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Yamin Sayyari

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An Extension of m-Convex Function and Applications in Tsallis Entropy

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Abstract. The purpose of this research is to incorporate the new class of functions called (m, δ)-convex functions. In addition, we establish several inequalities for (m, δ)-convex functions. Furthermore, we give some applications of the main results in Tsallis entropy and analysis.

1. basic notions

The applications of convex functions are widespread in numerous fields, including machine learning, control theory, signal processing, network design, image processing, entropy, game theory, economics, finance and robotics and path planning, due to their inherent mathematical properties, which facilitate effective modeling and optimization of complex real-world problems. We refer to [2, 4, 5], [14]-[21] to see the applications of convexity in information theory. Therefore, several generalizations of convex functions, such as m-convex [22], harmonically convex [9, 19], geometrically convex [11], quasi-convex, GA-convex function [1, 11, 12] and Generalized convex function [3, 24], have been defined so far.

In this paper, we introduce a family of functions that are an extension of the m-convex functions. Also, we prove some important inequalities related to this family. In addition, we present applications of the introduced functions in entropy and mean theory.

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Let $I \subseteq \mathbb{R}$ be an interval with $0 \in I$, and let $m \in [0,1]$. The function $f : I \longrightarrow \mathbb{R}$ is called to be m-convex if

$$f((1-t)x + mty) \leq (1-t)f(x) + mtf(y).$$

for all $x, y \in I$ and all $t \in [0,1]$.

The following theorem gives a Jensen type inequality for m-convex functions.

Theorem 1.1 ([13, Theorem 3.2])**.** *Suppose that $I \subseteq \mathbb{R}$ is an interval and $0 \in I$. Let $\sum_{i=1}^{n} t_i = 1$ where $0 \leq t_i \leq 1 (1 \leq i \leq n)$, $0 \leq m \leq 1$ and $f : I \to \mathbb{R}$ be an m-convex. Then*

$$f\left(m \sum_{i=1}^{n} t_i x_i\right) \leq m \sum_{i=1}^{n} t_i f\left(x_i\right).$$

In recent years, several researchers proved some types of Hermite-Hadamard and Jensen inequalities of m-convex functions. A number of the principal types of inequalities are itemized in the following.

Theorem 1.2 ([13, Corollary 4.1])**.** *If $a \leq 0 \leq b$ and $0 < m \leq 1$ and $f : [a,b] \to \mathbb{R}$ is a continuous m-convex function, then*

$$\frac{1}{m} f\left(m\frac{b+a}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{mb-a}{2(b-a)}f(a) + \frac{b-ma}{2(b-a)}f(b).$$

Theorem 1.3 ([7, Theorem 5])**.** *Assume that $0 \leq a < mb$ and f is a real m-convex function on $[a,b]$ and $m \in (0,1]$. If $f \in L_1[ma,b]$, then*

$$\frac{1}{m+1}\left[\int_a^{mb} f(x)dx + \frac{mb-a}{b-ma}\int_{ma}^{b} f(x)dx\right] \leq (mb-a)\frac{f(a)+f(b)}{2}.$$

Lemma 1.4 ([7, Lemma 2])**.** *If $0 \leq n \leq m \leq 1$ and $f : [a,b] \to \mathbb{R}$ is an m-convex and $f(0) \leq 0$. Then f is n-convex.*

2. Main Results

The purpose of this section is to generalize the notion of m-convex functions.

Definition 2.1. Let $0 \leq m \leq 1$ and $\delta \geq 0$. The function $f : [0,\infty) \longrightarrow \mathbb{R}$ is called (m,δ)-convex if the inequality

$$f((1-t)x + mty + t(1-m)\delta) \leq (1-t)f(x) + mtf(y) \tag{2.1}$$

holds for every $x, y \in [0,\infty)$ and every $t \in [0,1]$.

Note that (m,δ)-convex function f can be defined on $(0,\infty)$.

Remark 2.2. 1) If $m = 1$, then f is convex function on $[0,\infty)$.

2) If $\delta = 0$, then f is m- convex function.

Remark 2.3. 1) If f, g are (m, δ)-convex functions, then $f + g$ is (m, δ)-convex.

2) If f is (m, δ)-convex function and $c > 0$, then cf is (m, δ)-convex.

3) If g is (m, δ)-convex function and f is increasing and convex, then $f \circ g$ is (m, δ)-convex.

Throughout this work, the notation $\mathcal{K}_{m,\delta}$ is used for the set of the (m, δ)-convex functions on $[0, \infty)$ that $f(0) \leq 0$.

Theorem 2.4. *Let $f : [0, \infty) \to \mathbb{R}$ be (m, δ)-convex. Then*

$$f\left(m\sum_{i=1}^{n} t_i x_i + (1-m)\delta\right) \leq m\sum_{i=1}^{n} t_i f(x_i) \tag{2.2}$$

where $0 \leq t_i \leq 1, x_i \geq 0; 1 \leq i \leq n$ and $\sum_{i=1}^{n} t_i = 1$.

Proof. The proof is by induction on n. The inequality 2.2 in the case $n = 1$ follows from (2.1) by $t = 1$; assume it holds for $n - 1$. Let $0 \leq t_i \leq 1, x_i \geq 0$; $1 \leq i \leq n$ and $\sum_{i=1}^{n} t_i = 1$. Without loss of generality, let $t_1 < 1$. So by (2.1) and the inductive hypothesis, we have

$$\begin{aligned}
&f\left(m\sum_{i=1}^{n} t_i x_i + (1-m)\delta\right) \\
&= f\left(mt_1 x_1 + (1-t_1)\sum_{i=2}^{n} \frac{t_i}{1-t_1} m x_i + (1-m)\delta\right) \\
&= f\left(mt_1 x_1 + (1-t_1)\left(\sum_{i=2}^{n} \frac{t_i}{1-t_1} m x_i + (1-m)\delta\right) + (1-m)t_1\delta\right) \\
&\leq mt_1 f(x_1) + (1-t_1) f\left(\sum_{i=2}^{n} \frac{t_i}{1-t_1} m x_i + (1-m)\delta\right) \\
&\leq mt_1 f(x_1) + m\sum_{i=2}^{n} t_i f(x_i) = m\sum_{i=1}^{n} t_i f(x_i).
\end{aligned}$$

This proves (2.2). □

Theorem 2.5. *Let $f : [0, \infty) \to \mathbb{R}$ be (m, δ)-convex and $0 \leq a < b$. Then*

(i) $\frac{1}{b-a}\int_{ma}^{mb} f(u+(1-m)\delta)du \leq m^2\frac{f(a)+f(b)}{2}$.
(ii) $f\left(m^2\frac{a+b}{2}+(1-m)\delta\right) \leq \frac{1}{b-a}\int_{ma}^{mb} f(u)du$.

Proof. (i) By Theorem 2.4, we have

$$f\left(mta+m(1-t)b+(1-m)\delta\right) \leq mtf(a)+m(1-t)f(b)$$

for every $t \in [0,1]$. The inequality follows by integrating the above inequality over $[0,1]$ with respect to t and substituting $u = mta+m(1-t)b$.

(ii) For $x, y \in [a,b]$, we have

$$f\left(m^2\frac{x+y}{2}+(1-m)\delta\right) \leq m\frac{f(mx)+f(my)}{2}. \tag{2.3}$$

Substitute $x = ta+(1-t)b$ and $y = (1-t)a+tb$ in (2.3). Then integrating the obtained inequality with respect to t over $[0,1]$, we obtain the desired inequality, since

$$\begin{aligned}\int_0^1 f(mta+m(1-t)b)dt &= \int_0^1 f(m(1-t)a+mtb)dt \\ &= \frac{1}{b-a}\int_{ma}^{mb} f(u)du.\end{aligned}$$

This completes the proof. □

Theorem 2.6. *Let $f : [0,\infty) \to \mathbb{R}$ be (m,δ)-convex. Then*

$$f\left(t_1x_1+m\sum_{i=2}^{n} t_ix_i+(1-t_1)(1-m)\delta\right) \leq t_1f(x_1)+m\sum_{i=2}^{n} t_if\left(x_i\right) \tag{2.4}$$

where $0 \leq t_i \leq 1, x_i \geq 0; 1 \leq i \leq n$ and $\sum_{i=1}^{n} t_i = 1$.

Proof. We proceed by induction on n. The inequality 2.4 in the case $n = 2$ follows from (2.1) by $t := 1-t_1$, $x := x_1$ and $y := x_2$. Suppose that $n > 2$ and (2.4) holds for $n-1$. Let $0 \leq t_i \leq 1, x_i \geq 0$; $1 \leq i \leq n$ and $\sum_{i=1}^{n} t_i = 1$. Without loss in generality, let $t_n < 1$. So by (2.1) and the inductive hypothesis, we have

$$\begin{aligned}&f\left(t_1x_1+m\sum_{i=2}^{n} t_ix_i+(1-t_1)(1-m)\delta\right) \\ &= f\left((1-t_n)\left(\frac{t_1}{1-t_n}x_1+m\sum_{i=2}^{n-1}\frac{t_i}{1-t_n}x_i+\frac{1-t_1-t_n}{1-t_n}(1-m)\delta\right)\right.\end{aligned}$$

$$
\begin{aligned}
&+ t_n m x_n + (1-m)t_n\delta\Bigg) \\
&\leq (1-t_n)\, f\left(\frac{t_1}{1-t_n}x_1 + m\sum_{i=2}^{n-1}\frac{t_i}{1-t_n}x_i + \frac{1-t_1-t_n}{1-t_n}(1-m)\delta\right) \\
&\quad + t_n m f(x_n) \\
&\leq t_1 f(x_1) + m\sum_{i=2}^{n} t_i f\left(x_i\right),
\end{aligned}
$$

which proves (2.4). □

Lemma 2.7. *Let $f : [0,\infty) \to \mathbb{R}$ be (m,δ)-convex and $0 \leq a \leq x, y \leq mb + (1-m)\delta$ with*

$$x + y = a + mb + (1-m)\delta.$$

Then

$$f\left(x\right) \leq f(a) + mf(b) - f(y). \tag{2.5}$$

Proof. Since $0 \leq a \leq x \leq mb + (1-m)\delta$, there exists $t \in [0,1]$ such that

$$x = (1-t)a + t(mb + (1-m)\delta).$$

Therefore,

$$
\begin{aligned}
f\left(x\right) &= f\left((1-t)a + tmb + (1-m)t\delta\right) \\
&\leq (1-t)f(a) + mtf(b) \\
&= f(a) + mf(b) - (tf(a) + (1-t)mf(b)) \\
&\leq f(a) + mf(b) - f\left(ta + (1-t)mb + (1-m)(1-t)\delta\right) \\
&= f(a) + mf(b) - f(y),
\end{aligned}
$$

which proves (2.5). □

Theorem 2.8. *Let $f : [0,\infty) \to \mathbb{R}$ be (m,δ)-convex and $0 \leq a \leq x_i \leq mb + (1-m)\delta$; $i = 1,\ldots,n$. Then*

$$
\begin{aligned}
f\left(ma + m^2 b - m\sum_{i=1}^{n} t_i x_i + (1-m^2)\delta\right) \\
\leq mf(a) + m^2 f(b) - m\sum_{i=1}^{n} t_i f(x_i)
\end{aligned} \tag{2.6}
$$

where $t_i \geq 0$; $i = 1,\ldots,n$ and $\sum_{i=1}^{n} t_i = 1$.

Proof. By (2.2) and (2.5), we arrive at

$$\begin{aligned}
&f\left(ma+m^2b-m\sum_{i=1}^{n}t_ix_i+(1-m^2)\delta\right)\\
&=f\left(m\sum_{i=1}^{n}t_i\left(a+mb-x_i+(1-m)\delta\right)+(1-m)\delta\right)\\
&\leq m\sum_{i=1}^{n}t_if\left(a+mb-x_i+(1-m)\delta\right)\\
&\leq m\sum_{i=1}^{n}t_i\left(f(a)+mf(b)-f(x_i)\right)\\
&=mf(a)+m^2f(b)-m\sum_{i=1}^{n}t_if(x_i)
\end{aligned}$$

and the proof of (2.6) is complete. □

Theorem 2.9. *Let $f:[0,\infty)\to\mathbb{R}$ be (m,δ)-convex and $0\leq a\leq x_i\leq mb+(1-m)\delta$; $i=1,\ldots,n$ and $0\leq a\leq x\leq mb+(1-m)\delta$. Let $t_i\geq 0$; $i=1,\ldots,n$, $0\leq p\leq 1$ and $\sum_{i=1}^{n}t_i=p:=1-r$. Then*

$$\begin{aligned}
&f\left((r+mp)\left(a+mb\right)-rx-m\sum_{i=1}^{n}t_ix_i+(1+mp)(1-m)\delta\right)\\
&\leq(r+mp)\left(f(a)+mf(b)\right)-rf(x)-m\sum_{i=1}^{n}t_if(x_i).
\end{aligned}$$

Proof. Using(2.4) and (2.5), we obtain

$$\begin{aligned}
&f\left((r+mp)a+\left(rm+m^2p\right)b-rx-m\sum_{i=1}^{n}t_ix_i+(1+mp)(1-m)\delta\right)\\
&=f\Bigg(r\left(a+mb-x+(1-m)\delta\right)+m\sum_{i=1}^{n}t_i\left(a+mb-x_i+(1-m)\delta\right)\\
&\quad+(1-r)(1-m)\delta\Bigg)\\
&\leq rf\left(a+mb-x+(1-m)\delta\right)+m\sum_{i=1}^{n}t_if\left(a+mb-x_i+(1-m)\delta\right)\\
&\leq r\left(f(a)+mf(b)-f(x)\right)+m\sum_{i=1}^{n}t_i\left(f(a)+mf(b)-f(x_i)\right)
\end{aligned}$$

$$= (r+mp)f(a) + \left(rm + m^2p\right) f(b) - rf(x) - m\sum_{i=1}^{n} t_i f(x_i),$$

which proves the theorem □

Lemma 2.10. *If $f \in \mathcal{K}_{m,\delta}$, $t \in [0,1]$ and $x \geq 0$, then $f(tx + (1-t)(1-m)\delta) \leq tf(x)$.*

Proof. Let $t \in [0,1]$ and $x \geq 0$. Then

$$\begin{aligned} f(tx + (1-t)(1-m)\delta) &= f(tx + m(1-t).0 + (1-t)(1-m)\delta) \\ &\leq tf(x) + m(1-t)f(0) \leq tf(x). \end{aligned}$$

□

Theorem 2.11. *If $0 \leq n < m < 1$ and $f \in \mathcal{K}_{m,\delta} \bigcap \mathcal{K}_{m,\frac{\delta}{1-m}}$, then $f \in \mathcal{K}_{n,\delta}$.*

Proof. Let $t \in [0,1]$ and $x, y \geq 0$. Then

$$\begin{aligned} &f((1-t)x + nty + t(1-n)\delta) \\ &\quad = f\left((1-t)x + m.\frac{n}{m}ty + t(1-n)\delta\right) \\ &\quad = f\left((1-t)x + mt\left(\frac{n}{m}y + \left(1 - \frac{n}{m}\right)\delta\right) + t(1-m)\delta\right) \\ &\quad \leq (1-t)f(x) + mtf\left(\frac{n}{m}y + \left(1 - \frac{n}{m}\right)\delta\right) \\ &\quad = (1-t)f(x) + mtf\left(\frac{n}{m}y + \left(1 - \frac{n}{m}\right)(1-m).\frac{\delta}{1-m}\right) \\ &\quad \leq (1-t)f(x) + ntf(y). \end{aligned}$$

This completes the proof. □

Proposition 2.12. *If $0 \leq n \leq 1$, $f(0) \leq 0$ and f is convex on $[0,\infty)$, then $f \in \mathcal{K}_{n,\delta}$ for every $\delta \geq 0$ with $f(\delta) \leq 0$.*

Proof. Let $t \in [0,1]$ and $x, y \geq 0$. Then

$$\begin{aligned} f((1-t)x + nty + t(1-n)\delta) &= f\left((1-t)x + t\left(ny + (1-n)\delta\right)\right) \\ &\leq (1-t)f(x) + tf\left(ny + (1-n)\delta\right) \\ &\leq (1-t)f(x) + tnf\left(y\right) + t(1-n)f\left(\delta\right) \\ &\leq (1-t)f(x) + ntf(y), \end{aligned}$$

which completes the proof. □

Theorem 2.13. *Let f be (m,δ)-convex function on $[0,\infty)$, $0 \leq a < mb + (1-m)\delta$ and $f \in L_1[ma, b + (1-m)\delta]$. Then*

$$\frac{1}{1+m}\left(\int_a^{mb+(1-m)\delta} f(x)dx + \frac{mb - a + (1-m)\delta}{b - ma - (1-m)\delta}\int_{ma+(1-m)\delta}^{b} f(x)dx\right)$$

$$\leq (mb - a + (1-m)\delta)\frac{f(a)+f(b)}{2}.$$

Proof. Since f is (m,δ)-convex, we have

$$\begin{aligned}
&f((1-t)a + mtb + t(1-m)\delta) \leq (1-t)f(a) + mtf(b),\\
&f(m(1-t)a + tb + (1-t)(1-m)\delta) \leq m(1-t)f(a) + tf(b),\\
&f((1-t)b + mta + t(1-m)\delta) \leq (1-t)f(b) + mtf(a),\\
&f(m(1-t)b + ta + (1-t)(1-m)\delta) \leq m(1-t)f(b) + tf(a).
\end{aligned}$$

Now, add the above inequalities and integrate both sides with respect to t over $[0,1]$, to obtain

$$\begin{aligned}
&\frac{2}{mb - a + (1-m)\delta}\int_a^{mb+(1-m)\delta} f(x)dx\\
&\quad + \frac{2}{b - ma - (1-m)\delta}\int_{ma+(1-m)\delta}^{b} f(x)dx\\
&\leq (1+m)(f(a)+f(b)),
\end{aligned}$$

because

$$\begin{aligned}
&\int_0^1 f((1-t)a + mtb + t(1-m)\delta)dt\\
&= \int_0^1 f(m(1-t)b + ta + (1-t)(1-m)\delta)dt\\
&= \frac{1}{mb - a + (1-m)\delta}\int_a^{mb+(1-m)\delta} f(x)dx,
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 f(m(1-t)a + tb + (1-t)(1-m)\delta)dt\\
&= \int_0^1 f((1-t)b + mta + t(1-m)\delta)dt\\
&= \frac{1}{b - ma - (1-m)\delta}\int_{ma+(1-m)\delta}^{b} f(x)dx.
\end{aligned}$$

This completes the proof. □

Corollary 2.14. *If we set $\delta = 0$ in Theorem 2.13, we conclude Theorem 1.*

For $a, \delta \geq 0$, $0 \leq m \leq 1$ and $f : [0,\infty) \to \mathbb{R}$, consider the function $p^{\delta}_{a,m}(x)$ defined by

$$p^{\delta}_{a,m}(x) : [0,\infty)\backslash\{ma + (1-m)\delta\} \to \mathbb{R}$$

$$p^{\delta}_{a,m}(x) = \frac{f(x) - mf(a)}{x - ma - (1-m)\delta}.$$

Also, for $x, y, z \geq 0$ with $z \neq y$, $y \neq mx+(1-m)\delta$ and $z \neq mx+(1-m)\delta$ define

$$r_{m,\delta}(x,y,z) := \frac{\begin{vmatrix} 1 & 1 & 1 \\ m(x-\delta) & y-\delta & z-\delta \\ mf(x) & f(y) & f(z) \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ m(x-\delta) & y-\delta & z-\delta \\ m^2(x-\delta)^2 & (y-\delta)^2 & (z-\delta)^2 \end{vmatrix}}.$$

Theorem 2.15. *Suppose that $f : [0,\infty) \to \mathbb{R}$ is function and $a, \delta \geq 0$ and $0 \leq m \leq 1$. Then the following are equivalent:*

1. *$f((1-t)x + mty + t(1-m)\delta) \leq (1-t)f(x) + mtf(y)$ for every $t \in [0,1]$ and every $x \in [0,\infty)\backslash\{ma + (1-m)\delta\}$.*
2. *The function $p^{\delta}_{a,m}$ is increasing on $[0,\infty)\backslash\{ma + (1-m)\delta\}$.*
3. *If $x, y, z \geq 0$ with $z \neq y \neq mx+(1-m)\delta \neq z$, then $r_{m,\delta}(x,y,z) \geq 0$.*

Proof. (1) $\Rightarrow$ (2) : If $ma + (1-m)\delta < x < y$, then

$$x = ty + m(1-t)a + (1-m)(1-t)\delta.$$

for some $t \in [0,1]$. Therefore,

$$\begin{aligned} \frac{f(x) - mf(a)}{x - ma - (1-m)\delta} &= \frac{f(ty + m(1-t)a + (1-m)(1-t)\delta) - mf(a)}{ty + m(1-t)a + (1-m)(1-t)\delta - ma - (1-m)\delta} \\ &\leq \frac{tf(y) + m(1-t)f(a) - mf(a)}{t(y - ma) - t(1-m)\delta} \\ &= \frac{t(f(y) - mf(a))}{t(y - ma) - t(1-m)\delta} \\ &= \frac{f(y) - mf(a)}{(y - ma) - (1-m)\delta}. \end{aligned}$$

Similarly, if $y < x < ma + (1-m)\delta$, then there is $t \in [0,1]$ such that

$$x = ty + m(1-t)a + (1-m)(1-t)\delta$$

for some $t \in [0,1]$. Therefore, since $x < ma + (1-m)\delta$,

$$\begin{aligned} \frac{f(x) - mf(a)}{x - ma - (1-m)\delta} &= \frac{f(ty + m(1-t)a + (1-m)(1-t)\delta) - mf(a)}{ty + m(1-t)a + (1-m)(1-t)\delta - ma - (1-m)\delta} \\ &\geq \frac{tf(y) + m(1-t)f(a) - mf(a)}{t(y - ma) - t(1-m)\delta} \\ &= \frac{t(f(y) - mf(a))}{t(y - ma) - t(1-m)\delta} \end{aligned}$$

$$= \frac{f(y) - mf(a)}{y - ma - (1-m)\delta}.$$

(2) $\Leftrightarrow$ (3) : Since

$$r_{m,\delta}(x, y, z) = \frac{p^{\delta}_{x,m}(z) - p^{\delta}_{x,m}(y)}{z - y},$$

(2) and (3) are equivalent.

(3) $\Rightarrow$ (1) : Suppose that $x, z \in [0, \infty)\backslash\{ma + (1-m)\delta\}$ and $y = (1-t)x + mtz + t(1-m)\delta$ for some $t \in (0, 1)$. Thus,

$$\begin{aligned} r_{m,\delta}(z, y, x) &= \frac{p^{\delta}_{z,m}(y) - p^{\delta}_{z,m}(x)}{y - x} \\ &= \frac{(1-t)f(x) + tmf(z) - f((1-t)x + mtz + t(1-m)\delta)}{t(1-t)((1-m)\delta + mz - x)^2}. \end{aligned}$$

Hence, if (3) holds, then (1) holds. □

Corollary 2.16. *Let f be a differentiable function on $[0, \infty)$. Then f is (m, δ)-convex if and only if*

$$f'(x)(x - ma - (1-m)\delta) \geq f(x) - mf(a),$$

for every $x, a \geq 0$ that $x \neq ma + (1-m)\delta$.

Proof. This follows from Theorem 2.15 and this fact that $p^{\delta}_{a,m}(x)$ is increasing if and only if ${p^{\delta}_{a,m}}'(x) \geq 0$. □

3. Examples and Applications

In this section, some examples and applications of the results are given.

Example 3.1. If $\delta \geq 0$ and $a\delta + b \leq 0$, then the function $f(x) = ax + b$ is (m, δ)-convex on $[0, \infty)$ for every $m \in [0, 1]$.

Example 3.2. Let $0 < q \leq 1$, then the function $f(x) = 1 - x^q$ is $\left(\frac{1}{2}, \delta\right)$-convex on $(0, \infty)$ for every $\delta \geq 1$.

Proof. Let $x, y > 0$ be fixed. Define the function $g : [0, 1] \to \mathbb{R}$ by

$$\begin{aligned} g(t) &= \left((1-t)x + \frac{t}{2}y + \frac{t}{2}\delta\right)^q - (1-t)x^q - \frac{t}{2}y^q - \frac{t}{2} \\ &= \left((1-t)x + \frac{t}{2}y + \frac{t}{2}\delta\right)^q - 1 + (1 - x^q)(1-t) + \frac{t(1-y^q)}{2} \end{aligned}$$

We have $g(0) = 0$ and

$$g(1) = \frac{(y+\delta)^q}{2^q} - \frac{y^q + 1}{2}.$$

On the other hand, since u^q is concave in $(0,\infty)$, by Hermite-Hadamard inequality for concave functions, we get

$$\frac{y^q+1}{2} \leq \frac{(y+\delta)^q}{2^q}.$$

Therefore $g(1) \geq 0$.

Also,

$$g''(t) = (q-1)\frac{(\delta+y-2x)^2}{4}\left((1-t)x+\frac{t}{2}y+\frac{t}{2}\delta\right)^{q-2} \leq 0.$$

Therefore, $g(t) \geq 0$ for every $t \in [0,1]$, and thus $f(x) = 1-x^q$ is $\left(\frac{1}{2},\delta\right)$-convex on $(0,\infty)$. □

Example 3.3. The function $f(x) = -\log(x)$ is $\left(\frac{1}{2},\delta\right)$-convex on $(0,\infty)$ for every $\delta \geq 1$.

Proof. Let $x, y > 0$ be fixed. Define the function $g : [0,1] \to \mathbb{R}$ by

$$g(t) = \log\left((1-t)x+\frac{t}{2}y+\frac{t}{2}\delta\right) - (1-t)\log(x) - \frac{t}{2}\log(y).$$

We have $g(0) = 0$ and

$$g(1) = \log\left(\frac{y+\delta}{2\sqrt{y}}\right) = \log\left(1+\frac{(\sqrt{y}-1)^2+\delta-1}{2\sqrt{y}}\right) \geq 0$$

Also,

$$g''(t) = -\frac{(y+\delta-2x)^2}{(2(1-t)x+ty+t\delta)^2} \leq 0.$$

Therefore, $g(t) \geq 0$ for every $t \in [0,1]$, and thus $f(x) = -\log(x)$ is $\left(\frac{1}{2},\delta\right)$-convex on $(0,\infty)$. □

Definition 3.4. If $Q = \{t_i\}_{i=1}^n$ is a probability distribution, then the Tsallis entropy $T_q(Q)$ is defined by (see [23])

$$T_q(Q) := -\sum_{i=1}^n t_i^{1-q}\log_q t_i, \quad (q \neq 0)$$

where $\log_q x := \dfrac{x^q-1}{q}$, $(x > 0, q \neq 0)$ is the q–logarithmic function. Furthermore, the Shannon entropy of Q is defined by

$$H(Q) := -\sum_{i=1}^n t_i \log t_i.$$

Lemma 3.5. *The function* $\log_q x = \dfrac{x^q-1}{q}$ *is* $\left(\frac{1}{2},\delta\right)$*-convex on* $(0,\infty)$ *for every* $\delta \geq 1$.

Proof. If f is (m,δ)-convex and $c>0$, then cf is also (m,δ)-convex. Thus, the desired result follows from the Example 3.2. □

Proposition 3.6. *Let $Q=(t_1,\dots,t_n)$ be a positive probability distribution and $\delta\geq 1$. Then*

$$T_q(Q)\leq \frac{(n+\delta)^q-2^q}{q\times 2^{q-1}}.$$

Proof. Applying Theorem 2.4 with $f(x)=\dfrac{1-x^q}{q}$, $x_i:=\frac{1}{t_i}$; $i=1,\dots,n$, $m=\frac{1}{2}$ and using Lemma 3.5. □

Proposition 3.7. *If $Q=(t_1,\dots,t_n)$ is a positive probability distribution, then*

$$H(Q)\leq 2\log\left(\frac{n+\delta}{2}\right).$$

for every $\delta\geq 1$.

Proof. Using Theorem 2.4 with $f(x)=-\log(x)$, $x_i:=\frac{1}{t_i}$; $i=1,\dots,n$, $m=\frac{1}{2}$ and applying Example 3.3. □

Proposition 3.8. *If $Q=(t_1,\dots,t_n)$ is a positive probability distribution, then*

$$H(Q)\leq \min_{1\leq i\leq n}\left\{2\log\left(\frac{n+1+(1-t_i)\,\delta}{2}\right)+t_i\log(t_i)\right\}. \tag{3.1}$$

for every $\delta\geq 1$.

Proof. Suppose that $r\in\{1,\dots,n\}$ is arbitrary. Set $f(x)=-\log(x)$, $x_i:=\frac{1}{t_i}$; $i=1,\dots,n$, $m=\frac{1}{2}$ in Theorem 2.6 and by Example 3.2, we obtain

$$\begin{aligned}-\log\left(\frac{n+1+(1-t_r)\,\delta}{2}\right)&\leq t_r\log(t_r)+\frac{1}{2}\sum_{i\neq r}^{n}t_i\log(t_i)\\&=\frac{t_r\log(t_r)-H(Q)}{2}.\end{aligned}$$

Since $r\in\{1,\dots,n\}$ is arbitrary, (3.1) holds. □

Proposition 3.9. *If $Q=(t_1,\dots,t_n)$ is a positive probability distribution and $0\leq q\leq 1$, then*

$$T_q(Q)\leq \min_{1\leq i\leq n}\left\{\frac{(n+1+(1-t_i)\delta)^q-2^q}{q\times 2^{q-1}}+\frac{t_i-t_i^{1-q}}{q}\right\} \tag{3.2}$$

for every $\delta\geq 1$.

Proof. Assume that $r \in \{1, \dots, n\}$ is fixed. Putting $f(x) = \dfrac{1-x^q}{q}$, $x_i := \frac{1}{t_i}$; $i = 1, \dots, n$, $m = \frac{1}{2}$ in Theorem 2.6 and by Lemma 3.5, we get

$$\begin{aligned}\frac{1-\left(\frac{n+1+(1-t_1)\delta}{2}\right)^q}{q} &\leq \frac{t_r - t_r^{1-q}}{q} + \frac{1}{2}\sum_{i\neq r}^{n}\frac{t_i - t_i^{1-q}}{q} \\ &= \frac{t_r - t_r^{1-q}}{2q} - \frac{1}{2}T_q(Q).\end{aligned}$$

Thus,

$$T_q(Q) \leq \frac{2\left(\frac{n+1+(1-t_1)\delta}{2}\right)^q - 2}{q} + \frac{t_r - t_r^{1-q}}{q}.$$

Hence (3.2) holds, because $r \in \{1, \dots, n\}$ is arbitrary. □

Let $t_i \geq 0$, $\sum_{i=1}^{n} t_i = 1$, $0 < a \leq x_i \leq mb + (1-m)\delta$; $i = 1, \dots, n$ and

$$x = \{x_i | 1 \leq i \leq n\}.$$

Define

$$\tilde{G}_m(x) = \frac{a^m b^{m^2}}{[G(x)]^m}, \qquad \tilde{A}_{m,\delta}(x) = ma + m^2 b - mA(x) + (1-m^2)\delta$$

where

$$G(x) = \prod_{i=1}^{n} x_i^{t_i}, \qquad A(x) = \sum_{i=1}^{n} t_i x_i$$

are the usual geometric and arithmetic means, respectively.

Proposition 3.10. *Let $1 \leq \delta$. Then $\tilde{G}_{\frac{1}{2}}(x) \leq \tilde{A}_{\frac{1}{2},\delta}(x)$.*

Proof. Applying Theorem 2.8 with $f(x) = -\log(x)$, $m = \frac{1}{2}$ and use Example 3.3. □

Example 3.11. If $x \geq 0$, set $x_1 := x$, $x_2 := x^2$ in Proposition 3.10, then we have $\tilde{G}_{\frac{1}{2}}(x) = \sqrt[4]{x}$ and $\tilde{A}_{\frac{1}{2},1}(x) = \frac{1}{4}x + \frac{3}{4}$. Thus $\sqrt[4]{x} \leq \frac{1}{4}x + \frac{3}{4}$ for every $x \geq 0$.

Proposition 3.12. *Let $\delta \geq 1$ and $0 \leq a \leq x_i \leq \frac{b+\delta}{2}$; $i = 1, \dots, n$ and $0 \leq a \leq x \leq \frac{b+\delta}{2}$. Let $t_i \geq 0$; $i = 1, \dots, n$, $0 \leq p \leq 1$ and $\sum_{i=1}^{n} t_i = p := 1 - r$. Then*

$$\frac{a^{r+\frac{1}{2}p} \times b^{\frac{1}{2}r+\frac{1}{4}p}}{x^r \times \sqrt{G(x)}} \leq \frac{(2r+p)(2a+b)}{4} - rx - \frac{1}{2}A(x) + \frac{\delta(2+p)}{4}.$$

Proof. The result follows by the use of Example 3.3 and Theorem 2.9 with $f(x) = -\log(x)$ and $m = \frac{1}{2}$. □

4. Conclusion

In this paper, we introduced a new class of functions termed (m, δ)-convex functions, which generalizes the classical notion of m-convexity. We established fundamental properties of these functions and proved several important inequalities for them, including Jensen-type, Hermite-Hadamard-type, and other integral inequalities. Furthermore, we demonstrated the connection between these functions and Tsallis entropy, as well as arithmetic and geometric means, providing practical applications in information theory and analysis.

The obtained results represent a step forward in generalizing convex function theory and extending its applicability to optimization and entropy-related problems. For future research, the following directions are suggested: 1-Extending this concept to multi-dimensional spaces and functions of several variables. 2- Using the term $\delta(x, y)$ instead of δ in the definition of (m, δ)-convex functions.

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Department of Mathematics, Sirjan University of Technology, P.O.Box 78137-33385, Sirjan, Iran.
Email address: ysayyari@gmail.com, y.sayyari@sirjantech.ac.ir