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Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 23
Number: 2
Pages: 157-172

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2026.2063735.2225



Sahand Communications in Mathematical Analysis (SCMA) Vol. 23 No. 2 (2026), 157-172
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2026.2063735.2225

A Study on the Convergence Analysis of Adomian Decomposition Method for Solving Fuzzy Delay Mixed Volterra–Fredholm Integrodifferential Equations

Chintalachervu Subba Reddy[1], Yookesh Tiruchengode Lakshmanasamy[2], Sriramulu Sabarinathan[3*] and Vaidhiyanathan Suvitha[4]

Abstract. Fuzzy delay mixed Volterra-Fredholm integrodifferential equations arise naturally in the mathematical modeling of real–world phenomena involving uncertainty, memory effects, and time delays in engineering and applied sciences. This paper investigates the convergence behavior of the Adomian Decomposition Method (ADM) when applied to a class of such equations with fuzzy parameters and delay terms. Fuzzy set theory is employed to rigorously represent parametric uncertainty, thereby enhancing the realism and flexibility of the mathematical model. Sufficient conditions guaranteeing the convergence of the ADM series solution are derived using fixed–point arguments in appropriate function spaces. The theoretical analysis is supported by numerical examples that demonstrate the accuracy and efficiency of the proposed approach. The results confirm that ADM provides a reliable and effective framework for solving fuzzy delay mixed Volterra–Fredholm integrodifferential equations.

1. Introduction

1.1. **Review of Historical Literature.** Volterra integral and integrodifferential equations were first introduced by V. Volterra in the early twentieth century and have since played a fundamental role in the mathematical modeling of numerous problems arising in science and engineering [24, 25]. Among these, Volterra–Fredholm integrodifferential

2020 *Mathematics Subject Classification.* 45G10, 34K40, 03E72.
Key words and phrases. Adomian Decomposition Method, convergence analysis, Fuzzy delay mixed Volterra–Fredholm integrodifferential equations, Numerical examples.
Received: 2025-06-15, Accepted: 2026-01-04.
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equations (VFIDEs) constitute a distinguished hybrid class, combining variable integration limits typical of Volterra operators with fixed limits characteristic of Fredholm operators. Owing to this mixed structure, VFIDEs naturally arise in applications such as control theory, biological systems, viscoelasticity, and fluid dynamics.

Considerable attention has been devoted to the development of efficient numerical methods for solving VFIDEs. Hassan et al. [14] employed the Adomian Decomposition Method (ADM) to obtain approximate solutions for two–dimensional mixed inhomogeneous VFIDEs, demonstrating the effectiveness of ADM for spatially complex models. Savla and Sharmila [21] enhanced the ADM through the Shehu transform for fuzzy–fractional VFIDEs, illustrating its robustness in handling uncertainty. More recently, HamaRashid et al. [13] proposed novel computational algorithms aimed at improving solution accuracy for nonlinear cases. These works have stimulated sustained interest in adaptable and hybrid numerical approaches. In addition, alternative techniques such as the Laplace discrete decomposition method [12], homotopy perturbation techniques, and He's variational iteration method [10] have been successfully applied to VFIDEs, highlighting the necessity of reliable solution strategies for mixed Volterra–Fredholm structures.

Mixed Volterra–Fredholm integrodifferential equations are particularly effective in modeling systems exhibiting memory effects and distributed interactions. The inclusion of uncertain parameters and time delays further enhances modeling accuracy, while simultaneously increasing analytical and computational complexity. The ADM has proved to be well suited for such problems due to its systematic treatment of nonlinearities and uncertainties. In the context of stability analysis, Alasadi and Hamoud [6] investigated existence and Ulam–Hyers stability results for fractional VFIDEs under mixed environments, thereby establishing a foundation for delay–dependent formulations. Moreover, Afiatdoust et al. [4] introduced a hybrid numerical approach, showing that the combination of multiple techniques can significantly improve the performance of ADM–based schemes.

In practical applications, system parameters are often imprecise or only partially known, a limitation that classical deterministic models cannot fully address. Fuzzy set theory, introduced by Zadeh, provides a rigorous mathematical framework for representing such uncertainty. The resulting fuzzy VFIDEs yield more realistic models for systems with vague or imprecise data. The incorporation of delay terms further increases model fidelity, since delays frequently occur in signal transmission, population dynamics, and chemical reaction processes.

1.2. **Review of Related Literature.** Extensive research has been conducted on the numerical analysis of fuzzy and delay integrodifferential equations. Chang and Zadeh [11] introduced the concept of fuzzy mappings, while Kaleva [15] and Seikkala [22] developed the fundamental theory of fuzzy differential equations, enabling the systematic integration of fuzziness into dynamical systems. Allahviranloo et al. [9] proposed an expansion method for approximating solutions of linear and nonlinear fuzzy Volterra integrodifferential equations, emphasizing the importance of uncertainty in kernel and source terms. A detailed survey of ADM applications to nonlinear VFIDEs was reported by Rafik and Farooq [19], confirming the method's effectiveness for a broad range of nonlinear and delay–dependent models.

The ADM was originally developed by Adomian [1], who established its theoretical and algorithmic foundation for nonlinear problems in physics and engineering. Adomian and Rach [2] extended the method to stochastic differential–delay systems, demonstrating its capability to address randomness and temporal delays. Subsequent work by Adomian on stochastic systems [3] has influenced numerous modern fuzzy and fractional models. More recently, Yookesh et al. [27] investigated an iterative filtering approach for fuzzy Volterra integral equations with delay, with applications to material science and uncertainty quantification. Amirali and Acar [8] analyzed stability inequalities and numerical solutions for neutral delay VFIDEs, thereby improving the understanding of long–term system behavior.

Further developments specifically addressing fuzzy delay VFIDEs were presented by Reddy and Lakshmanasamy [20], which closely align with the objectives of the present study and confirm the versatility of ADM in treating both fuzzy and classical delay components. In parallel, the rapid growth of fractional calculus has stimulated significant advances in integrodifferential equations. Existence and uniqueness results for fractional integrodifferential equations were investigated in [5, 23], while efficient numerical schemes such as Bell polynomial collocation methods [28] and finite difference approaches [26] have been proposed. In addition, Alikhanov et al. [7] developed a second–order fast numerical scheme for Cauchy problems involving integrodifferential operators.

1.3. **Novelty and Contributions of the Study.** This work addresses existing gaps in the convergence analysis of fuzzy delay mixed Volterra–Fredholm integrodifferential equations. The main contributions of the present study are summarized as follows:

(1) The Adomian Decomposition Method is applied to a class of fuzzy delay mixed Volterra–Fredholm integrodifferential equations for which rigorous convergence results have not been previously established.
(2) A structured convergence analysis is developed under standard Lipschitz and boundedness assumptions, yielding practical conditions for the existence and uniqueness of solutions.
(3) By integrating fuzzy set theory with delay effects, a unified mathematical framework is obtained for representing uncertainty and memory phenomena within a single model.
(4) Several numerical examples are presented to validate the theoretical results and to demonstrate the accuracy, stability, and computational efficiency of the proposed approach.

These contributions enhance the theoretical understanding of ADM in fuzzy delay environments and provide a foundation for future investigations of uncertainty–driven dynamical systems.

1.4. **Fundamental Concepts of the Study.** In this subsection, we recall the fundamental concepts and mathematical tools required for the development of the proposed methodology. These notions are well established in the existing literature and form the analytical basis of the present study.

Definition 1.1. Using Zadeh's extension principle, we represent fuzzy numbers and define the corresponding arithmetic operations such as fuzzy addition and scalar multiplication. Let $\tilde{u}(t)$ denote a fuzzy–valued function. The fuzzy delay mixed Volterra–Fredholm integrodifferential equation (FDMVFIDE) considered in this work is given by

(1.1)
$$\tilde{u}'(t) = \tilde{f}(t, \tilde{u}(t)) + \int_a^t \tilde{K}_1(t, s, \tilde{u}(s))\, ds + \int_a^b \tilde{K}_2(t, s, \tilde{u}(s))\, ds + \tilde{g}(t, \tilde{u}(t-\tau)),$$

subject to appropriate initial or boundary conditions, where $\tau > 0$ denotes the delay parameter.

The general formulation (1.1) provides the theoretical foundation for the specific fuzzy delay mixed Volterra–Fredholm integrodifferential equations analyzed in Sections 2 and 3.

Definition 1.2. The Adomian Decomposition Method (ADM) seeks an approximate solution of (1.1) by decomposing the unknown function $\tilde{u}(t)$ into an infinite series of the form

(1.2)
$$\tilde{u}(t) = \sum_{n=0}^{\infty} \tilde{u}_n(t).$$

The nonlinear operator is expressed in terms of Adomian polynomials. Substituting the series representation (1.2) into the governing equation and equating terms of identical order yields a recursive sequence for the components $\tilde{u}_n(t)$. The convergence of this series depends on the analytical properties of the kernel functions and the structure of the nonlinear terms.

Definition 1.3 ([20])**.** A triangular fuzzy number (TFN) is a widely used class of fuzzy numbers due to its computational simplicity and interpretability. A TFN is characterized by a piecewise linear membership function defined by three parameters (α, a, β), where $\alpha \leq a \leq \beta$. Here, α and β represent the lower and upper bounds, respectively, and a denotes the modal value.

The membership function $\mu_T(x)$ of a TFN $T = (\alpha, a, \beta)$ is defined by

$$\mu_T(x) = \begin{cases} 0, & x \leq \alpha, \\ \dfrac{x-\alpha}{a-\alpha}, & \alpha < x \leq a, \\ \dfrac{\beta - x}{\beta - a}, & a < x < \beta, \\ 0, & x \geq \beta. \end{cases} \tag{1.3}$$

This membership function increases linearly from zero to unity as x varies from α to a, and subsequently decreases linearly to zero as x approaches β. Such a structure provides a simple yet effective representation of uncertainty, as illustrated in Figure 1.

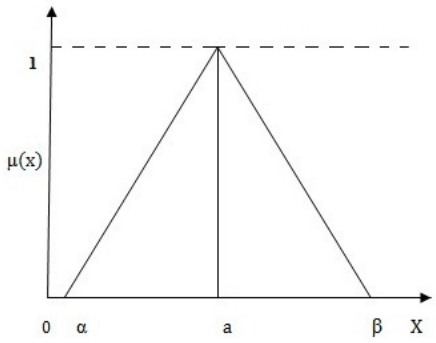


FIGURE 1. Graphical representation of a triangular fuzzy number

Definition 1.4. To quantify the difference between two fuzzy numbers, the Hausdorff distance is defined by

$$\mathcal{D}(\alpha,\beta) = \sup_{T\in[0,1]} \max\left\{\left|\alpha_L(T)-\beta_L(T)\right|, \left|\alpha_U(T)-\beta_U(T)\right|\right\}, \tag{1.4}$$

where $\alpha_L(T)$ and $\alpha_U(T)$ denote the lower and upper boundary functions, respectively.

The metric $\mathcal{D}(\alpha,\beta)$ measures the maximum deviation between the corresponding α–cut boundaries and provides a rigorous tool for comparing fuzzy quantities [20].

Theorem 1.5 ([20])**.** *Let $g(x)$ be a fuzzy–valued function defined on $[p,\infty)$ such that $g(x) = (l(x), c(x), u(x))$ is a triangular fuzzy number for each $x \in [p,q] \subseteq [p,\infty)$, with $l(x) \le c(x) \le u(x)$. Then the definite integral of g over $[p,q]$ is defined by*

$$\int_p^q g(x)\,dx = \left(\int_p^q l(x)\,dx,\ \int_p^q c(x)\,dx,\ \int_p^q u(x)\,dx\right).$$

Theorem 1.6 ([20])**.** *Let $(\mathcal{X},d)$ be a complete metric space and let $\phi : \mathcal{X} \to \mathcal{X}$ be a contraction mapping. Then there exists a unique fixed point $x^* \in \mathcal{X}$ such that $\phi(x^*) = x^*$. Moreover, the iterative sequence generated by ϕ converges to x^*.*

1.5. **Role of the Banach Fixed-Point Theorem.** The convergence analysis of the Adomian Decomposition Method is fundamentally based on fixed–point theory in complete metric spaces. By reformulating the fuzzy delay mixed Volterra–Fredholm integrodifferential equation as an equivalent integral equation, a contraction operator can be constructed under appropriate Lipschitz–type conditions. The Banach fixed–point theorem then ensures the existence, uniqueness, and convergence of the ADM solution in fuzzy–valued function spaces [16].

1.6. **Structure of the Study.** The remainder of the paper is organized as follows.

(1) Section 2 presents the mathematical formulation of the problem and establishes the required preliminary results.
(2) Section 3 describes the implementation of the proposed Adomian Decomposition Method.
(3) Section 4 is devoted to a rigorous convergence analysis of the method.
(4) Section 5 provides numerical examples illustrating the accuracy and effectiveness of the proposed approach.
(5) Section **??** summarizes the main findings and outlines possible directions for future research.

2. Fuzzy Delay Mixed Volterra-Fredholm Integrodifferential Equations

In this section, we formulate the fuzzy delay mixed Volterra–Fredholm integrodifferential equation following the analytical framework introduced in [8]. The governing model is established understandard smoothness and analyticity assumptions on the kernel and nonlinear functions. This formulation provides the mathematical foundation required for the application of the Adomian Decomposition Method and for the subsequent analysis of convergence and uniqueness.

The fuzzy delay mixed Volterra–Fredholm integrodifferential equation under consideration is given by

$$\tilde{B}'(P) = \tilde{f}(P) + \lambda \int_0^{P-\tau} k_1(P,\mathcal{L})\, F_1\Big(\tilde{B}(\mathcal{L})\Big)\, d\mathcal{L}, \tag{2.1}$$

subject to the initial condition

$$\tilde{B}(0) = \tilde{B}_0. \tag{2.2}$$

For given real parameters (λ, μ), the functions f and F_1 are assumed to be analytic, and the kernel k_1 is defined on $\mathbb{D}([0,P]^2)$ with values in $\mathbb{R}^+$. The fuzzy solution $\tilde{B}(P)$ is expressed in the series form

$$\tilde{B}(P) = \sum_{i=0}^{\infty} \tilde{B}_i(P). \tag{2.3}$$

Using the α–cut representation of fuzzy numbers, we write

$$\tilde{B}(P,\mathcal{L}) = \big(\underline{B}(P,\mathcal{L}),\, \overline{B}(P,\mathcal{L})\big), \qquad \tilde{f}(P,\mathcal{L}) = \big(\underline{f}(P,\mathcal{L}),\, \overline{f}(P,\mathcal{L})\big).$$

Accordingly, equation (2.1) can be rewritten in α–cut form as

$$\tilde{B}'(P,\mathcal{L}) = \tilde{f}(P,\mathcal{L}) + \lambda \int_0^{P-\tau} \int_0^{P} k_1(P,v)\, F_1\Big(\tilde{B}(v,\mathcal{L})\Big)\, dv. \tag{2.4}$$

Remark 2.1. To simplify the fuzzy system and facilitate the convergence analysis, we employ the standard center–difference decomposition of fuzzy–valued functions. For

$$\tilde{B}(P,\mathcal{L}) = \big(\underline{B}(P,\mathcal{L}),\, \overline{B}(P,\mathcal{L})\big),$$

the corresponding center and difference components are defined by

$$B^c(P,\mathcal{L}) = \frac{\underline{B}(P,\mathcal{L}) + \overline{B}(P,\mathcal{L})}{2}, \qquad B^d(P,\mathcal{L}) = \frac{\underline{B}(P,\mathcal{L}) - \overline{B}(P,\mathcal{L})}{2}.$$

The center and difference components of $\tilde{f}$ are defined analogously.

Using Remark 2.1, equation (2.4) is equivalently transformed into the following coupled system:

$$B^{c\prime}(P,\mathcal{L}) = f^{c}(P,\mathcal{L}) + \lambda \int_0^{P-\tau} \int_0^{P} k_1(P,v)\, F_1(B^{c}(v,\mathcal{L}))\, dv, \tag{2.5}$$

$$B^{d\prime}(P,\mathcal{L}) = f^{d}(P,\mathcal{L}) + \lambda \int_0^{P-\tau} \int_0^{P} k_1(P,v)\, F_1\Big(B^{d}(v,\mathcal{L})\Big)\, dv, \tag{2.6}$$

with initial conditions

$$B^{c}(0,\mathcal{L}) = \frac{\underline{B}(0,\mathcal{L}) + \overline{B}(0,\mathcal{L})}{2}, \qquad B^{d}(0,\mathcal{L}) = \frac{\underline{B}(0,\mathcal{L}) - \overline{B}(0,\mathcal{L})}{2}. \tag{2.7}$$

3. Methodological Overview of the Study

In this section, we present the Adomian Decomposition Method (ADM) [5] for solving the nonlinear fuzzy delay mixed Volterra-Fredholm integrodifferential equation considered in this study [20]. The method is applied separately to the lower and upper α-cut representations of the fuzzy solution in order to preserve the intrinsic structure of fuzzy-valued functions. Accordingly, the governing equation is written in terms of its lower and upper α–cut components as

$$\begin{aligned} \underline{B}'(P,L) &= \underline{f}(P,L) + \lambda \int_0^{P-\tau} \int_0^{P} k_1(P,\mathcal{T})\, \mathcal{F}_1(\underline{B}(\mathcal{T},L))\, d\mathcal{T}, \\ \overline{B}'(P,L) &= \overline{f}(P,L) + \lambda \int_0^{P-\tau} \int_0^{P} k_1(P,\mathcal{T})\, \mathcal{F}_1\big(\overline{B}(\mathcal{T},L)\big)\, d\mathcal{T}. \end{aligned} \tag{3.1}$$

Equations (3.1) describe, respectively, the lower and upper α–cut branches of the fuzzy delay mixed Volterra–Fredholm integrodifferential equation. Although structurally similar, these relations represent distinct components of the fuzzy solution.

Let T denote the first–order differential operator with respect to the independent variable P. Using this operator notation, system (3.1) can be rewritten in the operator form

$$\begin{aligned} T\underline{B}(P,L) &= \underline{f}(P,L) + \lambda \int_0^{P-\tau} \int_0^{P} k_1(P,\mathcal{T})\, \mathcal{F}_1(\underline{B}(\mathcal{T},L))\, d\mathcal{T}, \\ T\overline{B}(P,L) &= \overline{f}(P,L) + \lambda \int_0^{P-\tau} \int_0^{P} k_1(P,\mathcal{T})\, \mathcal{F}_1\big(\overline{B}(\mathcal{T},L)\big)\, d\mathcal{T}. \end{aligned} \tag{3.2}$$

Let $G[\cdot]$ and $H[\cdot]$ denote the nonlinear operators corresponding to the lower and upper fuzzy components, respectively. Applying the inverse operator T^{-1} to (3.2), we obtain the equivalent integral form

$$\underline{B}(P,L) = T^{-1}\underline{f}(P,L), \tag{3.3}$$

$$\overline{B}(P,L) = T^{-1}\overline{f}(P,L),$$

where

$$T^{-1}\underline{f}(P,L) = \lambda \int_0^{P-\tau} \int_0^{P} k_1(P,\mathcal{T})\, \mathcal{F}_1(\underline{B}(\mathcal{T},L))\, d\mathcal{T}, \tag{3.4}$$

and

$$T^{-1}\overline{f}(P,L) = \lambda \int_0^{P-\tau} \int_0^{P} k_1(P,\mathcal{T})\, \mathcal{F}_1\big(\overline{B}(\mathcal{T},L)\big)\, d\mathcal{T}. \tag{3.5}$$

According to ADM, the unknown functions $\underline{B}(P,L)$ and $\overline{B}(P,L)$ are expanded as infinite series

$$\underline{B}(P,L) = \sum_{i=0}^{\infty} \underline{B}_i(P,L), \qquad \overline{B}(P,L) = \sum_{i=0}^{\infty} \overline{B}_i(P,L). \tag{3.6}$$

The nonlinear operators in (3.4)–(3.5) are decomposed in terms of Adomian polynomials as

$$G[\underline{B}] = \sum_{n=0}^{\infty} \underline{A}_n, \qquad H[\overline{B}] = \sum_{n=0}^{\infty} \overline{A}_n, \tag{3.7}$$

where

$$\begin{aligned} \underline{A}_n &= \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} G\left(\sum_{i=0}^{n} \lambda^i \underline{B}_i\right)\right]_{\lambda=0}, \\ \overline{A}_n &= \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} H\left(\sum_{i=0}^{n} \lambda^i \overline{B}_i\right)\right]_{\lambda=0}. \end{aligned} \tag{3.8}$$

Substituting (3.6)–(3.8) into (3.3), the ADM recursive scheme is obtained as

$$\tilde{B}_0(P,L) = \tilde{g}(P,L), \qquad \tilde{B}_{n+1}(P,L) = T^{-1}\left(\tilde{A}_n\right), \quad n \geq 0. \tag{3.9}$$

The n–term approximate solution is given by

$$\tilde{\phi}_n(P,L) = \sum_{i=0}^{n-1} \tilde{B}_i(P,L),$$

and as $n \to \infty$, the ADM approximation converges to the exact fuzzy solution

$$\lim_{n\to\infty} \tilde{\phi}_n(P,L) = \tilde{B}(P,L).$$

4. Convergence Analysis

To analyze convergence, we impose standard assumptions on the kernel functions and nonlinear operators, including Lipschitz continuity and boundedness. By employing fixed–point theory, it can be shown that the series $\sum_{n=0}^{\infty} \tilde{B}_n(P)$ converges uniformly under these conditions [20]. Recent developments in fuzzy fixed–point theory further support the validity of contraction mappings in uncertain environments [17]. Based on these results, sufficient conditions for convergence and corresponding error estimates for truncated series are established.

4.1. **Uniqueness Theorem.**

Theorem 4.1. *The integral equation*

$$\tilde{B}(P) = \tilde{f}(P) + \lambda \int_0^{P-\tau} \int_0^{P} k_1(P,t)\, \tilde{\mathcal{F}}_1\Big(\tilde{B}(t)\Big)\, dt$$

admits a unique solution provided that $0 < L < 1$*, where*

$$L = (MT)(b-a).$$

Proof. Assume that $\tilde{B}$ and $\tilde{B}^*$ are two solutions of the above integral equation. Then

$$|\tilde{B} - \tilde{B}^*| = \left| \lambda \int_0^{P-\tau} \int_0^{P} k_1(P,t) \left[\mathcal{F}_1(\tilde{B}(t)) - \mathcal{F}_1(\tilde{B}^*(t)) \right] dt \right|.$$

Using the Lipschitz continuity of $\mathcal{F}_1$ and the boundedness of the kernel k_1, we obtain

$$|\tilde{B} - \tilde{B}^*| \leq L\, |\tilde{B} - \tilde{B}^*|.$$

Since $0 < L < 1$, it follows that $(1-L)|\tilde{B} - \tilde{B}^*| \leq 0$, which implies $\tilde{B} = \tilde{B}^*$. Hence, the solution is unique. □

4.2. **Convergence Theorem.**

Theorem 4.2. *If the series solution*

$$\tilde{B}(P,\gamma) = \sum_{i=0}^{\infty} \tilde{B}_i(P,\gamma)$$

obtained via the ADM converges and satisfies $\left\| \tilde{B}_1(P,\gamma) \right\| < \infty$*, then it converges to the exact solution of equation* (2.1) *for* $0 < L < 1$*.*

Proof. Let $(C[a,b], \|\cdot\|)$ be the Banach space of continuous functions on $[a,b]$. Assume that $\left\|\tilde{B}_1(P,\gamma)\right\| < \infty$ and define the partial sums $\overline{P}_n = \sum_{i=0}^{n} \tilde{B}_i(P,\gamma)$. For $m < n$,

$$\left\|\overline{P}_n - \overline{P}_m\right\| \leq \sum_{i=m+1}^{n} \left\|\tilde{B}_i(P,\gamma)\right\|.$$

Using bounds on the kernels and Lipschitz continuity of the nonlinear operators, we obtain

$$\left\|\overline{P}_n - \overline{P}_m\right\| \leq L \left\|\overline{P}_{n-1} - \overline{P}_{m-1}\right\|,$$

where $L = (M_1T_1 + M_2T_2)(b-a) < 1$. Iterating this inequality yields

$$\left\|\overline{P}_n - \overline{P}_m\right\| \leq L^m \left\|\overline{P}_{n-m} - \overline{P}_0\right\|.$$

Since $0 < L < 1$, it follows that $\{\overline{P}_n\}$ is a Cauchy sequence and hence converges in $C[a,b]$ to $\tilde{B}(P,\gamma)$, which is the exact solution of the problem. □

5. Numerical Example

In this section, a numerical example is presented to demonstrate the accuracy and effectiveness of the Adomian Decomposition Method (ADM) when applied to a fuzzy delay mixed Volterra–Fredholm integrodifferential equation. The ADM-based approximate solutions are compared with the corresponding exact solutions in order to examine convergence behavior and solution accuracy.

Example 5.1. Consider the following fuzzy delay mixed Volterra Fredholm integrodifferential equation:

$$\tilde{k}'(x) = 6 + 29x - \frac{7}{2}x^2 + \int_0^{x-\tau} \int_0^1 (r-t)\,\tilde{u}(t)\,dt\,dr, \tag{5.1}$$

subject to the initial condition

$$\tilde{k}(0) = (1-a,\ a-1), \qquad \tau = 0.3. \tag{5.2}$$

The corresponding lower and upper α–cut components of (5.1) are given by

$$\overline{k}'(x) = 6 + 29x - \frac{7}{2}x^2 + \int_0^{x-\tau} \int_0^1 (r-t)\,\overline{u}(t)\,dt\,dr, \tag{5.3}$$

$$\underline{k}'(x) = 6 + 29x - \frac{7}{2}x^2 + \int_0^{x-\tau} \int_0^1 (r-t)\,\underline{u}(t)\,dt\,dr, \tag{5.4}$$

with initial values

$$\overline{k}(0) = a - 1, \qquad \underline{k}(0) = 1 - a.$$

The exact solutions corresponding to (5.3) and (5.4) are obtained as

$$\overline{k}(x) = 6x + \frac{29}{2}x^2 - \frac{7}{6}x^3 + a - 1, \qquad \underline{k}(x) = 6x + \frac{29}{2}x^2 - \frac{7}{6}x^3 + 1 - a.$$

Using the ADM recursive scheme developed in Section 3, equations (5.3) and (5.4) are solved iteratively. Numerical approximations obtained at selected values of the independent variable are reported in Table 1, together with the corresponding exact solutions. A graphical comparison between the exact and approximate solutions is illustrated in Figure 2-4.

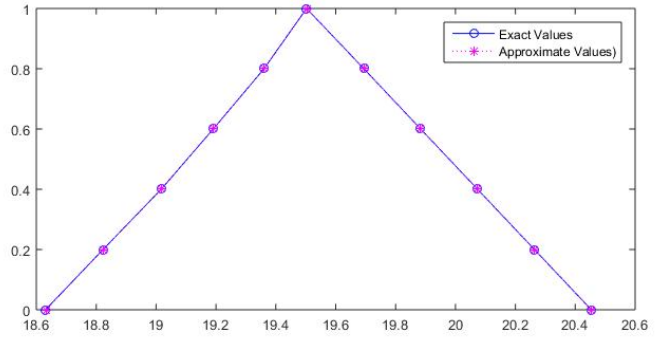


FIGURE 2. Comparison of exact and ADM approximate solutions for the lower fuzzy component.

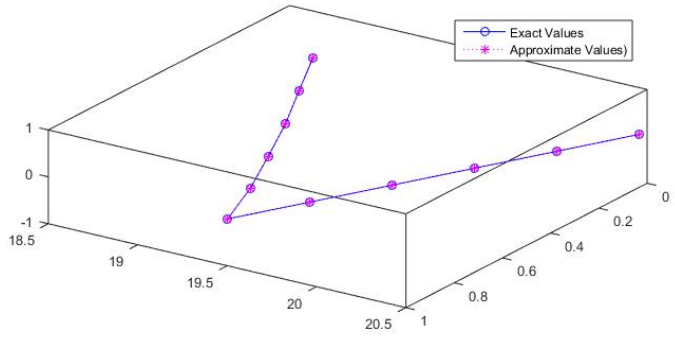


FIGURE 3. Comparison of exact and ADM approximate solutions for the upper fuzzy component.

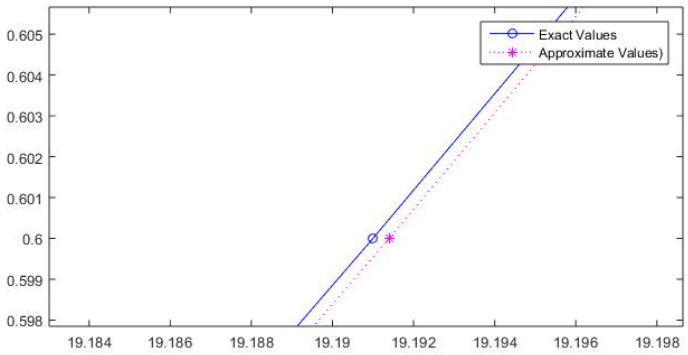


FIGURE 4. Three–dimensional convergence behavior of the ADM approximation toward the exact fuzzy solution.

From Table 1, it is evident that the ADM approximations converge rapidly to the corresponding exact solutions. Figure 2 illustrates the convergence behavior of the lower fuzzy component, while Figure 3 presents the corresponding convergence of the upper fuzzy component. In both cases, the absolute difference between the approximate and exact values decreases monotonically as the iteration index increases, confirming the uniform convergence predicted by the theoretical analysis. Furthermore, the three–dimensional convergence behavior of the ADM solution is depicted in Figure 4. All numerical computations and graphical illustrations in this section were carried out using MATLAB.

TABLE 1. Comparison of exact and ADM approximate solutions

T	$\underline{k}_{\text{exact}}(x)$	$\overline{k}_{\text{exact}}(x)$	$\underline{k}_{\text{ADM}}(x)$	$\overline{k}_{\text{ADM}}(x)$
0.0	18.6306	20.4536	18.6327	20.4539
0.2	18.8229	20.2636	18.8231	20.2639
0.4	19.0171	20.0736	19.0176	20.0741
0.6	19.1910	19.8836	19.1914	19.8838
0.8	19.3605	19.6939	19.3607	19.6942
1.0	19.5026	19.5026	19.5030	19.5030

6. CONCLUSION

In this work, the Adomian Decomposition Method (ADM) has been shown to provide a reliable and accurate analytical framework for solving

fuzzy delay mixed Volterra–Fredholm integrodifferential equations. By constructing a suitably convergent ADM scheme, the combined effects of fuzzy uncertainty and time delay are incorporated in a mathematically consistent manner that remains amenable to rigorous analysis. The convergence results, derived under standard assumptions on the kernel functions and nonlinear operators, confirm the theoretical soundness of the proposed approach.

The numerical experiments, implemented using MATLAB, demonstrate a very close agreement between the ADM approximations and the corresponding exact or benchmark solutions. In particular, the approximate solutions exhibit rapid and stable convergence for both the lower and upper fuzzy components, in full accordance with the theoretical convergence analysis. These findings highlight the robustness of ADM in effectively capturing uncertainty and memory effects that naturally arise in fuzzy delay systems.

Future research directions include the extension of the present methodology to more general classes of problems, such as fractional-order and higher-order fuzzy delay integrodifferential equations, multidimensional models, and hybrid fuzzy-stochastic systems. In addition, coupling ADM with spectral techniques or data-driven approaches may further enhance computational efficiency and broaden its applicability to real-time problems in engineering and applied sciences.

Acknowledgment. The authors express their sincere gratitude to the anonymous reviewers for their careful reading of the manuscript and for their valuable comments and suggestions, which significantly improved the quality and clarity of this work. The authors also acknowledge the support provided by their respective institutions in carrying out this research.

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