on $\mathbb{R}_+$.*

Theorem 2.7 ([1])**.** *Let B be a nonempty, bounded, closed, and convex subset of the Banach algebra E. Suppose $G, H : B \to E$ are continuous operators such that $G\left(B\right)$ and $H\left(B\right)$ are bounded, and define $A : B \to B$ by $A = G \cdot H$. If for every nonempty subset $Y \subset B$ one has*

$$\nu_c\left(G(Y)\right) \leq \psi_1\left(\nu_c(Y)\right), \qquad \nu_c\left(H(Y)\right) \leq \psi_2\left(\nu_c(Y)\right),$$

where ν_c is a measure of noncompactness fulfilling condition (m) and $\psi_1, \psi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are nondecreasing functions such that

$$\lim_{n\to\infty} \psi_1^n\left(s\right) = 0,$$
$$\lim_{n\to\infty} \psi_2^n\left(s\right) = 0,$$
$$\lim_{n\to\infty} \left(\|G(B)\|\psi_2 + \|H\left(B\right)\|\psi_1\right)^n\left(s\right) = 0,$$

for all $s \geq 0$, then A has at least one fixed point in B.

3. Main Results

The existence is studied under the following assumptions.

(A_1) $h \in BC(\mathbb{R}_+)$.

(A_2) There exist upper semicontinuous, nondecreasing functions $\psi_1, \psi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\lim_{n\to\infty} \psi_1^n(s) = \lim_{n\to\infty} \psi_2^n\left(s\right) = 0, \quad (s \in \mathbb{R}_+),$$

and, for all $s \in \mathbb{R}_+$ and $x_1, x_2 \in \mathbb{R}$,

$$|p(s, x_1) - p(s, x_2)\,| \leq \psi_1(|x_1 - x_2|),$$
$$|g(s, x_1) - g(s, x_2)| \leq \psi_2(|x_1 - x_2|).$$

Moreover, for $s, \zeta \in \mathbb{R}_+$ and $j = 1, 2$,

$$\psi_j\left(s\right) + \psi_j\left(\zeta\right) \leq \psi_j\left(s + \zeta\right).$$

(A_3) $F : I \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and there exists $k : \mathbb{R}_+ \to \mathbb{R}_+$ (nonnegative, continuous, bounded) such that

$$|F(s, x_1) - F(s, x_2)| \leq k\left(s\right)|x_1 - x_2|, \quad (s \in \mathbb{R}_+,\ x_1, x_2 \in \mathbb{R}_+).$$

(A_4) $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$|u(s, \zeta, x)| \leq \Phi(|x|),$$

where $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing.

(A_5) $q : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist continuous $l, m : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\lim_{s\to\infty} l(s) \int_0^s m(\zeta)\, d\zeta = 0,$$

and, for $0 \leq \zeta \leq s$ and $x \in \mathbb{R}$,

$$|q(s, \zeta, x)| \leq l(s)\, m(\zeta).$$

(A_6) Let

$$M_2 = \sup_{s\geq 0} \left\{ |g(s, 0)| + l(s) \int_0^s m(\zeta)\, d\zeta \right\}.$$

There exists $r_0 > 0$ such that

$$\left(\Psi_1(\|x\|) + \frac{A}{\Gamma(\beta+1)} \Phi(\|x\|) + M_1 \right) \times (\|h\| + \Psi_2(\|x\|) + M_2) \leq r$$

admits the positive solution $r = r_0$. In addition, for all $s \geq 0$,

$$\left(\Psi_1(r_0) + \tfrac{A}{\Gamma(\beta+1)} \Phi(r_0) + M_1 \right) \Psi_2(s) + (\|h\| + \Psi_2(r_0) + M_2)\, \Psi_1(s) < s.$$

(A_7) Define $a', b' : \mathbb{R}_+ \to \mathbb{R}$ by

$$a'(s) = l_2(s)\, s^{\alpha\beta},$$
$$b'(s) = k_2(s)\, s^{\delta},$$

which are bounded on $\mathbb{R}_+$ and satisfy

$$\lim_{s\to\infty} a'(s) = 0.$$

Lemma 3.1. *Suppose $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing and upper semi-continuous. Then the following are equivalent:*

(i) $\lim_{n\to\infty} \psi^n(s) = 0$ *for each* $s \geq 0$*;*

(ii) $\psi(s) < s$ *for every* $s > 0$.

Lemma 3.2. *Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be given by $g(s) = s^\alpha$. If $0 < \alpha < 1$ and $s_2 > s_1 \geq 0$, then*

$$s_2^\alpha - s_1^\alpha \leq (s_2 - s_1)^\alpha.$$

Theorem 3.3. *Under assumptions (A_1)–(A_8), equation (1.1) admits at least one solution $x \in B_{r_0}$.*

Proof. For $x \in BC(\mathbb{R}_+)$, define the operator Q by

$$(Qx)(s) = (Tx)(s)(Hx)(s), \qquad s \in \mathbb{R}_+,$$

where

$$\begin{aligned}(Tx)(s) &= p(s, x(\tilde{a}(s))) \\ &\quad + F\left(s, \frac{\alpha}{\Gamma(\beta)} \int_0^s \frac{\zeta^{\alpha-1} u(s, \zeta, x(\tilde{c}(\zeta)))}{(s^\alpha - \zeta^\alpha)^{1-\beta}} d\zeta\right),\end{aligned}$$

and

$$(Hx)(s) = h(s) + g\left(s, x\left(\tilde{b}(s)\right)\right) + \int_0^s q\left(s, \zeta, x\left(\tilde{d}(\zeta)\right)\right) d\zeta.$$

Step 1: We show that $T, H : BC(\mathbb{R}_+) \to BC(\mathbb{R}_+)$ are well-defined. It suffices to prove that the auxiliary functions

$$\begin{aligned}\kappa(s) &= \frac{\alpha}{\Gamma(\beta)} \int_0^s \frac{\zeta^{\alpha-1} u(s, \zeta, x(\tilde{c}(\zeta)))}{(s^\alpha - \zeta^\alpha)^{1-\beta}} d\zeta, \\ \tau(s) &= \int_0^s q\left(s, \zeta, x\left(\tilde{d}(\zeta)\right)\right) d\zeta\end{aligned}$$

belong to $BC(\mathbb{R}_+)$.

Fix $\varepsilon > 0$ and take $0 \le s_1 < s_2 \le L$ with $|s_2 - s_1| \le \varepsilon$. Then, by the assumptions on u and q (continuity and the growth bounds in (A_4)–(A_5)), standard estimates yield

$$|\kappa(s_2) - \kappa(s_1)| \le \frac{\alpha}{\Gamma(\beta)} \left(C_1(L)\, \omega_u(\varepsilon) + C_2(L)\, \varepsilon^\beta\right),$$

and

$$|\tau(s_2) - \tau(s_1)| \le \omega_q(\varepsilon) + \|l\|_{[0,L]} \|m\|_{L^1(0,L)} \varepsilon,$$

where ω_u, ω_q are moduli of continuity in s (uniform on bounded intervals) and $C_1(L), C_2(L)$ depend only on L, α, and β. Hence both κ and τ are uniformly continuous on $[0, L]$ for every $L > 0$. Boundedness on $\mathbb{R}_+$ follows from (A_4)–(A_5). Therefore $\kappa, \tau \in BC(\mathbb{R}_+)$, and thus T, H map $BC(\mathbb{R}_+)$ into itself.

The proof proceeds with the measure-of-noncompactness framework from Section 2 and the product estimate under condition (m) to show Q has a fixed point in B_{r_0}.

$$\begin{aligned}|\kappa(s_2) - \kappa(s_1)| &= \left| \frac{\alpha}{\Gamma(\beta)} \int_0^{s_2} \frac{\zeta^{\alpha-1} u(s_2, \zeta, x(\tilde{c}(\zeta)))}{(s_2^\alpha - \zeta^\alpha)^{1-\beta}} d\zeta \right. \\ &\quad \left. - \frac{\alpha}{\Gamma(\beta)} \int_0^{s_1} \frac{\zeta^{\alpha-1} u(s_1, \zeta, x(\tilde{c}(\zeta)))}{(s_1^\alpha - \zeta^\alpha)^{1-\beta}} d\zeta \right| \\ &\le \frac{\alpha}{\Gamma(\beta)} \int_0^{s_2} \frac{\zeta^{\alpha-1} |u(s_2, \zeta, x(\tilde{c}(\zeta))) - u(s_1, \zeta, x(\tilde{c}(\zeta)))|}{(s_2^\alpha - \zeta^\alpha)^{1-\beta}} d\zeta\end{aligned}$$

$$+\frac{\alpha}{\Gamma(\beta)}\int_{s_1}^{s_2}\frac{\zeta^{\alpha-1}\left|u(s_1,\zeta,x(\tilde{c}(\zeta)))\right|}{(s_2^{\alpha}-\zeta^{\alpha})^{1-\beta}}d\zeta$$
$$+\frac{\alpha}{\Gamma(\beta)}\int_{0}^{s_1}\zeta^{\alpha-1}\left|u(s_1,\zeta,x(\tilde{c}(\zeta)))\right|$$
$$\times\left|(s_2^{\alpha}-\zeta^{\alpha})^{\beta-1}-(s_1^{\alpha}-\zeta^{\alpha})^{\beta-1}\right|d\zeta.$$

Using the modulus of continuity in s (uniform on $[0,L]$),

$$w\,(u,\varepsilon)=\sup\{|u(s_2,\zeta,x)-u(s_1,\zeta,x)|:\ s_1,s_2\in[0,L]\,,\ |s_2-s_1|\le\varepsilon,\ \zeta\in[0,L]\,,\ |x|\le r_0\}.$$

and the bound $|u(s_1,\zeta,x(\tilde{c}(\zeta)))|\le\Phi(\|x\|)$, we get

(3.1)
$$|\kappa(s_2)-\kappa(s_1)|\le\frac{\alpha w(u,\varepsilon)}{\Gamma(\beta)}\int_0^{s_2}\frac{\zeta^{\alpha-1}}{(s_2^{\alpha}-\zeta^{\alpha})^{1-\beta}}d\zeta$$
$$+\frac{\alpha\Phi(\|x\|)}{\Gamma(\beta)}\left\{\int_0^{s_1}\zeta^{\alpha-1}\left|(s_2^{\alpha}-\zeta^{\alpha})^{\beta-1}-(s_1^{\alpha}-\zeta^{\alpha})^{\beta-1}\right|d\zeta\right.$$
$$\left.+\int_{s_1}^{s_2}\frac{\zeta^{\alpha-1}}{(s_2^{\alpha}-\zeta^{\alpha})^{1-\beta}}d\zeta\right\}.$$

The standard EK-kernel primitive gives

$$\int_0^{s}\frac{\zeta^{\alpha-1}}{(s^{\alpha}-\zeta^{\alpha})^{1-\beta}}d\zeta=\frac{s^{\alpha\beta}}{\alpha\beta}=\frac{s^{\alpha\beta}\Gamma(\beta)}{\alpha\Gamma(\beta+1)},$$

hence, from (3.1),

(3.2)
$$|\kappa(s_2)-\kappa(s_1)|\le\frac{w\,(u,\varepsilon)\,s_2^{\alpha\beta}}{\Gamma\,(\beta+1)}+\frac{\Phi(\|x\|)}{\Gamma\,(\beta+1)}\left[s_1^{\alpha\beta}-s_2^{\alpha\beta}+2(s_2^{\alpha}-s_1^{\alpha})^{\beta}\right].$$

By Lemma 3.2. with $0<\alpha<1$ and $\beta\in(0,1]$,

$$(s_2^{\alpha}-s_1^{\alpha})^{\beta}\le(s_2-s_1)^{\alpha\beta},$$

so (3.2) yields

$$|\kappa(s_2)-\kappa(s_1)|\le\frac{w(u,\varepsilon)s_2^{\alpha\beta}}{\Gamma(\beta+1)}+\frac{\Phi(\|x\|)}{\Gamma(\beta+1)}(s_2-s_1)^{\alpha\beta}.\tag{3.3}$$

Since u is uniformly continuous on $[0,L]\times[0,L]\times[-r_0,r_0]$, we have $w\,(u,\varepsilon)\to0$ as $\varepsilon\to0$. Thus, as $s_1\to s_2$, the right-hand side of (3.3) tends to 0, proving that κ is continuous on $[0,L]$. Analogously, τ is

continuous on $[0, L]$. Hence Tx and Hx are continuous on $\mathbb{R}_+$.
Step 2: Let $x \in BC(\mathbb{R}_+)$ with $\|x\| \leq r_0$. Then, for fixed $s \in \mathbb{R}_+$,

(3.4)
$$\begin{aligned}
|(Tx)(s)| &= \left| p(s, x(\tilde{a}(s))) + F\left(s, \frac{\alpha}{\Gamma(\beta)} \int_0^s \frac{\zeta^{\alpha-1} u(s, \zeta, x(\tilde{c}(\zeta)))}{(s^\alpha - \zeta^\alpha)^{1-\beta}} d\zeta\right) \right| \\
&\leq |p(s, x(\tilde{a}(s))) - p(s, 0)| + |p(s, 0)| \\
&\quad + \left| F\left(s, \frac{\alpha}{\Gamma(\beta)} \int_0^s \frac{\zeta^{\alpha-1} u(s, \zeta, x(\tilde{c}(\zeta)))}{(s^\alpha - \zeta^\alpha)^{1-\beta}} d\zeta\right) - F(s, 0) \right| \\
&\quad + |F(s, 0)| \\
&\leq \Psi_1(|x(\tilde{a}(s))|) + \frac{l_2(s)\alpha}{\Gamma(\beta)} \int_0^s \frac{\zeta^{\alpha-1} |u(s, \zeta, x(\tilde{c}(\zeta)))|}{(s^\alpha - \zeta^\alpha)^{1-\beta}} d\zeta + M_1 \\
&\leq \Psi_1(\|x\|) + \frac{l_2(s) s^{\alpha\beta}}{\Gamma(\beta+1)} \Phi(\|x\|) + M_1,
\end{aligned}$$

where
$$M_1 = \sup_{s \geq 0} \left(|p(s, 0)| + |F(s, 0)|\right).$$

$$\|Tx\| \leq \Psi_1(\|x\|) + \frac{A}{\Gamma(\beta+1)} \Phi(\|x\|) + M_1. \tag{3.5}$$

Hence, T maps $BC(\mathbb{R}_+) \to BC(\mathbb{R}_+)$. By the imposed assumptions, T also maps $B_{r_0} \to B_{r_0}$.

(3.6)
$$\begin{aligned}
|(Hx)(s)| &= \left| h(s) + g\left(s, x\left(\tilde{b}(s)\right)\right) + \int_0^s q\left(s, \zeta, x\left(\tilde{d}(\zeta)\right)\right) d\zeta \right| \\
&\leq |h(s)| + \left| g\left(s, x\left(\tilde{b}(s)\right)\right) - g(s, 0) \right| + |g(s, 0)| \\
&\quad + \int_0^s \left| q\left(s, \zeta, x\left(\tilde{d}(\zeta)\right)\right) \right| d\zeta \\
&\leq \|h\| + \Psi_2\left(\left|x\left(\tilde{b}(s)\right)\right|\right) + |g(s, 0)| + l(s) \int_0^s m(\zeta)\, d\zeta \\
&\leq \|h\| + \Psi_2\left(\left|x\left(\tilde{b}(s)\right)\right|\right) + M_2,
\end{aligned}$$

where
$$M_2 = \sup_{s \geq 0} \left\{ |g(s, 0)| + l(s) \int_0^s m(\zeta) d\zeta \right\}.$$

Taking sup over $s \geq 0$ in (3.6) yields

$$\|Hx\| \leq \|h\| + \Psi_2(\|x\|) + M_2. \tag{3.7}$$

Hence H maps $BC(\mathbb{R}_+) \to BC(\mathbb{R}_+)$, and by the assumptions, H maps $B_{r_0} \to B_{r_0}$.

Now, using (3.5)–(3.7), we obtain

$$\begin{aligned}\|Qx\| &= \|Tx\cdot Hx\|\\ &\le \|Tx\|\|Hx\|\\ &\le \left(\Psi_1(\|x\|)+\frac{A}{\Gamma(\beta+1)}\Phi(\|x\|)+M_1\right)(\|h\|+\Psi_2(\|x\|)+M_2),\end{aligned}$$

which implies $\|Qx\|\le r_0$ by assumption (A_7) for any $x\in B_{r_0}$. Consequently, $Q: B_{r_0}\to B_{r_0}$ is well-defined.

Step 3: Fix $\varepsilon>0$ and let $\tilde{x}_1,\tilde{x}_2\in B_{r_0}$ satisfy $\|\tilde{x}_1-\tilde{x}_2\|\le\varepsilon$. For $s\in[0,L]$,

(3.8)

$$\begin{aligned}|(T\tilde{x}_1)(s)-(T\tilde{x}_2)(s)| &= \Bigg| p\left(s,\tilde{x}_1(\tilde{a}(s))\right)-p\left(s,\tilde{x}_2(\tilde{a}(s))\right)\\ &\quad + F\left(s,\frac{\alpha}{\Gamma(\beta)}\int_0^s\frac{\zeta^{\alpha-1}u\left(s,\zeta,\tilde{x}_1(\tilde{c}(\zeta))\right)}{(s^\alpha-\zeta^\alpha)^{1-\beta}}d\zeta\right)\\ &\quad - F\left(s,\frac{\alpha}{\Gamma(\beta)}\int_0^s\frac{\zeta^{\alpha-1}u\left(s,\zeta,\tilde{x}_2(\tilde{c}(\zeta))\right)}{(s^\alpha-\zeta^\alpha)^{1-\beta}}d\zeta\right)\Bigg|\\ &\le \Psi_1\left(|\tilde{x}_1(\tilde{a}(s))-\tilde{x}_2(\tilde{a}(s))|\right)\\ &\quad + \frac{\alpha l_2(s)}{\Gamma(\beta)}\int_0^s\frac{\zeta^{\alpha-1}\gamma(s,\zeta,\tilde{x}_1,\tilde{x}_2)}{(s^\alpha-\zeta^\alpha)^{1-\beta}}d\zeta.\end{aligned}$$

where, $\gamma(s,\zeta,\tilde{x}_1,\tilde{x}_2)=|u(s,\zeta,\tilde{x}_1(\tilde{c}(\zeta)))-u(s,\zeta,\tilde{x}_2(\tilde{c}(\zeta)))|$. Thus

$$\begin{aligned}|(T\tilde{x}_1)(s)-(T\tilde{x}_2)(s)| &\le \Psi_1(\|\tilde{x}_1-\tilde{x}_2\|)+\frac{l_2(s)s^{\alpha\beta}}{\Gamma(\beta+1)}\delta_u(\varepsilon)\\ &\le \Psi_1(\varepsilon)+\frac{A\delta_u(\varepsilon)}{\Gamma(\beta+1)}.\end{aligned}$$

where

$$\begin{aligned}\delta_u(\varepsilon)=\sup\{&|u(s,\zeta,\xi)-u(s,\zeta,\eta)| : s,\zeta\in[0,L],\\ &\xi,\eta\in[-r_0,r_0],\ |\xi-\eta|\le\varepsilon\}.\end{aligned}$$

By the uniform continuity of u in $[0,L]\times[0,L]\times[-r_0,r_0]$, we have $\delta_u(\varepsilon)\to 0$ as $\varepsilon\to 0$. Hence T is continuous on B_{r_0}.

Similarly, for $s\in[0,L]$,

(3.9)

$$\begin{aligned}&|(H\tilde{x}_1)(s)-(H\tilde{x}_2)(s)|\\ &= \Bigg| g\left(s,\tilde{x}_1\left(\tilde{b}(s)\right)\right)-g\left(s,\tilde{x}_2\left(\tilde{b}(s)\right)\right)+\int_0^s\left(q\left(s,\zeta,\tilde{x}_1\left(\tilde{d}(\zeta)\right)\right)\right.\end{aligned}$$

$$-q\left(s,\zeta,\tilde{x}_2\left(\tilde{d}\left(\zeta\right)\right)\right)d\zeta\Bigg|$$

$$\leq \Psi_2\left(\left|\tilde{x}_1\left(\tilde{b}\left(s\right)\right)-\tilde{x}_2\left(\tilde{b}\left(s\right)\right)\right|\right)$$

$$+\int_0^s\left|q\left(s,\zeta,\tilde{x}_1\left(\tilde{d}\left(\zeta\right)\right)\right)-q\left(s,\zeta,\tilde{x}_2\left(\tilde{d}\left(\zeta\right)\right)\right)\right|d\zeta.$$

and, using the growth bound from (A_5) and $\|\tilde{x}_1-\tilde{x}_2\|\leq\varepsilon$,

$$|\left(H\tilde{x}_1\right)(s)-\left(H\tilde{x}_2\right)(s)|\leq\Psi_2(\varepsilon)+\|l\|_{[0,L]}\|m\|_{L^1(0,L)}\varepsilon.$$

Therefore H is continuous on B_{r_0}.

$$|\left(H\tilde{x}_1\right)(s)-\left(H\tilde{x}_2\right)(s)|\leq\Psi_2(\varepsilon)+2l(s)\int_0^s m\left(\zeta\right)d\zeta. \tag{3.10}$$

By (A_6), there exists $L>0$ such that, for every $s\geq L$,

$$2l\left(s\right)\int_0^s m\left(\zeta\right)d\zeta\leq\varepsilon. \tag{3.11}$$

Hence, combining (3.10) and (3.11), for any $s\geq L$,

$$|(H\tilde{x}_1)(s)-\left(H\tilde{x}_2\right)(s)|\leq\Psi_2(\varepsilon)+\varepsilon\leq 2\varepsilon. \tag{3.12}$$

Define the local modulus (on bounded boxes)

$$\delta_q\left(\varepsilon\right)=\sup\{|q\left(s,\zeta,\xi\right)-q\left(s,\zeta,\eta\right)|:s,\zeta\in[0,L],\ \xi,\eta\in[-r_0,r_0],|\xi-\eta|\leq\varepsilon\}.$$

By uniform continuity of q on $[0,L]\times[0,L]\times[-r_0,r_0]$, $\delta_q\left(\varepsilon\right)\to 0$ as $\varepsilon\to 0$. From (3.9), for fixed $s\in[0,L]$,

$$|\left(H\tilde{x}_1\right)(s)-\left(H\tilde{x}_2\right)(s)|\leq\Psi_2\left(\varepsilon\right)+\int_0^L\delta_q\left(\varepsilon\right)d\zeta=\Psi_2\left(\varepsilon\right)+L\delta_q\left(\varepsilon\right).$$

Together with (3.12), this proves H is continuous on B_{r_0}.

Step 4: Let $X(\neq\varnothing)\subset B_{r_0}$, fix $\varepsilon>0$, $x\in X$, and choose $L>0$. For $s_1,s_2\in[0,L]$ with $s_1<s_2$ and $|s_2-s_1|\leq\varepsilon$,

$$\begin{aligned}|\left(Tx\right)(s_2)-\left(Tx\right)(s_1)|&\leq|p\left(s_2,x(\tilde{a}(s_2))\right)-p\left(s_1,x(\tilde{a}(s_1))\right)|\\&+\left|F\left(s_2,\frac{\alpha}{\Gamma(\beta)}\int_0^{s_2}\frac{\zeta^{\alpha-1}u\left(s_2,\zeta,x(\tilde{c}(\zeta))\right)}{(s_2^\alpha-\zeta^\alpha)^{1-\beta}}d\zeta\right)\right.\\&\left.-F\left(s_2,\frac{\alpha}{\Gamma(\beta)}\int_0^{s_1}\frac{\zeta^{\alpha-1}u\left(s_1,\zeta,x(\tilde{c}(\zeta))\right)}{(s_1^\alpha-\zeta^\alpha)^{1-\beta}}d\zeta\right)\right|\\&+\left|F\left(s_2,\frac{\alpha}{\Gamma(\beta)}\int_0^{s_1}\frac{\zeta^{\alpha-1}u\left(s_1,\zeta,x(\tilde{c}(\zeta))\right)}{(s_1^\alpha-\zeta^\alpha)^{1-\beta}}d\zeta\right)\right.\\&\left.-F\left(s_1,\frac{\alpha}{\Gamma(\beta)}\int_0^{s_1}\frac{\zeta^{\alpha-1}u\left(s_1,\zeta,x(\tilde{c}(\zeta))\right)}{(s_1^\alpha-\zeta^\alpha)^{1-\beta}}d\zeta\right)\right|.\end{aligned}$$

Using the moduli of continuity $w\left(p,\varepsilon\right)$, $w\left(F,\varepsilon\right)$ in the first and last terms, and the Lipschitz-type bound on F in its second argument with coefficient $l_2\left(s\right)$, we obtain

$$\begin{aligned}|(Tx)(s_2)-(Tx)(s_1)| \leq\ & w\left(p,\varepsilon\right)+\Psi_1\left(|x(\tilde{a}(s_2))-x(\tilde{a}(s_1))|\right)\\ &+\frac{\alpha l_2(s)}{\Gamma(\beta)}\left|\int_0^{s_2}\frac{\zeta^{\alpha-1}u\left(s_2,\zeta,x(\tilde{c}(\zeta))\right)}{(s_2^{\alpha}-\zeta^{\alpha})^{1-\beta}}d\zeta\right.\\ &\left.-\int_0^{s_1}\frac{\zeta^{\alpha-1}u\left(s_1,\zeta,x(\tilde{c}(\zeta))\right)}{(s_1^{\alpha}-\zeta^{\alpha})^{1-\beta}}d\zeta\right|+w(F,\varepsilon).\end{aligned}$$

Splitting the integral difference as in Step 1 and applying the estimates used there,

$$\begin{aligned}|(Tx)(s_2)-(Tx)(s_1)| \leq\ & w(p,\varepsilon)+\Psi_1\left(w(x,\varepsilon)\right)\\ &+\frac{l_2(s)w(u,\varepsilon)s_2^{\alpha\beta}}{\Gamma(\beta+1)}+\frac{l_2(s)\Psi(\|x\|)}{\Gamma(\beta+1)}(s_2-s_1)^{\alpha\beta}\\ &+w(F,\varepsilon),\end{aligned}$$

where $w\left(x,\varepsilon\right)$ is the modulus of continuity of x on $[0,L]$.

Since $w\left(p,\varepsilon\right)$, $w\left(F,\varepsilon\right)$, $w\left(u,\varepsilon\right)\to 0$ as $\varepsilon\to 0$, follows $w^L\left(Tx,\varepsilon\right)\to 0$. Hence $w_0\left(TX\right)=0$ on bounded intervals, which yields the desired estimate for $w_0\left(TX\right)$.
Taking $\varepsilon\to 0$ and using the uniform continuity of p,u,F, we obtain

$$w_0(TX)\leq \lim_{\varepsilon\to 0}\Psi_1\left(w(X,\varepsilon)\right).$$

Since Ψ_1 is upper semicontinuous, it follows that

$$w_0(TX)\leq \Psi_1\left(w_0(X)\right). \tag{3.13}$$

Step 5: Let $X(\neq\varnothing)\subset B_{r_0}$ and $x,y\in X$. For $s\in\mathbb{R}_+$, by (3.8) and the imposed assumptions,

$$\begin{aligned}|(Tx)(s)-(Ty)(s)| &\leq \Psi_1\left(|x(\tilde{a}(s))-y\left(\tilde{a}(s)\right)|\right)\\ &\quad+\frac{\alpha l_2(s)}{\Gamma(\beta)}\int_0^{s}\frac{\zeta^{\alpha-1}\gamma(s,\zeta,x,y)}{(s^{\alpha}-\zeta^{\alpha})^{1-\beta}}d\zeta\\ &\leq \Psi_1\left(\mathrm{diam}X(s)\right)\\ &\quad+\frac{\alpha l_2(s)}{\Gamma(\beta)}\left(\int_0^{s}\frac{\zeta^{\alpha-1}|u(s,\zeta,x(\tilde{c}(\zeta)))|}{(s^{\alpha}-\zeta^{\alpha})^{1-\beta}}d\zeta\right.\\ &\quad\left.+\int_0^{s}\frac{\zeta^{\alpha-1}\left|u(s,\zeta,y(\tilde{c}(\zeta)))\right|}{(s^{\alpha}-\zeta^{\alpha})^{1-\beta}}d\zeta\right)\\ &\leq \Psi_1\left(\mathrm{diam}X(s)\right)+\frac{2l_2(s)s^{\alpha\beta}}{\Gamma(\beta+1)}\Psi(r_0).\end{aligned}$$

Here, $\int_0^s \frac{\zeta^{\alpha-1}}{(s^\alpha-\zeta^\alpha)^{1-\beta}}d\zeta = \frac{s^{\alpha\beta}}{\alpha\beta} = \frac{s^{\alpha\beta}\Gamma(\beta)}{\alpha\Gamma(\beta+1)}$ and $|u(s,\zeta,\xi)| \leq \Psi(|\xi|) \leq \Psi(r_0)$. Hence

$$\operatorname{diam}(TX)(s) \leq \Psi_1\left(\operatorname{diam}X(s)\right) + \frac{2l_2(s)s^{\alpha\beta}}{\Gamma(\beta+1)}\Psi(r_0).$$

If, by (A_6) (or the stated growth bounds), $l_2(s)s^{\alpha\beta} \to 0$ as $s \to \infty$, then

$$\lim_{s\to\infty}\sup \operatorname{diam}(TX)(s) \leq \Psi_1\left(\lim_{s\to\infty}\sup \operatorname{diam}X(s)\right). \tag{3.14}$$

Combining (3.13) and (3.14), we get

$$\begin{aligned}\nu_c(TX) &= w_0(TX) + \lim_{s\to\infty}\sup \operatorname{diam}(TX)(s)\\ &\leq \Psi_1\left(w_0(X)\right) + \Psi_1\left(\lim_{s\to\infty}\sup \operatorname{diam}X(s)\right).\end{aligned}$$

Since $\nu_c\left(X\right) = w_0\left(X\right) + \lim_{s\to\infty}\sup \operatorname{diam}X\left(s\right)$ and Ψ_1 is nondecreasing,

$$\nu_c(TX) \leq \Psi_1\left(\nu_c(X)\right). \tag{3.15}$$

Step 6: Let $X(\neq \varnothing) \subset B_{r_0}$, fix $\varepsilon > 0$, and pick $x \in X$. For $L > 0$ choose $s_1, s_2 \in [0, L]$ with $s_1 < s_2$ and $|s_2 - s_1| \leq \varepsilon$. Then

$$\begin{aligned}&|(Hx)(s_2) - (Hx)(s_1)|\\ &\quad= |h(s_2) - h(s_1)| + |g\left(s_2, x(\tilde{b}(s_2))\right) - g\left(s_1, x(\tilde{b}(s_1))\right)|\\ &\qquad+ \left|\int_0^{s_2} q\left(s_2,\zeta,x(\tilde{d}(\zeta))\right)d\zeta - \int_0^{s_1} q\left(s_1,\zeta,x(\tilde{d}(\zeta))\right)d\zeta\right|\\ &\quad\leq w(h,\varepsilon) + |g\left(s_2, x(\tilde{b}(s_2))\right) - g\left(s_2, x(\tilde{b}(s_1))\right)| + w(g,\varepsilon)\\ &\qquad+ \int_0^{s_2} |q\left(s_2,\zeta,x(\tilde{d}(\zeta))\right) - q\left(s_1,\zeta,x(\tilde{d}(\zeta))\right)|d\zeta\\ &\qquad+ \int_{s_1}^{s_2} |q\left(s_1,\zeta,x(\tilde{d}(\zeta))\right)|d\zeta\\ &\quad\leq w(h,\varepsilon) + \Psi_2\left(|x(\tilde{b}(s_2)) - x(\tilde{b}(s_1))|\right) + w(g,\varepsilon)\\ &\qquad+ Lw(q,\varepsilon) + \sup_{[0,L]} l \int_{s_1}^{s_2} m(\zeta)d\zeta\\ &\quad\leq w(h,\varepsilon) + \Psi_2\left(w(x,\varepsilon)\right) + w(g,\varepsilon) + Lw(q,\varepsilon)\\ &\qquad+ \varepsilon\sup\{l(s_1)m(s_2) : s_1, s_2 \in [0,L]\}.\end{aligned}$$

where

$$\begin{aligned}w(h,\varepsilon) &= \sup\{|h(s_2) - h(s_1)| : s_1, s_2 \in [0,L]\,, |s_2 - s_1| \leq \varepsilon\},\\ w(g,\varepsilon) &= \sup\{|g(s_2,\xi) - g(s_1,\xi)| : s_1, s_2 \in [0,L],\end{aligned}$$

$$\xi \in [-r_0, r_0], |s_2 - s_1| \leq \varepsilon\},$$
$$w(q,\varepsilon) = \sup\{|q(s_2,\zeta,\xi) - q(s_1,\zeta,\xi)| : s_1, s_2 \in [0,L],\ \zeta \in [0,L]\,,$$
$$\xi \in [-r_0, r_0], |s_2 - s_1| \leq \varepsilon\}.$$

By uniform continuity of h on $[0,L]$, g on $[0,L] \times [-r_0, r_0]$, and q on $[0,L] \times [0,L] \times [-r_0, r_0]$, we have $w\,(h,\varepsilon)\,, w\,(g,\varepsilon)\,, w\,(q,\varepsilon) \to 0$ as $\varepsilon \to 0$. Also, since l,m are continuous on $\mathbb{R}_+$, $\sup\{l(s_1)m(s_2) : s_1, s_2 \in [0,L]\} < \infty$. Hence

$$w_0^L(HX) \leq \lim_{\varepsilon\to 0} \Psi_2\left(w^L(X,\varepsilon)\right),$$

and, by upper semicontinuity of Ψ_2,

$$w_0^L(HX) \leq \Psi_2\left(w_0^L(X)\right).$$

Letting $L \to \infty$,

$$w_0^\infty(HX) \leq \Psi_2\left(w_0^\infty(X)\right). \tag{3.16}$$

Step 7: For $X(\neq \varnothing) \subset B_{r_0}$ and $x, y \in X$,

$$\begin{aligned}
&|(Hx)(s) - (Hy)(s)| \\
&\quad \leq \left|g\left(s, x(\tilde{b}(s))\right) - g\left(s, y(\tilde{b}(s))\right)\right| \\
&\qquad + \int_0^s \left(\left|q\left(s,\zeta,x(\tilde{d}(\zeta))\right)\right| + \left|q\left(s,\zeta,y(\tilde{d}(\zeta))\right)\right|\right) d\zeta \\
&\quad \leq \Psi_2\left(\left|x(\tilde{b}(s)) - y(\tilde{b}(s))\right|\right) + 2l(s)\int_0^s m(\zeta)d\zeta.
\end{aligned}$$

when

$$\operatorname{diam}\left(HX\right)(s) \leq \Psi_2\left(\operatorname{diam} X(s)\right) + 2l(s)\int_0^s m(\zeta)d\zeta.$$

If $l(s)\int_0^s m(\zeta)d\zeta \to 0$ as $s \to \infty$ (assumption (A_5)), then

$$\lim_{s\to\infty}\sup \operatorname{diam}(HX)(s) \leq \Psi_2\left(\lim_{s\to\infty}\sup \operatorname{diam} X(s)\right). \tag{3.17}$$

Linking (3.16) and (3.17), we obtain

$$\begin{aligned}
\nu_c(HX) &= w_0^\infty(HX) + \lim_{s\to\infty}\sup \operatorname{diam}(HX)(s) \\
&\leq \Psi_2\left(w_0^\infty(X)\right) + \Psi_2\left(\lim_{s\to\infty}\sup \operatorname{diam} X(s)\right) \\
&\leq \Psi_2\left(w_0^\infty(X) + \lim_{s\to\infty}\sup \operatorname{diam} X(s)\right) \\
&= \Psi_2\left(\nu_c(X)\right).
\end{aligned}$$

Hence

$$\nu_c(HX) \leq \Psi_2\left(\nu_c(X)\right). \tag{3.18}$$

Now the inequality in assumption (A_7) takes the form

(3.19)

$$\begin{aligned}\left(\|TB_{r_0}\|\Psi_2+\|HB_{r_0}\|\Psi_1\right)(s) &\le \Big(\Big(\Psi_1(r_0)+\frac{A}{\Gamma(\beta+1)}\Phi(r_0)+M_1\Big)\Psi_2\\ &\quad+\left(\|h\|+\Psi_2(r_0)+M_2\right)\Psi_1\Big)(s)\\ &< s.\end{aligned}$$

for all $s\in\mathbb{R}_+$. By Lemma 3.1. and (3.19),

$$\lim_{n\to\infty}\left(\|TB_{r_0}\|\Psi_2+\|HB_{r_0}\|\Psi_1\right)^n(s)=0.$$

Therefore, all the conditions of Theorem 2.7. are satisfied, and equation (1.1) has at least one solution in the space $BC(\mathbb{R}_+)$. □

4. Numerical Example

We scrutinize the following example to validate our key outcomes:

(4.1)

$$\begin{aligned}x(s)=&\left(\frac{|x|^{1/4}}{2(1+s^2)}+e^{-s}\arctan\left(\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^s\frac{\frac{5}{2}\zeta^{3/2}e^{-s\zeta}\cos|x|}{\left(s^{5/2}-\zeta^{5/2}\right)^{1/2}}d\zeta\right)\right)\\ &\times\left(e^{-(s-3)^2}+\frac{\sin(2s)}{1+2\cos s}\ln(1+|x|)+\int_0^s\frac{\zeta|\cos x|e^{-s^2}}{1+|x|^2}d\zeta\right),\end{aligned}$$

where $s\in\mathbb{R}_+$, $0<\beta<1$, and $\alpha>0$. It is a special case of (1.1) with

$$\begin{aligned}&p(s,x)=\frac{|x|^{1/4}}{2(1+s^2)},\\ &F(s,x)=e^{-s}\arctan(x),\\ &g(s,x)=\frac{\sin(2s)}{1+2\cos s}\ln(1+|x|),\\ &u(s,\zeta,x)=e^{-s\zeta}\cos|x|,\\ &q(s,\zeta,x)=\frac{\zeta|\cos x|e^{-s^2}}{1+|x|^2},\\ &h(s)=e^{-(s-3)^2},\\ &\tilde{a}(s)=\tilde{b}(s)=\tilde{c}(s)=\tilde{d}(s)=s,\\ &\alpha=\frac{5}{2},\\ &\beta=\frac{1}{2}.\end{aligned}$$

Clearly, $h\in BC(\mathbb{R}_+)$ with $\|h\|=1$.

For any fixed $x, y \in \mathbb{R}_+$ with $|y| \le |x|$ and $s > 0$,

$$|p(s,x) - p(s,y)| = \frac{1}{2(1+s^2)} \left| |x|^{1/4} - |y|^{1/4} \right| \le \frac{1}{2}|x-y|^{1/2}.$$

(The case $|x| \le |y|$ is analogous.) Thus we can take

$$\Psi_1(s) = \frac{1}{2}s^{1/2},$$

which is nondecreasing, concave on $\mathbb{R}_+$, and satisfies $\Psi_1(s) < s$ for all $s > 0$. Moreover,

$$\begin{aligned} |g(s,x) - g(s,y)| &= \left| \frac{\sin(2s)}{1 + 2\cos s} \right| |\ln(1+|x|) - \ln(1+|y|)| \\ &\le \ln\left(1 + \frac{||x| - |y||}{1+|y|}\right) \\ &\le \ln(1 + |x-y|). \end{aligned}$$

So we may take

$$\Psi_2(s) = \ln(1+s),$$

also nondecreasing, concave on $\mathbb{R}_+$, with $\Psi_2(s) < s$ for all $s > 0$.
Assumption (A_5). We have $|q(s,\zeta,x)| \le e^{-s^2}\zeta$, hence (A_5) holds with

$$l(s) = e^{-s^2}, \qquad m(\zeta) = \zeta, \qquad \lim_{s\to\infty} l(s) \int_0^s m(\zeta)\, d\zeta = \lim_{s\to\infty} e^{-s^2} \cdot \frac{s^2}{2} = 0,$$

and $\Phi(x) = \cos x$.
Constant M_2 in (A_6).

$$M_2 = \sup_{s\ge 0} \left\{ |g(s,0)| + l(s) \int_0^s m(\zeta)\, d\zeta \right\} = \sup_{s\ge 0} \frac{s^2 e^{-s^2}}{2} = \frac{e^{-1}}{2} \approx 0.1839.$$

Verification of (A_7). The inequality becomes

$$\left(\frac{1}{2}\sqrt{r} + 0.0830\cos r\right)\left(\frac{1}{6} + \ln(1+r) + 0.1839\right) \le r,$$

which is satisfied for $r = 1$. The second inequality in (A_7),

$$\begin{aligned} &\left((\Psi_1(r_0) + \frac{A}{\Gamma(\beta+1)}\Phi(r_0) + M_1)\Psi_2 + (\|h\| + \Psi_2(r_0) + M_2)\Psi_1 \right)(s) \\ &\quad < s, \quad s \in \mathbb{R}_+, \end{aligned}$$

also holds for $r_0 = 1$.

All assumptions (A_1)–(A_8) are fulfilled. By Theorem 3.3., the fractional integral equation (4.1) admits at least one solution $x(s) \in B_1 \subset BC(\mathbb{R}_+)$.

5. Conclusion

In this paper, a sufficient condition for the existence of solutions, concerning the nonlinear FIE associated with a weakly singular kernel, has been minutely derived. More precisely, it deals with the product of operators in a Banach algebra, which is a basic structure in functional analysis. Having made use of the properties of the Banach algebra in this paper, we have established a rigorous setting under which the solution to the nonlinear FIE is guaranteed.

The added complexity of the weakly singular kernel at certain points, hence its singularity, demands great care and subtlety so that the solutions are defined and exist under the conditions. All these intricacies were considered in our derivation, which therefore provides a robust criterion guaranteeing the existence of solutions.

An illustrative example of applying the derived sufficient condition to a specific nonlinear FIE is given to further substantiate and elucidate our theoretical findings. Not only does the example illustrate the applicability of our theoretical results in practice, but it also turns out that the sufficient condition is very effective and reliable for real-world problems connected with such equations. We want to illustrate through this example how abstract conditions turn into concrete solutions and thus validate theoretical contributions from our work.

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