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Fixed Point Theorems Using w-Distance in Rectangular Quasi b-Metric Spaces

Savita Rathee[1] and Nisha Balhara[2*]

Abstract. The objective of this paper is to prove fixed point results in rectangular quasi-b metric spaces using w-distance function. We establish some results for the existence and uniqueness of fixed points and introduce a new class of contractive mappings to generalize. We also analyze the existence of solutions to integral equations of the volterra type in order to show the significance of our theoretical results. Additionally, examples are provided. Furthermore, we applied one of our results to determine the existence of a solution to an integral equation.

1. Introduction

In nonlinear analysis, fixed point theory is one of the most effective and powerful tools. There are numerous ideas for Banach's contraction mapping idea in fixed point theory [11]. Various studies have expanded and generalized the Banach contraction concept in different ways [2, 3, 9, 10, 12–16, 19, 24, 26, 27]. The growing interest in modern fixed point theory has encouraged researchers to explore generalized distance frameworks, fractional structures and refined contraction conditions, resulting in versatile methods for analyzing nonlinear problems [4–7, 30, 32]. Recent investigations in fixed point theory have expanded toward richer metric frameworks such as complex-valued, b-metric and generalized b-metric-like structures, providing powerful tools for studying nonlinear mappings and integral or linear equations [18, 20, 21, 29, 31]. The concept of a rectangular quasi-metric space was first proposed by Lin et al. [28] in 2014. They also provided some

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fixed point results for the Meir-Keeler contractive-type mappings in the context of rectangular quasi-metric spaces. In 2014, Karapinar and Lakzian [23] defined (α, ψ)-contractive mapping and proved fixed point theorems in a rectangular quasi-metric space.

Rectangular b-metric space was introduced by George et al. [19] in 2015 as a generalization of metric, b-metric space and rectangle metric space. Several authors established and investigated numerous fixed point theorems that are in different spaces. In rectangular b-metric space, Alharbi et al. [8] established fixed point theorems and defined (α)-contractive mapping. Subsequently, various research publications have been published regarding fixed point results for single-valued and multivalued mappings within the context of rectangular quasi-metric spaces. Khuangsatung *et al.* [25] recently examined fixed point theorems for ψ-contraction mappings and developed the concept of ψ-contraction in entire rectangular quasi-metric spaces, proving the existence and uniqueness of fixed points.

Abagaro et al. [1] introduced the class of rectangular quasi b-metric spaces as a generalization of rectangular metric spaces, rectangular quasi-metric spaces and rectangular b-metric spaces. The works of Khuangsatung et al. [25], Alharbi et al. [8] and Karapinar and Lakzian [23] served as their inspiration and motivation. Additionally, they investigated fixed point outcomes for the mappings presented within the framework of rectangular quasi b-metric spaces and developed generalized (α, ψ)-contraction mappings.

For various uses in mathematical analysis, the w-distance functions are defined on metric spaces with an additional distance function. Kada et al. [22] proposed w-distance functions in 1996 to establish fixed point results and non-convex minimization. Fixed point theorems for functions satisfying w-distance inequalities rather than metric inequalities were derived using it in numerous other research. w-distance functions could be used for the classes of new family functions that were not covered by fixed point theorems in regular metric spaces. These initiatives represented a developing field of study and greatly improved fixed point theory.

2. Preliminiaries

We need the following symbols and class of functions:

- $R^+ = [0, \infty)$;
- R is the set of real numbers;
- N is the set of natural numbers;

- $\Psi = \{\psi : [0,\infty) \to [0,\infty)$, such that,$\psi$ is non decreasing, continuous, $\sum_{k=1}^{\infty} s^k\psi^k(t) < \infty$, $s\psi(t) < t$ for $t > 0$ and $\psi(0) = 0$ if and only if $t = 0$, where ψ^k is the kth iterate if ψ and $s \geq 1\}$.

Definition 1 ([15])**.** Let Ξ be a nonempty set and $s \geq 1$ be a given real number. A function $\delta : \Xi \times \Xi \to [0,\infty)$ is a b-metric if and only if for all $\theta, y, z \in \Xi$, the following conditions are satisfied:

(i) $\delta(\theta, y) = 0$ if and only if $\theta = y$;
(ii) $\delta(\theta, y) = \delta(y, \theta)$;
(iii) $\delta(\theta, y) \leq s[\delta(\theta, z) + \delta(z, y)]$.

The pair (Ξ, δ) is called a b-metric space.

Definition 2 ([33])**.** Let Ξ be a non empty and $s \geq 1$. Let $\delta : \Xi \times \Xi \to [0,\infty)$ be a function which satisfies:

(i) $\delta(\theta, y) = 0$ if and only if $\theta = y$;
(ii) $\delta(\theta, y) \leq s[\delta(\theta, z) + \delta(z, y)]$.

Then δ is called a quasi b-metric and the pair (Ξ, δ) is called a quasi b-metric space. The number s is called the coefficient of (Ξ, δ).

Remark. Any quasi metric space or any b-metric is a quasi b-metric space, but the converse is not true in general.

Example. Let $\Xi = \{1, 2, 3\}$ be a set and $\delta : X \times X \to [0,\infty)$ by

$$\delta(1,1) = \delta(2,2) = \delta(3,3) = 0;$$
$$\delta(1,2) = \delta(1,3) = \delta(2,3) = 6;$$
$$\delta(2,1) = \delta(3,1) = \delta(3,2) = 5.$$

Then (Ξ, δ) is a quasi b-metric space with $s \geq 1$ but it is not a b-metric space.

Definition 3 ([13])**.** Let Ξ be a nonempty set and $\delta : \Xi \times \Xi \to [0,\infty)$ be a function satisfying the following conditions:

(i) $\delta(\theta, y) = 0$ if and only if $\theta = y$;
(ii) $\delta(\theta, y) = \delta(y, \theta)$;
(iii) $\delta(\theta, y) \leq \delta(\theta, u) + \delta(u, v) + \delta(v, y)$ for all $\theta, y \in \Xi$ and all distinct points $u, v \in \Xi$.

Then δ is called a rectangular metric on Ξ and the pair (Ξ, δ) is called a rectangular metric space.

Definition 4 ([19])**.** Let Ξ be a non empty set, $s \geq 1$ be a given real number and $\delta : \Xi \times \Xi \to [0,\infty)$ be a function satisfying the following conditions:

(i) $\delta(\theta, y) = 0$ if and only if $\theta = y$;

(ii) $\delta(\theta, y) = \delta(y, \theta)$;
(iii) $\delta(\theta, y) \leq s[\delta(\theta, u) + \delta(u, v) + \delta(v, y)]$ for all $\theta, y \in \Xi$ and all distinct point $u, v \in \Xi$.

Then δ is called a rectangular b-metric on Ξ and the pair (Ξ, δ) is called a rectangular b-metric space.

Example ([19])**.** Let $\Xi = N$, $\alpha > 0$ and $\delta : \Xi \times \Xi \to [0, \infty)$ such that:

(i) $\delta(\theta, y) = \delta(y, \theta)$ for all $\theta, y \in \Xi$;
(ii) $\delta(\theta, y) = 0$, if and only if $\theta \neq y$;
(iii) $\delta(\theta, y) = 4\alpha$, if $\theta, y \in \{1, 2\}$ and $\theta \neq y$;
(iv) $\delta(\theta, y) = \alpha$, if $\theta, y \notin \{1, 2\}$ and $\theta \neq y$.

Then (Ξ, δ) is a rectangular b-metric space with coefficient $s = \frac{4}{3} > 1$, but (Ξ, δ) is not a rectangular metric space, as

$$\delta(1, 2) = 4\alpha \nleq 3\alpha = \delta(1, 3) + \delta(3, 4) + \delta(4, 2).$$

The following is the definition of the notion of rectangular quasi metric space.

Definition 5 ([28])**.** Let Ξ be a nonempty set, $s \geq 1$ and $\delta : \Xi \times \Xi \to [0, \infty)$ be a function satisfying the following conditions:

(i) $\delta(\theta, y) = 0$ if and only if $\theta = y$;
(ii) $\delta(\theta, y) \leq \delta(\theta, u) + \delta(u, v) + \delta(v, y)$ for all $\theta, y \in \Xi$ and all distinct point $u, v \in \Xi$.

Then, δ is called a rectangular quasi metric on Ξ and the pair (Ξ, δ) is called a rectangular quasi metric space.

Definition 6 ([1])**.** Let Ξ be a nonempty set, $s \geq 1$ and $\delta : \Xi \times \Xi \to [0, \infty)$ be a function satisfying the following conditions:

(i) $\delta(\theta, y) = 0$ if and only if $\theta = y$;
(ii) $\delta(\theta, y) \leq s\{\delta(\theta, u) + \delta(u, v) + \delta(v, y)\}$ for all $\theta, y \in \Xi$ and all distinct point $u, v \in \Xi$.

Then δ is called a rectangular quasi b-metric on Ξ and the pair (Ξ, δ) is called a rectangular quasi b-metric space.

Definition 7 ([1])**.** Let (Ξ, δ) be a rectangular quasi b-metric space and $\Upsilon : \Xi \to \Xi$ be a (α, ψ)-contraction mapping if there exist two functions $\alpha : \Xi \times \Xi \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(\theta, y)\delta(\Upsilon\theta, \Upsilon y) \leq \psi(M(\theta, y)), \quad \text{for all } \theta, y \in X,$$

where

$$M(\theta, y) = \max\left\{\delta(\theta, y), \frac{\delta(\theta, \Upsilon\theta)\delta(\theta, \Upsilon y)}{1 + \delta(\theta, \Upsilon y) + \delta(y, \Upsilon\theta)}, \delta(\theta, \Upsilon\theta), \delta(y, \Upsilon y)\right\}.$$

Definition 8 ([22])**.** Let (Ξ, δ) be a metric space and a function $\mathfrak{p}_t : \Xi \times \Xi \to [0, \infty)$ is called a w-distance on Ξ if it satisfies the following condition for each $\theta, y, z \in \Xi$:

(i) $\mathfrak{p}_t(\theta, z) \leq \mathfrak{p}_t(\theta, y) + \mathfrak{p}_t(y, z)$;
(ii) a mapping $\mathfrak{p}_t(\theta, .) : \Xi \to [0, \infty)$ is a lower semi continuous, i.e., if $\theta \in \Xi$ and $y_n \to y$ in Ξ, then $\mathfrak{p}_t(\theta, y) \leq \liminf_{n\to\infty} \mathfrak{p}_t(\theta, y_n)$;
(iii) for any $\epsilon > 0$, there exist $\delta > 0$ such that $\mathfrak{p}_t(z, \theta) \leq \delta$ and $\mathfrak{p}_t(z, y) \leq \delta$ imply $\delta(\theta, y) \leq \epsilon$.

Lemma 1 ([22])**.** *Let (Ξ, δ) be a metric space and $\mathfrak{p}_t$ be a w-distance function on Ξ. Suppose that $\{\theta_{\check{n}}\}$and $\{y_n\}$ are sequences in Ξ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, \infty)$ converging to 0. Then, the following holds for every $\theta, y, z \in \Xi$.*

(i) *if $\mathfrak{p}_t(\theta_{\check{n}}, y) \leq \alpha_n$ and $\mathfrak{p}_t(\theta_{\check{n}}, z) \leq \beta_n$ for any $n \in N$, then $y = z$; in particular, if $\mathfrak{p}_t(\theta, y) = 0$ and $\mathfrak{p}_t(\theta, z) = 0$, then $y = z$;*
(ii) *$\mathfrak{p}_t(\theta_{\check{n}}, y_n) \leq \alpha_n$ and $\mathfrak{p}_t(\theta_{\check{n}}, z) \leq \beta_n$ for any $n \in N$, then $\{y_n\}$ converges to z;*
(iii) *if $\mathfrak{p}_t(\theta_{\check{n}}, \theta_m) \leq \alpha_n$ for any $n, m \in N$ with $m > n$, then $\{\theta_{\check{n}}\}$ is a cauchy sequence;*
(iv) *if $\mathfrak{p}_t(y, \theta_{\check{n}}) \leq \alpha_n$ for every $n \in N$, then $\{\theta_{\check{n}}\}$ is a cauchy sequence.*

Definition 9 ([17])**.** Let (Ξ, δ) be a metric space and $\mathfrak{p}_0$ be a w_0-distance function on Ξ if $\mathfrak{p}_0(\theta, x) = 0$ for all $x \in \Xi$.

Remark. We obtain that, in general, for $\theta, y \in \Xi$, $\mathfrak{p}_t(\theta, y) \neq \mathfrak{p}_t(y, \theta)$ and neither of the implication $\mathfrak{p}_t(\theta, y) = 0 \Longleftrightarrow \theta = y$ necessarily holds.

3. Main Results

Definition 10. Let (Ξ, δ) be a rectangular quasi b-metric space, $\mathfrak{p}_t$ be a w-distance on Ξ and $\Upsilon : \Xi \to \Xi$ be a $\mathfrak{p}_t(\alpha, \psi)$-contraction mapping if $\exists$ two functions $\alpha : \Xi \times \Xi \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(\theta, y)\mathfrak{p}_t(\Upsilon\theta, \Upsilon y) \leq \psi(M(\theta, y)), \quad \text{for all } \theta, y \in \Xi,$$

where

$$M(\theta, y) = \max\left\{\mathfrak{p}_t(\theta, y), \frac{\mathfrak{p}_t(\theta, \Upsilon\theta)\mathfrak{p}_t(\theta, \Upsilon y)}{1 + \mathfrak{p}_t(\theta, \Upsilon y) + \mathfrak{p}_t(y, \Upsilon\theta)}, \mathfrak{p}_t(\theta, \Upsilon\theta), \mathfrak{p}_t(y, \Upsilon y)\right\}.$$

Theorem 1. *Let (Ξ, δ) be a complete rectangular quasi-b-metric space, with $\mathfrak{p}_t$ denoting an w-distance function on Ξ and $\Upsilon : \Xi \to \Xi$ be generalized (α, ψ)- contraction mapping. Suppose that*

(1) *Υ is an α-admissible mapping;*
(2) *there exists $\theta_0 \in \Xi$ such that $\alpha(\theta_0, \Upsilon\theta_0) \geq 1$, $\alpha(\Upsilon\theta_0, \theta_0) \geq 1$, $\alpha(\theta_0, \Upsilon^2\theta_0) \geq 1$ and $\alpha\left(\Upsilon^2\theta_0, \theta_0\right) \geq 1$;*
(3) *Υ is continuous;*

(4) *if for every* $y \in \Xi$ *with* $y \notin \Upsilon(y)$*, one has* $\inf\{\mathfrak{p}_t(\theta, y) + \mathfrak{p}_t(\theta, \Upsilon\theta) : \theta \in \Xi\} > 0$.

Then, Υ *admits a unique fixed point.*

Proof. For $\theta_0 \in \Xi$ and $\theta_1 \in \Upsilon\theta_0$ in (2), by definition $\alpha(\theta_0, \Upsilon\theta_0) \geq 1$, $\alpha(\Upsilon\theta_0, \theta_0) \geq 1$. Now, we proceed to construct a sequence $\{\theta_{\check{n}}\}$ in Ξ by $\theta_{\check{n}+1} = \Upsilon\theta_{\check{n}} = \Upsilon^{n+1}\theta_0$ for all $\check{n} \geq 0$. Suppose that $\theta_{\check{n0}} = \theta_{\check{n0+1}}$ for some $n_0 \geq 0$. Since $\Upsilon\theta_{\check{n0}} = \theta_{\check{n0+1}}$, $u = \theta_{\check{n0}}$ form a fixed point of Υ. Thus, the proof is complete.

We take $\theta_{\check{n}} \neq \theta_{\check{n}+1}\ \forall \check{n} \geq 0$. Since Υ is α-admissible mapping, it follows that

$$\alpha(\theta_0, \theta_1) = \alpha(\theta_0, \Upsilon\theta_0) \geq 1 \quad \Longrightarrow \quad \alpha(\Upsilon\theta_0, \Upsilon\theta_1) = \alpha(\theta_1, \Upsilon\theta_2) \geq 1.$$

Continuing like this, we get

$$\alpha\left(\theta_{\check{n}}, \theta_{\check{n}+1}\right) \geq 1, \quad \text{for all } \check{n} = 0, 1, 2, \ldots$$

and

$$\alpha(\theta_1, \theta_0) = \alpha(\Upsilon\theta_0, \theta_0) \geq 1 \quad \Longrightarrow \quad \alpha(\Upsilon\theta_1, \Upsilon\theta_0) = \alpha(\theta_2, \Upsilon\theta_1) \geq 1.$$

Again, continuing this, we obtain

$$\alpha\left(\theta_{\check{n}+1}, \theta_{\check{n}}\right) \geq 1, \quad \text{for all } n = 0, 1, 2, \ldots .$$

In a similar way, we get

$$\alpha(\theta_0, \theta_2) = \alpha(\theta_0, \Upsilon^2\theta_0) \geq 1 \quad \Longrightarrow \quad \alpha\left(\Upsilon\theta_0, \Upsilon\theta_2\right) = \alpha(\theta_1, \Upsilon\theta_3) \geq 1.$$

Hence, we obtain

$$\alpha(\theta_{\check{n}}, \theta_{\check{n}+2}) \geq 1, \quad \text{for all } n = 0, 1, 2, \ldots .$$

Now, it is evident that

$$\alpha(\theta_{\check{n}+2}, \theta_{\check{n}}) \geq 1, \quad \text{for all } \check{n} = 0, 1, 2, \ldots .$$

Here, we have to show that

$$\lim_{\check{n}\to\infty} \mathfrak{p}_t\left(\theta_{\check{n}}, \theta_{\check{n}+1}\right) = \lim_{\check{n}\to\infty} \mathfrak{p}_t\left(\theta_{\check{n}+1}, \theta_{\check{n}}\right) = 0.$$

For this, we have

$$\begin{aligned}
\mathfrak{p}_t\left(\theta_{\check{n}}, \theta_{\check{n}+1}\right) &= \mathfrak{p}_t(\Upsilon\theta_{\check{n}-1}, \Upsilon\theta_{\check{n}}) \\
&\leq \alpha\left(\theta_{\check{n}-1}, \theta_{\check{n}}\right) \mathfrak{p}_t(\Upsilon\theta_{\check{n}-1}, \Upsilon\theta_{\check{n}}) \\
&\leq \psi(M\left(\theta_{\check{n}-1}, \theta_{\check{n}}\right)), \quad \text{for all } \check{n} \geq 1,
\end{aligned}$$

where

$$M\left(\theta_{\check{n}-1}, \theta_{\check{n}}\right) = \max\left\{\mathfrak{p}_t\left(\theta_{\check{n}-1}, \theta_{\check{n}}\right), \frac{\mathfrak{p}_t(\theta_{\check{n}-1}, \Upsilon\theta_{\check{n}-1})\mathfrak{p}_t(\theta_{\check{n}-1}, \Upsilon\theta_{\check{n}})}{1 + \mathfrak{p}_t(\theta_{\check{n}-1}, \Upsilon\theta_{\check{n}}) + \mathfrak{p}_t(\theta_{\check{n}}, \Upsilon\theta_{\check{n}-1})}, \right.$$
$$\left. \mathfrak{p}_t(\theta_{\check{n}-1}, \Upsilon\theta_{\check{n}-1}), \mathfrak{p}_t(\theta_{\check{n}}, \Upsilon\theta_{\check{n}})\right\},$$

$$M\left(\theta_{\check{n}-1},\theta_{\check{n}}\right)=\max\left\{\mathfrak{p}_t\left(\theta_{\check{n}-1},\theta_{\check{n}}\right),\frac{\mathfrak{p}_t\left(\theta_{\check{n}-1},\theta_{\check{n}}\right)\mathfrak{p}_t(\theta_{\check{n}-1},\theta_{\check{n}+1})}{1+\mathfrak{p}_t(\theta_{\check{n}-1},\theta_{\check{n}+1})+\mathfrak{p}_t(\theta_{\check{n}},\theta_{\check{n}})},\right.$$
$$\left.\mathfrak{p}_t\left(\theta_{\check{n}-1},\theta_{\check{n}}\right),\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right)\right\}$$
$$=\max\left\{\mathfrak{p}_t\left(\theta_{\check{n}-1},\theta_{\check{n}}\right),\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right)\right\}.$$

If

$$M\left(\theta_{\check{n}-1},\theta_{\check{n}}\right)=\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right),$$

then

$$\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right)\leq\psi\{\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right)\}<\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right).$$

Which contradicts our earlier assumption. Therefore, $\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right)=0$ and then $\mathfrak{p}_t\left(\theta_{\check{n}-1},\theta_{\check{n}}\right)=0$ by the property of w distance $\theta_{\check{n}-1}$=$\theta_{\check{n}}$ implies that $\theta_{\check{n}-1}\in\Upsilon\theta_{\check{n}}$. But, if

$$M\left(\theta_{\check{n}-1},\theta_{\check{n}}\right)=\mathfrak{p}_t\left(\theta_{\check{n}-1},\theta_{\check{n}}\right)$$

then from above, we get

$$\begin{aligned}\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right)&\leq\psi\{\mathfrak{p}_t\left(\theta_{\check{n}-1},\theta_{\check{n}}\right)\}\\&=\psi\{\mathfrak{p}_t(\Upsilon\theta_{\check{n}-2},\Upsilon\theta_{\check{n}-1})\}\\&\leq\psi[\psi\{\mathfrak{p}_t(\theta_{\check{n}-2},\theta_{\check{n}-1})\}]\\&=\psi^2\{\mathfrak{p}_t(\theta_{\check{n}-2},\theta_{\check{n}-1})\}\\&\vdots\\&\leq\psi^n\{\mathfrak{p}_t(\theta_0,\theta_1)\}\to0\quad\text{as }\check{n}\to\infty.\end{aligned}\tag{3.1}$$

Also,

$$\begin{aligned}\mathfrak{p}_t\left(\theta_{\check{n}+1},\theta_{\check{n}}\right)&=\mathfrak{p}_t(\Upsilon\theta_{\check{n}},\Upsilon\theta_{\check{n}-1})\\&\leq\alpha\left(\theta_{\check{n}},\theta_{\check{n}-1}\right)\mathfrak{p}_t(\Upsilon\theta_{\check{n}},\Upsilon\theta_{\check{n}-1})\\&\leq\psi(M\left(\theta_{\check{n}},\theta_{\check{n}-1}\right)),\quad\text{for all }\check{n}\geq1,\end{aligned}\tag{3.2}$$

where

$$M\left(\theta_{\check{n}},\theta_{\check{n}-1}\right)=\max\left\{\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}-1}\right),\frac{\mathfrak{p}_t(\theta_{\check{n}},\Upsilon\theta_{\check{n}})\mathfrak{p}_t(\theta_{\check{n}},\Upsilon\theta_{\check{n}-1})}{1+\mathfrak{p}_t(\theta_{\check{n}},\Upsilon\theta_{\check{n}-1})+\mathfrak{p}_t(\theta_{\check{n}-1},\Upsilon\theta_{\check{n}})},\right.$$
$$\left.\mathfrak{p}_t(\theta_{\check{n}},\Upsilon\theta_{\check{n}}),\mathfrak{p}_t(\theta_{\check{n}-1},\Upsilon\theta_{\check{n}-1})\right\}$$
$$=\max\left\{\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}-1}\right),\frac{\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right)\mathfrak{p}_t(\theta_{\check{n}},\theta_{\check{n}})}{1+\mathfrak{p}_t(\theta_{\check{n}},\theta_{\check{n}})+\mathfrak{p}_t(\theta_{\check{n}-1},\theta_{\check{n}+1})},\right.$$
$$\left.\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right),\mathfrak{p}_t\left(\theta_{\check{n}-1},\theta_{\check{n}}\right)\right\}$$
$$=\max\{\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}-1}\right),\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right),\mathfrak{p}_t\left(\theta_{\check{n}-1},\theta_{\check{n}}\right)\}.$$

We partition the analysis into the following three cases:

Case 1: If

$$M\left(\theta_{\check{n}}, \theta_{\check{n}-1}\right) = \mathfrak{p}_t\left(\theta_{\check{n}-1}, \theta_{\check{n}}\right),$$

using equation (3.2), we obtain

$$\mathfrak{p}_t\left(\theta_{\check{n}+1}, \theta_{\check{n}}\right) \leq \psi(\mathfrak{p}_t\left(\theta_{\check{n}-1}, \theta_{\check{n}}\right)) \leq \psi^n \mathfrak{p}_t(\theta_0, \theta_1) \to 0 \text{ as } \check{n} \to \infty.$$

Case 2: If

$$M\left(\theta_{\check{n}}, \theta_{\check{n}-1}\right) = \mathfrak{p}_t\left(\theta_{\check{n}}, \theta_{\check{n}+1}\right),$$

by equation (3.2), we have

$$\mathfrak{p}_t\left(\theta_{\check{n}+1}, \theta_{\check{n}}\right) \leq \psi(\mathfrak{p}_t\left(\theta_{\check{n}}, \theta_{\check{n}+1}\right)) \leq \psi^n \mathfrak{p}_t(\theta_0, \theta_1) \to 0 \text{ as } \check{n} \to \infty.$$

Case 3: If

$$M\left(\theta_{\check{n}}, \theta_{\check{n}-1}\right) = \mathfrak{p}_t\left(\theta_{\check{n}}, \theta_{\check{n}-1}\right),$$

again applying equation (3.2), we derive

$$\begin{aligned}
\mathfrak{p}_t\left(\theta_{\check{n}+1}, \theta_{\check{n}}\right) &\leq \psi\{\mathfrak{p}_t\left(\theta_{\check{n}}, \theta_{\check{n}-1}\right)\} \\
&= \psi\{\mathfrak{p}_t(\Upsilon\theta_{\check{n}-1}, \Upsilon\theta_{\check{n}-2})\} \\
&\leq \psi\psi\{\mathfrak{p}_t(\theta_{\check{n}-1}, \theta_{\check{n}-2})\} \\
&= \psi^2\{\mathfrak{p}_t(\theta_{\check{n}-1}, \theta_{\check{n}-2})\} \\
&\vdots \\
&\leq \psi^n\{\mathfrak{p}_t(\theta_1, \theta_0)\} \to 0.
\end{aligned}$$

Letting $\check{n} \to \infty$ from Case 1 to Case 3, we get

$$\mathfrak{p}_t\left(\theta_{\check{n}+1}, \theta_{\check{n}}\right) \to 0 \quad \text{as } \check{n} \to \infty. \tag{3.3}$$

Then, from (3.1) and (3.3), we deduce that

$$\lim_{n\to\infty} \mathfrak{p}_t\left(\theta_{\check{n}}, \theta_{\check{n}+1}\right) = \lim_{n\to\infty} \mathfrak{p}_t\left(\theta_{\check{n}+1}, \theta_{\check{n}}\right) = 0.$$

Now, for $m, \check{n} \in N$, for which $\check{n} + p > \check{n}$, we have

$$\begin{aligned}
\mathfrak{p}_t(\theta_{\check{n}}, \theta_{\check{n}+p}) &\leq s\{\mathfrak{p}_t\left(\theta_{\check{n}}, \theta_{\check{n}+1}\right) + \mathfrak{p}_t(\theta_{\check{n}+1}, \theta_{\check{n}+2}) + \mathfrak{p}_t(\theta_{\check{n}+2}, \theta_{\check{n}+p})\} \\
\mathfrak{p}_t(\theta_{\check{n}}, \theta_{\check{n}+p}) &\leq s(\mathfrak{p}_t\left(\theta_{\check{n}}, \theta_{\check{n}+1}\right)) + s(\mathfrak{p}_t(\theta_{\check{n}+1}, \theta_{\check{n}+2})) \\
&\quad + s\{s\{\mathfrak{p}_t(\theta_{\check{n}+2}, \theta_{\check{n}+3}) + \mathfrak{p}_t(\theta_{\check{n}+3}, \theta_{\check{n}+4}) + \mathfrak{p}_t(\theta_{\check{n}+4}, \theta_{\check{n}+p})\} \\
\mathfrak{p}_t(\theta_{\check{n}}, \theta_{\check{n}+p}) &\leq s(\mathfrak{p}_t\left(\theta_{\check{n}}, \theta_{\check{n}+1}\right)) + s(\mathfrak{p}_t(\theta_{\check{n}+1}, \theta_{\check{n}+2})) + s^2\mathfrak{p}_t(\theta_{\check{n}+2}, \theta_{\check{n}+3}) \\
&\quad + s^2\mathfrak{p}_t(\theta_{\check{n}+3}, \theta_{\check{n}+4}) + s^2\mathfrak{p}_t(\theta_{\check{n}+4}, \theta_{\check{n}+p}) \\
\mathfrak{p}_t(\theta_{\check{n}}, \theta_{\check{n}+p}) &\leq s(\mathfrak{p}_t\left(\theta_{\check{n}}, \theta_{\check{n}+1}\right)) + s^2(\mathfrak{p}_t(\theta_{\check{n}+1}, \theta_{\check{n}+2})) + s^3\mathfrak{p}_t(\theta_{\check{n}+2}, \theta_{\check{n}+3}) \\
&\quad + s^4\mathfrak{p}_t(\theta_{\check{n}+3}, \theta_{\check{n}+4}) + \cdots + s^p\mathfrak{p}_t(\theta_{\check{n}+p-1}, \theta_{\check{n}+p})
\end{aligned}$$

$$\begin{aligned}
&\leq \psi^n s\{\mathfrak{p}_t(\theta_0,\theta_1)\} + \psi^{n+1}s^2\{\mathfrak{p}_t(\theta_0,\theta_1)\} \\
&\quad + \psi^{n+2}s^3\{\mathfrak{p}_t(\theta_0,\theta_1)\} + \cdots + \psi^{n+p-1}s^p\{\mathfrak{p}_t(\theta_0,\theta_1)\} \\
&\leq \frac{s\psi^n}{1-s\psi}\mathfrak{p}_t(\theta_0,\theta_1) \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Now, using the lemma, we get a cauchy sequence $\langle\theta_{\check{n}}\rangle$ in Ξ. Using the completeness of Ξ, we get $\langle\theta_{\check{n}}\rangle \to z$ for some $z \in \Xi$. Since $\mathfrak{p}_t(\theta_{\check{n}},.)$ is lower semi continuous, we have

$$\mathfrak{p}_t(\theta_{\check{n}},z) \leq \inf s.\mathfrak{p}_t(\theta_{\check{n}},\theta_{\check{n}+p}) \leq s^2\frac{\psi^n}{1-s\psi}\mathfrak{p}_t(\theta_0,\theta_1) \to 0 \quad \text{as } n \to \infty.$$

Now, if Υ is continuous, then $\theta_{\check{n}+1} = \Upsilon\theta_{\check{n}} \to \Upsilon z$ and so by uniqueness of limit $z = \Upsilon z$.

Assuming that $z \neq \Upsilon z$, then by hypothesis

$$\begin{aligned}
0 &< \inf\{\mathfrak{p}_t(\theta,z) + \mathfrak{p}_t(\theta,\Upsilon\theta) : \theta \in X\} \\
&\leq \inf\{\mathfrak{p}_t(\theta_{\check{n}},z) + \mathfrak{p}_t(\theta_{\check{n}},\Upsilon\theta_{\check{n}}) : \theta \in X\} \\
&\leq \inf\{\mathfrak{p}_t(\theta_{\check{n}},z) + \mathfrak{p}_t\,(\theta_{\check{n}},\theta_{\check{n}+1}) : \theta \in X\} \\
&\leq \inf\{\mathfrak{p}_t(\theta_{\check{n}},z) + \mathfrak{p}_t\,(\theta_{\check{n}},\theta_{\check{n}+1}) : \theta \in X\} \\
&\leq \frac{\psi^n}{1-\psi}\mathfrak{p}_t(\theta_0,\theta_1) \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

Clearly, this contradicts our earlier assumption. Therefore, $z = \Upsilon z$. Suppose u and v be two distinct fixed points of Υ. By given condition, $\alpha(\Upsilon u,\Upsilon v) = \alpha(u,v) \geq 1$ and $\alpha(\Upsilon v,\Upsilon u) = \alpha(v,u) \geq 1$. Now,by α-admissibility of Υ, we get

$$\mathfrak{p}_t(u,v) \leq \alpha(u,v)\mathfrak{p}_t(u,v) = \alpha(\Upsilon u,\Upsilon v)\mathfrak{p}_t(\Upsilon u,\Upsilon v) \leq \psi(M(u,v)),$$

where

$$\begin{aligned}
M(u,v) &= \max\left\{\mathfrak{p}_t(u,v), \frac{\mathfrak{p}_t(u,\Upsilon u)\mathfrak{p}_t(u,\Upsilon v)}{1+\mathfrak{p}_t(u,\Upsilon v)+\mathfrak{p}_t(v,\Upsilon u)}, \mathfrak{p}_t(u,\Upsilon u), \mathfrak{p}_t(v,\Upsilon v)\right\} \\
&= \max\left\{\mathfrak{p}_t(u,v), \frac{\mathfrak{p}_t(u,u)\mathfrak{p}_t(u,v)}{1+\mathfrak{p}_t(u,v)+\mathfrak{p}_t(v,u)}, \mathfrak{p}_t(u,u), \mathfrak{p}_t(v,v)\right\} \\
&= \mathfrak{p}_t(u,v).
\end{aligned}$$

By condition $s\psi(t) < t$, $\forall t > 0$, we obtain

$$\mathfrak{p}_t(u,v) \leq \psi(\mathfrak{p}_t(u,v)) \leq s\psi(\mathfrak{p}_t(u,v)) < \mathfrak{p}_t(u,v). \tag{3.4}$$

Then, this contradicts our earlier assumption. Similarly, we have

$$\mathfrak{p}_t(v,u) < \mathfrak{p}_t(v,u) \tag{3.5}$$

which is again a contradiction. By the inequalities (3.4) and (3.5), we get $\mathfrak{p}_t(u,v) = \mathfrak{p}_t(v,u)$ which follows that $u = v$. Therefore, Υ has a unique fixed point. □

Example 1. Let $\Xi = \{1, 2, 3\}$ and $\delta : \Xi \times \Xi \to [0, \infty)$ defined as

$$\begin{aligned}
&\delta(1,3) = \delta(2,1) = \delta(3,2) = 6,\\
&\delta(3,1) = 7,\\
&\delta(1,2) = \delta(2,3) = 8,\\
&\delta(1,1) = \delta(2,2) = \delta(3,3) = 0.
\end{aligned}$$

Now, we take $\Upsilon : \Xi \to \Xi$ defined by $\Upsilon(1) = 0$, $\Upsilon(2) = 0$ and $\Upsilon(3) = 1$, $\psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = \frac{t}{4}$ $\forall t \in [0, \infty)$ and $\alpha : \Xi \times \Xi \to [0, \infty)$ defined as

$$\alpha(\theta, y) = \begin{cases} 1, & \text{if } \theta, y \in \{1, 2\} \\ 0, & \text{otherwise.} \end{cases}$$

Consider the distance function $\mathfrak{p}_t$ in Ξ as $\mathfrak{p}_t(\theta, y) = y$. Then, we have the following cases

Case 1: If $x = 1$ and $y = 2$

$$\alpha(1,2)\mathfrak{p}_t(\Upsilon 1, \Upsilon 2) \leq \psi\bigg(\max\bigg\{\mathfrak{p}_t(1,2), \frac{\mathfrak{p}_t(1,\Upsilon 1)\mathfrak{p}_t(1,\Upsilon 2)}{1 + \mathfrak{p}_t(1,\Upsilon 2) + \mathfrak{p}_t(2,\Upsilon 1)}, \mathfrak{p}_t(1,\Upsilon 1), \mathfrak{p}_t(2,\Upsilon 2)\bigg\}\bigg)$$

then

$$\mathfrak{p}_t(0,0) \leq \psi \max\{2,0,0,0\} \quad \Longrightarrow \quad 0 \leq 2 \text{ which is true.}$$

Case 2: If $x = 2$ and $y = 3$

$$\alpha(2,3)\mathfrak{p}_t(\Upsilon 2, \Upsilon 3) \leq \psi\bigg(\max\bigg\{\mathfrak{p}_t(2,3), \frac{\mathfrak{p}_t(2,\Upsilon 2)\mathfrak{p}_t(2,\Upsilon 3)}{1 + \mathfrak{p}_t(2,\Upsilon 3) + \mathfrak{p}_t(3,\Upsilon 2)}, \mathfrak{p}_t(2,\Upsilon 2), \mathfrak{p}_t(3,\Upsilon 3)\bigg\}\bigg)$$

then

$$0 \leq \psi \max\{3,0,0,1\} \quad \Longrightarrow \quad 0 \leq 3.$$

Case 3: If $x = 3$ and $y = 1$

$$\alpha(3,1)\mathfrak{p}_t(\Upsilon 3, \Upsilon 1) \leq \psi\bigg(\max\bigg\{\mathfrak{p}_t(3,1), \frac{\mathfrak{p}_t(3,\Upsilon 3)\mathfrak{p}_t(3,\Upsilon 1)}{1 + \mathfrak{p}_t(3,\Upsilon 1) + \mathfrak{p}_t(1,\Upsilon 3)}, \mathfrak{p}_t(3,\Upsilon 3), \mathfrak{p}_t(1,\Upsilon 1)\bigg\}\bigg)$$

then

$$0 \leq \psi \max\{1,0,1,0\} \quad \Longrightarrow \quad 0 \leq 1.$$

Case 4: If $x = 1$ and $y = 3$

$$\alpha(1,3)\mathfrak{p}_t(\Upsilon 1, \Upsilon 3) \leq \psi\Bigg(\max\Bigg\{\mathfrak{p}_t(1,3), \frac{\mathfrak{p}_t(1,\Upsilon 1)\mathfrak{p}_t(1,\Upsilon 3)}{1+\mathfrak{p}_t(1,\Upsilon 3)+\mathfrak{p}_t(3,\Upsilon 1)}, \mathfrak{p}_t(1,\Upsilon 1), \mathfrak{p}_t(3,\Upsilon 3)\Bigg\}\Bigg)$$

then

$$0 \leq \psi \max\{3,0,0,1\} \quad \Longrightarrow \quad 0 \leq 3.$$

Case 5: If $x = 3$ and $y = 2$

$$\alpha(3,2)\mathfrak{p}_t(\Upsilon 3, \Upsilon 2) \leq \psi\Bigg(\max\Bigg\{\mathfrak{p}_t(3,2), \frac{\mathfrak{p}_t(3,\Upsilon 3)\mathfrak{p}_t(3,\Upsilon 2)}{1+\mathfrak{p}_t(3,\Upsilon 2)+\mathfrak{p}_t(2,\Upsilon 3)}, \mathfrak{p}_t(3,\Upsilon 3), \mathfrak{p}_t(2,\Upsilon 2)\Bigg\}\Bigg)$$

then

$$0 \leq \psi \max\{2,0,1,0\} \quad \Longrightarrow \quad 0 \leq 2.$$

Case 6: If $x = 2$ and $y = 1$

$$\alpha(2,1)\mathfrak{p}_t(\Upsilon 2, \Upsilon 1) \leq \psi\Bigg(\max\Bigg\{\mathfrak{p}_t(2,1), \frac{\mathfrak{p}_t(2,\Upsilon 2)\mathfrak{p}_t(2,\Upsilon 1)}{1+\mathfrak{p}_t(2,\Upsilon 1)+\mathfrak{p}_t(1,\Upsilon 2)}, \mathfrak{p}_t(2,\Upsilon 2), \mathfrak{p}_t(1,\Upsilon 1)\Bigg\}\Bigg)$$

then

$$0 \leq \psi \max\{1,0,0,0\} \quad \Longrightarrow \quad 0 \leq 1.$$

Theorem 2. *Let (Ξ, δ) be a complete rectangular quasi b-metric space, with $\mathfrak{p}_t$ denoting an w-distance function on Ξ and $\Upsilon : \Xi \to \Xi$ be generalized (α, ψ)- contraction mapping. Suppose that*

(i) *Υ is an α-admissible mapping;*
(ii) *there exists $\theta_0 \in \Xi$ such that $\alpha(\theta_0, \Upsilon\theta_0) \geq 1$, $\alpha(\Upsilon\theta_0, \theta_0) \geq 1$, $\alpha\left(\theta_0, \Upsilon^2\theta_0\right) \geq 1$ and $\alpha\left(\Upsilon^2\theta_0, \theta_0\right) \geq 1$;*
(iii) *$\{\theta_{\check{n}}\}$ is α-regular;*
(iv) *if for each $y \in \Xi$ with $y \notin \Upsilon(y)$, one has $\inf\{\mathfrak{p}_t(\theta, y) + \mathfrak{p}_t(\theta, \Upsilon\theta) : \theta \in \Xi\} > 0$.*

Then, Υ admits a unique fixed point.

Proof. If we take the sequence $\theta_{\check{n}}$ defined by $\Upsilon\theta_{\check{n}} = \theta_{\check{n}+1}$ $\forall \check{n} \geq 0$ then it follows from theorem that $\theta_{\check{n}}$ converges to a point u in Ξ. Now, we must show that Υ has a fixed point. By using the rectangular inequality, property of α-regularity and ψ, we obtain

$$\mathfrak{p}_t(u, \Upsilon u) \leq s\mathfrak{p}_t(u, \theta_{\check{n}}) + s\mathfrak{p}_t\left(\theta_{\check{n}}, \theta_{\check{n}+1}\right) + s\mathfrak{p}_t(\theta_{\check{n}+1}, \Upsilon u)$$

$$\begin{aligned}
&= s\mathfrak{p}_t(u,\theta_{\check{n}}) + s\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right) + s\mathfrak{p}_t(\Upsilon\theta_{\check{n}},\Upsilon u)\\
&\leq s\mathfrak{p}_t(u,\theta_{\check{n}}) + s\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right) + s\alpha(\theta_{\check{n}},u)\mathfrak{p}_t(\theta_{\check{n}+1},\Upsilon u)\\
&\leq s\mathfrak{p}_t(u,\theta_{\check{n}}) + s\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right) + s\psi(M(\theta_{\check{n}},u)),
\end{aligned}$$

where

$$\begin{aligned}
M(\theta_{\check{n}},u) &= \max\left\{\mathfrak{p}_t(\theta_{\check{n}},u), \frac{\mathfrak{p}_t(\theta_{\check{n}},\Upsilon\theta_{\check{n}})\mathfrak{p}_t(\theta_{\check{n}},\Upsilon u)}{1+\mathfrak{p}_t(\theta_{\check{n}},\Upsilon u)+\mathfrak{p}_t(u,\Upsilon\theta_{\check{n}})},\right.\\
&\qquad \left.\mathfrak{p}_t(\theta_{\check{n}},\Upsilon\theta_{\check{n}}), \mathfrak{p}_t(u,\Upsilon u)\right\}\\
&= \max\left\{\mathfrak{p}_t(\theta_{\check{n}},u), \frac{\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right)\mathfrak{p}_t(\theta_{\check{n}},\Upsilon u)}{1+\mathfrak{p}_t(\theta_{\check{n}},\Upsilon u)+\mathfrak{p}_t(u,\theta_{\check{n}+1})},\right.\\
&\qquad \left.\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right), \mathfrak{p}_t(u,\Upsilon u)\right\}\\
&= \max\{\mathfrak{p}_t(\theta_{\check{n}},u), \mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right), \mathfrak{p}_t(u,\Upsilon u)\}.
\end{aligned}$$

Now, the following cases arise accordingly:

Case 1: If $M(\theta_{\check{n}},u) = \mathfrak{p}_t(\theta_{\check{n}},u)$, then

$$\begin{aligned}
\mathfrak{p}_t(u,\Upsilon u) &\leq s\mathfrak{p}_t(u,\theta_{\check{n}}) + s\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right) + s\psi\mathfrak{p}_t(\theta_{\check{n}},u)\\
&< s\mathfrak{p}_t(u,\theta_{\check{n}}) + s\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right) + \mathfrak{p}_t(\theta_{\check{n}},u).
\end{aligned}$$

Taking the limit as $\check{n}\to\infty$ in the preceding inequality, we obtain

$$\mathfrak{p}_t(u,\Upsilon u)\leq 0.$$

Case 2: If $M(\theta_{\check{n}},u) = \mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right)$, then

$$\begin{aligned}
\mathfrak{p}_t(u,\Upsilon u) &\leq s\mathfrak{p}_t(u,\theta_{\check{n}}) + s\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right) + s\psi\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right)\\
&< s\mathfrak{p}_t(u,\theta_{\check{n}}) + s\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right) + \mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right).
\end{aligned}$$

If we take limit $\check{n}\to\infty$ in the preceding inequality, we get

$$\mathfrak{p}_t(u,\Upsilon u)\leq 0.$$

Case 3: If $M(\theta_{\check{n}},u) = \mathfrak{p}_t(u,\Upsilon u)$, then

$$\begin{aligned}
\mathfrak{p}_t(u,\Upsilon u) &\leq s\mathfrak{p}_t(u,\theta_{\check{n}}) + s\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right) + s\psi\mathfrak{p}_t(u,\Upsilon u)\\
&< s\mathfrak{p}_t(u,\theta_{\check{n}}) + s\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right) + \mathfrak{p}_t(u,\Upsilon u).
\end{aligned}$$

If we take the limit n$\to\infty$ in the preceding inequality, we get

$$\mathfrak{p}_t(u,\Upsilon u)\leq 0.$$

Clearly, $\mathfrak{p}_t(u,\Upsilon u)\geq 0$ implies that $\mathfrak{p}_t(u,\Upsilon u) = 0$. Also,

$$\begin{aligned}
\mathfrak{p}_t(\Upsilon u,u) &\leq s\mathfrak{p}_t(\Upsilon u,\theta_{\check{n}}) + s\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right) + s\mathfrak{p}_t(\theta_{\check{n}+1},u)\\
&= s\mathfrak{p}_t(\Upsilon u,\Upsilon\theta_{\check{n}-1}) + s\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right) + s\mathfrak{p}_t(\Upsilon\theta_{\check{n}+1},u)\\
&\leq s\alpha(u,\theta_{\check{n}-1})\mathfrak{p}_t(Tu,\Upsilon\theta_{\check{n}-1}) + s\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right) + s\mathfrak{p}_t(\theta_{\check{n}+1},u)
\end{aligned}$$

$$\leq s\psi(M(u,\theta_{\check{n}-1})) + s\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right) + s\mathfrak{p}_t(\theta_{\check{n}+1},u),$$

where

$$\begin{aligned}
M(u,\theta_{\check{n}-1}) &= \max\left\{\mathfrak{p}_t(u,\theta_{\check{n}-1}), \frac{\mathfrak{p}_t(u,\Upsilon u)\mathfrak{p}_t(u,\Upsilon\theta_{\check{n}-1})}{1+\mathfrak{p}_t(u,\Upsilon\theta_{\check{n}-1}}) + \mathfrak{p}_t(\Upsilon\theta_{\check{n}-1},u),\right.\\
&\qquad \left.\mathfrak{p}_t(u,\Upsilon u), \mathfrak{p}_t(\theta_{\check{n}-1},\Upsilon\theta_{\check{n}-1})\right\}\\
&= \max\left\{\mathfrak{p}_t(u,\theta_{\check{n}-1}), \frac{\mathfrak{p}_t(u,\Upsilon u)\mathfrak{p}_t(u,\theta_{\check{n}})}{1+\mathfrak{p}_t(u,\theta_{\check{n}})+\mathfrak{p}_t(\theta_{\check{n}},u)},\right.\\
&\qquad \left.\mathfrak{p}_t(u,\Upsilon u), \mathfrak{p}_t\left(\theta_{\check{n}-1},\theta_{\check{n}}\right)\right\}\\
&= \max\{\mathfrak{p}_t(u,\theta_{\check{n}-1}), \mathfrak{p}_t(u,\Upsilon u), \mathfrak{p}_t\left(\theta_{\check{n}-1},\theta_{\check{n}}\right).
\end{aligned}$$

Now, we consider the following three cases:

Case 1: If $M(u,\theta_{\check{n}-1}) = \mathfrak{p}_t(u,\theta_{\check{n}-1})$, then

$$\begin{aligned}
\mathfrak{p}_t(\Upsilon u,u) &\leq s\psi\mathfrak{p}_t(u,\theta_{\check{n}-1} + s\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right) + s\mathfrak{p}_t(\theta_{\check{n}+1},u)\\
&< \mathfrak{p}_t(u,\theta_{\check{n}-1} + s\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right) + s\mathfrak{p}_t(\theta_{\check{n}+1},u).
\end{aligned}$$

Letting $\check{n}\to\infty$, the above inequality becomes $\mathfrak{p}_t(\Upsilon u,u)\leq 0$.

Case 2: If $M(u,\theta_{\check{n}-1}) = \mathfrak{p}_t\left(\theta_{\check{n}-1},\theta_{\check{n}}\right)$ then

$$\begin{aligned}
\mathfrak{p}_t(\Upsilon u,u) &\leq s\psi\mathfrak{p}_t\left(\theta_{\check{n}-1},\theta_{\check{n}}\right) + s\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right) + s\mathfrak{p}_t(\theta_{\check{n}+1},u)\\
&< \mathfrak{p}_t\left(\theta_{\check{n}-1},\theta_{\check{n}}\right) + s\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right) + s\mathfrak{p}_t(\theta_{\check{n}+1},u)
\end{aligned}$$

now, we take limit $\check{n}\to\infty$, we get

$$\mathfrak{p}_t(\Upsilon u,u)\leq 0.$$

Case 3: If $M(u,\theta_{\check{n}-1}) = \mathfrak{p}_t(u,\Upsilon u)$ then

$$\begin{aligned}
\mathfrak{p}_t(\Upsilon u,u) &\leq s\psi\mathfrak{p}_t(u,\Upsilon u) + s\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right) + s\mathfrak{p}_t(\theta_{\check{n}+1},u)\\
&< \mathfrak{p}_t(u,\Upsilon u) + s\mathfrak{p}_t\left(\theta_{\check{n}},\theta_{\check{n}+1}\right) + s\mathfrak{p}_t(\theta_{\check{n}+1},u).
\end{aligned}$$

Now, we take limit $\check{n}\to\infty$, we get $\mathfrak{p}_t(\Upsilon u,u)\leq 0$ which gives $\mathfrak{p}_t(\Upsilon u,u)=0$. Hence,

$$\mathfrak{p}_t(u,\Upsilon u) = \mathfrak{p}_t(\Upsilon u,u) = 0.$$

So $\Upsilon u = u$. Therefore, Υ has a fixed point, namely u. □

4. Application

We establish an existence theorem for the solution of the integral equation

$$\theta(\tau) = \int_0^1 \mathcal{K}(\tau,r,\theta(r))dr,$$

where $\mathcal{K} : [0,1] \times [0,1] \times \mathcal{R} \to \mathcal{R}$ is continuous in all its variables. Let $\Xi = \mathcal{C}([0,1], \mathcal{R})$ denote the set of real continuous functions defined in the interval $[0,1]$. We now introduce a rectangular quasi b-metric $d : \Xi \times \Xi \to [0, \infty)$ defined by

$$\delta(\theta, y) = \begin{cases} \|(\theta - y)\|_\infty + \min\{\|\theta\|_\infty, \|y\|_\infty\}, & \text{if } \theta \neq y \\ 0, & \text{if } \theta = y, \end{cases}$$

where $\|\theta\|_\infty = \max \theta_{r \in [0,1]} |\theta(\tau)| \ \forall \theta \in \Xi$. Then, (Ξ, d) is a complete rectangular quasi-b metric space with $s = \frac{3}{2}$. Now we proceed to establish the following result.

Theorem 3. *Suppose*

(1) *let $\mathcal{K} : [0,1] \times [0,1] \times \mathcal{R} \to \mathcal{R}$ is continuous and non-decreasing in the third variable;*
(2) *$\Upsilon : \Xi \to \Xi$ defined by $\Upsilon(\theta(\tau)) = \int_0^1 \mathcal{K}(\tau, r, \theta(r))dr$;*
(3) *$\mathcal{K}$ satisfies the Lipschitz conditions, $\forall \theta, y \in \Xi$ with $\theta(\tau) \leq y(\tau)$, $\forall \tau \in [0,1]$ and for each $r \in [0,1]$, we have*

$$|\mathcal{K}(\tau, r, \theta) - \mathcal{K}(\tau, r, y)| \leq \frac{L|\theta - y|}{1 + |\theta| + |y|}, \text{ for } \tau, r \in [0,1] \text{ and } \theta, y \in \mathcal{R};$$

(4) *there exist $\theta_0 \in \Xi$ such that for all $\tau \in [0,1], \theta_0 \leq \int_0 \mathcal{K}(\tau, r, \theta(r))dr$;*
(5) *Consider a distance function $\mathfrak{p}_t : \Xi \times \Xi \to [0, \infty)$ defined by*

$$\mathfrak{p}_t(\theta, y) = \int_0^1 \frac{|\theta(\tau) - y(\tau)|}{1 + |\theta(\tau)| + |y(\tau)|}, \quad \text{for } \tau, r \in [0,1].$$

Therefore, the integral equation admits a solution in Ξ.

Proof. Let $\alpha : \Xi \times \Xi \to [0.\infty)$ defined by $\alpha(\theta, y) = 1$ for all $\theta, y \in \Xi$ and $\psi(\tau) = Lt$ with $0 < L < 1$. For $\theta, y \in \Xi$, we define $\theta \leq y$ iff $\theta(\tau) \leq y(\tau)$ $\forall \tau \in [0,1]$, where $\leq$ denotes usual order of real numbers. Let $\theta, y \in \Xi$, such that $\alpha(\theta, y) \geq 1$, so $\theta \leq y$, hence $\theta(\tau) \leq y(\tau)$ $\forall \tau \in [0,1]$. Thus, by condition (3)

$$\begin{aligned} |\Upsilon(\theta(\tau)) - \Upsilon(y(\tau))| &\leq \int_0^1 |\mathcal{K}(\tau, r, \theta(r)) - \mathcal{K}(\tau, r, y(r))|dr \\ &\leq L \int_0^1 \frac{|\theta(\tau) - y(\tau)|}{1 + |\theta(\tau)| + |y(\tau)|}. \end{aligned}$$

Taking the supremum over $\tau \in [0,1]$, we get

$$\begin{aligned} &\|\Upsilon\theta(\tau) - \Upsilon y(\tau)\|_\infty \leq L \int_0^1 \frac{|\theta(\tau) - y(\tau)|}{1 + |\theta(\tau)| + |y(\tau)|} \\ &\|\Upsilon\theta(\tau) - \Upsilon y(\tau)\|_\infty \leq L\mathfrak{p}_\tau(\theta, y) \\ &\mathfrak{p}_t(\Upsilon\theta, \Upsilon y) \leq L\mathfrak{p}_t(\theta, y) \leq \psi M(\theta, y). \end{aligned}$$

then

$$\begin{aligned}|\Upsilon\theta(\tau)| &\leq \int_0^1 |\mathcal{K}(\tau, r, \theta(r))|dr \\ &\leq L\frac{|\theta|}{1+|\theta|} \\ &\leq L\|\theta\|_\infty.\end{aligned}$$

Now, for all $\theta, y \in \Xi$ such that $\theta \leq y$,

$$\begin{aligned}\delta(\Upsilon\theta, \Upsilon y) &= \|(\Upsilon\theta - \Upsilon y)\|_\infty + \min\{\|\Upsilon\theta\|_\infty, \|\Upsilon y\|_\infty\} \\ &\leq k\|(\theta - y)\|_\infty + \min\{\|\theta\|_\infty, \|y\|_\infty\} \\ &\leq k\delta(\theta, y) = \psi(\delta(\theta, y)) \\ &\leq \psi M(\theta, y).\end{aligned}$$

Since the kernel $\mathcal{K}$ is non-decreasing with respect to its third variable, it follows that $\forall\theta, y \in \Xi$ with $\theta \leq y$, we have $\Upsilon\theta(\tau) \leq \Upsilon y(\tau)$ for each $\tau \in [0, 1]$, i.e., if $\alpha(\theta, y) \geq 1$, we get $\alpha(\Upsilon\theta, \Upsilon y) \geq 1$. Furthermore, from condition (4), there exists a function $\theta_0 \in \Xi$ such that $\alpha(\theta_0, \Upsilon\theta_0) \geq 1$, $\alpha(\Upsilon\theta_0, \theta_0) \geq 1$, $\alpha(\theta_0, \Upsilon^2\theta_0) \geq 1$ and $\alpha(\Upsilon^2\theta_0, \Upsilon\theta_0) \geq 1$. Therefore, all the hypotheses of the theorem are fulfilled with $s = \frac{3}{2}$ in the complete rectangular quasi b-metric space. Hence, the integral equation admits a solution and has a unique fixed point. □

5. Conclusion

In this paper, we obtained fixed point results in a generalized b-metric space equipped with a w-distance function. By employing $\mathfrak{p}_t(\alpha, \psi)$- contraction mapping, the presented theorems extend existing results and are supported through examples. The applicability of our main result was demonstrated by solving a nonlinear integral equation, showing that the w-distance framework provides a more flexible tool than the usual metric. These findings broaden the scope of fixed point theory in rectangular quasi b-metric spaces and motivate further study in nonlinear analysis.

6. Future Scope

The generalized fixed point framework developed in this paper can be extended to other distance structures, such as partial metric and G-metric settings. The results may also be applied to a wider class of nonlinear problems, including differential and integral equations with variable kernels. Further, research can explore weaker contraction conditions, multivalued mappings and stability analysis to deepen the

applicability of w-distance-based fixed point theory in nonlinear analysis.

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