Sahand Communications in Mathematical Analysis (SCMA) Vol. ⋯ No. ⋯ (20??), ⋯-⋯
http://scma.maragheh.ac.ir
DOI: ...

Properties of Subclasses of Meromorphic Functions Based on q-Derivative and Its Graphical Analysis

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Abstract. In this work, we introduce new subclasses $\Sigma^*_{S_q}(\sigma, \varrho)$ and $\Sigma^*_{C_q}(\sigma, \varrho)$ of meromorphic functions defined on a punctured unit disk and constructed using the q-derivative. We explore the properties coefficient estimates and the convexity of these classes. The study also includes the radius of meromorphically q-starlikeness, the radius of meromorphically q-convexity, modified Hadamard product and integral operators of class $\Sigma^*_{S_q}(\sigma, \varrho)$. We have provided a graphical explanation to clarify how changing the values of q affects the theoretical conclusions. This visual approach provides a clearer view of the variations that appear when q changes and supplements the analytical results. This work increases our understanding of the properties of meromorphic functions by integrating analytical findings with graphical representations.

1. Introduction

The aim of Geometric Function Theory(GFT) is to understand the geometric nature of analytic functions. Much of this theory depends mainly on functions defined on the open unit disk. Convex, starlike, univalent and close-to-convex functions belong to the conventional subclasses that GFT has mostly relied on. The class of meromorphic functions, which are analytic except for isolated poles, has now been added to these ideas. Due to their rich structure, meromorphic functions have remained a central focus of both classical and modern investigations in complex analysis. Examining meromorphic functions under specific geometric criteria, such as convexity or starlikeness, to identify important subclasses is an active field of research. In terms of coefficient limits,

2020 *Mathematics Subject Classification.* 30C45, 30C55.
Key words and phrases. q-derivative, meromorphic function, q-starlike, q-convex.
Received: 2025-08-21, Accepted: 2026-02-03.
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growth behavior and mapping qualities, these geometric properties often yield beautiful findings that are useful in other branches of mathematics and science, in addition to being theoretically interesting. Geometric function theory and q-calculus together provide a natural and potent extension of preexisting concepts, particularly when considering meromorphic functions.

Quantum calculus (q-calculus), is essentially a form of calculus that operates without relying on limits. It has several applications in applied mathematics. Jackson [11, 12, 13] first introduced the use of the q-calculus. In the study of geometric function theory, quantum calculus plays an important role. The q-calculus has been used to describe the q-analogue of the differential operator in geometric function theory. The classical derivative is replaced by the q-difference operator to define various new subclasses. These new classes show interesting geometric phenomena and generalize conventional geometric function theory. In recent studies, several novel subclasses of analytic functions related to the q-derivative operator have been proposed and examined; see [19, 21, 22, 23, 26, 27]. Several authors created new subclasses of meromorphic functions using the q-derivative operator, for example [2, 3, 4, 5, 6, 15, 20]. Ismail [10] used the q-derivative operator to investigate a class of starlike functions.

Our goal to create and analyze novel meromorphic subclasses defined using the q-derivative is motivated by recent advancements in q-based analytic and meromorphic subclasses, such as works on bi-starlike, biconvex and other q-operator families [8, 9, 24, 25]. Several subclasses of meromorphic functions using q-derivative have been introduced and studied in the literature. However, these studies rarely explore the geometric features of the mappings and are limited to theoretical studies. There hasn't been enough investigation or visualization of how image domains and radii behave as $q \to 1$, which corresponds to the transition from the q-analogue to the conventional situation. Motivated by these limitations, the present work provides a geometric interpretation of the behavior of new subclasses of meromorphic functions constructed using the q-derivative. In addition to extending previously known meromorphic q-classes, the investigation further explains the effect of the parameter q on the geometric properties of these functions.

This paper investigates a specific subclass within the family of meromorphic functions defined by the q-derivative operator. We focus on analyzing its geometric behavior, including coefficient estimates, Hadamard Product and radii of starlikeness or convexity. Special attention is paid to how these properties evolve as the parameter q varies. This not only extends several known results, but also reveals new geometric features driven by the q-calculus approach.

2. Basic Results and Definitions

Consider $\mathscr{A}$, the class of functions $\mathfrak{u}(z)$ given by

$$\mathfrak{u}(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{2.1}$$

in the interior of the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy the usual normalization condition $\mathfrak{u}(0) = \mathfrak{u}'(0) - 1 = 0$. Denote by S the family of functions $\mathfrak{u}(z)$ belonging to $\mathscr{A}$ which are all univalent in U.

If a function $\mathfrak{u} \in \mathscr{A}$ satisfies the following condition

$$\Re\left\{\frac{z\mathfrak{u}'(z)}{\mathfrak{u}(z)}\right\} > \upsilon, \quad (z \in U),$$

then it is a starlike function of the order $\upsilon,\ 0 \leq \upsilon < 1$, We denote this class by $S^*(\upsilon)$. If a function $u \in \mathscr{A}$ satisfies the following condition

$$\Re\left\{1 + \frac{z\mathfrak{u}''(z)}{\mathfrak{u}'(z)}\right\} > \upsilon, \quad (z \in U),$$

then it is a convex function of the order $\upsilon,\ 0 \leq \upsilon < 1$, We denote this class by $K(\upsilon)$. Observe that $S^*(0) = S^*$ and $K(0) = K$ align with the standard classes of starlike and convex functions in U, respectively. Let $\mathfrak{u} \in \mathscr{A}$ be given by (2.1) and $\phi(z)$ given by

$$\phi(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$

their convolution (or Hadamard product), denoted by $(\mathfrak{u} * \phi)$, is defined as

$$(\mathfrak{u} * \phi)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (\phi * \mathfrak{u})(z), \quad (z \in U).$$

Note that $\mathfrak{u} * \phi \in \mathscr{A}$.

Let Σ denote the class of functions of the form

$$l(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m,$$

which are analytic on the punctured unit disk $D = \{z : 0 < |z| < 1\}$ with a simple pole at the origin. Let $\Sigma_s, \Sigma^*(\alpha)$ and $\Sigma_k(\alpha)(0 \leq \alpha < 1)$ denote the subclasses of Σ that are univalent, meromorphically starlike of order α and meromorphically convex of order α, respectively.

The class $\Sigma^*(\alpha)$ and similar function classes have been studied in depth by Pommerenke [17], Clunie [7], Miller [16], Royster [18] and others.

A function $l \in \Sigma$ is called meromorphically starlike of order $\alpha(0 \leq \alpha < 1)$ if it satisfies the following

$$\Re\left(-\frac{zl'(z)}{l(z)}\right) > \alpha, \quad (z \in D).$$

A function $l \in \Sigma$ is called meromorphically convex of order $\alpha, (0 \leq \alpha < 1)$ if and only if

$$\Re\left\{-\left(1+\frac{zl''(z)}{l'(z)}\right)\right\} > \alpha, \quad (z \in D).$$

We use the notation $\Sigma^*(\alpha)$ to represent the set of starlike functions of order α and $\Sigma_k(\alpha)$ to represent the corresponding set of convex functions of that order. Let Σ_+ represent the family of meromorphic functions with a positive coefficient of the form

$$l(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m, \quad (a_m \geq 0), \tag{2.2}$$

that are analytic and univalent in D. This class was originally introduced and explored by Juneja and Reddy [14]. Let us discuss a few basic definitions from the q-calculus, which will be useful in this article.

Definition 2.1. The q-derivative of the function $\mathfrak{u} \in \mathscr{A}$ is defined as

$$\mathfrak{D}_q\mathfrak{u}(z) = \frac{\mathfrak{u}(qz)-\mathfrak{u}(z)}{(q-1)z}, \quad (z \neq 0,\ 0 < q < 1).$$

T. M. Seody [19] introduced the following definition of the class of q-starlike functions of order μ.

$$S_q^*(\mu) = \left\{\mathfrak{u} \in \mathscr{A} : \Re\left\{\frac{z\mathfrak{D}_q\mathfrak{u}(z)}{\mathfrak{u}(z)}\right\} > \mu, (z \in U; 0 \leq \mu < 1)\right\}.$$

The class of q-convex functions of order μ is defined as

$$C_q(\mu) = \left\{\mathfrak{u} \in \mathscr{A} : \Re\left\{\frac{\mathfrak{D}_q(z\mathfrak{D}_q\mathfrak{u}(z))}{\mathfrak{D}_q\mathfrak{u}(z)}\right\} > \mu, (z \in U; 0 \leq \mu < 1)\right\}.$$

AbuJarad et al. [1] studied certain subclasses of β-uniformly q-starlike and β-uniformly q-convex functions. These subclasses of β-uniformly q-starlike and β-uniformly q-convex functions of order α are defined as follows:

Definition 2.2. The function $\mathfrak{u} \in \mathscr{A}$ is said to be β-uniformly q-starlike of order α, if it satisfies the following inequality

$$\Re\left\{\frac{z\mathfrak{D}_q\mathfrak{u}(z))}{\mathfrak{u}(z)}-\alpha\right\}\geq\beta\left|\frac{z(\mathfrak{D}_q\mathfrak{u}(z))}{\mathfrak{u}(z)}-1\right|.$$

Definition 2.3. The function $\mathfrak{u} \in \mathscr{A}$ is said to be β-uniformly q-convex of order α, if it satisfies the following inequality

$$\Re\left\{\frac{\mathfrak{D}_q(z\mathfrak{D}_q\mathfrak{u}(z))}{\mathfrak{D}_q\mathfrak{u}(z)}-\alpha\right\}\geq\beta\left|\frac{\mathfrak{D}_q(z(\mathfrak{D}_q\mathfrak{u}(z))}{\mathfrak{D}_q\mathfrak{u}(z)}-1\right|.$$

Tang et al.[23] introduced the meromorphic version of the q-derivative. For $l \in \Sigma$, the q-derivative is given by

$$\mathfrak{D}_q l(z)=\frac{l(qz)-l(z)}{q-1}=\frac{1}{-qz^2}+\sum_{m=1}^{\infty}[m]_q a_m z^{m-1},\quad (0<q<1).$$

$$\mathfrak{D}_q(qz\mathfrak{D}_q l(z))=\frac{1}{qz^2}+\sum_{m=1}^{\infty}q[m]_q^2 a_m z^{m-1},\quad (0<q<1).$$

where $[m]_q$ is the q-number and is given by

$$[m]_q=\frac{q^m-1}{q-1},\quad (0<q<1).$$

$[m]_q \to m$ as $q \to 1^-$.

For $l \in \Sigma_+$ using the q-derivative, we define the subclasses $\Sigma_{S_q}(\nu)$ and $\Sigma_{C_q}(\nu)$ called subclass meromorphically q-starlike functions of order ν and meromorphically q-convex functions of order ν, respectively.

Definition 2.4.

$$\Sigma_{S_q}(\nu)=\left\{l\in\Sigma_+ : -\Re\left(\frac{qz\mathfrak{D}_q l(z)}{l(z)}\right)>\nu,(z\in D;\ 0\leq\nu<1)\right\}.$$

$$\Sigma_{C_q}(\nu)=\left\{l\in\Sigma_+ : -\Re\left(\frac{\mathfrak{D}_q(qz\mathfrak{D}_q l(z))}{\mathfrak{D}_q l(z)}\right)>\nu,(z\in D;\ 0\leq\nu<1)\right\}.$$

Following Abujarad's technique [1], we use the q-derivative to define subclasses $\Sigma^*_{S_q}(\sigma,\varrho)$ and $\Sigma^*_{C_q}(\sigma,\varrho)$ of Σ_+ as follows:

Definition 2.5. For $0\leq\sigma<1$, $\varrho\geq 0$, $\Sigma^*_{S_q}(\sigma,\varrho)$ is the subclass of Σ_+ that contains functions of the form (2.2) and satisfy the condition

$$-\Re\left\{\frac{qz\mathfrak{D}_q l(z)}{l(z)}+\sigma\right\}\geq\varrho\left|\frac{qz\mathfrak{D}_q l(z)}{l(z)}+1\right|.$$

Definition 2.6. For $0\leq\sigma<1$, $\varrho\geq 0$, $\Sigma^*_{C_q}(\sigma,\varrho)$ is the subclass of Σ_+ that contains functions of the form (2.2) and satisfy the condition

$$-\Re\left\{\frac{\mathfrak{D}_q(qz\mathfrak{D}_q l(z))}{\mathfrak{D}_q l(z)}+\sigma\right\}\geq\varrho\left|\frac{\mathfrak{D}_q(qz\mathfrak{D}_q l(z))}{\mathfrak{D}_q l(z)}+1\right|.$$

Theorem 2.7. *If function $l \in \Sigma^*_{S_q}(\sigma,\varrho)$ then $l \in \Sigma_{S_q}\left(\frac{\sigma+\varrho}{1+\sigma}\right)$, where $\varrho \geq 0$, $0 \leq \sigma < 1$ and $0 < q < 1$.*

Proof. Let $l \in \Sigma^*_{S_q}(\sigma,\varrho)$, by definition

$$-\Re\left\{\frac{qz\mathfrak{D}_q l(z)}{l(z)}+\sigma\right\} \geq \varrho\left|\frac{qz\mathfrak{D}_q l(z)}{l(z)}+1\right|.$$

$$Re(z) \leq |z|$$

$$\varrho\left|\frac{qz\mathfrak{D}_q l(z)}{l(z)}+1\right| \geq \varrho\Re\left\{\frac{qz\mathfrak{D}_q l(z)}{l(z)}+\sigma\right\},$$

which implies

$$-\Re\left\{\frac{qz\mathfrak{D}_q l(z)}{l(z)}+\sigma\right\} \geq \varrho\Re\left\{\frac{qz\mathfrak{D}_q l(z)}{l(z)}+\sigma\right\}.$$

$$-\Re\left\{\frac{qz\mathfrak{D}_q l(z)}{l(z)}\right\}(1+\sigma) \geq \varrho+\sigma.$$

That is

$$-\Re\left\{\frac{qz\mathfrak{D}_q l(z)}{l(z)}\right\} \geq \frac{\varrho+\sigma}{(1+\sigma)}.$$

This gives $l \in \Sigma_{S_q}\left(\frac{\sigma+\varrho}{1+\sigma}\right)$. □

Theorem 2.8. *If function $l \in \Sigma^*_{c_q}(\sigma,\varrho)$ then $l \in \Sigma_{c_q}\left(\frac{\sigma+\varrho}{1+\sigma}\right)$, where $\sigma \geq 0$, $0 \leq \varrho < 1$ and $0 < q < 1$.*

Proof. The proof is similar to the above Theorem 2.7. □

We begin by recalling the lemmas that will be used in the proofs of our results.

Lemma 2.9. *Let $\eta \in \mathbb{R}$ and $\Omega \in \mathbb{C}$. Then condition $-\Re(\Omega) \geq \eta$ is equivalent to $|\Omega + (1-\eta)| - |\Omega - (1+\eta)| \geq 0$.*

Lemma 2.10. *Let $\eta, k \in \mathbb{R}$ and $\Omega \in \mathbb{C}$. Then condition $-\Re(\Omega) \geq k|\Omega+1| + \eta$ is equivalent to*

$$-\Re\left(\Omega\left(1+ke^{i\theta}\right)+ke^{i\theta}\right) \geq \eta, \quad -\pi \leq \theta \leq \pi.$$

3. Coefficient Bounds

Theorem 3.1. *The function $l(z) \in \Sigma_+$ of the form (2.2) belongs to the class $\Sigma^*_{S_q}(\sigma,\varrho)$ for $z \in D$ if and only if it satisfies the inequality*

$$\sum_{m=1}^{\infty}[q[m]_q(1+\varrho)+\sigma+\varrho]a_m \leq (1-\sigma). \tag{3.1}$$

where, $\varrho \geq 0$, $0 \leq \sigma < 1$. This is exactly attained in the case of the function $l(z)$ given by

$$l(z) = \frac{1}{z} + \frac{1-\sigma}{[q[m]_q(1+\varrho)+(\sigma+\varrho)]}z^m.$$

Proof. Let $l(z) \in \Sigma$ satisfy condition (3.1), we have to show that

$$-\Re\left\{\frac{(qz(\mathfrak{D}_q l(z))}{l(z)} + \sigma\right\} \geq \varrho\left|\frac{(qz(\mathfrak{D}_q l(z))}{l(z)} + 1\right|.$$

Then by using Lemma (2.10) we get

$$-\Re\left\{\frac{qz(\mathfrak{D}_q l(z))}{l(z)}\left(1+\varrho e^{i\theta}\right) + \varrho e^{i\theta}\right\} \geq \sigma, \quad -\pi < \theta \leq \pi.$$

Let $P(z) = -(qz(\mathfrak{D}_q l(z)))\left(1+\varrho e^{i\theta}\right) - \varrho e^{i\theta} l(z)$ and $Q(z) = l(z)$. That is

$$\Re\left\{\frac{P(z)}{Q(z)}\right\} \geq \sigma.$$

By Lemma (2.9) it is enough to prove that

$$|P(z) + (1-\sigma)Q(z)| - |P(z) - (1+\sigma)Q(z)| \geq 0.$$

Following some simple calculations, we obtain

$$|P(z) + (1-\sigma)Q(z)| \geq (2-\sigma)\frac{1}{|z|} - \sum_{m=1}^{\infty}[q[m]_q(1+\varrho)+(\varrho+\sigma-1)]a_m |z|^m$$

and

$$|P(z) - (1+\varrho)Q(z)| \leq \sigma\frac{1}{|z|} - \sum_{m=1}^{\infty}[q[m]_q(1+\varrho)+(\varrho+\sigma+1)]a_m |z|^{m-1}.$$

So,

$$\begin{aligned}|P(z) + (1-\sigma)Q(z)| \; &- |P(z) - (1+\gamma)Q(z)| \\ &\geq 2(1-\sigma)\frac{1}{|z|} - 2\sum_{m=1}^{\infty}[q[m]_q(1+\varrho)+\varrho+\sigma]a_m |z|^{m-1} \\ &\geq 0,\end{aligned}$$

by the give inequality (3.1) Conversely, suppose that $l \in \Sigma^*_{S_q}(\sigma, \varrho)$, by Lemma (2.10) we have

$$-\Re\left\{\frac{qz(\mathfrak{D}_q l(z))}{l(z)}\left(1+\varrho e^{i\theta}\right) + \varrho e^{i\theta}\right\} \geq \sigma, \quad -\pi < \theta \leq \pi.$$

For z on the positive real axis, where $0 < |z| = r < 1$, we get

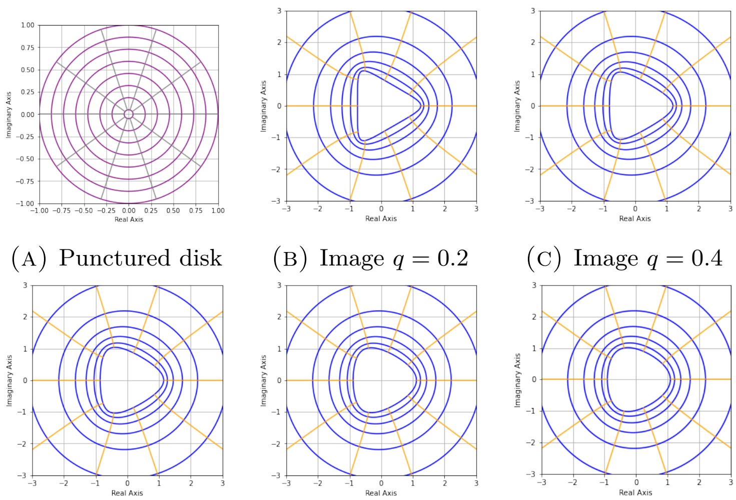
$$\Re\left\{\frac{(1-\sigma)\frac{1}{z}-\sum\limits_{m=1}^{\infty}[q[m]_q\left(1+\varrho e^{i\theta}\right)+(\varrho e^{i\theta}+\sigma)]a_m z^m}{\frac{1}{z}+\sum\limits_{m=1}^{\infty}a_m z^m}\right\}\geq 0.$$

Since $Re(-e^{i\theta})\geq -\left|e^{i\theta}\right| = -1$

$$\Re\left\{\frac{(1-\sigma)\frac{1}{r}-\sum\limits_{m=1}^{\infty}[q[m]_q(1+\varrho)+\varrho+\sigma]a_m r^m}{\frac{1}{r}+\sum\limits_{m=1}^{\infty}a_m r^m}\right\}\geq 0.$$

Letting $r\to 1^-$ we get the desired result. □

The following Figure 1 shows the transformation of the punctured unit disk under the function $l(z)=\frac{1}{z}+\frac{1-\sigma}{[q[m]_q(1+\varrho)+(\sigma+\varrho)]}z^m$ when the parameters are fixed as $\sigma=0.5$, $m=2$, $\varrho=1$ and the parameter q is varied. The figure shows variations in the mapped domain's structure. Each plot gives the effect of q on the behaviour of the function and the resulting image of the unit disk.

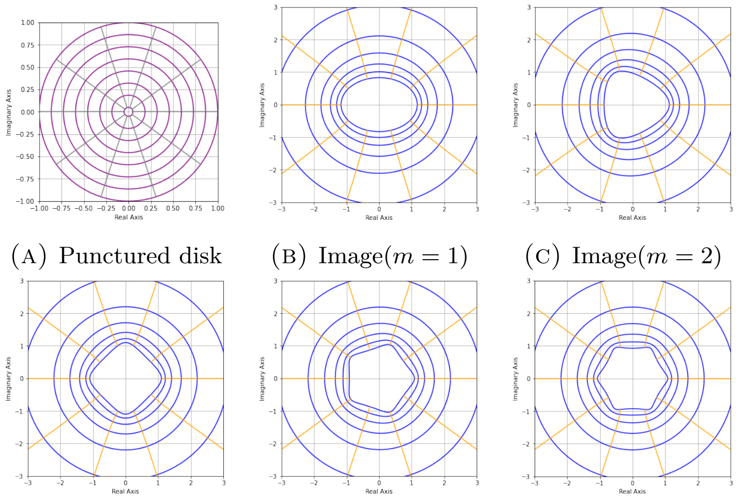


(A) Punctured disk (B) Image $q=0.2$ (C) Image $q=0.4$

(D) Image $q=0.6$ (E) Image $q=0.8$ (F) Image $q=1$

FIGURE 1. Image of the Punctured Unit Disk under $l(z)=\frac{1}{z}+\frac{1-\sigma}{q[m]_q(1+\varrho)+\sigma+\varrho}z^m$, for fixed parameters $(\sigma,m,\varrho)=(0.5,2,1)$ with varying values of q. Each subfigure shows the transformation of concentric circles and radial lines. As q increases, geometric distortion decreases.

For fixed parameters $q = 0.8$, $\sigma = 0.5$, $\varrho = 0.9$ and varying values of m, each subfigure in Figure 2 shows a distinct value of m. The resulting images show the effect of m on the geometric structure of the unit disk under the function l.



(A) Punctured disk (B) Image($m = 1$) (C) Image($m = 2$)

(D) Image($m = 3$) (E) Image ($m = 4$) (F) Image ($m = 5$)

FIGURE 2. Graphical representation of the punctured unit disk and image domains under the mapping $l(z) = \frac{1}{z} + \frac{1-\sigma}{q[m]_q(1+\varrho)+\sigma+\varrho} z^m$, for fixed parameters $(\sigma = 0.5, q = 0.8, \varrho = 0.9)$ with varying values of m. The first plot shows the reference punctured unit disk, while subsequent plots display its transformations for $m = 1, 2, 3, 4, 5$, illustrating how the mapping distorts circular and radial structures as m increases.

Corollary 3.2. *Consider the function l defined by* (2.2)*, which belongs to the class $\Sigma^*_{S_q}(\sigma, \varrho)$, then for each integer $m \geq 1$ the coefficient a_m satisfies the following*

$$a_m \leq \frac{1-\sigma}{[q[m]_q(1+\varrho)+\sigma+\varrho]}.$$

For fixed values $\sigma = 0.5$, $\varrho = 1$, the following graph demonstrates how the bounds of a_m change with values of m for a few selected values of q

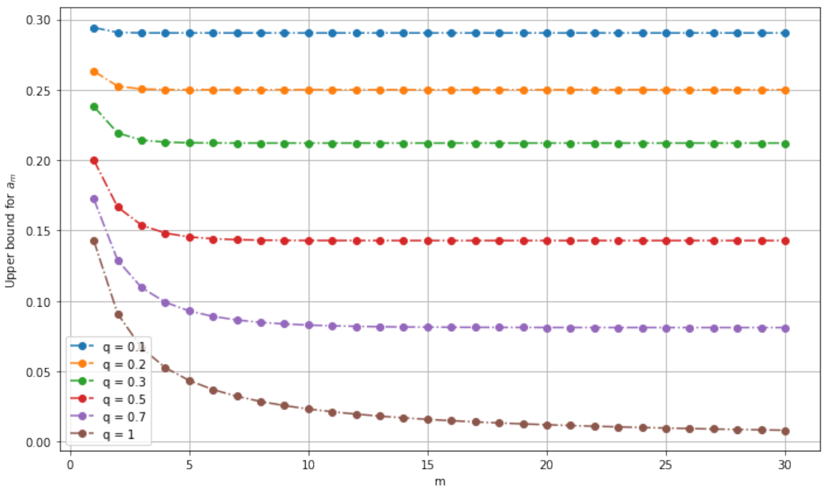


FIGURE 3. Variation of the coefficient bound $a_m \leq \frac{1-\sigma}{[q[m]_q(1+\varrho)+\sigma+\varrho]}$ with respect to m for different values of q and fixed parameters $(\sigma, \varrho) = (0.5, 1)$. The plot shows the influence of the q-parameter on the growth behavior of the coefficients, showing a smooth convergence to the classical case when $q = 1$.

Theorem 3.3. *The function $l(z)$ of the form (2.2) belongs to the class $\Sigma^*_{C_q}(\sigma, \varrho)$ for $z \in D$ if and only if it satisfies the inequality*

$$\sum_{m=1}^{\infty} q[m]_q(q[m]_q(1+\varrho)+\varrho+\sigma)a_m \leq (1-\sigma),$$

where $\varrho \geq 0$ and $0 \leq \sigma < 1$. The result attains equality for the function $l(z)$ given by

$$l(z) = \frac{1}{z} + \sum_{m=1}^{\infty} \frac{1-\sigma}{q[m]_q(q[m]_q(1+\varrho)+\varrho+\sigma)a_m} z^m.$$

Proof. We may use a similar argument to that in Theorem 3.1 to prove this theorem. □

Theorem 3.4. *The class $\Sigma^*_{S_q}(\sigma, \varrho)$ is a convex set.*

Proof. Let the functions

$$l_j(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_{m,j} z^m, \quad a_{m,j} \geq 0, \; j = 1, 2$$

be in the class $\Sigma^*_{S_q}(\sigma,\varrho)$. We have to show that the function $k(z)$ defined by $k(z)=\zeta l_1(z)+(1-\zeta)l_2(z),\ 0\le\zeta<1$ in the class $\Sigma^*_{S_q}(\sigma,\varrho)$.

$$k(z)=\frac{1}{z}+\sum_{m=1}^{\infty}[\zeta a_{m,1}+(1-\zeta)a_{m,2}]z^m.$$

Using Theorem 3.1 we get

$$\begin{aligned}&\sum_{m=1}^{\infty}[q[m]_q(1+\varrho)+\sigma+\varrho]\zeta a_{m,1}+\sum_{m=1}^{\infty}[q[m]_q(1+\varrho)+\sigma+\varrho](1-\zeta)a_{m,2}\\&\quad\le\zeta(1-\sigma)+(1-\zeta)(1-\sigma)\\&\quad\le(1-\sigma).\end{aligned}$$

This implies $k(z)\in\Sigma^*_{S_q}(\sigma,\varrho)$. Hence $\Sigma^*_{S_q}(\sigma,\varrho)$ is convex. □

Theorem 3.5. *The class $\Sigma^*_{C_q}(\sigma,\varrho)$ is a convex set.*

4. Radius of Meromorphically q-Starlikeness and q-Convexity

In [6], Alatawi et al determined the radius of q-starlikeness for the class $MS_q(k,A,B)$. Following a similar technique, we determined the radius of meromorphically q-starlikeness and the radius of meromorphically q-convexity for the class $\Sigma^*_{S_q}(\sigma,\varrho)$.

Theorem 4.1. *Let the function l defined by(2.2) be in the class $\Sigma^*_{S_q}(\beta,\varrho)$ then l is meromorphically q-starlike of order $\zeta(0\le\zeta<1)$ in the disk $|z|<r_1$, where*

$$r_1=\inf_{m\ge1}\left[\frac{(1-\zeta)[q[m]_q(\varrho+1)+(\varrho+\sigma)]}{(q[m]_q+2-\zeta)(1-\sigma)}\right]^{\frac{1}{m+1}},\quad(m\ge1).$$

Proof. Given $l\in\Sigma^*_{S_q}(\sigma,\varrho)$ and l is meromorphically q starlike of order ζ, we have

$$\left|\frac{qzD_ql(z)}{l(z)}+1\right|<(1-\zeta).\tag{4.1}$$

For the left hand side of (4.1) we have,

$$\left|\frac{qzD_ql(z)}{l(z)}+1\right|\le\frac{\sum\limits_{m=1}^{\infty}(q[m]_q+1)a_m\,|z|^{m+1}}{1-\sum\limits_{m=1}^{\infty}a_m\,|z|^{m+1}}$$

The last expression is less than $1-\zeta$ if

$$\sum_{m=1}^{\infty} \frac{q[m]_q + 2 - \zeta}{1-\zeta} a_m \left|z\right|^{m+1} < 1$$

using the fact that $l(z) \in \Sigma^*_{S_q}(\sigma, \varrho)$ if and only if

$$\sum_{m=1}^{\infty} \frac{[q[m]_q(\varrho+1) + (\varrho+\sigma)]}{1-\sigma} a_m \leq 1.$$

(4.1) will be true, if

$$\frac{q[m]_q + 2 - \zeta}{1-\zeta} \left|z\right|^{m+1} \leq \frac{[q[m]_q(\varrho+1) + (\varrho+\sigma)]}{1-\sigma}$$

equivalently

$$|z| \leq \left\{ \frac{(1-\zeta)[q[m]_q(\varrho+1) + (\varrho+\sigma)]}{(q[m]_q + 2 - \zeta)(1-\sigma)} \right\}^{\frac{1}{m+1}}$$

thus showing that the family is q-starlike. □

The figure below shows how the radius of meromorphically q-starlikeness of the function $l \in \Sigma^*_{S_q}(\sigma, \varrho)$ alters as we vary the parameter q, keeping other parameters fixed. The figure gives an idea of how the values of q affect the region where the function remains meromorphically q-starlike.

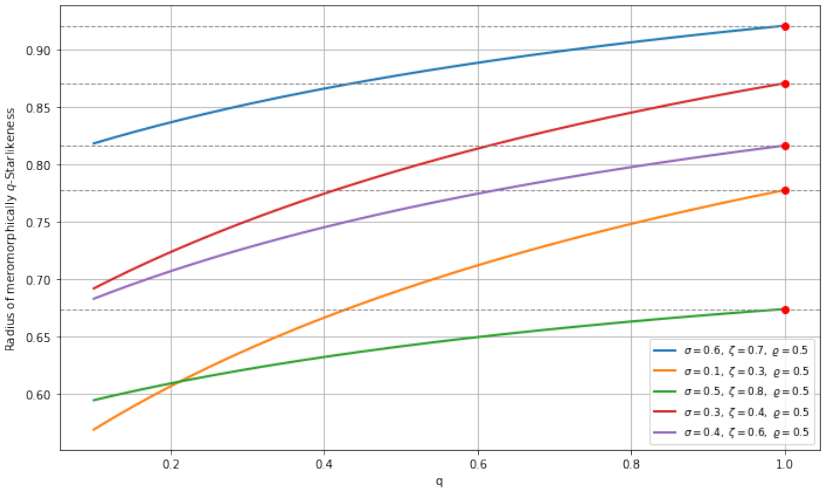


FIGURE 4. Variation of the radius of meromorphic q-starlikeness with respect to the parameter q for several parameter sets (σ, ζ, ϱ). Each curve corresponds to a specific parameter set, while the dashed horizontal lines and red dots mark the classical radius values at $q = 1$. The plot shows that the radius generally decreases for smaller q and gradually increases toward the classical limit as $q \to 1$.

Theorem 4.2. *Let the function l defined by* (2.2) *be in the class* $\Sigma^*_{S_q}(\sigma,\varrho)$ *then l is meromorphically q-convex of order $\eta(0\le\eta<1)$ on the disk $|z|<r_2$, where*

$$r_2=\inf_{m\ge 1}\left[\frac{(1-\eta)[q[m]_q(\varrho+1)+(\varrho+\sigma)]}{q[m]_q(q[m]_q+2-\eta)(1-\sigma)}\right]^{\frac{1}{m+1}},\quad (m\ge 1).$$

Proof. Given $l\in\Sigma^*_{S_q}(\sigma,\varrho)$ and l is meromorphically q convex of order η, we have

$$\left|\frac{D_q(qzD_ql(z))}{D_ql(z)}+1\right|<(1-\eta). \tag{4.2}$$

For the left hand side of (4.2) we have,

$$\left|\frac{D_q(qzD_ql(z))}{D_ql(z)}+1\right|\le\frac{\sum_{m=1}^{\infty}q[m]_q(q[m]_q+1)a_m|z|^{m+1}}{1-\sum_{m=1}^{\infty}a_m|z|^{m+1}}.$$

The last expression is less than $1-\eta$ if

$$\sum_{m=1}^{\infty}\frac{q[m]_q(q[m]_q+2-\eta)}{1-\eta}a_m|z|^{m+1}<1,$$

using the fact that $l(z)\in\Sigma^*_{S_q}(\sigma,\varrho)$ if and only if

$$\sum_{m=1}^{\infty}\frac{[q[m]_q(\varrho+1)+(\varrho+\sigma)]}{1-\sigma}a_m\le 1.$$

(4.2) will be true if

$$\frac{q[m]_q(q[m]_q+2-\eta)}{1-\eta}|z|^{m+1}\le\frac{[q[m]_q(\varrho+1)+(\varrho+\sigma)]}{1-\sigma}.$$

Equivalently

$$|z|\le\left\{\frac{(1-\eta)[q[m]_q(\varrho+1)+(\varrho+\sigma)]}{q[m]_q(q[m]_q+2-\eta)(1-\sigma)}\right\}^{\frac{1}{m+1}}.$$

that yields the q-convexity of the family. □

The figure below shows how the radius of meromorphically q-convexity of the function $l\in\Sigma^*_{S_q}(\sigma,\varrho)$ alters as we vary the parameter q, keeping other parameters fixed. The figure gives an idea of how the values of q affect the region where the function remains meromorphically q-convex.

In the figure indicating the radius of convexity and starlikeness, the infimum is considered in the limit as $m\to\infty$. Since it is not practical to compute this exactly, we approximate the infimum by evaluating the relevant expression for 100000 values of m.

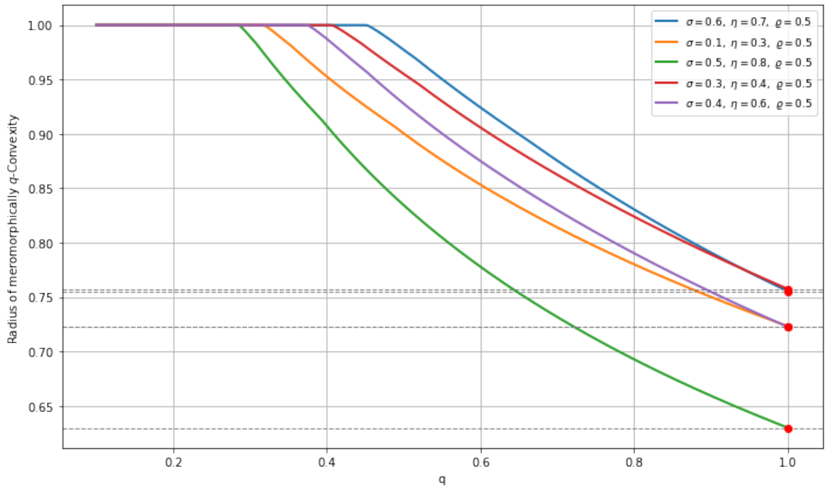


FIGURE 5. Dependence of the radius of meromorphic q-convexity on the q for various combinations of (σ, η, ϱ). Each curve represents a different parameter set, while the dashed lines and red markers denote the corresponding classical radii at $q = 1$. The figure demonstrates that the radius increases for smaller q and approaches its classical limit as $q \to 1$.

5. Modified Hadamard Product

Let the functions l(z) and h(z) be defined by

$$l(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m, \qquad h(z) = \frac{1}{z} + \sum_{m=1}^{\infty} b_m z^m, (z \in D). \tag{5.1}$$

The Hadamard product of $l(z)$ and $h(z)$ is defined by

$$(l * h)(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m b_m z^m.$$

Theorem 5.1. *Let the functions l and h given by (5.1) be in the class $\Sigma^*_{S_q}(\sigma, \varrho)$ then $(l * h)(z) \in \Sigma^*_{S_q}(\gamma, \varrho)$, where*

$$\gamma = 1 - \frac{(1-\sigma)^2(1+\varrho)(q+1)}{(1-\sigma)^2 + (q(1+\varrho) + \sigma + \varrho)^2}, \tag{5.2}$$

$0 < \beta \leq 1$, $0 \leq \gamma < 1$, $\varrho \geq 0$ for all z in D.

Proof. The functions l and h are in class $\Sigma^*_{S_q}(\sigma, \varrho)$ then we have

$$\frac{\sum_{m=1}^{\infty} (q[m]_q(1+\varrho) + \sigma + \varrho)}{1-\sigma} a_m \leq 1,$$

and

$$\frac{\sum\limits_{m=1}^{\infty}(q[m]_q(1+\varrho)+\sigma+\varrho)}{1-\sigma}b_m \leq 1.$$

By the Cauchy-Schwartz inequality we have

$$\frac{\sum\limits_{m=1}^{\infty}(q[m]_q(1+\varrho)+\sigma+\varrho)}{1-\sigma}\sqrt{a_m b_m} \leq 1.$$

We want to find the largest γ such that

$$\frac{\sum\limits_{m=1}^{\infty}(q[m]_q(1+\varrho)+\gamma+\varrho)}{1-\gamma}\,|a_m|\,|b_m| \leq 1.$$

It is enough to show that

$$\frac{(q[m]_q(1+\varrho)+\gamma+\varrho)}{1-\gamma}a_m b_m \leq \frac{(q[m]_q(1+\varrho)+\sigma+\varrho)}{1-\sigma}\sqrt{a_m b_m},$$

that is,

$$\sqrt{a_m b_m} \leq \frac{(q[m]_q(1+\varrho)+\sigma+\varrho)}{1-\sigma}\frac{1-\gamma}{(q[m]_q(1+\varrho)+\gamma+\varrho)}.$$

Note that

$$\sqrt{a_m b_m} \leq \frac{1-\sigma}{(q[m]_q(1+\varrho)+\sigma+\varrho)}.$$

We need to prove that

$$\frac{1-\sigma}{(q[m]_q(1+\varrho)+\sigma+\varrho)} \leq \frac{(q[m]_q(1+\varrho)+\sigma+\varrho)}{1-\sigma}\frac{1-\gamma}{(q[m]_q(1+\varrho)+\gamma+\varrho)}.$$

That is,

$$\frac{(1-\sigma)^2}{(q[m]_q(1+\varrho)+\sigma+\varrho)^2} \leq \frac{1-\gamma}{(q[m]_q(1+\varrho)+\gamma+\varrho)}.$$

This gives

$$\gamma \leq 1-\frac{(1-\sigma)^2(1+\varrho)(q[m]_q+1)}{(1-\sigma)^2+(q[m]_q(1+\varrho)+\sigma+\varrho)^2}.$$

Let

$$G(m) = 1-\frac{(1-\sigma)^2(1+\varrho)(q[m]_q+1)}{(1-\sigma)^2+(q[m]_q(1+\varrho)+\sigma+\varrho)^2}.$$

$G(m)$ is an increasing function of m. Taking m=1, we get (5.2). □

Theorem 5.2. *Let the functions l and h given by* (5.1) *be in the class* $\Sigma^*_{S_q}(\sigma,\varrho)$ *Then the function defined by* $g(z)=\frac{1}{z}+\sum\limits_{m=1}^{\infty}(a_m^2+b_m^2)z^m$ *is in the class* $\Sigma^*_{S_q}(\beta,\varrho)$, *where*

$$\beta \le 1-\frac{2(1-\sigma)^2(1+\varrho)(q+1)}{2(1-\sigma)^2+(q(1+\varrho)+\sigma+\varrho)^2},$$

$0\le\beta<1$, $0\le\sigma<1$, $\varrho\ge 0$ *for all z in D.*

Proof. We want to find the largest β such that

$$\frac{\sum\limits_{m=1}^{\infty}(q[m]_q(1+\varrho)+\beta+\varrho)}{1-\beta}(a_m^2+b_m^2)\le 1.$$

l and h given by (5.1) be in class $\Sigma^*_{S_q}(\sigma,\varrho)$. We have

$$\sum_{m=1}^{\infty}\left[\frac{(q[m]_q(1+\varrho)+\sigma+\varrho)}{1-\sigma}\right]^2 a_m^2 \le \sum_{m=1}^{\infty}\left[\frac{(q[m]_q(1+\varrho)+\sigma+\varrho)}{1-\sigma}a_m\right]^2\le 1.$$

$$\sum_{m=1}^{\infty}\left[\frac{(q[m]_q(1+\varrho)+\sigma+\varrho)}{1-\sigma}\right]^2 b_m^2 \le \sum_{m=1}^{\infty}\left[\frac{(q[m]_q(1+\varrho)+\sigma+\varrho)}{1-\sigma}b_m\right]^2\le 1.$$

Hence

$$\sum_{m=1}^{\infty}\frac{1}{2}\left[\frac{(q[m]_q(1+\varrho)+\sigma+\varrho)}{1-\sigma}\right]^2(a_m^2+b_m^2)\le 1.$$

$$\frac{(q[m]_q(1+\varrho)+\beta+\varrho)}{1-\beta}\le\frac{1}{2}\left[\frac{(q[m]_q(1+\varrho)+\sigma+\varrho)}{1-\sigma}\right]^2.$$

That is,

$$\beta\le 1-\frac{2(1-\sigma)^2(1+\varrho)(q[m]_q+1)}{2(1-\sigma)^2+(q[m]_q(1+\varrho)+\sigma+\varrho)^2}.$$

Since the right hand side of the above equation increases as m increases, setting $m=1$ gives the minimum value. This gives

$$\beta\le 1-\frac{2(1-\sigma)^2(1+\varrho)(q+1)}{2(1-\sigma)^2+(q(1+\varrho)+\sigma+\varrho)^2}.\qquad\square$$

6. Integral Operator

In this section, let us discuss q-integral transforms of functions belonging to class $\Sigma^*_{S_q}(\sigma,\varrho)$ discussed by Alatawi et al [6].

Theorem 6.1. *Let $l(z)$ given by (2.2) be in the class $\Sigma^*_{S_q}(\sigma, \varrho)$. Then the integral transform*

$$\psi_c(z) = \frac{[c]_q}{z^{c+1}} \int_0^z t^c l(t) D_q t, \quad 0 < c < \infty,$$

is in the class $\Sigma^*_{S_q}(\sigma, \varrho)$.

Proof. We have $l(z) \in \Sigma^*_{S_q}(\sigma, \varrho)$. That implies

$$\frac{(q[m]_q(1+\varrho) + \sigma + \varrho)a_m}{1-\sigma} \leq 1.$$

For $l(z) \in \Sigma^*_{S_q}(\sigma, \varrho)$, the integral operator is

$$\psi_c(z) = \frac{[c]_q}{z^{c+1}} \int_0^z t^c l(t) D_q t = \frac{1}{z} + \sum_{m=1}^{\infty} \frac{[c]_q}{[m+c+1]_q} a_m z^{m+c+1}.$$

It is enough to show that

$$\sum_{m=1}^{\infty} \frac{[c]_q(q[m]_q(1+\varrho) + \sigma + \varrho)a_m}{(1-\sigma)[m+c+1]_q} \leq 1.$$

Note that $[c]_q < [c+m+1]_q$.

$$\sum_{m=1}^{\infty} \frac{[c]_q(q[m]_q(1+\varrho) + \sigma + \varrho)a_m}{(1-\sigma)[m+c+1]_q} \leq \frac{\sum_{m=1}^{\infty}(q[m]_q(1+\varrho) + \sigma + \varrho)}{1-\sigma} a_m \leq 1.$$

Thus, $\psi_c(z) \in \Sigma^*_{S_q}(\sigma, \varrho)$. □

7. Conclusions

In this work, we have discussed two subclasses, denoted by $\Sigma^*_{S_q}(\sigma, \varrho)$ and $\Sigma^*_{C_q}(\sigma, \varrho)$ of meromorphic functions associated with the q-derivative and studied geometric properties like coefficient bounds, radius of meromorphically q-starlikeness, radius of meromorphically q-convexity, Modified Hadamard product and Integral operators of the class $\Sigma^*_{S_q}(\sigma, \varrho)$. To support theoretical conclusions, we provided graphs showing how the bounds of a_m, radius of meromorphically q-starlikeness, radius of meromorphically q-convexity change with parameter q. These visualizations increase the understanding of the geometric behavior and demonstrate that the results agree with classic examples $q \to 1$.

Acknowledgment. The first author gratefully acknowledges the Government of Maharashtra for financial support through the Mahatma Jyotiba Phule Research Fellowship (MJPRF). The authors also thank the editors and reviewers for their valuable comments and suggestions, which improved this article.

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