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Certain Hermitian Toeplitz Determinants for Classes of Starlike Functions Subordinated to Sine Function

Ankit Kumar Rathore[1], Amit Soni[2], Deepak Bansal[3*] and Ravider Krishna Raina[4]

Abstract. Let $\mathcal{S}^*_{\sin}$ be the class of normalized analytic functions f defined on the unit disk such that $zf'(z)/f(z)$ lies in an eight-shaped region in the right-half plane which is the image of the unit disk under an entire function $1+\sin z$. For this class we determine the sharp estimates for certain Hermitian Toeplitz determinants whose elements are coefficients of starlike functions governed by the subordination relation to the Sine function.

1. Introduction and Preliminaries

Given $r>0$, let $\mathbb{D}_r := \{z\in\mathbb{C} : |z|<r\}$, $\mathbb{T}_r := \{z\in\mathbb{C} : |z|=r\}$, $\mathbb{D} := \mathbb{D}_1$, $\mathbb{T} := \mathbb{T}_1$ and $\overline{\mathbb{D}} := \{z\in\mathbb{C} : |z|\leq 1\}$. Also, let $\mathcal{H}$ be the class of all analytic functions in $\mathbb{D}$ and $\mathcal{A}$ be its subclass of functions f normalized by $f(0)=0$ and $f'(0)=1$, i.e., of the form

$$f(z)=\sum_{n=1}^{\infty} a_n z^n, \quad (\text{where } a_1=1 \text{ and } z\in\mathbb{D}.) \tag{1.1}$$

Further, let $\mathcal{S}$ be the subclass of $\mathcal{A}$, consisting of all univalent functions and $\mathcal{S}^*$ be the subclass of $\mathcal{S}$ of starlike functions ([6]), i.e., $f\in\mathcal{S}^*$ if $f\in\mathcal{A}$ and

$$\Re\frac{zf'(z)}{f(z)}>0, \quad z\in\mathbb{D}.$$

An analytic function f is said to be subordinate to another analytic function g, written as $f\prec g$, if there is an analytic function w with

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$|w(z)| \leq |z|$ and $w(0) = 0$ such that $f(z) = g(w(z))$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$.

Let $\mathcal{S}^*_{\sin}$ denote the class of functions defined by

$$\mathcal{S}^*_{\sin} = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \sin z \right\}, \tag{1.2}$$

then it follows that the function $f \in \mathcal{S}^*_{\sin}$ is such that $zf'(z)/f(z)$ lies in an eight-shaped region in the right-half plane which is the image of the unit disk under the function defined by $\phi(z) = 1 + \sin z$c(see Figure (a)). This class of functions was introduced by Cho et al. [2]. Using this definition, Cho et al. [2] gave the following structural formula: A function $f \in \mathcal{S}^*_{\sin}$ if and only if there exists an analytic function q, satisfying $q(z) \prec q_0(z) = 1 + \sin z$ $(z \in \mathbb{D})$, such that

$$f(z) = z \exp\left(\int_0^z \frac{q(t) - 1}{t} dt \right). \tag{1.3}$$

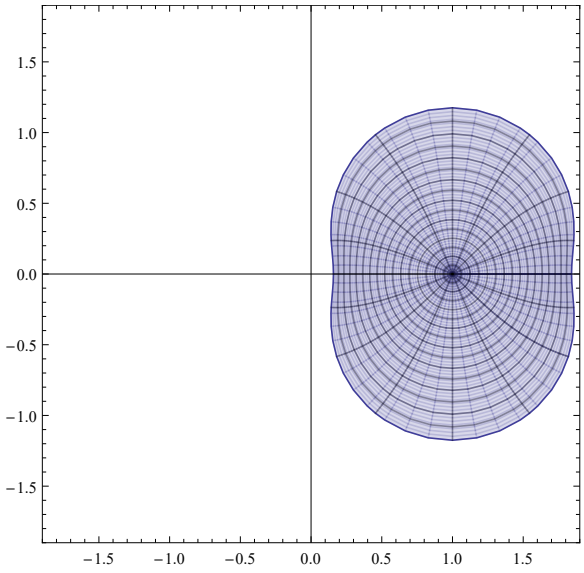


FIGURE 1. Image of $\mathbb{D}$ under $\phi(z) = 1 + \sin z$.

We give here a few examples of the functions which belong to the class $\mathcal{S}^*_{\sin}$. For this purpose, we set

$$\begin{aligned} &q_1(z) = 1 + \tfrac{z}{2}, \quad q_3(z) = \tfrac{5+3z}{5+z} \\ &q_2(z) = \tfrac{4+2z}{4+z}, \quad q_4(z) = 1 + az \quad (|a| \leq 1). \end{aligned} \tag{1.4}$$

Since q_0 is univalent in $\mathbb{D}$, $q_i(0) = q_0(0)$ $(i = 1, 2, 3, 4)$ and $q_i(\mathbb{D}) \subset q_0(\mathbb{D})$, it follows that each $q_i \prec q_0$. Hence from the structural formula (1.3), we obtain the following functions in the class $\mathcal{S}^*_{\sin}$ that correspond to each of the q_i 's $(i = 1, 2, 3, 4)$ given by (1.4), respectively (see also [2]):

$$\begin{aligned} &f_1(z) = ze^{z/2}, \quad f_3(z) = z\big(1 + \tfrac{z}{5}\big)^2 \\ &f_2(z) = z + \tfrac{z^2}{4}, \quad f_4(z) = ze^{az} \ (|a| \leqq 1). \end{aligned} \tag{1.5}$$

In particular, if $q(z) = q_0(z) = 1 + \sin z$ in (1.3), we then have

$$\Phi(z) = z \exp\left(\int_0^z \frac{\sin t}{t} dt\right). \tag{1.6}$$

We briefly mention the computation for evaluating the right-hand side of (1.6). We begin by applying Maclaurin's series to the function $\sin t$ and then using term-wise integration, we get

$$\int_0^z \frac{\sin t}{t} dt = z - \frac{z^3}{18} + \frac{z^5}{600} + \cdots = A(z). \tag{1.7}$$

Using (1.7) and collecting from the exponential series $\exp(A(z))$ the terms up to, say, z^5, we find after some elementary simplification that (1.3) finally gives

$$\begin{aligned} \Phi(z) &= z \exp\left(\int_0^z \frac{\sin t}{t} dt\right) \\ &= z + z^2 + \frac{z^3}{2} + \frac{z^4}{9} - \frac{z^5}{72} - \cdots . \end{aligned} \tag{1.8}$$

Using the definition of the class $\mathcal{S}^*_{\sin}$, one can easily deduce the following relationships between the initial coefficients of class $\mathcal{S}^*_{\sin}$ and class $\mathcal{P}$ consisting of functions whose power series representation is of the form (1.1)(see for details [7]):

$$a_2 = \frac{1}{2} c_1, \tag{1.9}$$

$$a_3 = \frac{1}{4} c_2, \tag{1.10}$$

$$a_4 = \frac{1}{6} c_3 - \frac{1}{144} c_1^3 - \frac{1}{24} c_1 c_2. \tag{1.11}$$

Applying the results mentioned in [5], some sharp upper bounds for $|a_2|$, $|a_3|$ and $|a_4|$ have been obtained in [7] and [2]. These sharp bounds are contained in the following lemma:

Lemma 1.1. *If $f \in \mathcal{S}^*_{\sin}$ has the series expansion of the form* (1.1)*, then*

$$|a_2| \leq 1, \tag{1.12}$$

$$|a_3| \leq \frac{1}{2}, \tag{1.13}$$

$$|a_4| \leq \frac{1}{3}. \tag{1.14}$$

Toeplitz determinants play an important role in several branches of mathematics and have many applications. For a summary of applications of Toeplitz determinants in a wide range of areas of pure and applied mathematics, we refer to [3, 9]. Recently, Thomas and Halim [8] introduced the symmetric Toeplitz determinant $T_q(n)[f]$ for analytic functions f of the form (1.1) by defining:

$$T_q(n)[f] := \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_n & \dots & a_{n+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \dots & a_n \end{vmatrix},$$

where $q, n \in \mathbb{N}$, and the authors in their paper investigated sharp upper bounds of the Toeplitz determinant $|T_q(n)|$ for functions belonging to the classes $\mathcal{S}^*$ and $\mathcal{K}$ for some small values of n and q. Further, in the same direction, Ali et al. [1] have found estimates for $|T_2(n)|$, $|T_3(1)|$, $|T_3(2)|$ and $|T_2(3)|$ for certain other subclasses of $\mathcal{A}$. In a recent paper by Kowalczyk et al. [4], the authors have introduced the concept of Hermitian Toeplitz determinant $T_{q,n}(f)$ for analytic functions of the form (1.1) for $q, n \in \mathbb{N}$ by defining

$$T_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ \overline{a}_{n+1} & a_n & \dots & a_{n+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a}_{n+q-1} & \overline{a}_{n+q-2} & \dots & a_n \end{vmatrix}, \tag{1.15}$$

(where $\overline{a}_k$ means conjugate of a_k) and investigated certain Hermitian Toeplitz determinants. Hermitian Toeplitz matrices play an important role in functional analysis, applied mathematics as well as in physics and technical sciences. In this paper, we obtain the sharp upper and lower bounds for the following:

$$T_{2,1}(f) := \begin{vmatrix} 1 & a_2 \\ \overline{a}_2 & 1 \end{vmatrix} = 1 - |a_2|^2, \tag{1.16}$$

$$T_{2,2}(f) := \begin{vmatrix} a_2 & a_3 \\ \overline{a}_3 & a_2 \end{vmatrix} = a_2^2 - |a_3|^2, \tag{1.17}$$

$$T_{2,3}(f) := \begin{vmatrix} a_3 & a_4 \\ \overline{a}_4 & a_3 \end{vmatrix} = a_3^2 - |a_4|^2, \tag{1.18}$$

$$T_{3,1}(f) := \begin{vmatrix} 1 & a_2 & a_3 \\ \overline{a}_2 & 1 & a_2 \\ \overline{a}_3 & \overline{a}_2 & 1 \end{vmatrix} = 2\Re\left(a_2^2\overline{a}_3\right) - 2\left|a_2\right|^2 - \left|a_3\right|^2 + 1, \tag{1.19}$$

and

$$\begin{aligned} T_{4,1}(f) &:= \begin{vmatrix} 1 & a_2 & a_3 & a_4 \\ \overline{a}_2 & 1 & a_2 & a_3 \\ \overline{a}_3 & \overline{a}_2 & 1 & a_2 \\ \overline{a}_4 & \overline{a}_3 & \overline{a}_2 & 1 \end{vmatrix} \\ &= 1 - 2\Re\left(a_2^3\overline{a}_4\right) + 4\Re\left(a_2^2\overline{a}_3\right) - 2\Re\left(a_2\overline{a}_3^2 a_4\right) + 4\Re\left(a_2 a_3\overline{a}_4\right) \\ &\quad + \left|a_2\right|^4 - 3\left|a_2\right|^2 + \left|a_3\right|^4 - 2\left|a_3\right|^2 + \left|a_2\right|^2\left|a_4\right|^2 - 2\left|a_2\right|^2\left|a_3\right|^2 - \left|a_4\right|^2. \end{aligned} \tag{1.20}$$

Let $\mathcal{P}$ denote the class of all functions $p \in \mathcal{H}$ of the form:

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad (z \in \mathbb{D}) \tag{1.21}$$

satisfying the condition that $\Re\{p(z)\} > 0$ in $\mathbb{D}$.

To prove our main results, we need the following preliminary result for functions in $\mathcal{P}$:

Lemma 1.2. ([6, p. 166]) *If $p \in \mathcal{P}$ is of the form* (1.21)*, then*

$$|c_n| \leq 2, \quad n \in \mathbb{N}. \tag{1.22}$$

Moreover,

$$c_1 = 2\zeta_1, \tag{1.23}$$

$$c_2 = 2\zeta_1^2 + 2\left(1 - \left|\zeta_1\right|^2\right)\zeta_2, \tag{1.24}$$

$$\begin{aligned} c_3 &= 2\zeta_1^3 + 4\left(1 - \left|\zeta_1\right|^2\right)\zeta_1\zeta_2 - 2\left(1 - \left|\zeta_1\right|^2\right)\overline{\zeta_1}\zeta_2^2 \\ &\quad + 2\left(1 - \left|\zeta_1\right|^2\right)\left(1 - \left|\zeta_2\right|^2\right)\zeta_3, \end{aligned} \tag{1.25}$$

for some $\zeta_i \in \overline{\mathbb{D}}$, $i \in \{1, 2, 3\}$.
For $\zeta_1 \in \mathbb{T}$, there is a unique function $f \in \mathcal{P}$ with c_1 as in (1.22)*, namely,*

$$p(z) = \frac{1 + \zeta_1 z}{1 - \zeta_1 z}, \quad z \in \mathbb{D}. \tag{1.26}$$

For $\zeta_1 \in \mathbb{D}$ and $\zeta_2 \in \mathbb{T}$, there is a unique function $f \in \mathcal{P}$ with c_1 and c_2 as in (1.22) *and* (1.23)*, namely,*

$$p(z) = \frac{1 + \left(\overline{\zeta_1}\zeta_2 + \zeta_1\right)z + \zeta_2 z^2}{1 + \left(\overline{\zeta_1}\zeta_2 - \zeta_1\right)z - \zeta_2 z^2}, \quad z \in \mathbb{D}. \tag{1.27}$$

For $\zeta_1, \zeta_2 \in \mathbb{D}$ and $\zeta_3 \in \mathbb{T}$, there is a unique function $f \in \mathcal{P}$ with c_1, c_2 and c_3 as in (1.22), (1.23) *and* (1.24), *namely,*

(1.28)
$$p(z) = \frac{1 + \left(\overline{\zeta_2}\zeta_3 + \overline{\zeta_1}\zeta_2 + \zeta_1\right) z + \left(\overline{\zeta_1}\zeta_3 + \zeta_1\overline{\zeta_2}\zeta_3 + \zeta_2\right) z^2 + \zeta_3 z^3}{1 + \left(\overline{\zeta_2}\zeta_3 + \overline{\zeta_1}\zeta_2 - \zeta_1\right) z + \left(\overline{\zeta_1}\zeta_3 - \zeta_1\overline{\zeta_2}\zeta_3 - \zeta_2\right) z^2 - \zeta_3 z^3}, \quad z \in \mathbb{D}.$$

2. Main Results

In the following results, we investigate the sharp bounds for $T_{q,n}(f)$ for $n = 1$ and $q = 2, 3$ for functions belonging to class $\mathcal{S}^*_{\sin}$ defined by (1.2).

Theorem 2.1. *If $f \in \mathcal{S}^*_{\sin}$, then*

$$0 \leq T_{2,1}(f) \leq 1. \tag{2.1}$$

Both inequalities are sharp.

Proof. In view of (1.12)), we note that $0 \leq |a_2| \leq 1$. Consequently, $0 \leq |a_2|^2 \leq 1$. This implies that $0 \geq -|a_2|^2 \geq -1$, and hence $1 \geq 1 - |a_2|^2 \geq 0$. The desired bounds (2.1) follow at once by using the defining relation (1.16).

It is easy to see that the lower bound in (2.1) is attained for the function $\Phi(z)$ given by (1.8) and its upper bound is attained for the function given by

$$\Phi_1(z) = z \exp\left(\int_0^z \frac{\sin(t^2)}{t} dt\right) = z + \frac{z^3}{2} + \cdots, \tag{2.2}$$

which follows in the same manner as in the computation for (1.6). □

We next compute the upper and lower bounds of $T_{2,2}(f)$.

Theorem 2.2. *If $f \in \mathcal{S}^*_{\sin}$, then*

$$0 \leq |T_{2,2}(f)| \leq \frac{5}{4}. \tag{2.3}$$

Both inequalities are sharp.

Proof. From the definition (1.17), we have

$$0 \leq |T_{2,2}(f)| = \left|-a_2^2 + |a_3|^2\right| \leq \frac{5}{4}. \tag{2.4}$$

It is easy to observe that lower bound (2.3) is attained for the function $\Phi_2(z)$ obtained in the same manner as explained in (1.6) and is given by

$$\Phi_2(z) = z \exp\left(\int_0^z \frac{\sin(t^3)}{t} dt\right) = z + \frac{z^4}{3} + \cdots. \tag{2.5}$$

The upper bound of (2.3) is attained for the function given by

(2.6)
$$\Phi_3(z) = z\exp\left(\int_0^z \frac{\sin(it)}{t}dt\right) = z + iz^2 - \frac{z^3}{2} - \frac{iz^4}{9} - \frac{z^5}{72} - \frac{4iz^6}{225} + \cdots .$$

□

We next compute the upper and lower bounds of $T_{2,3}(f)$.

Theorem 2.3. *If $f \in \mathcal{S}^*_{\sin}$, then*

$$0 \leq |T_{2,3}(f)| \leq \frac{13}{36}. \tag{2.7}$$

The lower bound is sharp.

Proof. From the definition (1.18), we have on using (1.13) and (1.14):

$$0 \leq |T_{2,3}(f)| = \left|-a_3^2 + |a_4|^2\right| \leq |-a_3|^2 + |a_4|^2 \leq \frac{13}{36}. \tag{2.8}$$

The lower bound is attained for the function given by

$$\Phi_4(z) = z\exp\left(\int_0^z \frac{\sin(t^4)}{t}dt\right) = z + \frac{z^5}{4} + \cdots . \tag{2.9}$$

□

Our next two results (Theorems 2.4 and 2.5) give the upper and lower bounds of $T_{3,1}(f)$.

Theorem 2.4. *If $f \in \mathcal{S}^*_{\sin}$, then*

$$T_{3,1}(f) \leq 1. \tag{2.10}$$

The inequality is sharp.

Proof. Since from Lemma 1.1, we have $|a_2| \leq 1$, $|a_3| \leq \frac{1}{2}$ and further $\Re\left(a_2^2\overline{a}_3\right) \leq \left|a_2^2 a_3\right|$. Therefore from (1.19), we obtain

$$T_{3,1}(f) \leq F_1(|a_2|, |a_3|), \tag{2.11}$$

where

$$F_1(x,y) = 2x^2y - 2x^2 - y^2 + 1, \quad (x,y) \in [0,1] \times \left[0, \frac{1}{2}\right]. \tag{2.12}$$

Observe that the following system of equation has no solution in $(0,1) \times \left(0, \frac{1}{2}\right)$:

$$\frac{\partial F_1}{\partial x} = 4x(y-1) = 0, \qquad \frac{\partial F_1}{\partial y} = 2(x^2 - y) = 0.$$

On the boundary of $[0,1] \times \left[0, \frac{1}{2}\right]$, we have

(i) $F_1(0,y) = 1 - y^2 \leq 1, \quad y \in \left[0, \frac{1}{2}\right]$,
(ii) $F_1(1,y) = 2y - y^2 - 1 \leq -\frac{1}{4}, y \in \left[0, \frac{1}{2}\right]$,
(iii) $F_1(x,0) = 1 - 2x^2 \leq 1, \quad x \in [0,1]$,

(iv) $F_1(x, \frac{1}{2}) = -\frac{3}{2} - x^2 \leq -\frac{3}{2}, \quad x \in [0, 1]$.

Therefore, the inequality $F_1(x, y) \leq 1$ holds for all $(x, y) \in [0, 1] \times \left[0, \frac{1}{2}\right]$, which in view of (2.11) establishes (2.10).

Equality in (2.10) holds for the function $\Phi_2(z)$ defined by (2.5). □

Theorem 2.5. *If $f \in \mathcal{S}^*_{\sin}$, then*

$$T_{3,1}(f) \geq -\frac{1}{4}. \tag{2.13}$$

The inequality is sharp.

Proof. On substituting (1.23) and (1.24) into (1.9) and (1.10) , we have

$$a_2 = \zeta_1, \qquad a_3 = \frac{1}{2}\left(\zeta_1^2 + \left(1 - |\zeta_1|^2\right)\zeta_2\right). \tag{2.14}$$

Using now the equation (2.14), we can express

$$\begin{aligned}\Re\left(a_2^2\overline{a}_3\right) &= \frac{1}{2}\Re\left(\zeta_1^2\left(\overline{\zeta_1^2} + \left(1 - |\zeta_1|^2\right)\overline{\zeta_2}\right)\right) \\ &= \frac{1}{2}\left[|\zeta_1|^4 + \left(1 - |\zeta_1|^2\right)\Re\left(\zeta_1^2\overline{\zeta_2}\right)\right],\end{aligned}$$

and

$$\begin{aligned}|a_3|^2 &= \frac{1}{4}\left|\zeta_1^2 + \left(1 - |\zeta_1|^2\right)\zeta_2\right|^2 \\ &= \frac{1}{4}\left[|\zeta_1|^4 + 2\left(1 - |\zeta_1|^2\right)\Re\left(\zeta_1^2\overline{\zeta_2}\right) + \left(1 - |\zeta_1|^2\right)^2|\zeta_2|^2\right].\end{aligned}$$

It follows then from (1.19) that

$$\begin{aligned}T_{3,1}(f) &= 2\Re\left(a_2^2\overline{a}_3\right) - 2|a_2|^2 - |a_3|^2 + 1 \\ &= \left[|\zeta_1|^4 + \left(1 - |\zeta_1|^2\right)\Re\left(\zeta_1^2\overline{\zeta_2}\right)\right] - 2|\zeta_1|^2 \\ &\quad - \frac{1}{4}\left[|\zeta_1|^4 + 2\left(1 - |\zeta_1|^2\right)\Re\left(\zeta_1^2\overline{\zeta_2}\right) + \left(1 - |\zeta_1|^2\right)^2|\zeta_2|^2\right] + 1 \\ &= \frac{1}{4}\Big[4 + 3|\zeta_1|^4 - 8|\zeta_1|^2 - \left(1 - |\zeta_1|^2\right)^2|\zeta_2|^2 + 2\left(1 - |\zeta_1|^2\right)\Re\left(\zeta_1^2\overline{\zeta_2}\right)\Big] \\ &= \frac{1}{4}\left(\psi_1 + \psi_2\right),\end{aligned}$$

where

$$\begin{aligned}\psi_1 &= 4 + 3|\zeta_1|^4 - 8|\zeta_1|^2 - \left(1 - |\zeta_1|^2\right)^2|\zeta_2|^2, \\ \psi_2 &= 2\left(1 - |\zeta_1|^2\right)\Re\left(\zeta_1^2\overline{\zeta_2}\right).\end{aligned} \tag{2.15}$$

(i) If $\zeta_1 = 0$, then

$$\psi_1 = 4 - |\zeta_2|^2, \qquad \psi_2 = 0,$$

and therefore

$$T_{3,1}(f) = \frac{1}{4}\left(4 - |\zeta_2|^2\right) \geq \frac{3}{4}. \tag{2.16}$$

(ii) If $\zeta_2 = 0$, then

$$\psi_1 = 4 + 3|\zeta_1|^4 - 8|\zeta_1|^2, \quad \psi_2 = 0,$$

and therefore

$$T_{3,1}(f) = \frac{1}{4}\left[4 + 3|\zeta_1|^4 - 8|\zeta_1|^2\right] \geq -\frac{1}{4}. \tag{2.17}$$

(iii) Suppose that $\zeta_1\zeta_2 \neq 0$. Then $\zeta_1 = re^{i\theta}$ and $\zeta_2 = se^{i\psi}$, where $r, s \in (0, 1]$ and $\theta, \psi \in [0, 2\pi)$. We therefore have the following

$$\psi_2 = 2(1 - r^2)r^2 s\cos(2\theta - \psi) \geq -2(1 - r^2)r^2 s. \tag{2.18}$$

Hence, from (2.15) and (2.18) for $(r, s) \in (0, 1]\times(0, 1]$, we obtain

$$T_{3,1}(f) = \frac{1}{4}(\psi_1 + \psi_2) \geq F_2(r, s), \tag{2.19}$$

where

$$F_2(x, y) = \frac{1}{4}\left[4 + 3x^4 - 8x^2 - \left(1 - x^2\right)^2 y^2 - 2\left(1 - x^2\right)x^2 y\right], \quad (x, y) \in (0, 1]^2.$$

Since

$$\frac{\partial F_2}{\partial y} = -\frac{1}{2}(1 - x^2)\left[(1 - x^2)y + x^2\right] \leq 0, \quad (x, y) \in (0, 1]^2,$$

we infer that

$$\begin{aligned} F_2(x, y) &\geq F_2(x, 1) \\ &= \frac{1}{4}\left[4 + 3x^4 - 8x^2 - (1 - x^2)^2 - 2(1 - x^2)x^2\right] \\ &\geq F_2(1, 1) = -\frac{1}{4}, \qquad (x, y) \in (0, 1]^2. \end{aligned} \tag{2.20}$$

From the aforementioned details, the above last inequality together with (2.16), (2.17) and (2.19) yields the desired result (2.13).

Equality in (2.13) holds for the function $\Phi(z)$ given by (1.8). □

Lastly, we compute the upper bounds of $T_{4,1}(f)$.

Theorem 2.6. *If $f \in \mathcal{S}^*_{\sin}$, then*

$$T_{4,1}(f) \leq \frac{25}{16}. \tag{2.21}$$

Proof. In view of the following inequalities of Lemma 1.1:

$$|a_2| \leq 1, \qquad |a_3| \leq \frac{1}{2}, \qquad |a_4| \leq \frac{1}{3},$$

and noting also that

$$-\Re\left(a_2^3\overline{a}_4\right) \leq \left|a_2^3 a_4\right|, \qquad \Re\left(a_2^2\overline{a}_3\right) \leq \left|a_2^2 a_3\right|$$

and

$$-\Re\left(a_2\overline{a}_3^2 a_4\right) \leq \left|a_2 a_3^2 a_4\right|, \qquad \Re\left(a_2 a_3 \overline{a}_4\right) \leq \left|a_2 a_3 a_4\right|,$$

we therefore obtain from (1.20) that

$$T_{4,1}(f) \leq F_3\left(|a_2|, |a_3|, |a_4|\right), \tag{2.22}$$

where

$$\begin{aligned} F_3(x,y,z) = {} & 2x^3z + 4x^2y + 2xy^2z + 4xyz + x^4 - 3x^2 \\ & + y^4 - 2y^2 + x^2z^2 - 2x^2y^2 - z^2 + 1, \end{aligned}$$

$(x,y,z) \in [0,1] \times \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{3}\right]$.

It may be noted that the following system of equations has no solution in $(0,1) \times \left(0, \frac{1}{2}\right) \times \left(0, \frac{1}{3}\right)$:

$$\begin{aligned} \frac{\partial F_3}{\partial x} &= -6x + 4x^3 + 8xy - 4xy^2 + 6x^2z + 4yz + 2y^2z + 2xz^2 = 0, \\ \frac{\partial F_3}{\partial y} &= 4(x^2 + xyz + xz + y^3 - y - x^2y) = 0, \\ \frac{\partial F_3}{\partial z} &= 2x^3 + 2xy^2 + 4xy + 2x^2z - 2z = 0. \end{aligned}$$

On the boundary of $[0,1] \times \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{3}\right]$, we have the following considerations:

(i) For $F_3(0,y,z) = y^4 - 2y^2 - z^2 + 1 = G_1(y,z), \quad y \in \left[0, \frac{1}{2}\right], z \in \left[0, \frac{1}{3}\right]$,
we note that

$$\frac{\partial G_1}{\partial z} = -2z \leq 0,$$

which asserts that G_1 is a decreasing function of z. Hence,

$$G_1(y,z) \leq G_1(y,0) = y^4 - 2y^2 + 1 = (1-y^2)^2 \leq 1,$$

which implies that

$$F_3(0,y,z) \leq 1, \quad y \in \left[0, \frac{1}{2}\right], z \in \left[0, \frac{1}{3}\right].$$

(ii) Next, for $x = 1$ we have

$$F_3(1,y,z) = 2z + 4y + 2y^2z + 4yz + y^4 - 4y^2 - 1 =: G_2(y,z),$$

where $y \in \left[0, \frac{1}{2}\right]$ and $z \in \left[0, \frac{1}{3}\right]$. We observe that

$$\frac{\partial G_2}{\partial z} = 2(1+y)^2 \geq 0,$$

which asserts that G_2 is an increasing function of z. Hence,

$$G_2(y,z) \leq G_2\left(y, \tfrac{1}{3}\right) = y^4 - \tfrac{10}{3}y^2 + \tfrac{16}{3}y - \tfrac{1}{3} \leq \tfrac{25}{16},$$

which implies that

$$F_3(1,y,z) \leq \tfrac{25}{16}, \quad y \in \left[0, \tfrac{1}{2}\right], \ z \in \left[0, \tfrac{1}{3}\right].$$

(iii) When y=0, we have

$$F_3(x,0,z) = x^4 + 2x^3z + x^2z^2 - 3x^2 - z^2 + 1 =: G_3(x,z),$$

where $x \in [0,1]$ and $z \in \left[0, \frac{1}{3}\right]$. It is observed that for the function $G_3(x,z)$, the following system of equations has no solution in $(x,z) = (0,1) \times \left(0, \frac{1}{3}\right)$:

$$\frac{\partial G_3}{\partial x} = 4x^3 + 6x^2z + 2xz^2 - 6x = 0,$$
$$\frac{\partial G_3}{\partial z} = 2x^3 - 2z(1-x^2) = 0.$$

On the boundary of $(x,z) = [0,1] \times \left[0, \frac{1}{3}\right]$, we have the following inequalities for the function $G_3(x,z)$:

(a) $G_3(0,z) = 1 - z^2 \leq 1, z \in \left[0, \frac{1}{3}\right]$,
(b) $G_3(1,z) = -1 + 2z \leq -\frac{1}{3}, z \in \left[0, \frac{1}{3}\right]$,
(c) $G_3(x,0) = x^4 - 3x^2 + 1 \leq 1, x \in [0,1]$
(d) $G_3(x, \frac{1}{3}) = x^4 + \frac{2}{3}x^3 - \frac{26}{9}x^2 + \frac{8}{9} \leq \frac{8}{9}, x \in [0,1]$.

Clearly, therefore,

$$G_3(x,z) \leq 1, \quad x \in [0,1], z \in \left[0, \frac{1}{3}\right],$$

and hence,

$$F_3(x,0,z) \leq 1, \quad x \in [0,1], z \in \left[0, \frac{1}{3}\right].$$

(iv) For

$$F_3\left(x, \tfrac{1}{2}, z\right) = x^4 + 2x^3z + x^2z^2 - \tfrac{3}{2}x^2 + \tfrac{5}{2}xz - z^2 + \tfrac{9}{16}$$
$$=: G_4(x,z),$$

where $x \in [0,1]$ and $z \in \left[0, \frac{1}{3}\right]$.
We observe that the following system of equations has no solution in $(x,z) = (0,1) \times \left(0, \frac{1}{3}\right)$:

$$\frac{\partial G_4}{\partial x} = 4x^3 + 6x^2z + 2xz^2 - 3x + \tfrac{5}{2}z = 0,$$

$$\frac{\partial G_4}{\partial z} = 2x^3 + 2x^2 z + \tfrac{5}{2}x - 2z = 0.$$

Again, on the boundary of $(x, z) = [0,1] \times \left[0, \frac{1}{3}\right]$, we have

(a) $G_4(0, z) = \frac{9}{16} - z^2 \leq \frac{9}{16}, z \in \left[0, \frac{1}{3}\right]$,
(b) $G_4(1, z) = \frac{1}{16} + \frac{9}{2}z \leq \frac{25}{16}, z \in \left[0, \frac{1}{3}\right]$,
(c) $G_4(x, 0) = x^4 - \frac{3}{2}x^2 + \frac{9}{16} \leq \frac{9}{16}, x \in [0,1]$
(d) $G_4(x, \frac{1}{3}) = x^4 + \frac{2}{3}x^3 - \frac{25}{18}x^2 + \frac{5}{6}x + \frac{65}{144} \leq \frac{25}{16}, x \in [0,1]$.

Evidently, it follows that

$$G_4(x, z) \leq \tfrac{25}{16}, \quad x \in [0,1],\ z \in \left[0, \tfrac{1}{3}\right],$$
$$F_3\left(x, \tfrac{1}{2}, z\right) \leq \tfrac{25}{16}, \quad x \in [0,1],\ z \in \left[0, \tfrac{1}{3}\right].$$

(v) Next, for the function

$$\begin{aligned} F_3(x, y, 0) &= 4x^2 y + x^4 - 3x^2 + y^4 - 2y^2 - 2x^2 y^2 + 1 \\ &=: G_5(x, y), \end{aligned}$$

where $x \in [0,1]$ and $y \in \left[0, \frac{1}{2}\right]$. We note that the following system of equations has no solution in $(x, y) \in (0,1) \times \left(0, \frac{1}{2}\right)$:

$$\frac{\partial G_5}{\partial x} = 8xy + 4x^3 - 6x - 4xy^2 = 0,$$
$$\frac{\partial G_5}{\partial y} = 4x^2 + 4y^3 - 4y - 4x^2 y = 0.$$

On the boundary of $(x, y) = [0,1] \times \left[0, \frac{1}{2}\right]$, we have

(a) $G_5(0, y) = (1 - y^2)^2 \leq 1, y \in \left[0, \frac{1}{2}\right]$,
(b) $G_5(1, y) = y^4 - 4y^2 + 4y - 1 \leq \frac{1}{2}(5\sqrt{5} - 11) = 0.09017, y \in \left[0, \frac{1}{2}\right]$.
(c) $G_5(x, 0) = x^4 - 3x^2 + 1 \leq 1, x \in [0,1]$,
(d) $G_5(x, \frac{1}{2}) = x^4 - \frac{3}{2}x^2 + \frac{9}{16} \leq \frac{9}{16}, x \in [0,1]$.

It is clear that

$$G_5(x, y) \leq 1, \quad x \in [0,1], y \in \left[0, \frac{1}{2}\right],$$

and therefore,

$$F_3(x, y, 0) \leq 1, \quad x \in [0,1], y \in \left[0, \frac{1}{2}\right].$$

(vi) For the function

$$\begin{aligned} F_3\left(x, y, \tfrac{1}{3}\right) &= \tfrac{2}{3}x^3 + 4x^2 y + \tfrac{2}{3}xy^2 + \tfrac{4}{3}xy + x^4 + y^4 \\ &\quad - 2y^2 - \tfrac{26}{9}x^2 - 2x^2 y^2 + \tfrac{8}{9} \\ &=: G_6(x, y), \end{aligned}$$

where $x \in [0,1]$ and $y \in \left[0, \frac{1}{2}\right]$. we observe that the following system of equations has no solution in $(x,y) = (0,1) \times \left(0, \frac{1}{2}\right)$:

$$\frac{\partial G_6}{\partial x} = 2x^2 + 8xy + \frac{2}{3}y^2 + \frac{4}{3}y + 4x^3 - \frac{52}{9}x - 4xy^2 = 0,$$
$$\frac{\partial G_6}{\partial y} = 4x^2 + \frac{4}{3}xy + \frac{4}{3}x + 4y^3 - 4y - 4x^2y = 0.$$

On the boundary of $(x,y) = [0,1] \times \left[0, \frac{1}{2}\right]$, we have

(a) $G_6(0,y) = y^4 - 2y^2 + \frac{8}{9} \leq \frac{8}{9}, y \in \left[0, \frac{1}{2}\right]$,
(b) $G_6(1,y) = y^4 - \frac{10}{3}y^2 + \frac{16}{3}y - \frac{1}{3} \leq \frac{25}{16}, y \in \left[0, \frac{1}{2}\right]$,
(c) $G_6(x,0) = x^4 + \frac{2}{3}x^3 - \frac{26}{9}x^2 + \frac{8}{9} \leq \frac{8}{9}, x \in [0,1]$,m[and]
(d) $G_6(x, \frac{1}{2}) = x^4 + \frac{2}{3}x^3 - \frac{25}{18}x^2 + \frac{5}{6}x + \frac{65}{144} \leq \frac{25}{16}, x \in [0,1]$.

It follows that

$$G_6(x,y) \leq \tfrac{25}{16}, \quad x \in [0,1],\ y \in \left[0, \tfrac{1}{2}\right],$$
$$F_3\left(x,y,\tfrac{1}{3}\right) \leq \tfrac{25}{16}, \quad x \in [0,1],\ y \in \left[0, \tfrac{1}{2}\right].$$

Therefore, we conclude that the inequality $F_3(x,y,z) \leq \frac{25}{16}$ holds for all $(x,y,z) \in [0,1] \times \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{3}\right]$, which in view of (2.22) establishes (2.21).

□

3. Conclusion

In this work, we established the sharp upper bounds of the Hermitian Toeplitz determinants $T_{2,n}$ for $n = 1, 2, 3$ and also derived the bounds for $T_{3,1}$ and $T_{4,1}$. A natural direction for future research is to investigate the sharp bounds of $T_{2,n}$ for general n, as well as extend the analysis to higher-order determinants such as $T_{3,2}$, $T_{3,3}$, $T_{4,2}$, $T_{4,3}$, and beyond.

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