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Characterization of Hilbert-Schmidt Composition-Differentiation Operators Induced by Radial Derivative

Jahanbakhsh Habibevand[1] and Ali Abkar[2*]

Abstract. We use the notion of radial derivative of analytic functions to introduce a new composition-differentiation operator on the classical spaces of analytic functions on the unit disk. These classical spaces of analytic functions include the Hardy space, the Bergman space, and the Dirichlet space. We then obtain necessary and sufficient conditions on the symbol function to ensure that the induced operator is Hilbert-Schmidt.

1. Introduction

Let X be a Banach space consisting of analytic functions on a region in the complex plane; for instance, X can be the Hardy space, the Bergman space, or the Dirichlet space on the unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

The space of analytic functions on the unit disk is denoted by $\mathrm{Hol}(\mathbb{D})$. The Hardy space H^2 consists of all functions $f \in \mathrm{Hol}(\mathbb{D})$ such that

$$\|f\| = \left(\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right)\right|^2 d\theta \right)^{1/2} < \infty.$$

H^2 is a Hilbert space with the inner product

$$\langle f, g\rangle = \frac{1}{2\pi} \int_0^{2\pi} f\left(e^{i\theta}\right) \overline{g\left(e^{i\theta}\right)} d\theta$$

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where

$$f\left(e^{i\theta}\right)=\lim_{r\to 1^-} f\left(re^{i\theta}\right)$$

is the boundary function of f (the radial limit of every function in the Hardy space exists almost everywhere on the unit circle [13, Chapter 17]). It now follows that if $f,g\in H^2$, and if $\{a_n\}$ and $\{b_n\}$ are Taylor coefficients of f and g, respectively, then

$$\langle f,g\rangle=\sum_{n=0}^{\infty} a_n\overline{b_n},$$

in particular,

$$\|f\|_{H^2}^2=\sum_{n=0}^{\infty}|a_n|^2.$$

The Bergman space A^2 consists of functions $f\in \mathrm{Hol}(\mathbb{D})$ that satisfy

$$\|f\|_{A^2}^2=\int_{\mathbb{D}}|f(z)|^2dA(z)<\infty,$$

where $dA(z)=\pi^{-1}dxdy$ is the normalized area measure in the unit disk. A^2 is a Hilbert space whose inner product is defined by

$$\langle f,g\rangle=\int_{\mathbb{D}} f(z)\overline{g(z)}dA(z).$$

This implies that if $f\in A^2$ has Taylor expansion $f(z)=\sum\limits_{n=0}^{\infty}a_nz^n$, then

$$\|f\|_{A^2}^2=\sum_{n=0}^{\infty}\frac{|a_n|^2}{n+1}.$$

The Dirichlet space $\mathcal{D}$ consists of analytic functions f in the unit disk with the property $f'\in A^2$; that is,

$$\int_{\mathbb{D}}\left|f'(z)\right|^2dA(z)<\infty.$$

The norm of a function $f\in\mathcal{D}$ is given by

$$\|f\|_{\mathcal{D}}=\left(|f(0)|^2+\int_{\mathbb{D}}\left|f'(z)\right|^2dA(z)\right)^{1/2}.$$

Let $\varphi:\mathbb{D}\to\mathbb{D}$ be an analytic function. The *composition operator* $C_\varphi:X\to X$ is defined by the following relation:

$$C_\varphi(f)=f\circ\varphi.$$

The analytic self-map φ is called the *symbol* function for C_φ. It is well known that the composition operator is bounded on the Hardy space, as well as on the Bergman space (see the books [5] or [18]); but for the

Dirichlet space there are instances in which the composition operator is unbounded [5].

Similarly, the *composition-differentiation* operator $D_\varphi : X \to X$ is defined by

$$D_\varphi(f) = f' \circ \varphi.$$

We should point out that $D(f) = f'$ is not bounded even on the Hardy space H^2; however, in some instances, $D_\varphi : H^2 \to H^2$ is bounded. The same argument applies to the Bergman and the Dirichlet space.

Several authors have studied the boundedness, compactness, and Hilbert-Schmidtness of this operator (see [4, 12]). We recall that an operator T on a separable Hilbert space $\mathcal{H}$ is said to be *Hilbert-Schmidt* if there exists an orthonormal basis $\{e_n\}$ in $\mathcal{H}$ such that

$$\|T\|_{HS}^2 := \sum_{n=0}^{\infty} \|Te_n\|^2 < \infty.$$

It is known that the Hilbert-Schmidt norm of T, $\|T\|_{HS}$, does not depend on the choice of the orthonormal basis. Some authors use the notation $C_\varphi D$ for D_φ (see [11]); indeed,

$$C_\varphi D(f) = C_\varphi\left(f'\right) = f' \circ \varphi = D_\varphi(f).$$

This notation justifies the operator $DC\varphi$ defined by

$$DC\varphi(f) = \left(f' \circ \varphi\right)\varphi'.$$

This latter was extensively studied by Hibschweiler and Portnoy [10].

In this paper we intend to study a new composition-differentiation operator E_φ. Given an analytic self-map φ on the unit disk, we define

$$E_\varphi(f) = (\mathcal{R}f) \circ \varphi = \varphi \cdot \left(f' \circ \varphi\right),$$

where

$$\mathcal{R}f(z) = zf'(z)$$

is the radial derivative of f. In this way, we may also consider

$$E_\varphi^2(f) = \left(\mathcal{R}f'\right) \circ \varphi = \varphi \cdot \left(f'' \circ \varphi\right),$$

and for a positive integer k,

$$E_\varphi^k(f) = \varphi \cdot \left(f^{(k)} \circ \varphi\right).$$

As for the origin of this operator, we should recall that for a holomorphic function $f(z_1, \ldots, z_n)$ in $\mathbb{C}^n$, the radial derivative of f is defined by

$$\mathcal{R}f(z) = z_1 \frac{\partial f}{\partial z_1}(z) + \cdots + z_n \frac{\partial f}{\partial z_n}(z), \quad z = (z_1, \ldots, z_n).$$

In several complex variables, the radial derivative and the gradient are regarded as suitable alternatives to replace the differentiation operator

for defining the Bloch-type spaces as well as the Besov spaces of holomorphic functions [19]. This justifies the study of the operator induced by the radial derivative in one complex variable. The advantage of this approach is that it opens the way to study operators induced by fractional radial derivatives. For instance, if f is an analytic function in the unit ball of $\mathbb{C}^n$, we may define for $t \in \mathbb{R}$,

$$\mathcal{R}^t f(z) = \sum_{k=1}^{\infty} k^t f_k(z), \qquad f(z) = \sum_{k=1}^{\infty} f_k(z),$$

where f is represented by its homogeneous expansion. The radial differential operator $\mathcal{R}^t$ is invertible on $\mathrm{Hol}(\mathbb{D})$ modulo constants (see [19, page 18]). We will not discuss $\mathcal{R}^t$ and its inverse in this paper; we just mention that they have lots of applications in the development of the theory in different areas of pure and applied analysis; see for instance [2, 15].

The study of Hilbert-Schmidtness of composition operator C_φ is classical. As for the composition-differentiation operator, it was proved in [4] that D_φ is Hilbert-Schmidt on H^2 if and only if

$$\sup_{0<r<1} \int_0^{2\pi} \frac{d\theta}{\left(1 - |\varphi\left(re^{i\theta}\right)|^2\right)^3} < \infty.$$

Similar result was also obtained in the setting of Bergman spaces; that is D_φ is Hilbert-Schmidt on A^2 if and only if

$$\int_{\mathbb{D}} \frac{dA}{\left(1 - |\varphi|^2\right)^4} < \infty.$$

Necessary and sufficient condition for the Hilbert-Schmidtness of D_φ on the Dirichlet space was provided in a recent paper by Allen et al [3]. The boundedness and spectral properties of this operator constitutes the content of the papers [7, 8].

In this paper, we focus on the Hilbert-Schmidtness of E_φ acting on the Hardy space, the Bergman space, and the Dirichlet space. We will find concrete equivalent conditions to ensure that E_φ is Hilbert-Schmidt.

In §2, we study E_φ and all its variants on H^2. In §3, we will take up the Bergman space, and finally in the last section we study E_φ on the Dirichlet space. We close this section by mentioning that a good reference for composition operators are the books [5, 16]. See also the relevant papers on this topic in [1, 14, 17].

2. Hardy Space

This section is devoted to the study of Hilbert-Schmidt composition-differentiation operator E_φ on the Hardy space. An analytic function f in the unit disk is said to belong to the Hardy space H^2 if

$$\|f\|^2 = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right)\right|^2 d\theta < \infty.$$

Since the evaluation functionals are bounded, the Hardy space is a reproducing kernel Hilbert space; this means that for each $w \in \mathbb{D}$, there is a function

$$K_w(z) = \frac{1}{1-\overline{w}z} \in H^2$$

satisfying

$$f(w) = \langle f, K_w \rangle .$$

An easy computation shows that the sequence $\{z^n\}_{n=0}^{\infty}$ constitutes an orthonormal basis for the Hardy space H^2. For a detailed account on the theory of Hardy spaces, we refer the reader to [6], [9], and [13].

It is reasonable to start our discussion with a theorem on the boundedness of our new operator, however, we prove the stronger result that E_φ is compact if $\|\varphi\|_\infty = \sup\{|\varphi(z)| : z \in \mathbb{D}\} < 1$.

Theorem 2.1. *Let φ be an analytic self-map of the unit disk such that $\|\varphi\|_\infty < 1$. Then $E_\varphi : H^2 \to H^2$ is compact.*

Proof. For $f(z) = \sum\limits_{k=0}^{\infty} a_k z^k$ we have

$$E_\varphi(f)(z) = \sum_{k=0}^{\infty} k a_k \varphi^k(z), \quad z \in \mathbb{D}.$$

We now define (for $f \in H^2$)

$$T_n(f) = \sum_{k=0}^{n} k a_k \varphi^k, \quad n = 0, 1, 2, \ldots$$

Note that T_n maps H^2 into the linear span of $\{1, \varphi, \ldots, \varphi^n\} \subset H^2$. It is clear that T_n is linear and

$$\begin{aligned} \|T_n f\|_{H^2}^2 &= \left\| \sum_{k=0}^{n} k a_k \varphi^k \right\|_{H^2}^2 \\ &\leq (n+1) \sum_{k=0}^{n} \left\| k a_k \varphi^k \right\|_{H^2}^2 \end{aligned}$$

$$\leq (n+1)\sum_{k=0}^{n} k^2|a_k|^2$$
$$\leq (n+1)n^2 \|f\|_{H^2}^2 ,$$

from which we obtain $\|T_n\|_{H^2} \leq n\sqrt{n+1}$. On the other hand, each T_n is a finite-rank operator so that it is compact. To prove that E_φ is compact, it suffices to verify that $T_n \to E_\varphi$ as $n \to \infty$. To this end we write (for $f \in H^2$):

$$\begin{aligned}
\|(E_\varphi - T_n) f\|_{H^2} &= \left\| \sum_{k=n+1}^{\infty} k a_k \varphi^k \right\|_{H^2} \\
&\leq \sum_{k=n+1}^{\infty} k|a_k| \left\|\varphi^k\right\|_{H^2} \\
&\leq \sum_{k=n+1}^{\infty} k|a_k| \, \|\varphi\|_\infty^k \\
&\leq \left(\sum_{k=n+1}^{\infty} |a_k|^2 \right)^{1/2} \left(\sum_{k=n+1}^{\infty} k^2 \|\varphi\|_\infty^{2k} \right)^{1/2},
\end{aligned}$$

which leads to

$$\|E_\varphi - T_n\|_{H^2} \leq \left(\sum_{k=n+1}^{\infty} k^2 \|\varphi\|_\infty^{2k} \right)^{1/2}.$$

Thus, E_φ is compact if the series

$$\sum_{k=0}^{\infty} k^2 \|\varphi\|_\infty^{2k}$$

converges and this is true as

$$\sum_{k=0}^{\infty} k^2 \|\varphi\|_\infty^{2k} = \frac{\|\varphi\|_\infty^2 + \|\varphi\|_\infty^4}{\left(1 - \|\varphi\|_\infty^2\right)^3}. \qquad \square$$

We now turn to the main topic of this paper.

Theorem 2.2. *Let φ be an analytic self-map of the unit disk. Then $E_\varphi : H^2 \to H^2$ is Hilbert-Schmidt if and only if*

$$\sup_{0<r<1} \int_0^{2\pi} \frac{\left|\varphi\left(re^{i\theta}\right)\right|^2}{\left(1 - \left|\varphi\left(re^{i\theta}\right)\right|^2\right)^3} d\theta < \infty.$$

Proof. We first recall that $\{z^k\}_{k=0}^{\infty}$ is an orthonormal basis in the Hardy space. We then have

$$\begin{aligned}\sum_{n=0}^{\infty}\left\|E_\varphi\left(z^k\right)\right\|^2 &= \sum_{k=0}^{\infty}\sup_{0<r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|k\varphi\left(re^{i\theta}\right)^k\right|^2 d\theta\right\}\\ &= \sup_{0<r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\sum_{k=0}^{\infty}k^2\left|\varphi\left(re^{i\theta}\right)\right|^{2k} d\theta\right\}\\ &\leq \sup_{0<r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{2\left|\varphi\left(re^{i\theta}\right)\right|^2}{\left(1-\left|\varphi\left(re^{i\theta}\right)\right|^2\right)^3}d\theta\right\}\\ &< \infty\end{aligned}$$

provided that

$$\sup_{0<r<1}\int_0^{2\pi}\frac{\left|\varphi\left(re^{i\theta}\right)\right|^2}{\left(1-\left|\varphi\left(re^{i\theta}\right)\right|^2\right)^3}d\theta<\infty.$$

Conversely, if E_φ is Hilbert-Schmidt, it follows from the third equality above that

$$\begin{aligned}\sup_{0<r<1}\int_0^{2\pi}\frac{\left|\varphi\left(re^{i\theta}\right)\right|^2}{\left(1-\left|\varphi\left(re^{i\theta}\right)\right|^2\right)^3}d\theta &\leq \sup_{0<r<1}\int_0^{2\pi}\frac{\left|\varphi\left(re^{i\theta}\right)\right|^2+\left|\varphi\left(re^{i\theta}\right)\right|^4}{\left(1-\left|\varphi\left(re^{i\theta}\right)\right|^2\right)^3}d\theta\\ &= 2\pi\sum_{k=0}^{\infty}\left\|E_\varphi\left(z^k\right)\right\|^2\\ &< \infty.\end{aligned}$$

□

Theorem 2.3. *Let φ be an analytic self-map of the unit disk. Then $E_\varphi^2 : H^2 \to H^2$ is Hilbert-Schmidt if and only if*

$$\sup_{0<r<1}\int_0^{2\pi}\frac{\left|\varphi\left(re^{i\theta}\right)\right|^2}{\left(1-\left|\varphi\left(re^{i\theta}\right)\right|^2\right)^5}d\theta<\infty. \tag{2.1}$$

Proof. Let the above supremum be finite. Considering

$$E_\varphi^2\left(z^k\right)\left(re^{i\theta}\right)=k(k-1)\varphi\left(re^{i\theta}\right)^{k-1}$$

together with

$$\sum_{k=1}^{\infty}k^2(k+1)^2t^k=\frac{4t^3+16t^2+4t}{(1-t)^5},\quad |t|<1,$$

we have

$$\begin{aligned}\sum_{k=0}^{\infty}\left\|E_{\varphi}^{2}\left(z^{k}\right)\right\|^{2} &= \sum_{k=2}^{\infty}\sup_{0<r<1}\left\{\frac{1}{2\pi}\int_{0}^{2\pi}\left|k(k-1)\varphi\left(re^{i\theta}\right)^{k-1}\right|^{2}d\theta\right\}\\ &= \sup_{0<r<1}\left\{\frac{1}{2\pi}\int_{0}^{2\pi}\sum_{k=1}^{\infty}k^{2}(k+1)^{2}\left|\varphi\left(re^{i\theta}\right)\right|^{2k}d\theta\right\}\\ &\leq \sup_{0<r<1}\left\{\frac{1}{2\pi}\int_{0}^{2\pi}\frac{24\left|\varphi\left(re^{i\theta}\right)\right|^{2}}{\left(1-\left|\varphi\left(re^{i\theta}\right)\right|^{2}\right)^{5}}d\theta\right\}\\ &< \infty.\end{aligned}$$

Now, assume that E_{φ}^{2} is Hilbert-Schmidt. We see that

$$\begin{aligned}&\sup_{0<r<1}\int_{0}^{2\pi}\frac{\left|\varphi\left(re^{i\theta}\right)\right|^{2}}{\left(1-\left|\varphi\left(re^{i\theta}\right)\right|^{2}\right)^{5}}d\theta\\ &\leq \sup_{0<r<1}\int_{0}^{2\pi}\frac{4\left|\varphi\left(re^{i\theta}\right)\right|^{2}+16\left|\varphi\left(re^{i\theta}\right)\right|^{4}+4\left|\varphi\left(re^{i\theta}\right)\right|^{6}}{\left(1-\left|\varphi\left(re^{i\theta}\right)\right|^{2}\right)^{5}}d\theta\\ &= \sum_{k=2}^{\infty}\sup_{0<r<1}\int_{0}^{2\pi}\left|k(k-1)\varphi\left(re^{i\theta}\right)^{k-1}\right|^{2}d\theta\\ &= 2\pi\sum_{n=1}^{\infty}\left\|E_{\varphi}^{2}\left(z^{k}\right)\right\|^{2}\\ &< \infty.\end{aligned}$$

which implies that (2.1) is satisfied. □

Theorem 2.4. *Let φ be an analytic self-map of the unit disk. Then E_{φ}^{3} is a Hilbert-Schmidt operator on H^{2} if and only if*

$$\sup_{0<r<1}\int_{0}^{2\pi}\frac{\left|\varphi\left(re^{i\theta}\right)\right|^{2}}{\left(1-\left|\varphi\left(re^{i\theta}\right)\right|^{2}\right)^{7}}d\theta<\infty. \tag{2.2}$$

Proof. We assume that the above supremum is finite, and write

$$E_{\varphi}^{3}\left(z^{k}\right)\left(re^{i\theta}\right)=k(k-1)(k-2)\varphi\left(re^{i\theta}\right)^{k-2}.$$

We see that

$$\sum_{k=0}^{\infty}\left\|E_{\varphi}^{3}\left(z^{k}\right)\right\|^{2}=\sup_{0<r<1}\left\{\frac{1}{2\pi}\int_{0}^{2\pi}\sum_{k=1}^{\infty}k^{2}(k+1)^{2}(k+2)^{2}\left|\varphi\left(re^{i\theta}\right)\right|^{2k}d\theta\right\}.$$

Note that

$$\sum_{k=1}^{\infty} k^2(k+1)^2(k+2)^2 t^k = \frac{36(t^4+9t^3+9t^2+3t)}{(1-t)^7}, \quad |t|<1, \tag{2.3}$$

so that

$$\begin{aligned}
&\sum_{k=0}^{\infty} \left\|E_\varphi^3\left(z^k\right)\right\|^2 \\
&= \sup_{0<r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{36\left(\left|\varphi\left(re^{i\theta}\right)\right|^8+9\left|\varphi\left(re^{i\theta}\right)\right|^6+9\left|\varphi\left(re^{i\theta}\right)\right|^4+3\left|\varphi\left(re^{i\theta}\right)\right|^2\right)}{\left(1-\left|\varphi\left(re^{i\theta}\right)\right|^2\right)^7} d\theta\right\} \\
&\le \sup_{0<r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{(36)(22)\left|\varphi\left(re^{i\theta}\right)\right|^2}{\left(1-\left|\varphi\left(re^{i\theta}\right)\right|^2\right)^7} d\theta\right\} \\
&< \infty.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\sup_{0<r<1}\int_0^{2\pi} \frac{\left|\varphi\left(re^{i\theta}\right)\right|^2}{\left(1-\left|\varphi\left(re^{i\theta}\right)\right|^2\right)^7} d\theta \\
&\le \sup_{0<r<1}\int_0^{2\pi} \frac{36\left(\left|\varphi\left(re^{i\theta}\right)\right|^8+9\left|\varphi\left(re^{i\theta}\right)\right|^6+9\left|\varphi\left(re^{i\theta}\right)\right|^4+3\left|\varphi\left(re^{i\theta}\right)\right|^2\right)}{\left(1-\left|\varphi\left(re^{i\theta}\right)\right|^2\right)^7} d\theta \\
&= 2\pi\sum_{k=0}^{\infty} \left\|E_\varphi^3\left(z^k\right)\right\|^2 \\
&< \infty.
\end{aligned}$$

□

Finally, we present another criterion for Hilbert-Schmidtness of E_φ on the Hardy space H^2 in terms of the area integral on the unit disk.

Theorem 2.5. *Let φ be an analytic self-map of the unit disk. Then $E_\varphi : H^2 \to H^2$ is Hilbert-Schmidt if and only if*

$$\int_{\mathbb{D}} \frac{|\varphi'|^2}{\left(1-|\varphi|^2\right)^5}\left(1-|z|^2\right) dA < \infty.$$

Proof. We first note that by the Littlewood-Paley identity,

$$\|f\|_{H^2}^2 = |f(0)|^2 + 2\int_{\mathbb{D}} \left|f'(z)\right|^2 \log\frac{1}{|z|} dA(z).$$

On the other hand,

$$\log \frac{1}{|z|^2} \approx 1 - |z|^2, \quad |z| \to 1^-.$$

Here we use the notation $f(x) \approx g(x)$ to indicate that f and g are non-negative functions in some vicinity of x, and there are two positive constants a and b such that for the values of x in the vicinity we have

$$a\, f(x) \leq g(x) \leq b\, f(x).$$

Thus

$$\|f\|_{H^2}^2 \approx |f(0)|^2 + \int_{\mathbb{D}} \left|f'(z)\right|^2 \left(1 - |z|^2\right) dA(z).$$

For simplicity assume that $\varphi(0) = 0$. Therefore we have

$$\begin{aligned} \sum_{k=0}^{\infty} \left\| E_\varphi\left(z^k\right)\right\|_{H^2}^2 &= \sum_{k=0}^{\infty} \int_{\mathbb{D}} \left|k^2 \varphi^{k-1} \varphi'\right|^2 \left(1 - |z|^2\right) dA \\ &= \int_{\mathbb{D}} \sum_{k=0}^{\infty} (k+1)^4 |\varphi|^{2k} |\varphi'|^2 \left(1 - |z|^2\right) dA. \end{aligned} \tag{2.4}$$

Differentiating the geometric series four times we obtain

$$\sum_{k=0}^{\infty} (k+1)(k+2)(k+3)(k+4) t^k = \frac{24}{(1-t)^5}, \quad |t| < 1. \tag{2.5}$$

On the other hand,

$$\frac{1}{24} \leq \frac{(k+1)^3}{(k+2)(k+3)(k+4)} \leq 1, \quad k \geq 0. \tag{2.6}$$

It now follows from (2.5) and (2.6) that

$$\frac{1}{24\,(1-t)^5} \leq \sum_{k=0}^{\infty} (k+1)^4 t^k \leq \frac{1}{(1-t)^5}, \quad |t| < 1.$$

This together with (2.4) implies that

$$\sum_{k=0}^{\infty} \left\| E_\varphi\left(z^k\right)\right\|_{H^2}^2$$

is finite if and only if

$$\int_{\mathbb{D}} \frac{|\varphi'|^2}{\left(1 - |\varphi|^2\right)^5} \left(1 - |z|^2\right) dA < \infty. \qquad \square$$

3. Bergman Space

The weighted Bergman space, denoted by A^2_α, where $\alpha > -1$ is fixed, consists of analytic functions f in the unit disk that satisfy the integrability condition

$$\|f\|^2_{A^2_\alpha} = (\alpha+1)\int_{\mathbb{D}} |f(z)|^2 \left(1-|z|^2\right)^\alpha dA(z) < \infty,$$

where $dA(z) = \frac{1}{\pi} dxdy$ is the normalized area measure in the unit disk. Note that A^2_α is a Hilbert space, and in case that $\alpha = 0$, we get the usual Bergman space A^2. Now, assume that $f \in A^2$ has Taylor expansion $f(z) = \sum\limits_{k=0}^{\infty} a_k z^k$. Then

$$\|f\|_{A^2} = \left(\sum_{k=0}^{\infty} \frac{|a_k|^2}{k+1}\right)^{1/2}.$$

Therefore, $\left\{\sqrt{k+1}\, z^k\right\}_{k=0}^{\infty}$ is an orthonormal basis for A^2. It follows from the Riesz representation theorem that A^2 is a reproducing kernel Hilbert space whose kernel is given by

$$K_z(w) = \frac{1}{(1-\overline{z}w)^2}, \quad (z,w) \in \mathbb{D}\times\mathbb{D}.$$

Similar computation shows that the vectors

$$e_k(z) = \left(\frac{\Gamma(k+\alpha+2)}{k!\,\Gamma(\alpha+2)}\right)^{1/2} z^k, \quad k = 0,1,2,\ldots$$

form an orthonormal basis for A^2_α, and the reproducing kernel of A^2_α is given by

$$K^\alpha_z(w) = \frac{1}{(1-\overline{z}w)^{\alpha+2}}, \quad (z,w) \in \mathbb{D}\times\mathbb{D}.$$

We are now in a position to characterize Hilbert-Schmidt composition-differentiation operators on the Bergman space. Before we begin the main topic, we should mention that if $\|\varphi\|_\infty < 1$, then E_φ is bounded, and indeed compact.

Theorem 3.1. *Let φ be an analytic self-map of the unit disk such that $\|\varphi\|_\infty < 1$. Then $E_\varphi : A^2 \to A^2$ is compact.*

Proof. The proof is similar to that of Theorem 2.1, so we skip the details. □

Theorem 3.2. *Let φ be an analytic self-map of the unit disk. Then E_φ is Hilbert-Schmidt on A^2 if and only if*

$$\int_{\mathbb{D}} \frac{|\varphi|^2}{(1-|\varphi|^2)^4} dA < \infty.$$

Proof. Assume the above inequality holds true. Using the equation

$$\sum_{k=0}^{\infty} k^2(k+1)t^k = \frac{4t^2+2t}{(1-t)^4}, \quad |t|<1,$$

we have

$$\begin{aligned}\sum_{k=0}^{\infty} \left\| E_\varphi\left(\sqrt{k+1}\, z^k\right)\right\|_{A^2}^2 &= \sum_{k=1}^{\infty} \int_{\mathbb{D}} \left| k\sqrt{k+1}\, \varphi^k \right|^2 dA \\ &\le 6 \int_{\mathbb{D}} \frac{|\varphi|^2}{\left(1-|\varphi|^2\right)^4} dA \\ &< \infty.\end{aligned}$$

For the converse, note that

$$\begin{aligned}\int_{\mathbb{D}} \frac{|\varphi|^2}{\left(1-|\varphi|^2\right)^4} dA &\le \int_{\mathbb{D}} \frac{4|\varphi|^4+2|\varphi|^2}{\left(1-|\varphi|^2\right)^4} dA \\ &= \int_{\mathbb{D}} \sum_{k=0}^{\infty} k^2(k+1)\,|\varphi|^{2k}\, dA \\ &= \sum_{k=0}^{\infty} \left\| E_\varphi\left(\sqrt{k+1}\, z^k\right)\right\|_{A^2}^2 \\ &< \infty.\end{aligned}$$

□

Theorem 3.3. *Let φ be an analytic self-map of the unit disk. Then E_φ^2 is Hilbert-Schmidt on A^2 if and only if*

$$\int_{\mathbb{D}} \frac{|\varphi|^2}{\left(1-|\varphi|^2\right)^6} dA < \infty. \tag{3.1}$$

Proof. Let φ satisfy the condition (3.1). Then, using

$$\sum_{k=1}^{\infty} k^2(k+1)^2(k+2)t^k = \frac{36t^3+72t^2+12t}{(1-t)^6}, \quad |t|<1,$$

we have

$$\sum_{k=0}^{\infty} \left\| E_\varphi^2\left(\sqrt{k+1}\, z^k\right)\right\|^2 = \sum_{k=2}^{\infty} \int_{\mathbb{D}} \left| k(k-1)\sqrt{k+1}\, \varphi^{k-1} \right|^2 dA$$

$$\begin{aligned} &\leq \int_{\mathbb{D}} \frac{120|\varphi|^2}{(1-|\varphi|^2)^6} dA \\ &< \infty. \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_{\mathbb{D}} \frac{|\varphi|^2}{(1-|\varphi|^2)^6} dA &\leq \int_{\mathbb{D}} \frac{36|\varphi|^6 + 72|\varphi|^4 + 12|\varphi|^2}{(1-|\varphi|^2)^6} dA \\ &= \sum_{k=0}^{\infty} \left\| E_\varphi^2 \left(\sqrt{k+1}\, z^k \right) \right\|^2 \\ &< \infty. \end{aligned}$$

□

Theorem 3.4. *Let φ be an analytic self-map of the unit disk. Then E_φ^3 is Hilbert-Schmidt on A^2 if and only if*

$$\int_{\mathbb{D}} \frac{|\varphi|^2}{\left(1-|\varphi|^2\right)^8} dA < \infty.$$

Proof. We start by writing

$$\begin{aligned} \sum_{k=0}^{\infty} \left\| E_\varphi^3 \left(\sqrt{k+1}\, z^k \right) \right\|^2 &= \sum_{k=3}^{\infty} \int_{\mathbb{D}} \left| k(k-1)(k-2)\sqrt{k+1}\, \varphi^{k-2} \right|^2 dA \\ &= \int_{\mathbb{D}} \sum_{k=1}^{\infty} k^2(k+1)^2(k+2)^2(k+3)|\varphi|^{2k} dA. \end{aligned}$$

It follows from (2.3) that

$$\sum_{k=1}^{\infty} k^2(k+1)^2(k+2)^2(k+3)t^k = \frac{36(16t^4 + 72t^3 + 66t^2)}{(1-t)^8}, \quad |t| < 1.$$

This implies that

$$\begin{aligned} \sum_{k=0}^{\infty} \left\| E_\varphi^3 \left(\sqrt{k+1}\, z^k \right) \right\|^2 &= \int_{\mathbb{D}} \frac{36\, \left(16|\varphi|^8 + 72|\varphi|^6 + 66|\varphi|^4\right)}{(1-|\varphi|^2)^8} dA \\ &\leq (36)(154) \int_{\mathbb{D}} \frac{|\varphi|^2}{(1-|\varphi|^2)^8} dA \\ &< \infty. \end{aligned}$$

Since $0 \leq |\varphi| \leq 1$, the reverse implication follows easily. □

The last theorem of this section is devoted to the study of E_φ on the weighted Bergman space A_α^2.

Theorem 3.5. *Let φ be an analytic self-map of the unit disk. Then $E_\varphi : A^2_\alpha \to A^2_\alpha$ is Hilbert-Schmidt if and only if*

$$\int_{\mathbb{D}} \frac{|\varphi|^2}{\left(1-|\varphi|^2\right)^{\alpha+4}} dA_\alpha < \infty.$$

Proof. Assume that the above inequality holds. We use the following orthonormal basis for A^2_α:

$$e_k(z) = \left(\frac{\Gamma(k+\alpha+2)}{k!\,\Gamma(\alpha+2)}\right)^{1/2} z^k, \quad k \geq 0.$$

It is easy to see that

$$E_\varphi(e_k)(z) = k\left(\frac{\Gamma(k+\alpha+2)}{k!\,\Gamma(\alpha+2)}\right)^{1/2} \varphi^k, \quad k \geq 0.$$

Manipulating the series

$$\frac{1}{(1-t)^{\alpha+2}} = \sum_{k=0}^{\infty} \frac{\Gamma(k+\alpha+2)}{k!\,\Gamma(\alpha+2)} t^k, \quad |t| < 1,$$

we see that

$$\sum_{k=0}^{\infty} \frac{k^2\,\Gamma(k+\alpha+2)}{k!\,\Gamma(\alpha+2)} t^k = \frac{(\alpha+2)^2 t^2 + (\alpha+2)t}{(1-t)^{\alpha+4}}, \quad |t| < 1.$$

This implies that

$$\begin{aligned}
\sum_{k=0}^{\infty} \|E_\varphi(e_k)\|^2_{A^2_\alpha} &= \int_{\mathbb{D}} \sum_{k=0}^{\infty} \frac{k^2\Gamma(k+\alpha+2)}{k!\,\Gamma(\alpha+2)} |\varphi|^{2k} dA_\alpha \\
&= \int_{\mathbb{D}} \frac{(\alpha+2)^2|\varphi|^4 + (\alpha+2)|\varphi|^2}{\left(1-|\varphi|^2\right)^{\alpha+4}} dA_\alpha \\
&\leq (\alpha+2)^2 \int_{\mathbb{D}} \frac{|\varphi|^2}{\left(1-|\varphi|^2\right)^{\alpha+4}} dA_\alpha \\
&< \infty.
\end{aligned}$$

Conversely, if $E_\varphi : A^2_\alpha \to A^2_\alpha$ is Hilbert-Schmidt, then

$$\begin{aligned}
\int_{\mathbb{D}} \frac{|\varphi|^2}{\left(1-|\varphi|^2\right)^{\alpha+4}} dA_\alpha &\leq \int_{\mathbb{D}} \frac{(\alpha+2)^2|\varphi|^4 + (\alpha+2)|\varphi|^2}{\left(1-|\varphi|^2\right)^{\alpha+4}} dA_\alpha \\
&= \sum_{k=0}^{\infty} \|E_\varphi(e_k)\|^2_{A^2_\alpha} \\
&< \infty.
\end{aligned}$$

□

We remark that if $\alpha = 0$, the above theorem reduces to Theorem 3.2.

4. Dirichlet Space

The Dirichlet space consists of all analytic functions f on the unit disk such that $f' \in A^2$; that is, f' satisfies the condition

$$\int_{\mathbb{D}} \left|f'(z)\right|^2 dA(z) < \infty.$$

The Dirichlet space is denoted by $\mathcal{D}$, and the norm of a function $f \in \mathcal{D}$ is defined by

$$\|f\|_{\mathcal{D}}^2 = |f(0)|^2 + \int_{\mathbb{D}} \left|f'(z)\right|^2 dA(z).$$

The inner product in the Dirichlet space is given by

$$\langle f, g\rangle = f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)}dA(z).$$

It is easy to see that the functions

$$e_0(z) = 1, \qquad e_k(z) = \frac{z^k}{\sqrt{k}}, \quad k \geq 1,$$

form an orthonormal basis for $\mathcal{D}$.

Theorem 4.1. *Let φ be an analytic self-map of the unit disk. Then $E_\varphi : \mathcal{D} \to \mathcal{D}$ is Hilbert-Schmidt if and only if*

$$\int_{\mathbb{D}} \frac{|\varphi'|^2}{\left(1 - |\varphi|^2\right)^4} dA < \infty.$$

Proof. Assume that φ satisfies the above integrability condition. We first note that for the vectors $e_k = \frac{z^k}{\sqrt{k}}$, $k \geq 1$,

$$E_\varphi(e_k)(z) = e_k'(\varphi(z))\varphi(z) = \sqrt{k}\,\varphi^k(z).$$

Thus

$$\|E_\varphi(e_k)\|_{\mathcal{D}}^2 = k\,|\varphi(0)|^{2k} + \int_{\mathbb{D}} \left|k\sqrt{k}\,\varphi^{k-1}\varphi'\right|^2 dA,$$

and

$$\begin{aligned}\sum_{k=1}^{\infty} \|E_\varphi(e_k)\|_{\mathcal{D}}^2 &= \sum_{k=1}^{\infty} k|\varphi(0)|^{2k} + \sum_{k=1}^{\infty} \int_{\mathbb{D}} \left|k\sqrt{k}\,\varphi^{k-1}\right|^2 \left|\varphi'\right|^2 dA \\ &= \frac{|\varphi(0)|^2}{1 - |\varphi(0)|^2} + \int_{\mathbb{D}} |\varphi'|^2 \sum_{k=0}^{\infty} (k+1)^3 |\varphi|^{2k} dA\end{aligned}$$

It is easy to verify that

$$\sum_{k=0}^{\infty}(k+1)^3t^k=\frac{t^2+4t+1}{(1-t)^4},\quad |t|<1, \tag{4.1}$$

from which we obtain

$$\begin{aligned}\sum_{k=1}^{\infty}\|E_\varphi(e_k)\|_{\mathcal{D}}^2&=\frac{|\varphi(0)|^2}{1-|\varphi(0)|^2}+\int_{\mathbb{D}}\frac{|\varphi'|^2\left(|\varphi|^4+4|\varphi|^2+1\right)}{(1-|\varphi|^2)4}dA\\&\leq\frac{|\varphi(0)|^2}{1-|\varphi(0)|^2}+\int_{\mathbb{D}}\frac{6|\varphi'|^2}{(1-|\varphi|^2)4}dA\\&<\infty.\end{aligned}$$

The converse statement is a consequence of

$$1\leq|\varphi|^4+4|\varphi|^2+1\leq 6. \qquad \square$$

The following theorem describes Hilbert-Schmidtness of E_φ from the Hardy space to the Dirichlet space.

Theorem 4.2. *Let φ be an analytic self-map of the unit disk. Then $E_\varphi : H^2\to\mathcal{D}$ is Hilbert-Schmidt if and only if*

$$\int_{\mathbb{D}}\frac{|\varphi'|^2}{(1-|\varphi|^2)^5}dA<\infty.$$

Proof. Assume that φ satisfies the above assumption. For the orthonormal basis $\{z^k\}_{k=0}^{\infty}$ in H^2 we have

$$\begin{aligned}\sum_{k=0}^{\infty}\left\|E_\varphi\left(z^k\right)\right\|_{\mathcal{D}}^2&=\sum_{k=1}^{\infty}k^2|\varphi(0)|^{2k}+\sum_{k=1}^{\infty}\int_{\mathbb{D}}\left|k^2\,\varphi^{k-1}\varphi'\right|^2dA\\&=\frac{|\varphi(0)|^2+|\varphi(0)|^4}{(1-|\varphi(0)|^2)^3}+\int_{\mathbb{D}}\left|\varphi'\right|^2\sum_{k=0}^{\infty}(k+1)^4\left|\varphi\right|^{2k}dA.\end{aligned} \tag{4.2}$$

It follows from (4.1) that

$$\sum_{k=0}^{\infty}(k+1)^3t^{k+1}=\frac{t^3+4t^2+t}{(1-t)^4},\quad |t|<1,$$

and

$$\sum_{k=0}^{\infty}(k+1)^4t^k=\frac{t^3+11t^2+11t+1}{(1-t)^5},\quad |t|<1.$$

Thus,

$$\sum_{k=0}^{\infty}(k+1)^4|\varphi|^{2k}=\frac{|\varphi|^6+11|\varphi|^4+11|\varphi|^2+1}{(1-|\varphi|^2)^5} \tag{4.3}$$

$$\leq \frac{24}{(1-|\varphi|^2)^5},$$

and eventually,

$$\sum_{k=0}^{\infty}\left\|E_\varphi\left(z^k\right)\right\|_{\mathcal{D}}^2 \leq \frac{|\varphi(0)|^2+|\varphi(0)|^4}{(1-|\varphi(0)|^2)^3}+24\int_{\mathbb{D}}\frac{|\varphi'|^2}{(1-|\varphi|^2)^5}dA<\infty,$$

which means that E_φ is Hilbert-Schmidt. The converse follows from (4.2) and the first equality in (4.3). □

Theorem 4.3. *The operator $E_\varphi : \mathcal{D} \to H^2$ is Hilbert-Schmidt if and only if*

$$\sup_{0<r<1}\int_0^{2\pi}\frac{\left|\varphi\left(re^{i\theta}\right)\right|^2}{\left(1-\left|\varphi\left(re^{i\theta}\right)\right|^2\right)^2}d\theta<\infty.$$

The proof is similar, so we omit the details.

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