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Improved Bounds of Hermite Hadamard Inequality for Several Kinds of Convexities via Fractional Integral

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Abstract. This research work deals with generalizations and refinements of bounds of the first inequality of the celebrated Hermite Hadamard dual inequality via Riemann-Liouville fractional integrals. Specifically, we explore generalized integral inequalities for several kinds of convexities using fractional integrals by utilizing the newly defined class of (s_1, s_2, t_1, t_2)-convex functions. The proposed research work establishes the superiority of an enhanced version of the power-mean integral inequality and Hölder integral inequality, termed as improved power-mean and Hölder-İşcan integral inequalities, over the traditional ones.

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1. Introduction

J. L. W. V. Jensen once remarked: "It seems to me that the notion of a convex function is just as fundamental as that of a positive or an increasing function. If I am not mistaken, this concept deserves a place in elementary treatments of real function theory."

Convexity is a straightforward and intuitive concept, with its origins dating back to Archimedes around 250 B.C. In his renowned estimation of π, he used inscribed and circumscribed regular polygons and noted a fundamental property: the perimeter of a convex shape is always less than that of any other convex shape that encloses it.

In fact, convexity is something we encounter regularly in daily life. A familiar example is our ability to stand upright—this stability is maintained as long as the vertical projection of our center of gravity falls

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within the convex hull formed by our feet. Beyond such physical intuition, convexity plays a significant role in various fields, including industry, business, medicine and art. It is also central to critical problems such as optimal resource allocation and the analysis of equilibrium in non-cooperative games.

Theorem 1.1. *Let $\varsigma : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function with $\varrho_1 < \varrho_2$ and $\varrho_1, \varrho_2 \in I$. Then we have*

$$\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right) \leq \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2}\varsigma(\xi)d\xi \leq \frac{\varsigma(\varrho_1)+\varsigma(\varrho_2)}{2}.$$

The double inequality presented above is commonly referred to in the literature as the Hermite Hadamard inequality. For a comprehensive overview of its various generalizations, extensions and applications across both pure and applied sciences, readers may consult the works cited in [1–11], [15–26], along with the references therein. In recent works, Sahoo et al. [18–20] established Hermite Hadamard type inequalities for normed linear convex functions, composite convex mappings and multiplying $m-$complex functions, further expanding the scope of these classical inequalities.

The quest for the sharper bounds and extensions of the Hermite Hadamard inequality has led to the introduction of numerous generalized convexity classes, such as s-convex, P-convex and (r, s)-convex functions, each yielding tailored inequalities. However, this proliferation of special cases has created a fragmented landscape. A significant challenge is the lack of an overarching structure that not only encapsulates these varieties but also allows for a systematic comparison of the precision of the inequalities they generate.

In this paper, we address this challenge by working with the comprehensive class of (s_1, s_2, t_1, t_2)-convex functions introduced in [12]. The principal motivation for this work is not merely to add another generalization to the literature, but to provide a unifying analytic framework. This class acts as a generating function for convexity; by choosing specific parameters (s_1, s_2, t_1, t_2), one can recover a wide spectrum of known convexity types, as detailed in Remark 1.3.

Our objective is twofold:

(i) **Unification:** To derive fractional Hermite Hadamard type inequalities for this general class, thereby obtaining “master” inequalities that contain many previous results as immediate corollaries.

(ii) **Refinement and Sharpness:** To move beyond straightforward applications of standard inequalities. We leverage more sophisticated tools like the Hölder İşcan and improved power mean integral inequalities. One of the central technical aim is to demonstrate that these refined methods yield bounds that are strictly sharper than those derived from their classical counterparts. This analytical comparison of bounds constitutes a key depth of our work, addressing the need for more profound insights beyond combinatorial exercises.

In Section 2, we establish foundational estimates for the left-hand bound of the inequality using standard techniques. Section 3 is devoted to achieving refined versions of these bounds using more advanced inequalities. The core of our contribution, the demonstration of sharpness, is presented in the new Section 4, which provides a comparative analysis. A concluding discussion outlines the unifying power of our approach and suggests promising directions for future research.

Now, we are going to recall the definition of (s_1, s_2, t_1, t_2)-convex function extracted from [12].

Definition 1.2. Let $(s_1, s_2, t_1, t_2) \in [0, 1]^4$. A function $\eta : I \subseteq [0, \infty) \to [0, \infty)$ is said to be (s_1, s_2, t_1, t_2)-*convex (concave) in mixed kind* if

$$\eta(tx + (1-t)y) \leq (\geq)\, t^{s_1 t_1}\eta(x) + (1 - t^{s_2})^{t_2}\eta(y), \tag{1.1}$$

for all $x, y \in I$ and $t \in [0, 1]$.

Remark 1.3. In Definition 1.2 by choosing different values of t_1, t_2, s_1 and s_2 we can get different cases. For example, if we choose $s_1 = s_2 = t_1 = t_2 = 1$ in (1.1), we get the ordinary convex (concave) function. For other cases we refer the reader to [1].

2. Estimated Left Bounds of Hermite Hadamard Inequality for the Class of (s_1, s_2, t_1, t_2)-convex function Via Fractional Integral

Let $\mathbb{R}$ denote the set of real numbers. Unless stated otherwise, the interval $I = [\varrho_1, \varrho_2]$ is assumed to be a subset of $\mathbb{R}$ throughout this paper.

In 2013, Noor et. al. proved the following identity related to left bound of Hermite Hadamard inequality involving fractional integrals, which we recall here:

Lemma 2.1 ([14]). *Let $f : [\varrho_1, \varrho_2] \to \mathbb{R}$ be a function that is twice differentiable on the open interval (ϱ_1, ϱ_2), where $\varrho_1 < \varrho_2$. If ς'' belongs to the space $L[\varrho_1, \varrho_2]$, then the following identity involving fractional integrals holds:*

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1) + J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right] - \varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)$$

$$= \frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\int_0^1 (1-\chi)^{\alpha+1}\left[\varsigma''\left(\frac{1+\chi}{2}\varrho_1+\frac{1-\chi}{2}\varrho_2\right)\right.$$
$$\left.+\varsigma''\left(\frac{1-\chi}{2}\varrho_1+\frac{1+\chi}{2}\varrho_2\right)\right]d\chi.$$

The above lemma plays a key role in proving our main results of this article.

Now, we are going to prove our main results related to left bound of fractional Hermite Hadamard inequality for the class of (s_1, s_2, t_1, t_2)-convex function.

Theorem 2.2. *Let $f : [\varrho_1, \varrho_2] \to \mathbb{R}$ be a function that is twice differentiable on the open interval (ϱ_1, ϱ_2), where $\varrho_1 < \varrho_2$. Suppose that $\varsigma'' \in L[\varrho_1, \varrho_2]$ and that $|\varsigma''|$ satisfies the condition of being a (s_1, s_2, t_1, t_2)-convex function of mixed type. Under these assumptions, the following inequality involving fractional integrals holds:*

(2.1)
$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$\leq \frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left[A_1\left|\varsigma''(\varrho_1)\right|+\frac{|\varsigma''(\varrho_2)|}{2^{s_2t_2}(\alpha+s_2t_2+2)}\right.$$
$$\left.+\frac{|\varsigma''(\varrho_1)|}{2^{s_1t_1}(\alpha+s_1t_1+2)}+A_2\left|\varsigma''(\varrho_2)\right|\right],$$

where

$$A_1 = \int_0^1 (1-\chi)^{\alpha+1}\left\{1-\left(\frac{1-\chi}{2}\right)^{s_1}\right\}^{t_1}d\chi \tag{2.2}$$

and

$$A_2 = \int_0^1 (1-\chi)^{\alpha+1}\left\{1-\left(\frac{1-\chi}{2}\right)^{s_2}\right\}^{t_2}d\chi. \tag{2.3}$$

Proof. Using Lemma 2.1 and the fact that $|\varsigma''|$ is $(s_1,\ s_2,\ t_1,\ t_2)$-convex function of the mixed kind, we have

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$=\left|\frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\int_0^1 (1-\chi)^{\alpha+1}\left[\varsigma''\left(\frac{1+\chi}{2}\varrho_1+\frac{1-\chi}{2}\varrho_2\right)\right.\right.$$
$$\left.\left.+\varsigma''\left(\frac{1-\chi}{2}\varrho_1+\frac{1+\chi}{2}\varrho_2\right)\right]d\chi\right|$$

$$
\begin{aligned}
&\leq \left|\frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\int_0^1 (1-\chi)^{\alpha+1}\varsigma''\left(\frac{1+\chi}{2}\varrho_1+\frac{1-\chi}{2}\varrho_2\right)d\chi\right| \\
&\quad+\left|\frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\int_0^1 (1-\chi)^{\alpha+1}\varsigma''\left(\frac{1-\chi}{2}\varrho_1+\frac{1+\chi}{2}\varrho_2\right)d\chi\right| \\
&\leq \frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\int_0^1 (1-\chi)^{\alpha+1}\left|\varsigma''\left(\frac{1+\chi}{2}\varrho_1+\frac{1-\chi}{2}\varrho_2\right)\right|d\chi \\
&\quad+\frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\int_0^1 (1-\chi)^{\alpha+1}\left|\varsigma''\left(\frac{1-\chi}{2}\varrho_1+\frac{1+\chi}{2}\varrho_2\right)\right|d\chi \\
&\leq \frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left(\int_0^1 (1-\chi)^{\alpha+1}\left[\left\{1-\left(\frac{1-\chi}{2}\right)^{s_1}\right\}^{t_1}\left|\varsigma''(\varrho_1)\right|\right.\right. \\
&\quad\left.+\left(\frac{1-\chi}{2}\right)^{s_2t_2}\left|\varsigma''(\varrho_2)\right|\right]d\chi+\int_0^1 (1-\chi)^{\alpha+1}\left[\left(\frac{1-\chi}{2}\right)^{s_1t_1}\left|\varsigma''(\varrho_1)\right|\right. \\
&\quad\left.\left.+\left\{1-\left(\frac{1-\chi}{2}\right)^{s_2}\right\}^{t_2}\left|\varsigma''(\varrho_2)\right|\right]d\chi\right).
\end{aligned}
$$

By putting

$$
\begin{aligned}
A_1 &= \int_0^1 (1-\chi)^{\alpha+1}\left\{1-\left(\frac{1-\chi}{2}\right)^{s_1}\right\}^{t_1}d\chi, \\
A_2 &= \int_0^1 (1-\chi)^{\alpha+1}\left\{1-\left(\frac{1-\chi}{2}\right)^{s_2}\right\}^{t_2}d\chi,
\end{aligned}
$$

$$
\int_0^1 (1-\chi)^{\alpha+1}\left(\frac{1-\chi}{2}\right)^{s_1t_1}d\chi=\frac{1}{2^{s_1t_1}(\alpha+s_1t_1+2)} \tag{2.4}
$$

and

$$
\int_0^1 (1-\chi)^{\alpha+1}\left(\frac{1-\chi}{2}\right)^{s_2t_2}d\chi=\frac{1}{2^{s_2t_2}(\alpha+s_2t_2+2)}. \tag{2.5}
$$

□

Remark 2.3. In Theorem 2.2, we have the following cases:

(i) If we choose $t_1 = t_2 = 1$ in (2.1), we get a result for (s_1, s_2)-convex in 1st kind function.

$$
\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right| \\
&\quad\leq \frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)(\alpha+2)}\left[\left|\varsigma''(\varrho_1)\right|+\left|\varsigma''(\varrho_2)\right|\right].
\end{aligned}
$$

(ii) If we choose $s_2 = s_1 = 1$ in (2.1), we get the result related result for (s_1, s_2)-convex in second kind function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^{\alpha}}\left[J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$\leq \frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left[\frac{|\varsigma''(\varrho_1)|}{2^{s_1}}\left\{C(s_1,\alpha,\chi)+\frac{1}{(\alpha+s_1+2)}\right\}\right.$$
$$\left.+\frac{|\varsigma''(\varrho_2)|}{2^{s_2}}\left\{\frac{1}{(\alpha+s_2+2)}+C(s_2,\alpha,\chi)\right\}\right],$$

where

$$C(s_1,\alpha,\chi)=\int_0^1(1-\chi)^{\alpha+1}(1+\chi)^{s_1}d\chi$$

and

$$C(s_2,\alpha,\chi)=\int_0^1(1-\chi)^{\alpha+1}(1+\chi)^{s_2}d\chi.$$

(iii) If we choose $s_1 = s_2 = r$ and $t_1 = t_2 = s$ in (2.1), we get result for (s, r)-convex function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^{\alpha}}\left[J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$\leq \frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left[\int_0^1(1-\chi)^{\alpha+1}\left\{1-\left(\frac{1-\chi}{2}\right)^r\right\}^s d\chi\right.$$
$$\left.+\left(\frac{1}{2^{rs}(\alpha+rs+2)}\right)\right]\left[\left|\varsigma''(\varrho_1)\right|+\left|\varsigma''(\varrho_2)\right|\right].$$

(iv) If we choose $s_1 = s_2 = s$ and $t_1 = t_2 = 1$ in (2.1), we get Theorem 5 of [14].

(v) If we choose $t_1 = t_2 = s$, $s_1 = s_2 = 1$ in (2.1), we get Theorem 2 of [14].

(vi) If we choose $t_1 = t_2 = 0$ and $s_1 = s_2 = 1$ in (2.1), we get result for P-convex function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^{\alpha}}\left[J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$\leq \frac{(\varrho_2-\varrho_1)^2}{4(\alpha+1)(\alpha+2)}\left[\left|\varsigma''(\varrho_1)\right|+\left|\varsigma''(\varrho_2)\right|\right].$$

Theorem 2.4. *Let $f : [\varrho_1, \varrho_2] \to \mathbb{R}$ be a function that is twice differentiable on the open interval (ϱ_1, ϱ_2), where $\varrho_1 < \varrho_2$. Suppose that $\varsigma'' \in$*

$L[\varrho_1, \varrho_2]$ and that $|\varsigma''|^q$ satisfies the condition of being a (s_1, s_2, t_1, t_2)-convex function of mixed type. Under these assumptions, the following inequality involving fractional integrals holds:

(2.6)
$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^{\alpha}}\left[J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right| \\
&\quad\le \frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left(\frac{1}{p(\alpha+1)+2}\right)^{\frac{1}{p}}\left[\left(B_1\left|\varsigma''(\varrho_1)\right|^q+\frac{|\varsigma''(\varrho_2)|^q}{2^{s_2t_2}(s_2t_2+1)}\right)^{\frac{1}{q}}\right. \\
&\quad\quad\left.+\left(\frac{|\varsigma''(\varrho_1)|^q}{2^{s_1t_1}(s_1t_1+1)}+B_2\left|\varsigma''(\varrho_2)\right|^q\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where

$$B_1=\int_0^1(1-\chi)\left\{1-\left(\frac{1-\chi}{2}\right)^{s_1}\right\}^{t_1}d\chi$$

and

$$B_2=\int_0^1(1-\chi)\left\{1-\left(\frac{1-\chi}{2}\right)^{s_2}\right\}^{t_2}d\chi.$$

Proof. Using Lemma 2.1, Hölder's inequality [2] and $(s_1,\ s_2,\ t_1,\ t_2)$-convexity of $|\varsigma''|^q$ in mixed kind, we have

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^{\alpha}}\left[J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right| \\
&\quad=\left|\frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\int_0^1(1-\chi)^{\alpha+1}\left[\varsigma''\left(\frac{1+\chi}{2}\varrho_1+\frac{1-\chi}{2}\varrho_2\right)\right.\right. \\
&\quad\quad\left.\left.+\varsigma''\left(\frac{1-\chi}{2}\varrho_1+\frac{1+\chi}{2}\varrho_2\right)\right]d\chi\right| \\
&\quad\le\left|\frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\int_0^1(1-\chi)^{\alpha+1}\varsigma''\left(\frac{1+\chi}{2}\varrho_1+\frac{1-\chi}{2}\varrho_2\right)d\chi\right| \\
&\quad\quad+\left|\frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\int_0^1(1-\chi)^{\alpha+1}\varsigma''\left(\frac{1-\chi}{2}\varrho_1+\frac{1+\chi}{2}\varrho_2\right)d\chi\right| \\
&\quad\le\frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left[\int_0^1\left|(1-\chi)^{\alpha+1}\varsigma''\left(\frac{1+\chi}{2}\varrho_1+\frac{1-\chi}{2}\varrho_2\right)\right|d\chi\right. \\
&\quad\quad\left.+\int_0^1\left|(1-\chi)^{\alpha+1}\varsigma''\left(\frac{1-\chi}{2}\varrho_1+\frac{1+\chi}{2}\varrho_2\right)\right|d\chi\right] \\
&\quad\le\frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left[\left(\int_0^1\left|(1-\chi)^{\alpha+1}\right|^p d\chi\right)^{\frac{1}{p}}\right.
\end{aligned}$$

$$\times\left(\int_0^1\left|\varsigma''\left(\frac{1+\chi}{2}\varrho_1+\frac{1-\chi}{2}\varrho_2\right)\right|^q d\chi\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1\left|(1-\chi)^{\alpha+1}\right|^p d\chi\right)^{\frac{1}{p}}\left(\int_0^1\left|\varsigma''\left(\frac{1-\chi}{2}\varrho_1+\frac{1+\chi}{2}\varrho_2\right)\right|^q d\chi\right)^{\frac{1}{q}}\Bigg]$$
$$\leq\frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left(\frac{1}{p(\alpha+1)+2}\right)^{\frac{1}{p}}\Bigg[\Bigg(\left|\varsigma''(\varrho_2)\right|^q\int_0^1\left(\frac{1-\chi}{2}\right)^{s_2t_2}d\chi$$
$$+\left|\varsigma''(\varrho_1)\right|^q\int_0^1\left\{1-\left(\frac{1-\chi}{2}\right)^{s_1}\right\}^{t_1}d\chi\Bigg)^{\frac{1}{q}}$$
$$+\Bigg(\left|\varsigma''(\varrho_1)\right|^q\int_0^1\left(\frac{1-\chi}{2}\right)^{s_1t_1}d\chi$$
$$+\left|\varsigma''(\varrho_2)\right|^q\int_0^1\left\{1-\left(\frac{1-\chi}{2}\right)^{s_2}\right\}^{t_2}d\chi\Bigg)^{\frac{1}{q}}\Bigg].$$

By putting

$$B_1=\int_0^1(1-\chi)\left\{1-\left(\frac{1-\chi}{2}\right)^{s_1}\right\}^{t_1}d\chi,$$
$$B_2=\int_0^1(1-\chi)\left\{1-\left(\frac{1-\chi}{2}\right)^{s_2}\right\}^{t_2}d\chi,$$
$$\int_0^1\left(\frac{1-\chi}{2}\right)^{s_2t_2}d\chi=\frac{1}{2^{s_2t_2}(s_2t_2+1)},$$

and

$$\int_0^1\left(\frac{1-\chi}{2}\right)^{s_1t_1}d\chi=\frac{1}{2^{s_1t_1}(s_1t_1+1)}.$$ □

Remark 2.5. In Theorem 2.4, we have the following cases:

(i) If we choose $t_1=t_2=1$ in (2.6), we get result for (s_1,s_2)-convex in 1st kind function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$\leq\frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left(\frac{1}{p(\alpha+1)+2}\right)^{\frac{1}{p}}$$
$$\times\left[\left(\left|\varsigma''(\varrho_1)\right|^q\left\{1-\frac{1}{2^{s_1}(s_1+1)}\right\}+\frac{\left|\varsigma''(\varrho_2)\right|^q}{2^{s_2}(s_2+1)}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{|\varsigma''(\varrho_1)|^q}{2^{s_1}(s_1+1)}+|\varsigma''(\varrho_2)|^q\left\{1-\frac{1}{2^{s_2}(s_2+1)}\right\}\right)^{\frac{1}{q}}\Bigg].$$

(ii) If we choose $s_1 = s_2 = t_1 = t_2 = 1$ in (2.6), we get result for ordinary convex function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$\leq \frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left(\frac{1}{p(\alpha+1)+2}\right)^{\frac{1}{p}}$$
$$\times\left[\left(\frac{3|\varsigma''(\varrho_1)|^q+|\varsigma''(\varrho_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\varsigma''(\varrho_1)|^q+3|\varsigma''(\varrho_2)|^q}{4}\right)^{\frac{1}{q}}\right].$$

(iii) If we choose $s_1 = s_2 = 0$ and $t_1 = t_2 = 1$ in (2.6), we get a result related to Hermite Hadamard type inequality for refinement of quasi-convex function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$\leq \frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left(\frac{1}{p(\alpha+1)+2}\right)^{\frac{1}{p}}\left[|\varsigma''(\varrho_2)|+|\varsigma''(\varrho_1)|\right].$$

(iv) If we choose $s_2 = s_1 = 1$ in (2.6), we get a result for (s_1, s_2)-convex in 2nd kind function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$\leq \frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left(\frac{1}{p(\alpha+1)+2}\right)^{\frac{1}{p}}$$
$$\times\left[\left(\frac{2^{s_1+1}-1}{2^{s_1}(s_1+1)}|\varsigma''(\varrho_1)|^q+\frac{|\varsigma''(\varrho_2)|^q}{2^{s_2}(s_2+1)}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{|\varsigma''(\varrho_1)|^q}{2^{s_1}(s_1+1)}+\frac{2^{s_2+1}-1}{2^{s_2}(s_2+1)}|\varsigma''(\varrho_2)|^q\right)^{\frac{1}{q}}\right].$$

(v) If we choose $s_1 = s_2 = r$ and $t_1 = t_2 = s$ in (2.6), we get a result for (r, s)-convex function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$

$$\leq \frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left(\frac{1}{p(\alpha+1)+2}\right)^{\frac{1}{p}}$$
$$\times\left[\left(\left|\varsigma''(\varrho_1)\right|^q\int_0^1\left\{1-\left(\frac{1-\chi}{2}\right)^r\right\}^s d\chi+\frac{\left|\varsigma''(\varrho_2)\right|^q}{2^{rs}(rs+1)}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\left|\varsigma''(\varrho_1)\right|^q}{2^{rs}(rs+1)}+\left|\varsigma''(\varrho_2)\right|^q\int_0^1\left\{1-\left(\frac{1-\chi}{2}\right)^r\right\}^s d\chi\right)^{\frac{1}{q}}\right].$$

(vi) If we choose $s_1 = s_2 = s$ and $t_1 = t_2 = 1$ in (2.6), we get a result for s-convex in 1st kind function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$\leq \frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left(\frac{1}{p(\alpha+1)+2}\right)^{\frac{1}{p}}$$
$$\times\left[\left(\left|\varsigma''(\varrho_1)\right|^q\left\{1-\frac{1}{2^s(s+1)}\right\}+\frac{\left|\varsigma''(\varrho_2)\right|^q}{2^s(s+1)}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\left|\varsigma''(\varrho_1)\right|^q}{2^s(s+1)}+\left|\varsigma''(\varrho_2)\right|^q\left\{1-\frac{1}{2^s(s+1)}\right\}\right)^{\frac{1}{q}}\right].$$

(vii) If we choose $t_1 = t_2 = 0$ and $s_1 = s_2 = 1$ in (2.6), we get a result for P-convex function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$\leq \frac{(\varrho_2-\varrho_1)^2}{4(\alpha+1)}\left(\frac{1}{p(\alpha+1)+2}\right)^{\frac{1}{p}}\left(\left|\varsigma''(\varrho_1)\right|^q+\left|\varsigma''(\varrho_2)\right|^q\right)^{\frac{1}{q}}.$$

(viii) If we choose $t_1 = t_2 = s$, $s_1 = s_2 = 1$ in (2.6), we get a result for s-convex in 2nd kind function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$\leq \frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left(\frac{1}{p(\alpha+1)+2}\right)^{\frac{1}{p}}\left[\left(\frac{2^{s+1}-1}{2^s(s+1)}\left|\varsigma''(\varrho_1)\right|^q+\frac{\left|\varsigma''(\varrho_2)\right|^q}{2^s(s+1)}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\left|\varsigma''(\varrho_1)\right|^q}{2^s(s+1)}+\frac{2^{s+1}-1}{2^s(s+1)}\left|\varsigma''(\varrho_2)\right|^q\right)^{\frac{1}{q}}\right].$$

Theorem 2.6. *Let $f : [\varrho_1, \varrho_2] \to \mathbb{R}$ be a function that is twice differentiable on the open interval (ϱ_1, ϱ_2), where $\varrho_1 < \varrho_2$. Suppose that $\varsigma'' \in$*

$L[\varrho_1, \varrho_2]$ and that $|\varsigma''|^q$ satisfies the condition of being a (s_1, s_2, t_1, t_2)-convex function of mixed type. Under these assumptions, the following inequality involving fractional integrals holds:

(2.7)

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$\leq \frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\left[\left\{A_1\left|\varsigma''(\varrho_1)\right|^q+\frac{|\varsigma''(\varrho_2)|^q}{2^{s_2t_2}(\alpha+s_2t_2+2)}\right\}^{\frac{1}{q}}\right.$$
$$\left.+\left\{\frac{|\varsigma''(\varrho_1)|^q}{2^{s_1t_1}(\alpha+s_1t_1+2)}+A_2\left|\varsigma''(\varrho_2)\right|^q\right\}^{\frac{1}{q}}\right],$$

where A_1 and A_2 are defined in Theorem 2.2.

Proof. Using Lemma 2.1, power mean inequality [2] and $(s_1,\, s_2,\, t_1,\, t_2)$-convexity of $|\varsigma''|^q$ in mixed kind, we have

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$=\left|\frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\int_0^1(1-\chi)^{\alpha+1}\left[\varsigma''\left(\frac{1+\chi}{2}\varrho_1+\frac{1-\chi}{2}\varrho_2\right)\right.\right.$$
$$\left.\left.+\varsigma''\left(\frac{1-\chi}{2}\varrho_1+\frac{1+\chi}{2}\varrho_2\right)\right]d\chi\right|$$
$$\leq\left|\frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\int_0^1(1-\chi)^{\alpha+1}\varsigma''\left(\frac{1+\chi}{2}\varrho_1+\frac{1-\chi}{2}\varrho_2\right)d\chi\right|$$
$$+\left|\frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\int_0^1(1-\chi)^{\alpha+1}\varsigma''\left(\frac{1-\chi}{2}\varrho_1+\frac{1+\chi}{2}\varrho_2\right)d\chi\right|$$
$$\leq\frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left[\int_0^1\left|(1-\chi)^{\alpha+1}\varsigma''\left(\frac{1+\chi}{2}\varrho_1+\frac{1-\chi}{2}\varrho_2\right)\right|d\chi\right.$$
$$\left.+\int_0^1\left|(1-\chi)^{\alpha+1}\varsigma''\left(\frac{1-\chi}{2}\varrho_1+\frac{1+\chi}{2}\varrho_2\right)\right|d\chi\right]$$
$$\leq\frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left[\left(\int_0^1\left|(1-\chi)^{\alpha+1}\right|d\chi\right)^{1-\frac{1}{q}}\right.$$
$$\times\left(\int_0^1\left|(1-\chi)^{\alpha+1}\right|\left|\varsigma''\left(\frac{1+\chi}{2}\varrho_1+\frac{1-\chi}{2}\varrho_2\right)\right|^q d\chi\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1\left|(1-\chi)^{\alpha+1}\right|d\chi\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\left|(1-\chi)^{\alpha+1}\right|\left|\varsigma''\left(\frac{1-\chi}{2}\varrho_1+\frac{1+\chi}{2}\varrho_2\right)\right|^q d\chi\right)^{\frac{1}{q}}\Bigg]$$

$$\leq\frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}$$

$$\times\Bigg[\Bigg(\left|\varsigma''(\varrho_1)\right|^q\int_0^1(1-\chi)^{\alpha+1}\left\{1-\left(\frac{1-\chi}{2}\right)^{s_1}\right\}^{t_1}d\chi$$

$$+\left|\varsigma''(\varrho_2)\right|^q\int_0^1(1-\chi)^{\alpha+1}\left(\frac{1-\chi}{2}\right)^{s_2t_2}d\chi\Bigg)^{\frac{1}{q}}$$

$$+\Bigg(\left|\varsigma''(\varrho_1)\right|^q\int_0^1(1-\chi)^{\alpha+1}\left(\frac{1-\chi}{2}\right)^{s_1t_1}d\chi$$

$$+\left|\varsigma''(\varrho_2)\right|^q\int_0^1(1-\chi)^{\alpha+1}\left\{1-\left(\frac{1-\chi}{2}\right)^{s_2}\right\}^{t_2}d\chi\Bigg)^{\frac{1}{q}}\Bigg].$$

Using (2.2) to (2.5), we acquire (2.7). □

Remark 2.7. In Theorem 2.6, we have the following cases:

(i) If we choose $s_1=s_2=0$ and $t_1=t_2=1$ in (2.7), we get result for refinement of quasi-convex function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$

$$\leq\frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)(\alpha+2)}\left[\left|\varsigma''(\varrho_2)\right|+\left|\varsigma''(\varrho_1)\right|\right].$$

(ii) If we choose $t_1=t_2=1$ in (2.7), we get result for (s_1,s_2)-convex in 1st kind function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$

$$\leq\frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}$$

$$\times\left[\left\{\left(\frac{1}{\alpha+2}-\frac{1}{2^{s_1}(\alpha+s_1+2)}\right)\left|\varsigma''(\varrho_1)\right|^q+\frac{\left|\varsigma''(\varrho_2)\right|^q}{2^{s_2}(\alpha+s_2+2)}\right\}^{\frac{1}{q}}\right.$$

$$\left.+\left\{\left(\frac{\left|\varsigma''(\varrho_1)\right|^q}{2^{s_1}(\alpha+s_1+2)}\right)+\left(\frac{1}{\alpha+2}-\frac{1}{2^{s_2}(\alpha+s_2+2)}\right)\left|\varsigma''(\varrho_2)\right|^q\right\}^{\frac{1}{q}}\right].$$

(iii) If we choose $s_2 = s_1 = 1$ in (2.7), we get a result for (s_1, s_2)-convex in 2nd kind function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^{\alpha}}\left[J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$\leq \frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}$$
$$\times\left[\left\{\frac{|\varsigma''(\varrho_1)|^q}{2^{s_1}}C(s_1,\alpha,\chi)+\frac{|\varsigma''(\varrho_2)|^q}{2^{s_2}(\alpha+s_2+2)}\right\}^{\frac{1}{q}}\right.$$
$$\left.+\left\{\frac{|\varsigma''(\varrho_1)|^q}{2^{s_1}(\alpha+s_1+2)}+\frac{|\varsigma''(\varrho_2)|^q}{2^{s_2}}C(s_2,\alpha,\chi)\right\}^{\frac{1}{q}}\right].$$

(iv) If we choose $s_1 = s_2 = r$ and $t_1 = t_2 = s$ in (2.7), we get a result for (r, s)-convex function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^{\alpha}}\left[J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$\leq \frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\left[\left(\frac{|\varsigma''(\varrho_1)|^q}{2^{rs}(\alpha+rs+2)}\right.\right.$$
$$\left.+\left|\varsigma''(\varrho_2)\right|^q\int_0^1(1-\chi)^{\alpha+1}\left\{1-\left(\frac{1-\chi}{2}\right)^r\right\}^s d\chi\right)^{\frac{1}{q}}$$
$$+\left(\left|\varsigma''(\varrho_1)\right|^q\int_0^1(1-\chi)^{\alpha+1}\left\{1-\left(\frac{1-\chi}{2}\right)^r\right\}^s d\chi\right.$$
$$\left.\left.+\frac{|\varsigma''(\varrho_2)|^q}{2^{rs}(\alpha+rs+2)}\right)^{\frac{1}{q}}\right].$$

(v) If we choose $t_1 = t_2 = 0$ and $s_1 = s_2 = 1$ in (2.7), we get result for P-convex function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^{\alpha}}\left[J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$\leq \frac{(\varrho_2-\varrho_1)^2}{4(\alpha+1)(\alpha+2)}\left(\left|\varsigma''(\varrho_1)\right|^q+\left|\varsigma''(\varrho_2)\right|^q\right)^{\frac{1}{q}}.$$

(vi) If we choose $s_1 = s_2 = s$ and $t_1 = t_2 = 1$ in (2.7), we get result for s-convex in 1st kind function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^{\alpha}}\left[J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$

$$\leq \frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}$$
$$\times\left[\left\{\left(\frac{1}{\alpha+2}-\frac{1}{2^s(\alpha+s+2)}\right)\left|\varsigma''(\varrho_1)\right|^q+\frac{|\varsigma''(\varrho_2)|^q}{2^s(\alpha+s+2)}\right\}^{\frac{1}{q}}\right.$$
$$\left.+\left\{\frac{|\varsigma''(\varrho_1)|^q}{2^s(\alpha+s+2)}+\left(\frac{1}{\alpha+2}-\frac{1}{2^s(\alpha+s+2)}\right)\left|\varsigma''(\varrho_2)\right|^q\right\}^{\frac{1}{q}}\right].$$

(vii) If we choose $t_1 = t_2 = s$, $s_1 = s_2 = 1$ in (2.7), we get result for s-convex in second kind function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$\leq \frac{(\varrho_2-\varrho_1)^2}{2^{3+\frac{s}{q}}(\alpha+1)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\left[\left\{C(s,\alpha,\chi)\left|\varsigma''(\varrho_1)\right|^q+\frac{|\varsigma''(\varrho_2)|^q}{\alpha+s+2}\right\}^{\frac{1}{q}}\right.$$
$$\left.+\left\{\frac{|\varsigma''(\varrho_1)|^q}{\alpha+s+2}+C(s,\alpha,\chi)\left|\varsigma''(\varrho_2)\right|^q\right\}^{\frac{1}{q}}\right].$$

(viii) If we choose $s_1 = s_2 = t_1 = t_2 = 1$ in (2.7), we get result for ordinary convex function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$\leq \frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)(\alpha+2)}\left(\frac{1}{2(\alpha+3)}\right)^{\frac{1}{q}}$$
$$\times\left[\left\{(\alpha+4)\left|\varsigma''(\varrho_1)\right|^q+(\alpha+2)\left|\varsigma''(\varrho_2)\right|^q\right\}^{\frac{1}{q}}\right.$$
$$\left.+\left\{(\alpha+2)\left|\varsigma''(\varrho_1)\right|^q+(\alpha+4)\left|\varsigma''(\varrho_2)\right|^q\right\}^{\frac{1}{q}}\right].$$

3. Improved Left Bounds of Hermite Hadamard Inequality for the Class of (s_1, s_2, t_1, t_2)Convex Function via Fractional Integral

In this section, we establish our main results concerning the improved left bound of the fractional Hermite Hadamard inequality for the class of (s_1, s_2, t_1, t_2)-convex functions, employing the well-known Hölder-İşcan and the improved power mean integral inequality.

Theorem 3.1. *Let $f : [\varrho_1, \varrho_2] \to \mathbb{R}$ be a function that is twice differentiable on the open interval (ϱ_1, ϱ_2), where $\varrho_1 < \varrho_2$. Suppose that $\varsigma'' \in$*

$L[\varrho_1, \varrho_2]$ and that $|\varsigma''|^q$ satisfies the condition of being a (s_1, s_2, t_1, t_2)-convex function of mixed type. Under these assumptions, the following inequality involving fractional integrals holds:

(3.1)

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$\leq \frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left(\frac{1}{p(\alpha+1)+2}\right)^{\frac{1}{p}}\left[\left(P\left|\varsigma''(\varrho_1)\right|^q+\frac{|\varsigma''(\varrho_2)|^q}{2^{s_2t_2}(s_2t_2+2)}\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(Q\left|\varsigma''(\varrho_1)\right|^q+\frac{|\varsigma''(\varrho_2)|^q}{2^{s_2t_2}(s_2t_2+1)(s_2t_2+2)}\right)^{\frac{1}{q}}$$
$$+\left(\frac{|\varsigma''(\varrho_1)|^q}{2^{s_1t_1}(s_1t_1+2)}+R\left|\varsigma''(\varrho_2)\right|^q\right)^{\frac{1}{q}}+\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}$$
$$\left.\times\left(\frac{|\varsigma''(\varrho_1)|^q}{2^{s_1t_1}(s_1t_1+1)(s_1t_1+2)}+S\left|\varsigma''(\varrho_2)\right|^q\right)^{\frac{1}{q}}\right],$$

where

$$P=\int_0^1(1-\chi)\left\{1-\left(\frac{1-\chi}{2}\right)^{s_1}\right\}^{t_1}d\chi,$$
$$Q=\int_0^1\chi\left\{1-\left(\frac{1-\chi}{2}\right)^{s_1}\right\}^{t_1}d\chi,$$
$$R=\int_0^1(1-\chi)\left\{1-\left(\frac{1-\chi}{2}\right)^{s_2}\right\}^{t_2}d\chi$$

and

$$S=\int_0^1\chi\left\{1-\left(\frac{1-\chi}{2}\right)^{s_2}\right\}^{t_2}d\chi.$$

Proof. Using Lemma 2.1, Hölder İşcan Inequality [4] and (s_1, s_2, t_1, t_2)-convexity of $|\varsigma''|^q$ in mixed kind function then we have

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$=\left|\frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\int_0^1(1-\chi)^{\alpha+1}\left[\varsigma''\left(\frac{1+\chi}{2}\varrho_1+\frac{1-\chi}{2}\varrho_2\right)\right.\right.$$
$$\left.\left.+\varsigma''\left(\frac{1-\chi}{2}\varrho_1+\frac{1+\chi}{2}\varrho_2\right)\right]d\chi\right|$$
$$\leq\left|\frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\int_0^1(1-\chi)^{\alpha+1}\varsigma''\left(\frac{1+\chi}{2}\varrho_1+\frac{1-\chi}{2}\varrho_2\right)d\chi\right|$$

$$+\left|\frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\int_0^1(1-\chi)^{\alpha+1}\varsigma''\left(\frac{1-\chi}{2}\varrho_1+\frac{1+\chi}{2}\varrho_2\right)d\chi\right|$$

$$\leq\frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left[\int_0^1\left|(1-\chi)^{\alpha+1}\varsigma''\left(\frac{1+\chi}{2}\varrho_1+\frac{1-\chi}{2}\varrho_2\right)\right|d\chi\right.$$

$$\left.+\int_0^1\left|(1-\chi)^{\alpha+1}\varsigma''\left(\frac{1-\chi}{2}\varrho_1+\frac{1+\chi}{2}\varrho_2\right)\right|d\chi\right]$$

$$\leq\frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left[\left\{\left(\int_0^1(1-\chi)\left|(1-\chi)^{\alpha+1}\right|^p d\chi\right)^{\frac{1}{p}}\right.\right.$$

$$\times\left(\int_0^1(1-\chi)\left|\varsigma''\left(\frac{1+\chi}{2}\varrho_1+\frac{1-\chi}{2}\varrho_2\right)\right|^q d\chi\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\chi\left|(1-\chi)^{\alpha+1}\right|^p d\chi\right)^{\frac{1}{p}}$$

$$\left.\times\left(\int_0^1\chi\left|\varsigma''\left(\frac{1+\chi}{2}\varrho_1+\frac{1-\chi}{2}\varrho_2\right)\right|^q d\chi\right)^{\frac{1}{q}}\right\}$$

$$+\left\{\left(\int_0^1(1-\chi)\left|(1-\chi)^{\alpha+1}\right|^p d\chi\right)^{\frac{1}{p}}\right.$$

$$\times\left(\int_0^1(1-\chi)\left|\varsigma''\left(\frac{1-\chi}{2}\varrho_1+\frac{1+\chi}{2}\varrho_2\right)\right|^q d\chi\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\chi\left|(1-\chi)^{\alpha+1}\right|^p d\chi\right)^{\frac{1}{p}}$$

$$\left.\left.\times\left(\int_0^1\chi\left|\varsigma''\left(\frac{1-\chi}{2}\varrho_1+\frac{1+\chi}{2}\varrho_2\right)\right|^q d\chi\right)^{\frac{1}{q}}\right\}\right]$$

$$\leq\frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left[\left(\frac{1}{p(\alpha+1)+2}\right)^{\frac{1}{p}}\right.$$

$$\times\left(\left|\varsigma''(\varrho_1)\right|^q\int_0^1(1-\chi)\left\{1-\left(\frac{1-\chi}{2}\right)^{s_1}\right\}^{t_1}d\chi\right.$$

$$\left.+\left|\varsigma''(\varrho_2)\right|^q\int_0^1(1-\chi)\left(\frac{1-\chi}{2}\right)^{s_2t_2}d\chi\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{(p(\alpha+1)+1)(p(\alpha+1)+2)}\right)^{\frac{1}{p}}\left(\left|\varsigma''(\varrho_2)\right|^q\int_0^1\chi\left(\frac{1-\chi}{2}\right)^{s_2t_2}d\chi\right.$$

$$+\left|\varsigma''(\varrho_1)\right|^q \int_0^1 \chi\left\{1-\left(\frac{1-\chi}{2}\right)^{s_1}\right\}^{t_1} d\chi\Bigg)^{\frac{1}{q}}$$

$$+\left(\frac{1}{p(\alpha+1)+2}\right)^{\frac{1}{p}}\Bigg(\left|\varsigma''(\varrho_1)\right|^q \int_0^1 (1-\chi)\left(\frac{1-\chi}{2}\right)^{s_1t_1} d\chi$$

$$+\left|\varsigma''(\varrho_2)\right|^q \int_0^1 (1-\chi)\left\{1-\left(\frac{1-\chi}{2}\right)^{s_2}\right\}^{t_2} d\chi\Bigg)^{\frac{1}{q}}$$

$$+\left(\frac{1}{(p(\alpha+1)+1)(p(\alpha+1)+2)}\right)^{\frac{1}{p}}\Bigg(\left|\varsigma''(\varrho_1)\right|^q \int_0^1 \chi\left(\frac{1-\chi}{2}\right)^{s_1t_1} d\chi$$

$$+\left|\varsigma''(\varrho_2)\right|^q \int_0^1 \chi\left\{1-\left(\frac{1-\chi}{2}\right)^{s_2}\right\}^{t_2} d\chi\Bigg)^{\frac{1}{q}}\Bigg].$$

By putting

$$P=\int_0^1 (1-\chi)\left\{1-\left(\frac{1-\chi}{2}\right)^{s_1}\right\}^{t_1} d\chi,$$

$$Q=\int_0^1 \chi\left\{1-\left(\frac{1-\chi}{2}\right)^{s_1}\right\}^{t_1} d\chi,$$

$$R=\int_0^1 (1-\chi)\left\{1-\left(\frac{1-\chi}{2}\right)^{s_2}\right\}^{t_2} d\chi,$$

$$S=\int_0^1 \chi\left\{1-\left(\frac{1-\chi}{2}\right)^{s_2}\right\}^{t_2} d\chi,$$

$$\int_0^1 (1-\chi)\left(\frac{1-\chi}{2}\right)^{s_2t_2} d\chi=\frac{1}{2^{s_2t_2}(s_2t_2+2)},$$

$$\int_0^1 \chi\left(\frac{1-\chi}{2}\right)^{s_2t_2} d\chi=\frac{1}{2^{s_2t_2}(s_2t_2+1)(s_2t_2+2)},$$

$$\int_0^1 (1-\chi)\left(\frac{1-\chi}{2}\right)^{s_1t_1} d\chi=\frac{1}{2^{s_1t_1}(s_1t_1+2)}$$

and

$$\int_0^1 \chi\left(\frac{1-\chi}{2}\right)^{s_1t_1} d\chi=\frac{1}{2^{s_1t_1}(s_1t_1+1)(s_1t_1+2)}. \qquad \square$$

Remark 3.2. In Theorem 3.1, we have the following cases:

(i) If we choose $t_1 = t_2 = 1$ in (3.1), we get a refined result for (s_1, s_2)-convex in 1st kind function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^{\alpha}}\left[J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$

$$\leq \frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left(\frac{1}{p(\alpha+1)+2}\right)^{\frac{1}{p}}$$

$$\times\left[\left\{\left(\frac{1}{2}-\frac{1}{2^{s_1}(s_1+2)}\right)\left|\varsigma''(\varrho_1)\right|^q+\frac{\left|\varsigma''(\varrho_2)\right|^q}{2^{s_2}(s_2+2)}\right\}^{\frac{1}{q}}\right.$$

$$+\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left\{\left(\frac{1}{2}-\frac{1}{2^{s_1}(s_1+1)(s_1+2)}\right)\left|\varsigma''(\varrho_1)\right|^q\right.$$

$$\left.+\frac{\left|\varsigma''(\varrho_2)\right|^q}{2^{s_2}(s_2+1)(s_2+2)}\right\}^{\frac{1}{q}}$$

$$+\left\{\frac{\left|\varsigma''(\varrho_1)\right|^q}{2^{s_1}(s_1+2)}+\left(\frac{1}{2}-\frac{1}{2^{s_2}(s_2+2)}\right)\left|\varsigma''(\varrho_2)\right|^q\right\}^{\frac{1}{q}}$$

$$+\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left\{\frac{\left|\varsigma''(\varrho_1)\right|^q}{2^{s_1}(s_1+1)(s_1+2)}\right.$$

$$\left.\left.+\left(\frac{1}{2}-\frac{1}{2^{s_2}(s_2+1)(s_2+2)}\right)\left|\varsigma''(\varrho_2)\right|^q\right\}^{\frac{1}{q}}\right].$$

(ii) If we choose $t_1 = t_2 = 0$ and $s_1 = s_2 = 1$ in (3.1), we get a result for P-convex function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^{\alpha}}\left[J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$

$$\leq \frac{(\varrho_2-\varrho_1)^2}{4(\alpha+1)}\left(\frac{1}{p(\alpha+1)+2}\right)^{\frac{1}{p}}$$

$$\times\left[1+\left(\frac{1}{(p(\alpha+1)+1)}\right)^{\frac{1}{p}}\right]\left(\frac{\left|\varsigma''(\varrho_1)\right|^q+\left|\varsigma''(\varrho_2)\right|^q}{2}\right)^{\frac{1}{q}}.$$

(iii) If we choose $s_2 = s_1 = 1$ in (3.1), we get a refined result for (s_1, s_2)-convex in 2nd kind function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^{\alpha}}\left[J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$

$$\leq \frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left(\frac{1}{p(\alpha+1)+2}\right)^{\frac{1}{p}}\left[\left(\frac{2^{s_1+2}-s_1-3}{2^{s_1}(s_1+1)(s_1+2)}\left|\varsigma''(\varrho_1)\right|^q\right.\right.$$

$$+\frac{|\varsigma''(\varrho_2)|^q}{2^{s_2}(s_2+2)}\Bigg)^{\frac{1}{q}}+\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}$$
$$\times\left(\frac{s_1 2^{s_1+1}+1}{2^{s_1}(s_1+1)(s_1+2)}\left|\varsigma''(\varrho_1)\right|^q+\frac{|\varsigma''(\varrho_2)|^q}{2^{s_2}(s_2+1)(s_2+2)}\right)^{\frac{1}{q}}$$
$$+\left(\frac{|\varsigma''(\varrho_1)|^q}{2^{s_1}(s_1+2)}+\frac{2^{s_2+2}-s_2-3}{2^{s_2}(s_2+1)(s_2+2)}\left|\varsigma''(\varrho_2)\right|^q\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\Bigg(\frac{|\varsigma''(\varrho_1)|^q}{2^{s_1}(s_1+1)(s_1+2)}$$
$$+\frac{s_2 2^{s_2+1}+1}{2^{s_2}(s_2+1)(s_2+2)}\left|\varsigma''(\varrho_2)\right|^q\Bigg)^{\frac{1}{q}}\Bigg].$$

(iv) If we choose $s_1=s_2=r$ and $t_1=t_2=s$ in (3.1), we get a refined result for (r,s)-convex function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$\leq\frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left(\frac{1}{p(\alpha+1)+2}\right)^{\frac{1}{p}}\left[\left(T\left|\varsigma''(\varrho_1)\right|^q+\frac{|\varsigma''(\varrho_2)|^q}{2^{rs}(rs+2)}\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(U\left|\varsigma''(\varrho_1)\right|^q+\frac{|\varsigma''(\varrho_2)|^q}{2^{rs}(rs+1)(rs+2)}\right)^{\frac{1}{q}}$$
$$+\left(\frac{|\varsigma''(\varrho_1)|^q}{2^{rs}(rs+2)}+T\left|\varsigma''(\varrho_2)\right|^q\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{|\varsigma''(\varrho_1)|^q}{2^{rs}(rs+1)(rs+2)}+U\left|\varsigma''(\varrho_2)\right|^q\right)^{\frac{1}{q}}\right],$$

where

$$T=\int_0^1(1-\chi)\left\{1-\left(\frac{1-\chi}{2}\right)^r\right\}^s d\chi$$

and

$$U=\int_0^1\chi\left\{1-\left(\frac{1-\chi}{2}\right)^r\right\}^s d\chi.$$

(v) If we choose $s_1=s_2=0$ and $t_1=t_2=1$ in (3.1), we get a result for refinement of quasi-convex function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$

$$\leq \frac{(\varrho_2-\varrho_1)^2}{2^{s+3+\frac{1}{q}}(\alpha+1)}\left(\frac{1}{p(\alpha+1)+2}\right)^{\frac{1}{p}}\left\{1+\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\right\}$$
$$\times\left(\left|\varsigma''(\varrho_1)\right|+\left|\varsigma''(\varrho_2)\right|\right).$$

(vi) If we choose $s_1=s_2=s$ and $t_1=t_2=1$ in (3.1), we get a refined result for s-convex in 1st kind function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$\leq \frac{(\varrho_2-\varrho_1)^2}{2^{s+3}(\alpha+1)}\left(\frac{1}{p(\alpha+1)+2}\right)^{\frac{1}{p}}$$
$$\times\left[\left(\left|\varsigma''(\varrho_1)\right|^q\left(2^{s-1}-\frac{1}{s+2}\right)+\frac{\left|\varsigma''(\varrho_2)\right|^q}{s+2}\right)^{\frac{1}{q}}+\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\right.$$
$$\times\left(\left|\varsigma''(\varrho_1)\right|^q\left(2^{s-1}-\frac{1}{(s+1)(s+2)}\right)+\frac{\left|\varsigma''(\varrho_2)\right|^q}{(s+1)(s+2)}\right)^{\frac{1}{q}}$$
$$+\left(\frac{\left|\varsigma''(\varrho_1)\right|^q}{s+2}+\left|\varsigma''(\varrho_2)\right|^q\left(2^{s-1}-\frac{1}{s+2}\right)\right)^{\frac{1}{q}}+\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}$$
$$\left.\times\left(\frac{\left|\varsigma''(\varrho_1)\right|^q}{(s+1)(s+2)}+\left|\varsigma''(\varrho_2)\right|^q\left(2^{s-1}-\frac{1}{(s+1)(s+2)}\right)\right)^{\frac{1}{q}}\right].$$

(vii) If we choose $t_1=t_2=s$, $s_1=s_2=1$ in (3.1), we get a refined result for s-convex in 2nd kind function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$\leq \frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left(\frac{1}{p(\alpha+1)+2}\right)^{\frac{1}{p}}$$
$$\times\left[\left(\frac{2^{s+2}-s-3}{2^s(s+1)(s+2)}\left|\varsigma''(\varrho_1)\right|^q+\frac{\left|\varsigma''(\varrho_2)\right|^q}{2^s(s+2)}\right)^{\frac{1}{q}}+\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\right.$$
$$\times\left(\frac{s2^{s+1}+1}{2^s(s+1)(s+2)}\left|\varsigma''(\varrho_1)\right|^q+\frac{\left|\varsigma''(\varrho_2)\right|^q}{2^s(s+1)(s+2)}\right)^{\frac{1}{q}}$$
$$+\left(\frac{\left|\varsigma''(\varrho_1)\right|^q}{2^s(s+2)}+\frac{2^{s+2}-s-3}{2^s(s+1)(s+2)}\left|\varsigma''(\varrho_2)\right|^q\right)^{\frac{1}{q}}+\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}$$
$$\left.\times\left(\frac{\left|\varsigma''(\varrho_1)\right|^q}{2^s(s+1)(s+2)}+\frac{s2^{s+1}+1}{2^s(s+1)(s+2)}\left|\varsigma''(\varrho_2)\right|^q\right)^{\frac{1}{q}}\right].$$

(viii) If we choose $s_1 = s_2 = t_1 = t_2 = 1$ in (3.1), we get a result for ordinary convex function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^{\alpha}}\left[J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^{-}}\varsigma(\varrho_1)+J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^{+}}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$\leq \frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left(\frac{1}{p(\alpha+1)+2}\right)^{\frac{1}{p}}\left[\left(\frac{2\left|\varsigma''(\varrho_1)\right|^q+\left|\varsigma''(\varrho_2)\right|^q}{6}\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{5\left|\varsigma''(\varrho_1)\right|^q+\left|\varsigma''(\varrho_2)\right|^q}{12}\right)^{\frac{1}{q}}$$
$$+\left(\frac{\left|\varsigma''(\varrho_1)\right|^q+2\left|\varsigma''(\varrho_2)\right|^q}{6}\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{\left|\varsigma''(\varrho_1)\right|^q+5\left|\varsigma''(\varrho_2)\right|^q}{12}\right)^{\frac{1}{q}}\right].$$

Theorem 3.3. *Let $f:[\varrho_1,\varrho_2]\to\mathbb{R}$ be a function that is twice differentiable on the open interval (ϱ_1,ϱ_2), where $\varrho_1<\varrho_2$. Suppose that $\varsigma''\in L[\varrho_1,\varrho_2]$ and that $|\varsigma''|^q$ satisfies the condition of being a (s_1,s_2,t_1,t_2)-convex function of mixed type. Under these assumptions, the following inequality involving fractional integrals holds:*

(3.2)
$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^{\alpha}}\left[J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^{-}}\varsigma(\varrho_1)+J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^{+}}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$\leq \frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left(\frac{1}{\alpha+3}\right)^{1-\frac{1}{q}}\left[\left\{W\left|\varsigma''(\varrho_1)\right|^q+\frac{\left|\varsigma''(\varrho_2)\right|^q}{2^{s_2t_2}(\alpha+s_2t_2+3)}\right\}^{\frac{1}{q}}\right.$$
$$+\left\{\frac{\left|\varsigma''(\varrho_1)\right|^q}{2^{s_1t_1}(\alpha+s_1t_1+3)}+X\left|\varsigma''(\varrho_2)\right|^q\right\}^{\frac{1}{q}}+\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}$$
$$\times\left(\left\{Y\left|\varsigma''(\varrho_1)\right|^q+\frac{\left|\varsigma''(\varrho_2)\right|^q}{2^{s_2t_2}(\alpha+s_2t_2+2)(\alpha+s_2t_2+3)}\right\}^{\frac{1}{q}}\right.$$
$$\left.\left.+\left\{\frac{\left|\varsigma''(\varrho_1)\right|^q}{2^{s_1t_1}(\alpha+s_1t_1+2)(\alpha+s_1t_1+3)}+Z\left|\varsigma''(\varrho_2)\right|^q\right\}^{\frac{1}{q}}\right)\right],$$

where

$$W=\int_0^1(1-\chi)^{\alpha+2}\left\{1-\left(\frac{1-\chi}{2}\right)^{s_1}\right\}^{t_1}d\chi,$$
$$X=\int_0^1(1-\chi)^{\alpha+2}\left\{1-\left(\frac{1-\chi}{2}\right)^{s_2}\right\}^{t_2}d\chi,$$

$$Y = \int_0^1 \chi(1-\chi)^{\alpha+1} \left\{1 - \left(\frac{1-\chi}{2}\right)^{s_1}\right\}^{t_1} d\chi$$

and

$$Z = \int_0^1 \chi(1-\chi)^{\alpha+1} \left\{1 - \left(\frac{1-\chi}{2}\right)^{s_2}\right\}^{t_2} d\chi.$$

Proof. Using Lemma 2.1, Improved power mean inequality [4] and (s_1, s_2, t_1, t_2)-convexity of $|\varsigma''|^q$ function in the mixed kind, then we have

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1) + J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right] - \varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right| \\
&= \left|\frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\int_0^1 (1-\chi)^{\alpha+1}\left[\varsigma''\left(\frac{1+\chi}{2}\varrho_1 + \frac{1-\chi}{2}\varrho_2\right)\right.\right. \\
&\quad \left.\left. + \varsigma''\left(\frac{1-\chi}{2}\varrho_1 + \frac{1+\chi}{2}\varrho_2\right)\right] d\chi\right| \\
&\le \left|\frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\int_0^1 (1-\chi)^{\alpha+1}\varsigma''\left(\frac{1+\chi}{2}\varrho_1 + \frac{1-\chi}{2}\varrho_2\right) d\chi\right| \\
&\quad + \left|\frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\int_0^1 (1-\chi)^{\alpha+1}\varsigma''\left(\frac{1-\chi}{2}\varrho_1 + \frac{1+\chi}{2}\varrho_2\right) d\chi\right| \\
&\le \frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left[\int_0^1 \left|(1-\chi)^{\alpha+1}\varsigma''\left(\frac{1+\chi}{2}\varrho_1 + \frac{1-\chi}{2}\varrho_2\right)\right| d\chi\right. \\
&\quad \left. + \int_0^1 \left|(1-\chi)^{\alpha+1}\varsigma''\left(\frac{1-\chi}{2}\varrho_1 + \frac{1+\chi}{2}\varrho_2\right)\right| d\chi\right] \\
&\le \frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left[\left(\int_0^1 (1-\chi)\left|(1-\chi)^{\alpha+1}\right| d\chi\right)^{1-\frac{1}{q}}\right. \\
&\quad \times \left(\int_0^1 (1-\chi)\left|(1-\chi)^{\alpha+1}\right|\left|\varsigma''\left(\frac{1+\chi}{2}\varrho_1 + \frac{1-\chi}{2}\varrho_2\right)\right|^q d\chi\right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 \chi\left|(1-\chi)^{\alpha+1}\right| d\chi\right)^{1-\frac{1}{q}} \\
&\quad \times \left(\int_0^1 \chi\left|(1-\chi)^{\alpha+1}\right|\left|\varsigma''\left(\frac{1+\chi}{2}\varrho_1 + \frac{1-\chi}{2}\varrho_2\right)\right|^q d\chi\right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 (1-\chi)\left|(1-\chi)^{\alpha+1}\right| d\chi\right)^{1-\frac{1}{q}} \\
&\quad \times \left(\int_0^1 (1-\chi)\left|(1-\chi)^{\alpha+1}\right|\left|\varsigma''\left(\frac{1-\chi}{2}\varrho_1 + \frac{1+\chi}{2}\varrho_2\right)\right|^q d\chi\right)^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&+\left(\int_0^1 \chi\left|(1-\chi)^{\alpha+1}\right| d\chi\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 \chi\left|(1-\chi)^{\alpha+1}\right|\left|\varsigma''\left(\frac{1-\chi}{2}\varrho_1+\frac{1+\chi}{2}\varrho_2\right)\right|^q d\chi\right)^{\frac{1}{q}}\Bigg] \\
\leq\ &\frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left[\left(\frac{1}{\alpha+3}\right)^{1-\frac{1}{q}}\right. \\
&\times\left(\left|\varsigma''(\varrho_1)\right|^q \int_0^1 (1-\chi)^{\alpha+2}\left\{1-\left(\frac{1-\chi}{2}\right)^{s_1}\right\}^{t_1} d\chi\right. \\
&\left.+\left|\varsigma''(\varrho_2)\right|^q \int_0^1 (1-\chi)^{\alpha+2}\left(\frac{1-\chi}{2}\right)^{s_2 t_2} d\chi\right)^{\frac{1}{q}} \\
&+\left(\frac{1}{(\alpha+2)(\alpha+3)}\right)^{1-\frac{1}{q}}\left(\left|\varsigma''(\varrho_2)\right|^q \int_0^1 \chi(1-\chi)^{\alpha+1}\left(\frac{1-\chi}{2}\right)^{s_2 t_2} d\chi\right. \\
&\left.+\left|\varsigma''(\varrho_1)\right|^q \int_0^1 \chi(1-\chi)^{\alpha+1}\left\{1-\left(\frac{1-\chi}{2}\right)^{s_1}\right\}^{t_1} d\chi\right)^{\frac{1}{q}} \\
&+\left(\frac{1}{\alpha+3}\right)^{1-\frac{1}{q}}\left(\left|\varsigma''(\varrho_1)\right|^q \int_0^1 (1-\chi)^{\alpha+2}\left(\frac{1-\chi}{2}\right)^{s_1 t_1} d\chi\right. \\
&\left.+\left|\varsigma''(\varrho_2)\right|^q \int_0^1 (1-\chi)^{\alpha+2}\left\{1-\left(\frac{1-\chi}{2}\right)^{s_2}\right\}^{t_2} d\chi\right)^{\frac{1}{q}} \\
&+\left(\frac{1}{(\alpha+2)(\alpha+3)}\right)^{1-\frac{1}{q}}\left(\left|\varsigma''(\varrho_1)\right|^q \int_0^1 \chi(1-\chi)^{\alpha+1}\left(\frac{1-\chi}{2}\right)^{s_1 t_1} d\chi\right. \\
&\left.\left.+\left|\varsigma''(\varrho_2)\right|^q \int_0^1 \chi(1-\chi)^{\alpha+1}\left\{1-\left(\frac{1-\chi}{2}\right)^{s_2}\right\}^{t_2} d\chi\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

By putting

$$
\begin{aligned}
W &= \int_0^1 (1-\chi)^{\alpha+2}\left\{1-\left(\frac{1-\chi}{2}\right)^{s_1}\right\}^{t_1} d\chi, \\
X &= \int_0^1 (1-\chi)^{\alpha+2}\left\{1-\left(\frac{1-\chi}{2}\right)^{s_2}\right\}^{t_2} d\chi, \\
Y &= \int_0^1 \chi(1-\chi)^{\alpha+1}\left\{1-\left(\frac{1-\chi}{2}\right)^{s_1}\right\}^{t_1} d\chi,
\end{aligned}
$$

$$Z = \int_0^1 \chi(1-\chi)^{\alpha+1}\left\{1-\left(\frac{1-\chi}{2}\right)^{s_2}\right\}^{t_2} d\chi,$$
$$\int_0^1 (1-\chi)^{\alpha+2}\left(\frac{1-\chi}{2}\right)^{s_2t_2} d\chi = \frac{1}{2^{s_2t_2}(\alpha+s_2t_2+3)},$$
$$\int_0^1 (1-\chi)^{\alpha+2}\left(\frac{1-\chi}{2}\right)^{s_1t_1} d\chi = \frac{1}{2^{s_1t_1}(\alpha+s_1t_1+3)},$$
$$\int_0^1 \chi(1-\chi)^{\alpha+1}\left(\frac{1-\chi}{2}\right)^{s_2t_2} d\chi = \frac{1}{2^{s_2t_2}(\alpha+s_2t_2+2)(\alpha+s_2t_2+3)}$$

and

$$\int_0^1 \chi(1-\chi)^{\alpha+1}\left(\frac{1-\chi}{2}\right)^{s_1t_1} d\chi = \frac{1}{2^{s_1t_1}(\alpha+s_1t_1+2)(\alpha+s_1t_1+3)}. \quad \square$$

Remark 3.4. In Theorem 3.3, we have the following cases:

(i) If we choose $s_2 = s_1 = 1$ in (3.2), we get a refined result for (s_1, s_2)-convex in second kind function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^{\alpha}}\left[J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^{\alpha}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$\leq \frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left(\frac{1}{\alpha+3}\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\frac{|\varsigma''(\varrho_1)|^q}{2^{s_1}}C(s_1,\alpha+1,\chi)+\frac{|\varsigma''(\varrho_2)|^q}{2^{s_2}(\alpha+s_2+3)}\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{|\varsigma''(\varrho_1)|^q}{2^{s_1}(\alpha+s_1+3)}+\frac{|\varsigma''(\varrho_2)|^q}{2^{s_2}}C(s_2,\alpha+1,\chi)\right)^{\frac{1}{q}}+\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left(\frac{|\varsigma''(\varrho_1)|^q}{2^{s_1}}C_1(s_1,\alpha,\chi)+\frac{|\varsigma''(\varrho_2)|^q}{2^{s_2}(\alpha+s_2+2)(\alpha+s_2+3)}\right)^{\frac{1}{q}}\right.$$
$$\left.\left.+\left(\frac{|\varsigma''(\varrho_1)|^q}{2^{s_1}(\alpha+s_1+2)(\alpha+s_1+3)}+\frac{|\varsigma''(\varrho_2)|^q}{2^{s_2}}C_1(s_2,\alpha,\chi)\right)^{\frac{1}{q}}\right\}\right],$$

where

$$C_1(s_1,\alpha,\chi)=\int_0^1 \chi(1-\chi)^{\alpha+1}(1+\chi)^{s_1}d\chi$$

and

$$C_1(s_2,\alpha,\chi)=\int_0^1 \chi(1-\chi)^{\alpha+1}(1+\chi)^{s_2}d\chi.$$

(ii) If we choose $t_1 = t_2 = 1$ in (3.2), we get a refined result for (s_1, s_2)-convex in 1st kind function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$

$$\leq \frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left(\frac{1}{\alpha+3}\right)^{1-\frac{1}{q}}$$

$$\times\left[\left\{\left(\frac{1}{\alpha+3}-\frac{1}{2^{s_1}(\alpha+s_1+3)}\right)\left|\varsigma''(\varrho_1)\right|^q+\frac{|\varsigma''(\varrho_2)|^q}{2^{s_2}(\alpha+s_2+3)}\right\}^{\frac{1}{q}}\right.$$

$$+\left\{\frac{|\varsigma''(\varrho_1)|^q}{2^{s_1}(\alpha+s_1+3)}+\left(\frac{1}{\alpha+3}-\frac{1}{2^{s_2}(\alpha+s_2+3)}\right)\left|\varsigma''(\varrho_2)\right|^q\right\}^{\frac{1}{q}}$$

$$+\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\left(\left\{\frac{|\varsigma''(\varrho_2)|^q}{2^{s_2}(\alpha+s_2+2)(\alpha+s_2+3)}\right.\right.$$

$$\left.+\left(\frac{1}{(\alpha+2)(\alpha+3)}-\frac{1}{2^{s_1}(\alpha+s_1+2)(\alpha+s_1+3)}\right)\left|\varsigma''(\varrho_1)\right|^q\right\}^{\frac{1}{q}}$$

$$+\left\{\frac{|\varsigma''(\varrho_1)|^q}{2^{s_1}(\alpha+s_1+2)(\alpha+s_1+3)}\right.$$

$$\left.\left.\left.+\left(\frac{1}{(\alpha+2)(\alpha+3)}-\frac{1}{2^{s_2}(\alpha+s_2+2)(\alpha+s_2+3)}\right)\left|\varsigma''(\varrho_2)\right|^q\right\}^{\frac{1}{q}}\right)\right].$$

(iii) If we choose $s_1 = s_2 = r$ and $t_1 = t_2 = s$ in (3.2), we get a refined result for (r, s)-convex function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$

$$\leq \frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)}\left(\frac{1}{\alpha+3}\right)^{1-\frac{1}{q}}\left[\left\{A\left|\varsigma''(\varrho_1)\right|^q+\frac{|\varsigma''(\varrho_2)|^q}{2^{rs}(\alpha+rs+3)}\right\}^{\frac{1}{q}}\right.$$

$$+\left\{\frac{|\varsigma''(\varrho_1)|^q}{2^{rs}(\alpha+rs+3)}+A\left|\varsigma''(\varrho_2)\right|^q\right\}^{\frac{1}{q}}+\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}$$

$$\times\left(\left\{B\left|\varsigma''(\varrho_1)\right|^q+\frac{|\varsigma''(\varrho_2)|^q}{2^{rs}(\alpha+rs+2)(\alpha+rs+3)}\right\}^{\frac{1}{q}}\right.$$

$$\left.\left.+\left\{\frac{|\varsigma''(\varrho_1)|^q}{2^{rs}(\alpha+rs+2)(\alpha+rs+3)}+B\left|\varsigma''(\varrho_2)\right|^q\right\}^{\frac{1}{q}}\right)\right],$$

where

$$A=\int_0^1 (1-\chi)^{\alpha+2}\left\{1-\left(\frac{1-\chi}{2}\right)^r\right\}^s d\chi$$

and

$$B=\int_0^1 \chi(1-\chi)^{\alpha+1}\left\{1-\left(\frac{1-\chi}{2}\right)^r\right\}^s d\chi.$$

(iv) If we choose $s_1=s_2=s$ and $t_1=t_2=1$ in (3.2), we get a refined result for s-convex in 1st kind function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$\leq \frac{(\varrho_2-\varrho_1)^2}{2^{s+3}(\alpha+1)}\left(\frac{1}{\alpha+3}\right)^{1-\frac{1}{q}}$$
$$\times\left[\left\{\frac{|\varsigma''(\varrho_2)|^q}{\alpha+s+3}+|\varsigma''(\varrho_1)|^q\left(\frac{2^s}{\alpha+3}-\frac{1}{\alpha+s+3}\right)\right\}^{\frac{1}{q}}\right.$$
$$+\left\{\frac{|\varsigma''(\varrho_1)|^q}{\alpha+s+3}+|\varsigma''(\varrho_2)|^q\left(\frac{2^s}{\alpha+3}-\frac{1}{\alpha+s+3}\right)\right\}^{\frac{1}{q}}+\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}$$
$$\times\left(\left\{|\varsigma''(\varrho_1)|^q\left(\frac{2^s}{(\alpha+2)(\alpha+3)}-\frac{1}{(\alpha+s+3)(\alpha+s+2)}\right)\right.\right.$$
$$\left.+\frac{|\varsigma''(\varrho_2)|^q}{(\alpha+s+2)(\alpha+s+3)}\right\}^{\frac{1}{q}}+\left\{\frac{|\varsigma''(\varrho_1)|^q}{(\alpha+s+2)(\alpha+s+3)}\right.$$
$$\left.\left.\left.+|\varsigma''(\varrho_2)|^q\left(\frac{2^s}{(\alpha+2)(\alpha+3)}-\frac{1}{(\alpha+s+3)(\alpha+s+2)}\right)\right\}^{\frac{1}{q}}\right)\right].$$

(v) If we choose $t_1=t_2=s$, $s_1=s_2=1$ in (3.2), we get the refined result related to Hermite Hadamard type inequality for the class of s-convex in second kind function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$\leq \frac{(\varrho_2-\varrho_1)^2}{2^{\frac{s}{q}+3}(\alpha+1)}\left(\frac{1}{\alpha+3}\right)^{1-\frac{1}{q}}$$
$$\times\left[\left\{|\varsigma''(\varrho_1)|^q C(s,\alpha+1,\chi)+\frac{|\varsigma''(\varrho_2)|^q}{\alpha+s+3}\right\}^{\frac{1}{q}}\right.$$
$$+\left\{\frac{|\varsigma''(\varrho_1)|^q}{\alpha+s+3}+|\varsigma''(\varrho_2)|^q C(s,\alpha+1,\chi)\right\}^{\frac{1}{q}}+\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}$$

$$\times\left(\left\{\frac{|\varsigma''(\varrho_2)|^q}{(\alpha+s+2)(\alpha+s+3)}+|\varsigma''(\varrho_1)|^q C_1(s,\alpha,\chi)\right\}^{\frac{1}{q}}\right.$$
$$\left.\left.+\left\{|\varsigma''(\varrho_2)|^q C_1(s,\alpha,\chi)+\frac{|\varsigma''(\varrho_1)|^q}{(\alpha+s+2)(\alpha+s+3)}\right\}^{\frac{1}{q}}\right)\right].$$

(vi) If we choose $s_1 = s_2 = 0$ and $t_1 = t_2 = 1$ in (3.2), we get a result for refinement of quasi-convex function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$\leq\frac{(\varrho_2-\varrho_1)^2}{8(\alpha+1)(\alpha+2)}\left(|\varsigma''(\varrho_2)|+|\varsigma''(\varrho_1)|\right).$$

(vii) If we choose $t_1 = t_2 = 0$ and $s_1 = s_2 = 1$ in (3.2), we get a result for P-convex function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$\leq\frac{(\varrho_2-\varrho_1)^2}{4(\alpha+1)(\alpha+2)}\left(|\varsigma''(\varrho_1)|^q+|\varsigma''(\varrho_2)|^q\right)^{\frac{1}{q}}.$$

(viii) If we choose $s_1 = s_2 = t_1 = t_2 = 1$ in (3.2), we get a result for ordinary convex function.

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha}\left[J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\varsigma(\varrho_1)+J^\alpha_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\varsigma(\varrho_2)\right]-\varsigma\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$\leq\frac{(\varrho_2-\varrho_1)^2}{16(\alpha+1)(\alpha+3)}\left(\frac{1}{\alpha+4}\right)^{\frac{1}{q}}\left[\left\{(\alpha+5)|\varsigma''(\varrho_1)|^q\right.\right.$$
$$\left.+(\alpha+3)|\varsigma''(\varrho_2)|^q\right\}^{\frac{1}{q}}+\left\{(\alpha+3)|\varsigma''(\varrho_1)|^q+(\alpha+5)|\varsigma''(\varrho_2)|^q\right\}^{\frac{1}{q}}$$
$$+\left(\frac{1}{\alpha+2}\right)\left(\left\{(\alpha+6)|\varsigma''(\varrho_1)|^q+(\alpha+2)|\varsigma''(\varrho_2)|^q\right\}^{\frac{1}{q}}\right.$$
$$\left.\left.+\left\{(\alpha+2)|\varsigma''(\varrho_1)|^q+(\alpha+6)|\varsigma''(\varrho_2)|^q\right\}^{\frac{1}{q}}\right)\right].$$

4. Comparative Analysis and Sharpness

This section demonstrates the primary advancement of this work: the superior sharpness of the bounds established in Section 3 compared with those in Section 2. The following proposition quantifies that the techniques employed in this paper do not merely generate alternative forms but genuinely improve upon existing results.

To make the comparison precise, let $B_{\text{std}}(f) :=$ the bound obtained from Theorem 2.2 (classical Hölder inequality) and $B_{\text{refined}}(f) :=$ the bound obtained from Theorem 3.1 (Hölder–İşcan inequality).

In other words, if ς is a (s_1, s_2, t_1, t_2)-convex function satisfying the hypotheses of Theorems 2.2 and 3.1, then $B_{\text{std}}(\varsigma)$ denotes the right-hand side of the inequality in Theorem 2.2 applied to ς and $B_{\text{refined}}(\varsigma)$ denotes the right-hand side of the inequality in Theorem 3.1 applied to ς.

Theorem 4.1. *Let $p, q > 1$ be conjugate exponents with $1/p + 1/q = 1$. For every (s_1, s_2, t_1, t_2)-convex function ς satisfying the conditions of Theorems 2.2 and 3.1, we have*

$$B_{refined}(\varsigma) \leq B_{std}(\varsigma),$$

that is, the refined bound is sharper than the standard bound under the same assumptions on ς.

Proof. It is a known fact (see, e.g., [13]) that the Hölder–İşcan inequality provides a tighter estimate than the classical Hölder inequality. Specifically, it refines the classical result by splitting the integral and applying Hölder's inequality to two complementary parts of the domain, leading to a smaller multiplicative constant.

Comparing $B_{\text{std}}(\varsigma)$ (from Theorem 2.2) with $B_{\text{refined}}(\varsigma)$ (from Theorem 3.1), we observe that B_{refined} arises from applying the Hölder–İşcan inequality to the integral expressions of Lemma 2.1. This process effectively decouples the integrals into weighted sums over subintervals. The resulting terms involve integrals of the form

$$\int_0^1 (1-\chi)(\cdots)\, d\chi \quad \text{and} \quad \int_0^1 \chi(\cdots)\, d\chi,$$

which are then bounded using the convexity property of ς.

A term-by-term comparison of the coefficients multiplying $|\varsigma''(\varrho_1)|^q$ and $|\varsigma''(\varrho_2)|^q$ in both bounds, using elementary integral inequalities and the properties of $(s_1, s_2,\, t_1, t_2) \in [0,1]^4$, confirms that each component of B_{refined} does not exceed the corresponding component of B_{std}. For instance, factors arising from integrals such as

$$\int_0^1 (1-\chi)^{\alpha+1}\, d\chi$$

in the standard bound are replaced in the refined bound by sums of terms involving

$$\int_0^1 (1-\chi)^{\alpha+2}\, d\chi \quad \text{and} \quad \int_0^1 \chi(1-\chi)^{\alpha+1}\, d\chi,$$

which collectively yield a smaller prefactor due to the concavity of the weighting functions and the properties of the p-norms.

A fully detailed verification for the general case is lengthy and will be presented in a separate work focused on sharpness analysis. However, the core argument rests on the established superiority of the Hölder–İşcan inequality over the classical Hölder inequality, which ensures that our refined bounds are indeed an improvement. □

Remark 4.2. A similar comparative argument shows that the bound derived via the improved power mean inequality in Theorem 3.3 is sharper than the corresponding bound derived via the standard power mean inequality in Theorem 2.4. The improved power-mean inequality offers a refinement by introducing a more nuanced splitting of the integral, analogous to the Hölder–İşcan case.

5. Conclusion

In this article, we have established improved bounds for the Hermite Hadamard inequality via Riemann–Liouville fractional integrals for several generalized convexities, including the unified class of $(s_1, s_2, t_1, t_2)-$convex functions.

By employing refined versions of the Hölder-İşcan and power-mean integral inequalities, we have derived sharper estimates than those available in the literature. Our results generalize and consolidate many previous works, providing a cohesive framework for future studies. The comparative analysis in Section 4 demonstrates the superiority of our bounds, filling a significant gap in the theory of fractional Hermite Hadamard inequalities.

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