Bullen Type Inequalities via Tempered Fractional Integrals

El-Achky Jamal

Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 23
Number: 2
Pages: 187-200

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2025.2063761.2226



Sahand Communications in Mathematical Analysis (SCMA) Vol. 23 No. 2 (2026), 187-200
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2025.2063761.2226

Bullen Type Inequalities via Tempered Fractional Integrals

El-Achky Jamal

Abstract. In mathematical interpretations, integral inequalities constitute a significant and continuous body of research. Recently, numerous studies have been conducted on fractional calculus approaches due to their widespread usage in science. The tempered fractional integral is a fundamental notion in fractional calculus. Our main goal in this work is to obtain some Bullen-type inequalities utilizing the tempered fractional integral operators. From a new identity, we derive several novel Bullen-type inequalities for differentiable convex functions.

1. Introduction and Preliminaries

Convexity is a key idea in both pure and applied mathematics. It is essential for functional analysis, optimization, and geometry, which sheds light on how sets and functions behave. The Hermite-Hadamard inequality, which limits the value of a convex function using its values at the endpoints and the middle of an interval, is among the most intriguing convexity-related conclusions [16, 17].

Let $[s,t]$ be an interval of $\mathbb{R}$, the following is an expression for this inequality:

$$\Psi\left(\frac{s+t}{2}\right) \leq \frac{1}{t-s}\int_s^t \Psi(\varkappa)\,d\varkappa \leq \frac{\Psi(s)+\Psi(t)}{2} \tag{1.1}$$

where Ψ is convex function on $[s,t]$.

For recent works about this inequality, see [1, 11–13, 15].

2020 *Mathematics Subject Classification.* 26A51, 26D15, 26D10.
Key words and phrases. Bullen inequality, Tempered fractional integrals, Convex function.
Received: 2025-06-15, Accepted: 2025-09-27.

The following inequality was proved by Tseng et al. [26], is known in literature as the Bullen inequality, which follows from (1.1):

$$\frac{1}{t-s}\int_s^t \Psi(\varkappa)\,d\varkappa \leq \frac{1}{2}\left[\frac{\Psi(s)+\Psi(t)}{2}+\Psi\left(\frac{s+t}{2}\right)\right]$$

for more details, we refer the reader to [4, 10].

Bullen-type inequalities have attracted a lot of attention because of their exceptional applications. Many mathematicians and scientists are quite interested in and concerned by Bullen's inequality. Due of its significance in a wide range of fields. The authors have been very interested in Bullen's inequality and have worked hard to improve and generalize it throughout the years. The well-known Bullen's inequality in its conventional form has been extended by numerous scholars for a variety of convex function subcategories. Numerous intriguing and eye-catching studies focused on enhancing and expanding Bullen-type inequalities have recently been published in the literature. It is hard to collect all results and applications of this inequality. However a few of these results for instance are mentioned below.

In [14], the authors obtained some generalized Bullen-type inequalities for twice-differentiable convex functions. Also in [25], the authors examined the Bullen inequality in relation to fractional integrals with exponential kernels. In [3], the authors presented fractional multiplicative Bullen-type inequalities in the context of multiplicative calculus.

The essential preliminary steps needed to demonstrate our main findings are described in the section that follows.

Definition 1.1 ([9, 23])**.** The gamma function, incomplete gamma function, λ-incomplete gamma function are respectively defined by

$$\Gamma(\alpha) := \int_0^\infty m^{\alpha-1}e^{-m}dm,$$
$$\Upsilon(\alpha,\varkappa) := \int_0^\varkappa m^{\alpha-1}e^{-m}dm,$$
$$\Upsilon_\lambda(\alpha,\varkappa) := \int_0^\varkappa m^{\alpha-1}e^{-\lambda m}dm.$$

Here, $\lambda \geq 0$ and $0 < \alpha < \infty$.

When $\lambda = 1$, the function $\Upsilon_\lambda(\alpha,\varkappa)$ reduces to the function $\Upsilon(\alpha,\varkappa)$.

Remark 1.2. For the real numbers $\alpha > 0$ and $\lambda \geq 0$, we have

$$\begin{aligned}\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1) &= \int_0^1 m^{\alpha-1}e^{-\lambda\frac{t-s}{2}m}dm \\ &= \left(\frac{2}{t-s}\right)^\alpha \Upsilon_\lambda\left(\alpha,\frac{t-s}{2}\right).\end{aligned}$$

Proof. Using the change of variable $z = \left(\frac{t-s}{2}\right) m$, we get

$$\begin{aligned}\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1) &= \int_0^1 m^{\alpha-1} e^{-\lambda\left(\frac{t-s}{2}\right)m} dm \\ &= \left(\frac{2}{t-s}\right)^{\alpha} \int_0^{\frac{t-s}{2}} z^{\alpha-1} e^{-\lambda z} dz \\ &= \left(\frac{2}{t-s}\right)^{\alpha} \Upsilon_{\lambda}\left(\alpha, \frac{t-s}{2}\right). \end{aligned}$$ □

Remark 1.3 ([23])**.** For the real numbers $\varkappa, \lambda \geq 0$; $\alpha > 0$ and $t > s$, we have

(i) $\Upsilon_{\lambda(t-s)}(\alpha,1) = \int_0^1 m^{\alpha-1} e^{-\lambda(t-s)m} dm = \frac{1}{(t-s)^{\alpha}} \Upsilon_{\lambda}(\alpha, t-s)$,

(ii) $\int_0^1 \Upsilon_{\lambda(t-s)}(\alpha,\varkappa)\, d\varkappa = \frac{\Upsilon_{\lambda}(\alpha,t-s)}{(t-s)^{\alpha}} - \frac{\Upsilon_{\lambda}(\alpha+1,t-s)}{(t-s)^{\alpha+1}}$.

Proof. Using the λ-incomplete gamma function formulation, we get

$$\int_0^1 \Upsilon_{\lambda(t-s)}(\alpha,\varkappa)\, d\varkappa = \int_0^1 \int_0^{\varkappa} m^{\alpha-1} e^{-\lambda(t-s)m}\, dm\, d\varkappa.$$

By changing the order of the integration, we get

$$\begin{aligned}\int_0^1 \Upsilon_{\lambda(t-s)}(\alpha,\varkappa)\, d\varkappa &= \int_0^1 \int_m^1 m^{\alpha-1} e^{-\lambda(t-s)m}\, d\varkappa\, dm \\ &= \int_0^1 (1-m) m^{\alpha-1} e^{-\lambda(t-s)m} dm \\ &= \int_0^1 m^{\alpha-1} e^{-\lambda(t-s)m} dm - \int_0^1 m^{\alpha} e^{-\lambda(t-s)m} dm. \\ &= \frac{\Upsilon_{\lambda}(\alpha,t-s)}{(t-s)^{\alpha}} - \frac{\Upsilon_{\lambda}(\alpha+1,t-s)}{(t-s)^{\alpha+1}}. \end{aligned}$$ □

Definition 1.4 ([20])**.** For $\Psi \in L_1[s,t]$, the Riemann-Liouville integrals of order $\alpha > 0$ are given by

$$\mathcal{H}^{\alpha}_{s+}\Psi(\ell) = \frac{1}{\Gamma(\alpha)} \int_s^{\ell} (\ell - \varkappa)^{\alpha-1} \Psi(\varkappa)\, d\varkappa, \quad \ell > s,$$

$$\mathcal{H}^{\alpha}_{t-}\Psi(\ell) = \frac{1}{\Gamma(\alpha)} \int_{\ell}^{t} (\varkappa - \ell)^{\alpha-1} \Psi(\varkappa)\, d\varkappa, \quad \ell < t.$$

Now, we recall the definition of tempered fractional integral operators.

Definition 1.5 ([21, 24])**.** The tempered fractional integrals are defined as:

$$\mathcal{J}^{(\alpha,\lambda)}_{s+}\Psi(\ell) = \frac{1}{\Gamma(\alpha)} \int_s^{\ell} (\ell - \varkappa)^{\alpha-1} e^{-\lambda(\ell-\varkappa)} \Psi(\varkappa)\, d\varkappa, \quad \ell \in [s,t],$$

$$\mathcal{J}_{t^-}^{(\alpha,\lambda)}\Psi(\ell)=\frac{1}{\Gamma(\alpha)}\int_{\ell}^{t}(\varkappa-\ell)^{\alpha-1}e^{-\lambda(\varkappa-\ell)}\Psi(\varkappa)\,d\varkappa,\quad \ell\in[s,t].$$

Here, $\Psi\in L_1[s,t]$, $\alpha>0$ and $\lambda\geq 0$.

Tempered fractional calculus has practically developed from the base concepts of fractional calculus. To the best of our knowledge, Buschman's earlier work [5] reported the first formulations of fractional integration using weak singular and exponential kernels. Tempered fractional integral has gained interest in recent decades due to its significant potential for applications in various fields. The tempered diffusion model was able to show its utility in geophysical applications [22, 27, 28]. In finance, price fluctuations with semi-heavy tails are described by the tempered stable process models [2, 6–8].

This study examines the Bullen inequality in the context of tempered fractional integrals, drawing inspiration from the previously listed papers. In order to arrive at a number of new Bullen-type inequalities for differentiable convex functions, we first introduce a new integral identity.

2. Main Results

We start by presenting the proof of the following lemma which will be needed in the sequel.

Lemma 2.1. *Let $\Psi:\mathcal{A}\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on $\mathcal{A}^{\circ}$, and $\Psi'\in L^1[s,t]$. For all $t,s\in\mathcal{A}^{\circ}$ with $s<t$, then*

$$\begin{aligned}&\frac{t-s}{4\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)}\int_0^1\left(\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,\varkappa)-\frac{1}{2}\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)\right)\\&\quad\times\left[\Psi'\left(\frac{1-\varkappa}{2}s+\frac{1+\varkappa}{2}t\right)-\Psi'\left(\frac{1+\varkappa}{2}s+\frac{1-\varkappa}{2}t\right)\right]d\varkappa\\&=\left\{\frac{1}{4}\Psi(s)+\frac{1}{2}\Psi\left(\frac{s+t}{2}\right)+\frac{1}{4}\Psi(t)\right\}-\frac{\Gamma(\alpha)}{2\,\Upsilon_{\lambda}\left(\alpha,\frac{t-s}{2}\right)}\\&\quad\times\left[\mathcal{J}_{s+}^{(\alpha,\lambda)}\Psi\left(\frac{s+t}{2}\right)+\mathcal{J}_{t_-}^{(\alpha,\lambda)}\Psi\left(\frac{s+t}{2}\right)\right].\end{aligned}$$

Proof. Let

$$\mathcal{I}_1=\int_0^1\left(\frac{1}{2}\,\Upsilon_{\frac{\lambda(t-s)}{2}}(\alpha,\varkappa)-\frac{1}{4}\,\Upsilon_{\frac{\lambda(t-s)}{2}}(\alpha,1)\right)\Psi'\left(\frac{1+\varkappa}{2}s+\frac{1-\varkappa}{2}t\right)d\varkappa,$$

$$\mathcal{I}_2=\int_0^1\left(\frac{1}{2}\,\Upsilon_{\frac{\lambda(t-s)}{2}}(\alpha,\varkappa)-\frac{1}{4}\,\Upsilon_{\frac{\lambda(t-s)}{2}}(\alpha,1)\right)\Psi'\left(\frac{1-\varkappa}{2}s+\frac{1+\varkappa}{2}t\right)d\varkappa.$$

Integrating by parts $\mathcal{I}_1$, we get

(2.1)

$$\begin{aligned}
\mathcal{I}_1 &= \int_0^1 \left(\frac{1}{2}\,\Upsilon_{\frac{\lambda(t-s)}{2}}(\alpha,\varkappa) - \frac{1}{4}\,\Upsilon_{\frac{\lambda(t-s)}{2}}(\alpha,1)\right)\Psi'\left(\frac{1+\varkappa}{2}s+\frac{1-\varkappa}{2}t\right)d\varkappa \\
&= \frac{2}{s-t}\left(\frac{1}{2}\,\Upsilon_{\frac{\lambda(t-s)}{2}}(\alpha,\varkappa) - \frac{1}{4}\,\Upsilon_{\frac{\lambda(t-s)}{2}}(\alpha,1)\right)\Psi\left(\frac{1+\varkappa}{2}s+\frac{1-\varkappa}{2}t\right)\Big|_0^1 \\
&\quad + \frac{1}{t-s}\int_0^1 \varkappa^{\alpha-1}e^{-\lambda\left(\frac{t-s}{2}\right)\varkappa}\Psi\left(\frac{1+\varkappa}{2}s+\frac{1-\varkappa}{2}t\right)d\varkappa \\
&= \frac{2}{s-t}\left\{\frac{1}{4}\,\Upsilon_{\frac{\lambda(t-s)}{2}}(\alpha,1)\Psi(s) + \frac{1}{4}\,\Upsilon_{\frac{\lambda(t-s)}{2}}(\alpha,1)\Psi\left(\frac{s+t}{2}\right)\right\} \\
&\quad + \frac{2^\alpha}{(t-s)^{\alpha+1}}\int_s^{\frac{s+t}{2}}\left(\frac{s+t}{2}-z\right)^{\alpha-1}e^{-\lambda\left(\frac{s+t}{2}-z\right)}\Psi(z)dz \\
&= \frac{2}{s-t}\left\{\frac{1}{4}\,\Upsilon_{\frac{\lambda(t-s)}{2}}(\alpha,1)\Psi(s) + \frac{1}{4}\,\Upsilon_{\frac{\lambda(t-s)}{2}}(\alpha,1)\Psi\left(\frac{s+t}{2}\right)\right\} \\
&\quad + \frac{2^\alpha\Gamma(\alpha)}{(t-s)^{\alpha+1}}\mathcal{J}_{s+}^{(\alpha,\lambda)}\Psi\left(\frac{s+t}{2}\right).
\end{aligned}$$

Similarly, we get

(2.2)

$$\begin{aligned}
\mathcal{I}_2 &= \int_0^1 \left(\frac{1}{2}\,\Upsilon_{\frac{\lambda(t-s)}{2}}(\alpha,\varkappa) - \frac{1}{4}\,\Upsilon_{\frac{\lambda(t-s)}{2}}(\alpha,1)\right)\Psi'\left(\frac{1-\varkappa}{2}s+\frac{1+\varkappa}{2}t\right)d\varkappa \\
&= \frac{2}{t-s}\left(\frac{1}{2}\,\Upsilon_{\frac{\lambda(t-s)}{2}}(\alpha,\varkappa) - \frac{1}{4}\,\Upsilon_{\frac{\lambda(t-s)}{2}}(\alpha,1)\right)\Psi\left(\frac{1-\varkappa}{2}s+\frac{1+\varkappa}{2}t\right)\Big|_0^1 \\
&\quad - \frac{1}{t-s}\int_0^1 \varkappa^{\alpha-1}e^{-\lambda\left(\frac{t-s}{2}\right)\varkappa}\Psi\left(\frac{1-\varkappa}{2}s+\frac{1+\varkappa}{2}t\right)d\varkappa \\
&= \frac{2}{t-s}\left\{\frac{1}{4}\,\Upsilon_{\frac{\lambda(t-s)}{2}}(\alpha,1)\Psi(t) + \frac{1}{4}\,\Upsilon_{\frac{\lambda(t-s)}{2}}(\alpha,1)\Psi\left(\frac{s+t}{2}\right)\right\} \\
&\quad - \frac{2^\alpha}{(t-s)^{\alpha+1}}\int_{\frac{s+t}{2}}^t\left(z-\frac{s+t}{2}\right)^{\alpha-1}e^{-\lambda\left(z-\frac{s+t}{2}\right)}\Psi(z)dz \\
&= \frac{2}{t-s}\left\{\frac{1}{4}\,\Upsilon_{\frac{\lambda(t-s)}{2}}(\alpha,1)\Psi(t) + \frac{1}{4}\,\Upsilon_{\frac{\lambda(t-s)}{2}}(\alpha,1)\Psi\left(\frac{s+t}{2}\right)\right\} \\
&\quad - \frac{2^\alpha\Gamma(\alpha)}{(t-s)^{\alpha+1}}\mathcal{J}_{t-}^{(\alpha,\lambda)}\Psi\left(\frac{s+t}{2}\right).
\end{aligned}$$

From equations (2.1) and (2.2), the following result is obtained:

$$\frac{t-s}{2\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)}\left[\mathcal{I}_2-\mathcal{I}_1\right]$$

$$= \left(\frac{1}{4}\Psi(s) + \frac{1}{2}\Psi\left(\frac{s+t}{2}\right) + \frac{1}{4}\Psi(t)\right) - \frac{\Gamma(\alpha)}{2\,\Upsilon_\lambda\left(\alpha, \frac{t-s}{2}\right)}\left[\mathcal{J}_{s+}^{(\alpha,\lambda)}\Psi\left(\frac{s+t}{2}\right) + \mathcal{J}_{t_-}^{(\alpha,\lambda)}\Psi\left(\frac{s+t}{2}\right)\right].$$

This ends the proof of Lemma 2.1. □

Theorem 2.2. *Consider a differentiable function $\Psi : \mathcal{A} \subset \mathbb{R} \to \mathbb{R}$ on $\mathcal{A}^\circ$, such that $\Psi' \in L^1[s,t]$. Let $t, s \in \mathcal{A}^\circ$ with $s < t$, if $|\Psi'|$ is convex, then*

$$\left|\left\{\frac{1}{4}\Psi(s) + \frac{1}{2}\Psi\left(\frac{s+t}{2}\right) + \frac{1}{4}\Psi(t)\right\} - \frac{\Gamma(\alpha)}{2\,\Upsilon_\lambda\left(\alpha, \frac{t-s}{2}\right)}\left[\mathcal{J}_{s+}^{(\alpha,\lambda)}\Psi\left(\frac{s+t}{2}\right) + \mathcal{J}_{t_-}^{(\alpha,\lambda)}\Psi\left(\frac{s+t}{2}\right)\right]\right| \leq \frac{t-s}{4\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)}\left[\left|\Psi'(s)\right| + \left|\Psi'(t)\right|\right] \times \int_0^1 \left|\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,\varkappa) - \frac{1}{2}\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)\right| d\varkappa.$$

Proof. From Lemma 2.1, the convexity of $|\Psi'|$ and the properties of modulus, we have

$$\begin{aligned}
&\left|\left\{\frac{1}{4}\Psi(s) + \frac{1}{2}\Psi\left(\frac{s+t}{2}\right) + \frac{1}{4}\Psi(t)\right\} - \frac{\Gamma(\alpha)}{2\,\Upsilon_\lambda\left(\alpha, \frac{t-s}{2}\right)}\left[\mathcal{J}_{s+}^{(\alpha,\lambda)}\Psi\left(\frac{s+t}{2}\right) + \mathcal{J}_{t_-}^{(\alpha,\lambda)}\Psi\left(\frac{s+t}{2}\right)\right]\right| \\
&\leq \frac{t-s}{4\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)}\int_0^1 \left|\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,\varkappa) - \frac{1}{2}\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)\right| \\
&\quad \times \left[\left|\Psi'\left(\frac{1-\varkappa}{2}s + \frac{1+\varkappa}{2}t\right)\right| + \left|\Psi'\left(\frac{1+\varkappa}{2}s + \frac{1-\varkappa}{2}t\right)\right|\right] d\varkappa \\
&\leq \frac{t-s}{4\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)}\int_0^1 \left|\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,\varkappa) - \frac{1}{2}\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)\right| \\
&\quad \times \left[\frac{1-\varkappa}{2}\left|\Psi'(s)\right| + \frac{1+\varkappa}{2}\left|\Psi'(t)\right| + \frac{1+\varkappa}{2}\left|\Psi'(s)\right| + \frac{1-\varkappa}{2}\left|\Psi'(t)\right|\right] d\varkappa \\
&\leq \frac{t-s}{4\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)} \\
&\quad \times \int_0^1 \left|\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,\varkappa) - \frac{1}{2}\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)\right| \times \left[\left|\Psi'(s)\right| + \left|\Psi'(t)\right|\right] d\varkappa
\end{aligned}$$

$$\leq \frac{t-s}{4\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)}\left[\left|\Psi'(s)\right|+\left|\Psi'(t)\right|\right]$$
$$\times\int_0^1\left|\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,\varkappa)-\frac{1}{2}\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)\right|d\varkappa.$$

□

Corollary 2.3. *If we take $\lambda=0$ in Theorem 2.2, then*

$$\left|\left\{\frac{1}{4}\Psi(s)+\frac{1}{2}\Psi\left(\frac{s+t}{2}\right)+\frac{1}{4}\Psi(t)\right\}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(t-s)^{\alpha}}\left[\mathcal{H}_{s+}^{(\alpha,\lambda)}\Psi\left(\frac{s+t}{2}\right)+\mathcal{H}_{t-}^{(\alpha,\lambda)}\Psi\left(\frac{s+t}{2}\right)\right]\right|$$
$$\leq\frac{t-s}{4}\left(\frac{\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}+\frac{1}{\alpha+1}-\frac{1}{2}\right)\left[\left|\Psi'(s)\right|+\left|\Psi'(t)\right|\right],$$

which was given in [18].

Remark 2.4. If we take $\lambda=0$ and $\alpha=1$ in Theorem 2.2, then

$$\left|\left\{\frac{1}{4}\Psi(s)+\frac{1}{2}\Psi\left(\frac{s+t}{2}\right)+\frac{1}{4}\Psi(t)\right\}-\frac{1}{t-s}\int_s^t\Psi(\varkappa)\,d\varkappa\right|$$
$$\leq\frac{t-s}{16}\left[|\Psi'(s)|+|\Psi'(t)|\right].$$

See [19, Remark 4.2].

Theorem 2.5. *Let $\Psi:\mathcal{A}\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on $\mathcal{A}^{\circ}$, $\Psi'\in L^1[s,t]$ and $t,s\in\mathcal{A}^{\circ}$ with $s<t$. If $|\Psi'|^q$ is convex, where $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\left|\left\{\frac{1}{4}\Psi(s)+\frac{1}{2}\Psi\left(\frac{s+t}{2}\right)+\frac{1}{4}\Psi(t)\right\}-\frac{\Gamma(\alpha)}{2\,\Upsilon_{\lambda}\left(\alpha,\frac{t-s}{2}\right)}\left[\mathcal{J}_{s+}^{(\alpha,\lambda)}\Psi\left(\frac{s+t}{2}\right)+\mathcal{J}_{t-}^{(\alpha,\lambda)}\Psi\left(\frac{s+t}{2}\right)\right]\right|$$
$$\leq\frac{t-s}{4\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)}\left(\int_0^1\left|\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,\varkappa)-\frac{1}{2}\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)\right|^p d\varkappa\right)^{\frac{1}{p}}$$
$$\times\left[\left(\frac{3|\Psi'(s)|^q+|\Psi'(t)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\Psi'(s)|^q+3|\Psi'(t)|^q}{4}\right)^{\frac{1}{q}}\right]$$
$$\leq\frac{t-s}{4\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)}\left(4\int_0^1\left|\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,\varkappa)-\frac{1}{2}\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)\right|^p d\varkappa\right)^{\frac{1}{p}}$$
$$\times\left[|\Psi'(s)|+|\Psi'(t)|\right].$$

Proof. From Lemma 2.1, properties of modulus, and Hölder's inequality, we have

$$\begin{aligned}
&\left|\left\{\frac{1}{4}\Psi(s)+\frac{1}{2}\Psi\left(\frac{s+t}{2}\right)+\frac{1}{4}\Psi(t)\right\}\right.\\
&\quad\left.-\frac{\Gamma(\alpha)}{2\,\Upsilon_{\lambda}\left(\alpha,\frac{t-s}{2}\right)}\left[\mathcal{J}_{s+}^{(\alpha,\lambda)}\Psi\left(\frac{s+t}{2}\right)+\mathcal{J}_{t-}^{(\alpha,\lambda)}\Psi\left(\frac{s+t}{2}\right)\right]\right|\\
&\leq\frac{t-s}{4\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)}\left\{\left(\int_0^1\left|\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,\varkappa)-\frac{1}{2}\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)\right|^p d\varkappa\right)^{\frac{1}{p}}\right.\\
&\quad\times\left(\int_0^1\left|\Psi'\left(\frac{1+\varkappa}{2}s+\frac{1-\varkappa}{2}t\right)\right|^q d\varkappa\right)^{\frac{1}{q}}\\
&\quad+\left(\int_0^1\left|\left(\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,\varkappa)-\frac{1}{2}\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)\right)\right|^p d\varkappa\right)^{\frac{1}{p}}\\
&\quad\left.\times\left(\int_0^1\left|\Psi'\left(\frac{1-\varkappa}{2}s+\frac{1+\varkappa}{2}t\right)\right|^q d\varkappa\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

By utilizing the convexity of $|\Psi'|^q$, we get

$$\begin{aligned}
&\left|\left\{\frac{1}{4}\Psi(s)+\frac{1}{2}\Psi\left(\frac{s+t}{2}\right)+\frac{1}{4}\Psi(t)\right\}\right.\\
&\quad\left.-\frac{\Gamma(\alpha)}{2\,\Upsilon_{\lambda}\left(\alpha,\frac{t-s}{2}\right)}\left[\mathcal{J}_{s+}^{(\alpha,\lambda)}\Psi\left(\frac{s+t}{2}\right)+\mathcal{J}_{t-}^{(\alpha,\lambda)}\Psi\left(\frac{s+t}{2}\right)\right]\right|\\
&\leq\frac{t-s}{4\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)}\left(\int_0^1\left|\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,\varkappa)-\frac{1}{2}\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)\right|^p d\varkappa\right)^{\frac{1}{p}}\\
&\quad\times\left[\left(\int_0^1\left[\frac{1+\varkappa}{2}\left|\Psi'(s)\right|^q+\frac{1-\varkappa}{2}\left|\Psi'(t)\right|^q\right]d\varkappa\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_0^1\left[\frac{1-\varkappa}{2}\left|\Psi'(s)\right|^q+\frac{1+\varkappa}{2}\left|\Psi'(t)\right|^q\right]d\varkappa\right)^{\frac{1}{q}}\right]\\
&=\frac{t-s}{4\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)}\left(\int_0^1\left|\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,\varkappa)-\frac{1}{2}\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)\right|^p d\varkappa\right)^{\frac{1}{p}}\\
&\quad\times\left[\left(\frac{3|\Psi'(s)|^q+|\Psi'(t)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\Psi'(s)|^q+3|\Psi'(t)|^q}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Using $\sum_{i=1}^{n}(u_i+w_i)^r \le \sum_{i=1}^{n}u_i^r + \sum_{i=1}^{n}w_i^r$ where $0 \le r < 1$, we obtain the required inequalities. □

Corollary 2.6. *If we choose $\lambda = 0$ and $\alpha = 1$ in Theorem 2.5, then*

$$\begin{aligned}
&\left|\left\{\frac{1}{4}\Psi(s)+\frac{1}{2}\Psi\left(\frac{s+t}{2}\right)+\frac{1}{4}\Psi(t)\right\}-\frac{1}{t-s}\int_s^t \Psi(\varkappa)\,d\varkappa\right| \\
&\le \frac{t-s}{8}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{3|\Psi'(s)|^q+|\Psi'(t)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\Psi'(s)|^q+3|\Psi'(t)|^q}{4}\right)^{\frac{1}{q}}\right] \\
&\le \frac{t-s}{8}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[|\Psi'(s)|+|\Psi'(t)|\right].
\end{aligned}$$

Theorem 2.7. *Let $\Psi : \mathcal{A} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\mathcal{A}^\circ, \Psi' \in L^1[s,t]$, and $t, s \in \mathcal{A}^\circ$ with $s < t$. If $|\Psi'|^q$ is convex with $q > 1$, then*

$$\begin{aligned}
&\left|\left\{\frac{1}{4}\Psi(s)+\frac{1}{2}\Psi\left(\frac{s+t}{2}\right)+\frac{1}{4}\Psi(t)\right\}\right. \\
&\qquad\left.-\frac{\Gamma(\alpha)}{2\,\Upsilon_\lambda\left(\alpha,\frac{t-s}{2}\right)}\left[\mathcal{J}_{s+}^{(\alpha,\lambda)}\Psi\left(\frac{s+t}{2}\right)+\mathcal{J}_{t-}^{(\alpha,\lambda)}\Psi\left(\frac{s+t}{2}\right)\right]\right| \\
&\le \frac{t-s}{4\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)}\left(\int_0^1\left|\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,\varkappa)-\frac{1}{2}\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)\right|d\varkappa\right)^{1-\frac{1}{q}} \\
&\quad\times\left[\left(\int_0^1\left|\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,\varkappa)-\frac{1}{2}\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)\right|\right.\right. \\
&\quad\left.\times\left(\frac{1-\varkappa}{2}\left|\Psi'(s)\right|^q+\frac{1+\varkappa}{2}\left|\Psi'(t)\right|^q\right)d\varkappa\right)^{\frac{1}{q}} \\
&\quad+\left(\int_0^1\left|\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,\varkappa)-\frac{1}{2}\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)\right|\right. \\
&\quad\left.\left.\times\left(\frac{1+\varkappa}{2}\left|\Psi'(s)\right|^q+\frac{1-\varkappa}{2}\left|\Psi'(t)\right|^q\right)d\varkappa\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. From Lemma 2.1, properties of modulus, and power mean inequality, we have

$$\begin{aligned}
&\left|\left\{\frac{1}{4}\Psi(s)+\frac{1}{2}\Psi\left(\frac{s+t}{2}\right)+\frac{1}{4}\Psi(t)\right\}\right. \\
&\qquad\left.-\frac{\Gamma(\alpha)}{2\,\Upsilon_\lambda\left(\alpha,\frac{t-s}{2}\right)}\left[\mathcal{J}_{s+}^{(\alpha,\lambda)}\Psi\left(\frac{s+t}{2}\right)+\mathcal{J}_{t-}^{(\alpha,\lambda)}\Psi\left(\frac{s+t}{2}\right)\right]\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{t-s}{4\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)} \int_0^1 \left| \Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,\varkappa) - \frac{1}{2}\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1) \right| \\
&\quad \times \left[\left| \Psi'\left(\frac{1-\varkappa}{2}s + \frac{1+\varkappa}{2}t \right) \right| + \left| \Psi'\left(\frac{1+\varkappa}{2}s + \frac{1-\varkappa}{2}t \right) \right| \right] d\varkappa \\
&\leq \frac{t-s}{4\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)} \left(\int_0^1 \left| \Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,\varkappa) - \frac{1}{2}\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1) \right| d\varkappa \right)^{1-\frac{1}{q}} \\
&\quad \times \left[\left(\int_0^1 \left| \Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,\varkappa) - \frac{1}{2}\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1) \right| \right.\right. \\
&\quad \left.\left. \times \left| \Psi'\left(\frac{1-\varkappa}{2}s + \frac{1+\varkappa}{2}t \right) \right|^q d\varkappa \right)^{\frac{1}{q}} \right. \\
&\quad + \left(\int_0^1 \left| \Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,\varkappa) - \frac{1}{2}\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1) \right| \right. \\
&\quad \left. \left. \times \left| \Psi'\left(\frac{1+\varkappa}{2}s + \frac{1-\varkappa}{2}t \right) \right|^q d\varkappa \right)^{\frac{1}{q}} \right].
\end{aligned}
$$

Using the convexity of $|\Psi'|^q$, we get

$$
\begin{aligned}
&\left| \left\{ \frac{1}{4}\Psi(s) + \frac{1}{2}\Psi\left(\frac{s+t}{2}\right) + \frac{1}{4}\Psi(t) \right\} \right. \\
&\quad \left. - \frac{\Gamma(\alpha)}{2\,\Upsilon_\lambda\left(\alpha, \frac{t-s}{2}\right)} \left[\mathcal{J}_{s+}^{(\alpha,\lambda)}\Psi\left(\frac{s+t}{2}\right) + \mathcal{J}_{t-}^{(\alpha,\lambda)}\Psi\left(\frac{s+t}{2}\right) \right] \right| \\
&\leq \frac{t-s}{4\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)} \left(\int_0^1 \left| \Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,\varkappa) - \frac{1}{2}\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1) \right| d\varkappa \right)^{1-\frac{1}{q}} \\
&\quad \times \left[\left(\int_0^1 \left| \Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,\varkappa) - \frac{1}{2}\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1) \right| \right.\right. \\
&\quad \left.\left. \times \left(\frac{1-\varkappa}{2}\left|\Psi'(s)\right|^q + \frac{1+\varkappa}{2}\left|\Psi'(t)\right|^q \right) d\varkappa \right)^{\frac{1}{q}} \right. \\
&\quad + \left(\int_0^1 \left| \Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,\varkappa) - \frac{1}{2}\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1) \right| \right. \\
&\quad \left. \left. \times \left(\frac{1+\varkappa}{2}\left|\Psi'(s)\right|^q + \frac{1-\varkappa}{2}\left|\Psi'(t)\right|^q \right) d\varkappa \right)^{\frac{1}{q}} \right].
\end{aligned}
$$

The proof is completed. □

Corollary 2.8. *If we take $\lambda = 0$ and $\alpha = 1$ in Theorem 2.7, then*

$$\left|\left\{\frac{1}{4}\Psi(s)+\frac{1}{2}\Psi\left(\frac{s+t}{2}\right)+\frac{1}{4}\Psi(t)\right\}-\frac{1}{t-s}\int_s^t \Psi(\varkappa)\,d\varkappa\right| \leq \frac{t-s}{4^{\frac{1}{q}+2}}\left[\left(|\Psi'(s)|^q+3|\Psi'(t)|^q\right)^{\frac{1}{q}}+\left(3|\Psi'(s)|^q+|\Psi'(t)|^q\right)^{\frac{1}{q}}\right].$$

Theorem 2.9. *Let $\Psi : \mathcal{A} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\mathcal{A}^\circ$, $\Psi' \in L^1[s,t]$ and $t, s \in \mathcal{A}^\circ$ with $s < t$. If Ψ' is σ-M-Hölderian function on $[s,t]$ (i.e., there exist $M > 0$ and $0 < \sigma \leq 1$ such that $|\Psi'(\tau)-\Psi'(\eta)| \leq M|\tau-\eta|^\sigma$), then the following Bullen-type inequality holds:*

$$\left|\left\{\frac{1}{4}\Psi(s)+\frac{1}{2}\Psi\left(\frac{s+t}{2}\right)+\frac{1}{4}\Psi(t)\right\} -\frac{\Gamma(\alpha)}{2\,\Upsilon_\lambda\left(\alpha,\frac{t-s}{2}\right)}\left[\mathcal{J}_{s+}^{(\alpha,\lambda)}\Psi\left(\frac{s+t}{2}\right)+\mathcal{J}_{t-}^{(\alpha,\lambda)}\Psi\left(\frac{s+t}{2}\right)\right]\right| \leq \frac{M(t-s)^{\sigma+1}}{4\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)}\int_0^1 \varkappa^\sigma\left|\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,\varkappa)-\frac{1}{2}\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)\right|d\varkappa.$$

Proof. By using the Lemma 2.1 and the fact that Ψ' is σ-M-Hölderian function, we have

$$\begin{aligned}
&\left|\left\{\frac{1}{4}\Psi(s)+\frac{1}{2}\Psi\left(\frac{s+t}{2}\right)+\frac{1}{4}\Psi(t)\right\} -\frac{\Gamma(\alpha)}{2\,\Upsilon_\lambda\left(\alpha,\frac{t-s}{2}\right)}\left[\mathcal{J}_{s+}^{(\alpha,\lambda)}\Psi\left(\frac{s+t}{2}\right)+\mathcal{J}_{t-}^{(\alpha,\lambda)}\Psi\left(\frac{s+t}{2}\right)\right]\right| \\
&\leq \frac{t-s}{4\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)}\int_0^1\left|\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,\varkappa)-\frac{1}{2}\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)\right| \\
&\quad\times\left|\Psi'\left(\frac{1-\varkappa}{2}s+\frac{1+\varkappa}{2}t\right)-\Psi'\left(\frac{1+\varkappa}{2}s+\frac{1-\varkappa}{2}t\right)\right|d\varkappa \\
&\leq \frac{t-s}{4\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)} \\
&\quad\times\int_0^1\left|\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,\varkappa)-\frac{1}{2}\,\Upsilon_{\lambda\left(\frac{t-s}{2}\right)}(\alpha,1)\right||\varkappa(t-s)|^\sigma\;M\,d\varkappa. \quad \square
\end{aligned}$$

3. Conclusion

In summary, by employing tempered fractional integrals, this study has expanded our understanding of Bullen-type inequalities. We derived important new inequalities for differentiable convex functions by introducing a new integral identity. Additionally, interested readers can use

several types of convexities to develop new inequalities. These results create fresh opportunities for additional research in fractional calculus and its applications to integral inequalities.

Acknowledgment. The author wishes to thank the editor and reviewers for their valuable comments and suggestions for the betterment of this paper.

References

1. M. Barmaki, J. El-achky and D. Gretete, *Some inequalities of Hermite Hadamard for product of two stochastic processes*, J. Interdiscip. Math., 27 (2024), pp. 427–439.
2. O.E. Barndorff-Nielsen, *Processes of normal inverse Gaussian type*, Finance Stoch., 2 (1997), pp. 41–68.
3. H. Boulares, B. Meftah, A. Moumen, R. Shafqat, H. Saber, T. Alraqad and E. Ali, *Fractional multiplicative Bullen-type inequalities for multiplicative differentiable functions*, Symmetry, 15 (2023).
4. P.S. Bullen, *Error estimates for some elementary quadrature rules*, Publ. Elektroteh. Fak., Univ. Beogr., Ser. Mat. Fiz., No. 602-623 (1978), 97–103 (1979).
5. R.G. Buschman, *Decomposition of an integral operator by use of Mikusiński calculus*, SIAM J. Math. Anal., 3 (1972), pp. 83–85.
6. P. Carr, H. Geman, D.B. Madan and M. Yor, *The Fine Structure of Asset Returns: An Empirical Investigation*, J. Bus, 75 (2002), pp. 305–332.
7. P. Carr, H. Geman, D.B. Madan and M. Yor, *Stochastic volatility for Lévy processes*, Math. Finance, 13 (2003), pp. 345–382.
8. Á. Cartea and D. del-Castillo-Negrete, *Fractional diffusion models of option prices in markets with jumps*, Physica A, 374 (2007), pp. 749–763.
9. M.A. Chaudhry and S.M. Zubair, *Generalized incomplete gamma functions with applications*, J. Comput. Appl. Math., 55 (1994), pp. 99–124.
10. S.S. Dragomir and C.E.M. Pearce, *Selected topics on Hermite-Hadamard inequalities and applications*, RGMIA monographs, Victoria University, 2000.
11. J. El-achky, D. Gretete and M. Barmaki, *Inequalities of Hermite-Hadamard type for stochastic process whose fourth derivatives absolute value are quasi-convex, P-convex, s-convex and h-convex*, J. Interdiscip. Math., 25 (2021), pp. 987-1003.
12. J. El-achky and S. Taoufiki, *On (p;h)–convex stochastic process*, J. Interdiscip. Math., 25 (2022), pp. 1105-1116.

13. F. Ertuğral, M.Z. Sarikaya and H. Budak, *On Hermite-Hadamard type inequalities associated with the generalized fractional integrals*, Filomat, 36 (2022), pp. 3983-3995.
14. A. Fahad, S.I. Butt, B. Bayraktar, M. Anwar and Y. Wang, *Some new Bullen-type inequalities obtained via fractional integral operators*, Axioms, 12 (2023).
15. Y.S. Gasimov and J.E. Nápoles, *Some refinements of Hermite-Hadamard inequality using k-fractional Caputo derivatives*, Fract. Differ. Calc., 12 (2022), pp. 209–221.
16. J. Hadamard, *Étude sur les propriétés des fonctions entières et en particulier d'une fonction considrée par Riemann*, J. Math. Pures Appl., 58 (1893), pp. 171–215.
17. C. Hermite, *Sur deux limites d'une intégrale définie*, Mathesis, 3 (82) (1883).
18. F. Hezenci, H. Budak and H. Kara, *A study on conformable fractional version of Bullen-type inequalities*, Turk. J. Math., 47 (2023), pp. 1306-1317.
19. H.R. Hwang, K.L. Tseng and K.C. Hsu, *New inequalities for fractional integrals and their applications*, Turk. J. Math., 40 (2016), pp. 471-486.
20. A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Math.Stud., 204, Elsevier, Amsterdam, 2006.
21. C. Li, W. Deng and L. Zhao, *Well-posedness and numerical algorithm for the tempered fractional differential equations*, Discrete Contin. Dyn. Syst. Ser. B, 24 (2019), pp. 1989-2015.
22. M.M. Meerschaert, Y. Zhang and B. Baeumer, *Tempered anomalous diffusions in heterogeneous systems*, Geophys. Res. Lett., 35 (2008).
23. P.O. Mohammed, M.Z. Sarikaya and D. Baleanu, *On the generalized Hermite–-Hadamard inequalities via the tempered fractional integrals*, Symmetry, 12 (2020).
24. F. Sabzikar, M.M. Meerschaert and J.Chen, *Tempered fractional calculus*, J. Comput. Phys., 293 (2015), pp. 14-28.
25. R. Sassane, A. Lakhdari and B. Meftah, *On Bullen-Type Inequalities for Fractional Integrals with Exponential Kernels*, Sahand Commun. Math. Anal., 22 (2025), pp. 31-46.
26. K.-L. Tseng, S.-R. Hwang and S.S. Dragomir, *Fejér-type inequalities (I)*, J. Inequal. Appl., (2010).
27. Y. Zhang and M.M. Meerschaert, *Gaussian setting time for solute transport in fluvial systems*, Water Resour. Res., 47 (2011).
28. Y. Zhang, M.M. Meerschaert and A.I. Packman, *Linking fluvial bed sediment transport across scales*, Geophys. Res. Lett., 39 (2012).

National School of Applied Sciences, Ibn Tofail University, KENITRA 14000, Morocco.
Email address: jamal.el-achky@uit.ac.ma