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Existence and Uniqueness Results for Mixed Type Fractional Iterative Differential Equations with Deviating Argument

Mansoorali Kazi[1*], Vinod Kharat[2], Anand Reshimkar[3] and Machchhindra Gophane[4]

Abstract. This research paper investigates the solutions for mixed type fractional iterative differential equations with deviating argument subject to boundary conditions. To prove the existence and uniqueness of solutions we employ the Krasnoselskii's and Banach fixed point theorem. Also, Ulam–Hyers and Ulam–Hyers–Rassias stability were examined to analyze the solutions. An illustrative example is provided to demonstrate the applicability of the main theoretical results.

1. Introduction

Fractional calculus concerns the study of derivatives and integrals of non-integer orders [6, 20]. Over the years, fractional differential equations have attracted significant attention due to their wide range of applications in control systems, signal processing, groundwater flow, image processing, neuroscience, epidemiology and finance. Models formulated using mixed type fractional derivatives are highly effective in describing complex physical phenomena in real world problems such as anomalous diffusion in physics, viscoelastic materials in mechanics and biological processes with memory like population dynamics, signal analysis and economic models. Many researchers extensively examined various aspects of fractional differential equation applying initial value problems, boundary value problems, implicit and integral boundary conditions.

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Also, numerous studies discussed the solution of these problems investigating the Ulam–Hyers stability and Ulam–Hyers–Rassias stability.

Riemann–Liouville and Caputo derivatives are the most widely employed due to their accurate and realistic mathematical models for systems exhibiting widespread natural memory and hereditary properties. The significance of conformable derivative lies in its adherence to classical calculus properties which simplifies the analysis and solution of fractional differential equations in applied fields. Khalil et al. introduced the notion of the conformable fractional derivative in [14, 21], which attracted considerable attention of researchers. Subsequently, investigations were extended to fractional differential equations involving both left- and right-sided fractional derivatives. For instance, Lakoud et al. in [17] examined resonant mixed Caputo fractional differential equations incorporating left and right Caputo derivatives (see also [19]). The existence of solutions for mixed fractional boundary value problem(BVP) combining left Riemann–Liouville and right Caputo derivatives has been established in [16].

Iterative fractional differential equations have been the subject of extensive investigation in recent years [9–11]. In particular, Kharat and Jagtap [15] analyzed iterative fractional differential equations with nonlocal conditions. Moreover, fractional-order iterative functional differential equations involving parameters were examined in [25].

Motivated by the contributions above mentioned and followed by [3], [4, 8, 18, 23, 26], we consider equations involving iterative fractional differential equations with deviating argument subject to boundary conditions of the mixed type:

$$D^{\alpha}_{1^-}\left({}^{C}D^{\beta}_{0^+}f\right)(p) = h(p, f(\lambda f(p))) \tag{1.1}$$

$$f(0) = \rho\int_0^1 f(p)dp \tag{1.2}$$

$${}^{C}D^{\beta}_{0^+}f(1) = 0, \tag{1.3}$$

where $\lambda, \rho \in (0,1), \alpha, \beta \in (0,1]$. $D^{\alpha}_{1^-}$ represents the right conformable fractional derivative, ${}^{C}D^{\beta}_{0^+}$ indicates the left Caputo fractional derivative, f is the function to be determined and $h : [0,1]\times[0,1] \to [0,1]$ is a continuous function.

2. Preliminaries

In this section, we see some important definitions, lemmas and theorems on fractional calculus.

Definition 2.1 ([1, 14])**.** The left conformable fractional derivative starting from a is $f : [a, +\infty) \to \mathbb{R}$ of order $0 < \alpha \leq 1$ defined by

$$D^{\alpha}_{a^+} f(p) = \lim_{\epsilon \to 0} \frac{f\left(p + \epsilon(p-a)^{(1-\alpha)}\right) - f(p)}{\epsilon}, \quad \forall\ p > 0.$$

If $D^{\alpha}_{a^+} f(p)$ exists on (a, b) then $D^{\alpha}_{a^+} f(a) = \lim_{p \to a^+} f(p)$.
The right conformable fractional derivative of order $0 < \alpha \leq 1$ terminating at b of f is defined by

$$D^{\alpha}_{b^-} f(p) = -\lim_{\epsilon \to 0} \frac{f\left(p + \epsilon(b-p)^{(1-\alpha)}\right) - f(p)}{\epsilon}, \quad \forall\ p > 0.$$

If $D^{\alpha}_{b^-} f(p)$ exists on (a, b) then $D^{\alpha}_{b^-} f(b) = \lim_{t \to b^-} f(p)$.

Definition 2.2 ([1, 14])**.** The left and right conformable fractional integral of a function f of order $0 < \alpha \leq 1$ is defined respectively by

$$I^{\alpha}_{a^+} f(p) = \int_a^p (s-a)^{\alpha-1} f(s) ds, \qquad I^{\alpha}_{b^-} f(p) = \int_p^b (b-s)^{\alpha-1} f(s) ds.$$

Definition 2.3 ([12])**.** The left Caputo fractional derivative of order $0 < \beta \leq 1$ of an absolutely continuous function $f : [a, b] \to \mathbb{R}$ is given by

$${}^{C}D^{\beta}_{a^+} f(p) = \frac{1}{\Gamma(1-\beta)} \int_a^p (p-s)^{-\beta} f'(s) ds.$$

The right Caputo fractional derivative of order $0 < \beta \leq 1$ terminating at b of f is defined by

$${}^{C}D^{\beta}_{b^-} f(p) = -\frac{1}{\Gamma(1-\beta)} \int_p^b (s-p)^{-\beta} f'(s) ds.$$

Definition 2.4 ([12])**.** The left and right Riemann-Liouville fractional integral of order $0 < \beta \leq 1$ of a function f are defined respectively by

$$J^{\beta}_{a^+} f(p) = \frac{1}{\Gamma(\beta)} \int_a^p (p-s)^{\beta-1} f(s) ds,$$

$$J^{\beta}_{b^-} f(p) = \frac{1}{\Gamma(\beta)} \int_p^b (s-p)^{\beta-1} f(s) ds.$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Lemma 2.5 ([1, 12, 14])**.** (i) *If f is a continuous function on $(a, b]$ then*

$$\begin{aligned} D^{(\alpha)}_{b^-}\left(I^{\alpha}_{b^-} f(p)\right) &= D^{(\alpha)}_{a^+}\left(I^{\alpha}_{a^+} f(p)\right) \\ &=^{C} D^{\beta}_{b^-}\left(J^{\beta}_{b_-} f(p)\right) \\ &=^{C} D^{\beta}_{a^+}\left(J^{\beta}_{a_+} f(p)\right) \end{aligned}$$

$$= f(p).$$

(ii) *If* $D^{\alpha}_{a^+}f, D^{\alpha}_{b^-}f, {}^{C}D^{\beta}_{a^+}f, {}^{C}D^{\beta}_{b^-}f$ *are continuous on* (a,b) *then*

$$I^{\alpha}_{a^+}\left(D^{\alpha}_{a^+}f(p)\right) = J^{\beta}_{a^+}\left({}^{C}D^{\beta}_{a^+}f(p)\right) = f(p) - f(a),$$

$$I^{\alpha}_{b^-}\left(D^{\alpha}_{b^-}f(p)\right) = J^{\beta}_{b^-}\left({}^{C}D^{\beta}_{b^-}f(p)\right) = f(p) - f(b).$$

(iii) *If* f *is differentiable on* (a,b) *then*

$$D^{\alpha}_{a^+}f(p) = (p-a)^{1-\alpha}f'(p), \qquad D^{\alpha}_{b^-}f(p) = -(b-p)^{1-\alpha}f'(p).$$

Theorem 2.6 ([2, 5]Contraction Mapping Principle)**.** *Consider* $(E, \|\cdot\|$ *be a Banach space,* I *is a non-empty closed subset. If* $\mathcal{T} : I \to I$ *is a strictly contraction mapping, i.e. if there exists* $\mathcal{K} \in (0,1)$ *such that* $\|\mathcal{T}f - \mathcal{T}g\| \leq \mathcal{K}\|f-g\|, \forall\ f,g \in E$*. Then* $\mathcal{T}$ *has a fixed point.*

Theorem 2.7 ([13]Kransnoselskii Fixed Point Theorem)**.** *Let* I *be a non-empty and closed convex subset of a Banach space* E*. Let* $\mathcal{T}_1, \mathcal{T}_2$ *be the operators such that*

(i) $\mathcal{T}_1 f + \mathcal{T}_2 g \in I,\ \forall\ f,g \in E$
(ii) $\mathcal{T}_1$ *is continuous and compact*
(iii) $\mathcal{T}_2$ *is a contraction mapping.*

Then there exists $z \in I$ *such that* $z = \mathcal{T}_1(z) + \mathcal{T}_2(z)$*.*

3. Main Results

Lemma 3.1 ([7])**.** *Let* $\beta, \alpha \in (0,1]$ *and* $\rho \in (0,1)$*,* $h : [0,1]\times[0,1] \to [0,1]$ *be a continuous function. Then* $f \in E$ *is a solution of*

$$D^{\alpha}_{1^-}\left({}^{C}D^{\beta}_{0^+}f\right)(p) = h\left(p, f(p)\right)$$

$$f(0) = \rho\int_0^1 f(p)dp$$

$${}^{C}D^{\beta}_{0^+}f(1) = 0$$

iff $f \in E$ *satisfies*

$$f(p) = \int_0^1 G(p,s)h\left(s, f(s)\right)ds,$$

where

$$G(p,s) = \begin{cases} \left[\frac{\rho}{(1-\rho)(\beta+1)}\left(1-(1-s)^{\beta+1}\right) + p^{\beta} - (p-s)^{\beta}\right] & \\ \quad \times \frac{(1-s)^{\alpha-1}}{\Gamma(\beta+1)}, & \text{if } 0 \leq s \leq p \leq 1 \\ \left[\frac{\rho}{(1-\rho)(\beta+1)}\left(1-(1-s)^{\beta+1}\right) + p^{\beta}\right]\frac{(1-s)^{\alpha-1}}{\Gamma(\beta+1)}, & \text{if } 0 \leq s \leq p \leq 1. \end{cases}$$

Theorem 3.2. *Let $\beta, \alpha \in (0,1]$ and $\rho \in (0,1), h : [0,1] \times [0,1] \to [0,1]$ be continuous function. Then $f \in E$ is a solution of (1.1)-(1.3) iff $f \in E$ satisfies*

(3.1)

$$\begin{aligned} f(p) = &\frac{\rho}{(1-\rho)\Gamma(\beta+2)} \int_0^1 \left[(1-s)^{\alpha-1} - (1-s)^{\beta+\alpha}\right] h\left(s, f(\lambda f(s))\right) ds \\ &+ \frac{1}{\Gamma(\beta+1)} \int_0^1 p^{\beta}(1-s)^{\alpha-1} h\left(s, f(\lambda f(s))\right) ds \\ &- \frac{1}{\Gamma(\beta+1)} \int_0^p (p-s)^{\beta}(1-s)^{\alpha-1} h\left(s, f(\lambda f(s))\right) ds. \end{aligned}$$

Proof. Using the Lemma 3.1, rewriting the solution, we get the result. □

To proceed, we define following. Let us denote $E = C([0,1],[0,1])$,

$$C_{\beta} = \{f \in E : |f(p_1) - f(p_2)| \leq \mathcal{N}|p_1 - p_2|\}$$
$$C_{\beta,r} = \{f \in C_{\beta} : \|f\| \leq r\}$$

and

$$\Upsilon = \frac{1}{\Gamma(\beta+1)} \left[\frac{\rho}{(1-\rho)\alpha(\beta+\alpha+1)} + \frac{1}{\alpha} + \frac{1}{\beta+1}\right]. \tag{3.2}$$

Hypotheses:

($H1$) There exists a real number $\mathcal{L} > 0$ such that

$$\|h(p,f) - h(p,g)\| \leq \mathcal{L}\|f - g\|, \quad \forall\ p\ \in [0,1], \quad f, g \in [0,1]. \tag{3.3}$$

($H2$) There exists a $\zeta : [0,1] \to [0,1]$ a continuous function such that

$$\|h(p,f)\| \leq \zeta(p), \quad \forall (p,f) \in [0,1] \times [0,1]. \tag{3.4}$$

Theorem 3.3. *Suppose that $h \in C([0,1],[0,1])$ is a continuous function and satisfies (H1) with $\mathcal{L} < \dfrac{1}{\Upsilon\lambda(\mathcal{N}+1)}$. Then the BVP (1.1)-(1.3) has a unique solution.*

Proof. Let $\mathcal{M} = \sup_{p\in[0,1]} |h(p,0)|$, choose $r \geq \dfrac{\mathcal{M}\Upsilon}{1-\mathcal{L}\Upsilon}$. Define $\mathcal{T} : C_{\beta,r} \to C_{\beta,r}$ given by

(3.5)

$$\begin{aligned} (\mathcal{T}f)(p) = &\frac{\rho}{(1-\rho)\Gamma(\beta+2)} \int_0^1 \left[(1-s)^{\alpha-1} - (1-s)^{\beta+\alpha}\right] h\left(s, f(\lambda(f(s)))\right) ds \\ &+ \frac{1}{\Gamma(\beta+1)} \int_0^1 p^{\beta}(1-s)^{\alpha-1} h\left(s, f(\lambda(f(s)))\right) ds \end{aligned}$$

$$- \frac{1}{\Gamma(\beta+1)} \int_0^p (p-s)^{\beta} (1-s)^{\alpha-1} h\left(s, f(\lambda(f(s)))\right) ds.$$

Consider

$$\begin{aligned}
&\|(\mathcal{T}f)(p)\| \\
&\leq \frac{\rho}{(1-\rho)\Gamma(\beta+2)} \int_0^1 \left[(1-s)^{\alpha-1} - (1-s)^{\beta+\alpha}\right] \\
&\quad \times \|[h\left(s, f(\lambda(f(s)))\right) - h(s,0)] + h(s,0)\| \, ds \\
&\quad + \frac{1}{\Gamma(\beta+1)} \int_0^1 p^{\beta} (1-s)^{\alpha-1} \|[h\left(s, f(\lambda(f(s)))\right) - h(s,0)] + h(s,0)\| \, ds \\
&\quad + \frac{1}{\Gamma(\beta+1)} \int_0^p (p-s)^{\beta} (1-s)^{\alpha-1} \|[h\left(s, f(\lambda(f(s)))\right) - h(s,0)] + h(s,0)\| \, ds \\
&\leq \frac{\rho}{(1-\rho)\Gamma(\beta+2)} \int_0^1 \left[(1-s)^{\alpha-1} - (1-s)^{\beta+\alpha}\right] \|h\left(s, f(\lambda(f(s)))\right) - h(s,0)\| \, ds \\
&\quad + \frac{\rho}{(1-\rho)\Gamma(\beta+2)} \int_0^1 \left[(1-s)^{\alpha-1} - (1-s)^{\beta+\alpha}\right] \|h(s,0)\| \, ds \\
&\quad + \frac{1}{\Gamma(\beta+1)} \int_0^1 p^{\beta} (1-s)^{\alpha-1} \|h\left(s, f(\lambda(f(s)))\right) - h(s,0)\| \, ds \\
&\quad + \frac{1}{\Gamma(\beta+1)} \int_0^1 p^{\beta} (1-s)^{\alpha-1} \|h(s,0)\| \, ds \\
&\quad + \frac{1}{\Gamma(\beta+1)} \int_0^p (p-s)^{\beta} (1-s)^{\alpha-1} \|h\left(s, f(\lambda(f(s)))\right) - h(s,0)\| \, ds \\
&\quad + \frac{1}{\Gamma(\beta+1)} \int_0^p (p-s)^{\beta} (1-s)^{\alpha-1} \|h(s,0)\| \, ds \\
&\leq \frac{\mathcal{L}r + \mathcal{M}}{\Gamma(\beta+1)} \left[\frac{\rho}{(1-\rho)\alpha(\beta+\alpha+1)} + \frac{1}{\alpha} + \frac{1}{\beta+1}\right] \\
&\leq (\mathcal{L}r + \mathcal{M})\Upsilon \\
&\leq r,
\end{aligned}$$

where Υ is given by (3.2) and $r \geq \dfrac{\mathcal{M}\Upsilon}{1-\mathcal{L}\Upsilon}$. This proves that $(\mathcal{T}f) \in C_{\beta,r}$.
Consider

$$\begin{aligned}
&\|\mathcal{T}f - \mathcal{T}g\| \\
&\quad \leq \frac{\rho}{(1-\rho)\Gamma(\beta+2)} \int_0^1 \left[(1-s)^{\alpha-1} - (1-s)^{\beta+\alpha}\right] \\
&\qquad \times |h\left(s, f(\lambda(f(s))) - h\left(s, g(\lambda g(s))\right)\right| \, ds \\
&\qquad + \frac{1}{\Gamma(\beta+1)} \int_0^1 p^{\beta} (1-s)^{\alpha-1} |h\left(s, f(\lambda(f(s)))\right) - h\left(s, g(\lambda g(s))\right)| ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\beta+1)}\int_0^p (p-s)^{\beta}(1-s)^{\alpha-1}\left|h\left(s,f(\lambda(f(s)))\right)-h\left(s,g(\lambda g(s))\right)\right|ds\\
&\leq \frac{\mathcal{L}\rho}{(1-\rho)\Gamma(\beta+2)}\int_0^1 \left[(1-s)^{\alpha-1}-(1-s)^{\beta+\alpha}\right]\left|f(\lambda(f(s))-g(\lambda g(s))\right|ds\\
&+\frac{\mathcal{L}}{\Gamma(\beta+1)}\int_0^1 p^{\beta}(1-s)^{\alpha-1}\left|f(\lambda(f(s))-g(\lambda g(s))\right|ds\\
&+\frac{\mathcal{L}}{\Gamma(\beta+1)}\int_0^p (p-s)^{\beta}(1-s)^{\alpha-1}\left|f(\lambda(f(s))-g(\lambda g(s))\right|ds
\end{aligned}
$$

by using (H1)

$$
\begin{aligned}
&\leq \frac{\mathcal{L}\rho}{(1-\rho)\Gamma(\beta+2)}\int_0^1 \left[(1-s)^{\alpha-1}-(1-s)^{\beta+\alpha}\right]\\
&\quad\times\left|f(\lambda(f(s))-f(\lambda g(s))+f(\lambda g(s))-g(\lambda g(s))\right|ds\\
&\quad+\frac{\mathcal{L}}{\Gamma(\beta+1)}\int_0^1 p^{\beta}(1-s)^{\alpha-1}\\
&\quad\times\left|f(\lambda(f(s))-f(\lambda g(s))+f(\lambda g(s))-g(\lambda g(s))\right|ds\\
&\quad+\frac{\mathcal{L}}{\Gamma(\beta+1)}\int_0^p (p-s)^{\beta}(1-s)^{\alpha-1}\\
&\quad\times\left|f(\lambda(f(s))-f(\lambda g(s))+f(\lambda g(s))-g(\lambda g(s))\right|ds\\
&\leq \mathcal{L}\lambda(\mathcal{N}+1)\Upsilon\left\|f-g\right\|.
\end{aligned}
$$

As $\mathcal{L}<\dfrac{1}{\Upsilon\lambda(\mathcal{N}+1)}$, we get $\mathcal{T}$ is a contraction. Therefore by Banach fixed point theorem, the operator defined by (3.5) has unique solution. □

Theorem 3.4. *Consider jointly continuous function $h\in C(J\times J,J)$ mapping bounded sets of $[0,1]\times[0,1]$ to $[0,1]$ and the assumptions (H1)-(H2) hold with condition*

$$
Q=\frac{\lambda\mathcal{L}(\mathcal{N}+1)}{\Gamma(\beta+1)}\left[\frac{\rho}{(1-\rho)\alpha(\beta+\alpha+1)}+\frac{1}{\alpha}\right]<1, \tag{3.6}
$$

then, the BVP (1.1)-(1.3) has at least one solution in $[0,1]$.

Proof. Define $B_\delta=\{f\in C_{\beta,r}:||f||\leq\delta\}$,
$\sup_{t\in[0,1]}|\zeta(p)|=\|\zeta\|$ and fix

$$
\delta\geq\frac{\|\zeta\|}{\Gamma(\beta+1)}\left[\frac{1+(1-\rho)(\beta+1)+\alpha(1-\rho)}{(1-\rho)\alpha(\beta+1)}\right]. \tag{3.7}
$$

Define operators $\mathcal{T}_1,\mathcal{T}_2$ on B_δ by

$$
(\mathcal{T}_1 f)(p)=-\frac{1}{\Gamma(\beta+1)}\int_0^p (p-s)^{\beta}(1-s)^{\alpha-1}h\left(s,f(\lambda(f(s)))\right)ds
$$

$$(\mathcal{T}_2 f)(p) = \frac{\rho}{(1-\rho)\Gamma(\beta+2)} \int_0^1 \left[(1-s)^{\alpha-1} - (1-s)^{\beta+\alpha}\right] h\left(s, f(\lambda(f(s)))\right) ds + \frac{1}{\Gamma(\beta+1)} \int_0^1 p^{\beta}(1-s)^{\alpha-1} h\left(s, f(\lambda(f(s)))\right) ds.$$

Claim 1: $\mathcal{T}(B_\delta) \subset B_\delta$. For $f, g \in B_\delta$, consider

$$\begin{aligned}
&\|\mathcal{T}_1 f + \mathcal{T}_2 g\| \\
&\leq \left| - \frac{1}{\Gamma(\beta+1)} \int_0^p (p-s)^{\beta}(1-s)^{\alpha-1} h\left(s, f(\lambda(f(s)))\right) ds \right. \\
&\quad + \frac{\rho}{(1-\rho)\Gamma(\beta+2)} \int_0^1 \left[(1-s)^{\alpha-1} - (1-s)^{\beta+\alpha}\right] h\left(s, g(\lambda g(s))\right) ds \\
&\quad \left. + \frac{1}{\Gamma(\beta+1)} \int_0^1 p^{\beta}(1-s)^{\alpha-1} h\left(s, g(\lambda g(s))\right) ds \right| \\
&\leq \|\zeta\| \left[\frac{1}{\Gamma(\beta+1)} \int_0^p (p-s)^{\beta} ds \right. \\
&\quad + \frac{\rho}{(1-\rho)\Gamma(\beta+2)} \int_0^1 \left[(1-s)^{\alpha-1} - (1-s)^{\beta+\alpha}\right] ds \\
&\quad \left. + \frac{1}{\Gamma(\beta+1)} \int_0^1 (1-s)^{\alpha-1} ds \right] \\
&\leq \|\zeta\| \left\{ \frac{1}{\Gamma(\beta+1)} \left[\frac{\rho}{(1-\rho)\alpha(\beta+\alpha+1)} + \frac{1}{\alpha} + \frac{1}{\beta+1} \right] \right\} \\
&\leq \delta.
\end{aligned}$$

Hence, $\mathcal{T}_1 f + \mathcal{T}_2 g \in B_\delta$.
Claim 2: The hypothesis (H_1) and (3.6) that $\mathcal{T}_2$ is a contraction mapping. For

$$\begin{aligned}
&|(\mathcal{T}_2 f)(p) - (\mathcal{T}_2 g)(p)| \\
&\leq \left| \frac{\rho}{(1-\rho)\Gamma(\beta+2)} \int_0^1 \left[(1-s)^{\alpha-1} - (1-s)^{\beta+\alpha}\right] h\left(s, f(\lambda(f(s)))\right) ds \right. \\
&\quad + \frac{1}{\Gamma(\beta+1)} \int_0^1 p^{\beta}(1-s)^{\alpha-1} h\left(s, f(\lambda(f(s)))\right) ds \\
&\quad - \left[\frac{\rho}{(1-\rho)\Gamma(\beta+2)} \int_0^1 \left[(1-s)^{\alpha-1} - (1-s)^{\beta+\alpha}\right] h\left(s, g(\lambda g(s))\right) ds \right. \\
&\quad \left. \left. + \frac{1}{\Gamma(\beta+1)} \int_0^1 p^{\beta}(1-s)^{\alpha-1} h\left(s, g(\lambda g(s))\right) ds \right] \right|
\end{aligned}$$

$$\leq \frac{\rho}{(1-\rho)\Gamma(\beta+2)}\int_0^1 \left[(1-s)^{\alpha-1}-(1-s)^{\beta+\alpha}\right] \times \left|h\left(s,f(\lambda(f(s)))\right)-h\left(s,g(\lambda g(s))\right)\right| ds + \frac{1}{\Gamma(\beta+1)}\int_0^1 p^{\beta}(1-s)^{\alpha-1}\left|h\left(s,f(\lambda(f(s)))\right)-h\left(s,g(\lambda g(s))\right)\right| ds$$

Therefore,

$$\|\mathcal{T}_2(f)-\mathcal{T}_2(g)\| \leq \frac{\mathcal{L}\lambda(\mathcal{N}+1)}{\Gamma(\beta+1)}\left[\frac{\rho}{(1-\rho)\alpha(\beta+\alpha+1)}+\frac{1}{\alpha}\right]\|f-g\| \leq Q\|f-g\|,$$

where Q is given (3.6).

Also, h is continuous so that $\mathcal{T}_1$. Furthermore,, $\mathcal{T}_1$ is uniformly bounded on B_δ, since

$$\|\mathcal{T}_1 f\| \leq \|\zeta\|\frac{1}{\Gamma(\beta+1)}.$$

Claim 3: $\mathcal{T}_1$ is compact. For $p_1, p_2 \in [0,1]$ with $p_2 > p_1$ and $f \in B_\delta$. Consider

$$\begin{aligned}
&\|(\mathcal{T}_1 f)(p_2)-(\mathcal{T}_1 f)(p_1)\| \\
&= \left\| -\frac{1}{\Gamma(\beta+1)}\int_0^{p_2}(p_2-s)^{\beta}(1-s)^{\alpha-1}\,h\left(p,f(\lambda(f(s))\right)ds \right. \\
&\quad \left. -\left(-\frac{1}{\Gamma(\beta+1)}\int_0^{p_1}(p_1-s)^{\beta}(1-s)^{\alpha-1}h\left(s,f(\lambda(f(s)))\right)ds\right)\right\| \\
&\leq \frac{\|\zeta\|}{\Gamma(\beta+1)}\left|\int_0^{p_2}(p_2-s)^{\beta}ds-\int_0^{p_1}(p_1-s)^{\beta}ds\right| \\
&\leq \frac{\|\zeta\|}{\Gamma(\beta+1)}\left|\int_0^{p_1}\left((p_2-s)^{\beta}-(p_1-s)^{\beta}\right)ds+\int_{p_1}^{p_2}(p_2-s)^{\beta}ds\right| \\
&\leq \frac{\|\zeta\|}{\Gamma(\beta+1)}\left|2(p_2-p_1)^{\beta}+p_2^{\beta}-p_1^{\beta}\right|,
\end{aligned}$$

that is not dependent on f. Thus, $\mathcal{T}_1$ is uniformly continuous. Since h transforms bounded subsets into relatively compact subsets, it follows that $\mathcal{T}_1(B)(p)$ is a relatively compact subset of E for all p and for each bounded subset B of C_β. Therefore, $\mathcal{T}_1$ is considered relatively compact on B_δ. Therefore, all conditions of the Kransnoseleskii theorem are fulfilled, thus the BVP (1.1)-(1.3) has at least one solution. □

4. Stability Results

4.1. **Definitions.** Let $h : [0,1]\times[0,1] \to [0,1]$ be a continuous function and $\epsilon > 0$ be any real number. For the BVP (1.1)-(1.3), we concentrate

on the following inequalities.

$$\left|D^{\alpha}_{1-}\left({}^{C}D^{\beta}_{0+}g\right)(p) - h(p, g(\lambda g(p)))\right| \leq \epsilon, \quad p \in [0,1] \tag{4.1}$$

$$\left|D^{\alpha}_{1-}\left({}^{C}D^{\beta}_{0+}g\right)(p) - h(p, g(\lambda g(p)))\right| \leq \psi(p), \quad p \in [0,1] \tag{4.2}$$

$$\left|D^{\alpha}_{1-}\left({}^{C}D^{\beta}_{0+}g\right)(p) - h(p, g(\lambda g(p)))\right| < \epsilon\psi(p), \quad p \in [0,1]. \tag{4.3}$$

Definition 4.1 ([22, 24])**.** The BVP (1.1)-(1.3) exhibits Ulam-Hyers stability if there exist constants $c > 0$ such that for every $\epsilon > 0$ and for every solution $g \in E$ of the inequality (4.1), there is a solution $f \in E$ of the BVP (1.1)-(1.3) such that

$$|g(p) - f(p)| \leq c\epsilon, \quad p \in [0,1].$$

Definition 4.2 ([22, 24])**.** The BVP (1.1)-(1.3) is considered generalized Ulam-Hyers stable if there is a function $\rho \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $\rho(0) = 0$ such that for every $\epsilon > 0$ and for any solution $g \in E$ of the inequality (4.1), a corresponding solution $f \in E$ of the BVP (1.1)-(1.3) exists such that

$$|g(p) - f(p)| \leq \rho(\epsilon), \quad p \in [0,1].$$

Definition 4.3 ([22, 24])**.** The BVP (1.1)-(1.3) is Ulam-Hyers-Rassias stable concerning ψ if there exists a positive real number $d > 0$ for every $\epsilon > 0$ and for each solution $g \in E$ of the inequality (4.2), there is a solution $f \in E$ of the BVP (1.1)-(1.3) such that

$$|g(p) - f(p)| \leq d\epsilon\psi(p), \quad p \in [0,1].$$

Definition 4.4 ([22, 24])**.** The BVP (1.1)-(1.3) is said to be generalized Ulam-Hyers-Rassias stable with respect to ψ if there exists a real number $d > 0$ such that for every solution $g \in E$ of the inequality (4.2), there is a solution $f \in E$ of the BVP (1.1)-(1.3) such that

$$|g(p) - f(p)| \leq d\psi(p), \quad p \in [0,1].$$

Remark 4.5 ([24])**.** (i) Definition4.1 $\Rightarrow$ Definition 4.2.
(ii) Definition 4.3 $\Rightarrow$ Definition 4.4.

Remark 4.6 ([22, 24])**.** (i) A function $g \in E$ is a solution of inequality (4.1) if and only if there exists a function $u \in C([0,1], \mathbb{R}_+)$ such that
(a) $|u(p)| \leq \epsilon,\ p \in [0,1]$.
(b) $D^{\alpha}_{1-}\left({}^{C}D^{\beta}_{0+}g\right)(p) = h(p, g(\lambda g(p)))) + u(p),\ p \in [0,1]$.
(ii) Also a function $g \in E$ is a solution of the inequality (4.2) iff there exists $v \in C([0,1], \mathbb{R})$ such that
(a) $|v(p)| \leq \psi(p),\ p \in [0,1]$,

(b) $D_{1^-}^{\alpha}\left({}^{C}D_{0^+}^{\beta}g\right)(p) = h(p, g(\lambda g(p)))) + v(p),\ p \in [0,1]$.

(iii) For, (4.3), there exists a function $\Psi \in C([0,1],\mathbb{R})$ such that

(a) $|\Psi(p)| \leq \epsilon\psi(p),\ p \in [0,1]$,

(b) $D_{1^-}^{\alpha}\left({}^{C}D_{0^+}^{\beta}g\right)(p) = h(p, g(\lambda g(p)))) + \Psi(p),\ p \in [0,1]$.

4.2. **Ulam-Hyers Stability.** In this part, we give the fourth main result about Ulam-Hyers stability for the given BVP (1.1)-(1.3).

Theorem 4.7. *Suppose $h : [0,1] \times [0,1] \to [0,1]$ is continuous and satisfying (3.3) with $L < \dfrac{1}{\lambda\Upsilon(\mathcal{N}+1)}$. Then*

(i) *The BVP (1.1)-(1.3) is Ulam-Hyers stable and therefore generalized Ulam-Hyers stable.*

(ii) *If $\psi : [0,1] \to [0,1]$ is a differentiable and increasing function with $\psi(0) \neq 0$, then the BVP (1.1)-(1.3) exhibits Ulam-Hyers-Rassias stability and is therefore also generalized Ulam-Hyers-Rassias stable.*

Proof. (i) Let $g \in E$ be any solution of the inequality (4.1), then by Remark 4.6, we have

$$D_{1^-}^{\alpha}\left({}^{C}D_{0^+}^{\beta}g\right)(p) = h(p, g(\lambda g(p)))) + u(p).$$

Then

(4.4)

$$\begin{aligned} g(p) = {} & \frac{\rho}{(1-\rho)\Gamma(\beta+2)}\int_0^1 \left[(1-s)^{\alpha-1} - (1-s)^{\beta+\alpha}\right] h\left(s, g(\lambda g(s))\right) ds \\ & + \frac{1}{\Gamma(\beta+1)}\int_0^1 p^{\beta}(1-s)^{\alpha-1} h\left(s, g(\lambda g(s))\right) ds \\ & - \frac{1}{\Gamma(\beta+1)}\int_0^p (p-s)^{\beta}(1-s)^{\alpha-1} h\left(s, g(\lambda g(s))\right) ds \\ & + \frac{\rho}{(1-\rho)\Gamma(\beta+2)}\int_0^1 \left[(1-s)^{\alpha-1} - (1-s)^{\beta+\alpha}\right] u(s) ds \\ & + \frac{1}{\Gamma(\beta+1)}\int_0^1 p^{\beta}(1-s)^{\alpha-1} u(s) ds \\ & - \frac{1}{\Gamma(\beta+1)}\int_0^p (p-s)^{\beta}(1-s)^{\alpha-1} u(s) ds. \end{aligned}$$

Therefore,

$$\left| f(p) - \left\{ \frac{\rho}{(1-\rho)\Gamma(\beta+2)}\int_0^1 \left[(1-s)^{\alpha-1} - (1-s)^{\beta+\alpha}\right] \right.$$

$$
\begin{aligned}
&\times h\left(s, g(\lambda g(s))\right) ds + \frac{1}{\Gamma(\beta+1)} \int_0^1 p^{\beta}(1-s)^{\alpha-1} h\left(s, g(\lambda g(s))\right) ds \\
&- \frac{1}{\Gamma(\beta+1)} \int_0^p (p-s)^{\beta}(1-s)^{\alpha-1} h\left(s, g(\lambda g(s))\right) ds \Bigg\} \Bigg| \\
&= \left| \frac{\rho}{(1-\rho)\Gamma(\beta+2)} \int_0^1 \left[(1-s)^{\alpha-1} - (1-s)^{\beta+\alpha}\right] u(s) ds \right. \\
&+ \frac{1}{\Gamma(\beta+1)} \int_0^1 p^{\beta}(1-s)^{\alpha-1} u(s) ds \\
&\left. - \frac{1}{\Gamma(\beta+1)} \int_0^p (p-s)^{\beta}(1-s)^{\alpha-1} u(s) ds \right| \\
&\leq \Upsilon \epsilon.
\end{aligned}
$$

Let $f \in E$ be a solution of the BVP (1.1)-(1.3).
Consider

$$
\begin{aligned}
&|g(p) - f(p)| \\
&= \left| g(p) - \left\{ \frac{\rho}{(1-\rho)\Gamma(\beta+2)} \int_0^1 \left[(1-s)^{\alpha-1} - (1-s)^{\beta+\alpha}\right] h\left(s, f(\lambda(f(s)))\right) ds \right. \right. \\
&+ \frac{1}{\Gamma(\beta+1)} \int_0^1 p^{\beta}(1-s)^{\alpha-1} h\left(s, f(\lambda(f(s)))\right) ds \\
&\left. \left. - \frac{1}{\Gamma(\beta+1)} \int_0^p (p-s)^{\beta}(1-s)^{\alpha-1} h\left(s, f(\lambda(f(s)))\right) ds \right\} \right| \\
&\leq \left| g(p) - \left\{ \frac{\rho}{(1-\rho)\Gamma(\beta+2)} \int_0^1 \left[(1-s)^{\alpha-1} - (1-s)^{\beta+\alpha}\right] h\left(s, f(\lambda f(s))\right) ds \right. \right. \\
&+ \frac{1}{\Gamma(\beta+1)} \int_0^1 p^{\beta}(1-s)^{\alpha-1} h\left(s, f(\lambda f(s))\right) ds \\
&\left. \left. - \frac{1}{\Gamma(\beta+1)} \int_0^p (p-s)^{\beta}(1-s)^{\alpha-1} h\left(s, f(\lambda f(s))\right) ds \right\} \right| \\
&\leq \left| \frac{\rho}{(1-\rho)\Gamma(\beta+2)} \int_0^1 \left[(1-s)^{\alpha-1} - (1-s)^{\beta+\alpha}\right] h\left(s, g(\lambda g(s))\right) ds \right. \\
&+ \frac{1}{\Gamma(\beta+1)} \int_0^1 p^{\beta}(1-s)^{\alpha-1} h\left(s, g(\lambda g(s))\right) ds \\
&- \frac{1}{\Gamma(\beta+1)} \int_0^p (p-s)^{\beta}(1-s)^{\alpha-1} h\left(s, g(\lambda g(s))\right) ds \\
&- \left\{ \frac{\rho}{(1-\rho)\Gamma(\beta+2)} \int_0^1 \left[(1-s)^{\alpha-1} - (1-s)^{\beta+\alpha}\right] h\left(s, f(\lambda(f(s)))\right) ds \right. \\
&+ \frac{1}{\Gamma(\beta+1)} \int_0^1 p^{\beta}(1-s)^{\alpha-1} h\left(s, f(\lambda(f(s)))\right) ds \\
&\left. \left. - \frac{1}{\Gamma(\beta+1)} \int_0^p (p-s)^{\beta}(1-s)^{\alpha-1} h\left(s, f(\lambda(f(s)))\right) ds \right\} \right|
\end{aligned}
$$

$$\leq \Upsilon\epsilon + \lambda\mathcal{L}(\mathcal{N}+1)\Upsilon \|g-f\|$$

from equations (3.2) and (4.3)

$$\|g-f\| \leq \Upsilon\epsilon + \mathcal{L}\lambda(\mathcal{N}+1)\Upsilon \|g-f\|,$$

which implies that $\|g-f\| \leq \dfrac{\Upsilon}{[1-\lambda\mathcal{L}\Upsilon(\mathcal{N}+1)]}\epsilon$,

where $c = \dfrac{\Upsilon}{[1-\lambda\mathcal{L}\Upsilon(\mathcal{N}+1)]}\epsilon > 0$. Therefore, the problem (1.1)-(1.3) is Ulam-Hyers stable. Now, let us set $\eta(\epsilon) = \lambda\epsilon$, then the BVP (1.1)-(1.3) is generalized Ulam-Hyers-stable.

(ii) Let $g \in E$ be the solution of the inequality (4.2), then by Remark 4.6, we have

$$D^{\alpha}_{1^-}\left({}^{C}D^{\beta}_{0^+}g\right)(p) = h(p, g(\lambda g(p)))) + \Psi(p), \quad p \in [0,1].$$

Using Theorem 3.2, we get

(4.5)

$$\begin{aligned}
g(p) = {} & \frac{\rho}{(1-\rho)\Gamma(\beta+2)}\int_0^1 \left[(1-s)^{\alpha-1} - (1-s)^{\beta+\alpha}\right] h\left(s, g(\lambda g(s))\right) ds \\
& + \frac{1}{\Gamma(\beta+1)}\int_0^1 p^{\beta}(1-s)^{\alpha-1} h\left(s, g(\lambda g(s))\right) ds \\
& - \frac{1}{\Gamma(\beta+1)}\int_0^p (p-s)^{\beta}(1-s)^{\alpha-1} h\left(s, g(\lambda g(s))\right) ds \\
& + \frac{\rho}{(1-\rho)\Gamma(\beta+2)}\int_0^1 \left[(1-s)^{\alpha-1} - (1-s)^{\beta+\alpha}\right] \Psi(s) ds \\
& + \frac{1}{\Gamma(\beta+1)}\int_0^1 p^{\beta}(1-s)^{\alpha-1}\Psi(s) ds \\
& - \frac{1}{\Gamma(\beta+1)}\int_0^p (p-s)^{\beta}(1-s)^{\alpha-1}\Psi(s) ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \Bigg| f(p) - \Bigg\{ \frac{\rho}{(1-\rho)\Gamma(\beta+2)}\int_0^1 \left[(1-s)^{\alpha-1} - (1-s)^{\beta+\alpha}\right] \times h\left(s, g(\lambda g(s))\right) ds \\
& \quad + \frac{1}{\Gamma(\beta+1)}\int_0^1 p^{\beta}(1-s)^{\alpha-1} h\left(s, g(\lambda g(s))\right) ds \\
& \quad - \frac{1}{\Gamma(\beta+1)}\int_0^p (p-s)^{\beta}(1-s)^{\alpha-1} h\left(s, g(\lambda g(s))\right) ds \Bigg\} \Bigg| \\
& = \left| \frac{\rho}{(1-\rho)\Gamma(\beta+2)}\int_0^1 \left[(1-s)^{\alpha-1} - (1-s)^{\beta+\alpha}\right] \Psi(s) ds \right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\beta+1)}\int_0^1 p^{\beta}(1-s)^{\alpha-1}\Psi(s)ds\\
&-\frac{1}{\Gamma(\beta+1)}\int_0^p (p-s)^{\beta}(1-s)^{\alpha-1}\Psi(s)ds\Bigg|\\
&\leq \frac{\rho}{(1-\rho)\Gamma(\beta+2)}\int_0^1 \left[(1-s)^{\alpha-1}-(1-s)^{\beta+\alpha}\right]|\Psi(s)|\,ds\\
&+\frac{1}{\Gamma(\beta+1)}\int_0^1 p^{\beta}(1-s)^{\alpha-1}|\Psi(s)|\,ds\\
&+\frac{1}{\Gamma(\beta+1)}\int_0^p (p-s)^{\beta}(1-s)^{\alpha-1}|\Psi(s)|\,ds\\
&\leq \frac{\rho}{(1-\rho)\Gamma(\beta+2)}\int_0^1 \left[(1-s)^{\alpha-1}-(1-s)^{\beta+\alpha}\right]\psi(s)ds\\
&+\frac{1}{\Gamma(\beta+1)}\int_0^1 p^{\beta}(1-s)^{\alpha-1}\psi(s)ds\\
&+\frac{1}{\Gamma(\beta+1)}\int_0^p (p-s)^{\beta}(1-s)^{\alpha-1}\psi(s)ds.
\end{aligned}
$$

As $\psi:[0,1]\to[0,1]$ is increasing, so in $0\leq s\leq p$. Then,
$s\leq p \quad\Rightarrow\quad \psi(s)\leq\psi(p)$,

$$\int_0^p \left[(1-s)^{\alpha-1}-(1-s)^{\beta+\alpha}\right]\psi(s)ds \leq \psi(p)\int_0^p \left[(1-s)^{\alpha-1}-(1-s)^{\beta+\alpha}\right]ds,$$

$$\int_0^p (1-s)^{\alpha-1}\psi(s)ds \leq \psi(1)\int_0^p (1-s)^{\alpha-1}ds.$$

and $p\leq 1 \quad\Rightarrow\quad \psi(s)\leq\psi(1)$,

$$\int_p^1 \left[(1-s)^{\alpha-1}-(1-s)^{\beta+\alpha}\right]\psi(s)ds \leq \psi(1)\int_0^p \left[(1-s)^{\alpha-1}-(1-s)^{\beta+\alpha}\right]ds,$$

$$\int_p^1 (1-s)^{\alpha-1}\psi(s)ds \leq \psi(1)\int_p^1 (1-s)^{\alpha-1}ds.$$

Consider

$$
\begin{aligned}
&\Bigg|g(p)-\Bigg\{\frac{\rho}{(1-\rho)\Gamma(\beta+2)}\int_0^1 \left[(1-s)^{\alpha-1}-(1-s)^{\beta+\alpha}\right]h\left(s,g(\lambda g(s))\right)ds\\
&+\frac{1}{\Gamma(\beta+1)}\int_0^1 p^{\beta}(1-s)^{\alpha-1}h\left(s,g(\lambda g(s))\right)ds\\
&-\frac{1}{\Gamma(\beta+1)}\int_0^p (p-s)^{\beta}(1-s)^{\alpha-1}h\left(s,g(\lambda g(s))\right)ds\Bigg\}\Bigg|\\
&\leq \frac{\rho}{(1-\rho)\Gamma(\beta+2)}\int_0^1 \left[(1-s)^{\alpha-1}-(1-s)^{\beta+\alpha}\right]|\Psi(s)|\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\beta+1)}\int_0^1 p^{\beta}(1-s)^{\alpha-1}\left|\Psi(s)\right|ds\\
&+\frac{1}{\Gamma(\beta+1)}\int_0^p (p-s)^{\beta}(1-s)^{\alpha-1}\left|\Psi(s)\right|ds\\
&\leq \epsilon\frac{\rho}{(1-\rho)\Gamma(\beta+2)}\int_0^1\left[(1-s)^{\alpha-1}-(1-s)^{\beta+\alpha}\right]\psi(s)ds\\
&+\epsilon\frac{1}{\Gamma(\beta+1)}\int_0^1 p^{\beta}(1-s)^{\alpha-1}\psi(s)ds\\
&+\epsilon\frac{1}{\Gamma(\beta+1)}\int_0^p (p-s)^{\beta}(1-s)^{\alpha-1}\psi(s)ds\\
&\leq \epsilon\frac{\rho}{(1-\rho)\Gamma(\beta+2)}\psi(p)\int_0^p\left[(1-s)^{\alpha-1}-(1-s)^{\beta+\alpha}\right]ds\\
&+\epsilon\frac{\rho}{(1-\rho)\Gamma(\beta+2)}\psi(1)\int_p^1\left[(1-s)^{\alpha-1}-(1-s)^{\beta+\alpha}\right]ds\\
&+\frac{1}{\Gamma(\beta+1)}\epsilon\psi(p)\int_0^p(1-s)^{\alpha-1}ds+\frac{1}{\Gamma(\beta+1)}\epsilon\psi(1)\int_p^1(1-s)^{\alpha-1}ds\\
&+\frac{1}{\Gamma(\beta+1)}\epsilon\psi(p)\int_0^p(p-s)^{\beta}ds\\
&\leq \epsilon\psi(p)\frac{1}{\Gamma(\beta+1)}\left[\frac{\rho}{(1-\rho)\alpha(\beta+\alpha+1)}+\frac{1}{\alpha}+\frac{1}{\beta+1}\right]\\
&+\epsilon\psi(1)\frac{1}{\Gamma(\beta+1)}\left[\Upsilon-\frac{1}{\beta+1}\right].
\end{aligned}
$$

Define a function $\mu:[0,1]\to[0,1]$ by

$$\mu(p)=\Upsilon\psi(p)+\psi(1)\left[\Upsilon-\frac{1}{\beta+1}\right]-\frac{\psi(1)}{\psi(0)}\Upsilon\psi(p).$$

Then

$$\mu'(p)=\Upsilon\left[1-\frac{\psi(1)}{\psi(0)}\right]\psi'(p).$$

Since, ψ is increasing and differentiable, we have $\mu'(p)\leq 0$.

Also, $\mu(0)=0$. For any solution $f\in E$ of a BVP (1.1)-(1.3), then

$$
\begin{aligned}
&|g(p)-f(p)|\\
&\leq\Bigg|g(p)-\Bigg\{\frac{\rho}{(1-\rho)\Gamma(\beta+2)}\int_0^1\left[(1-s)^{\alpha-1}-(1-s)^{\beta+\alpha}\right]h\left(s,f(\lambda(f(s)))\right)ds\\
&+\frac{1}{\Gamma(\beta+1)}\int_0^1 p^{\beta}(1-s)^{\alpha-1}h\left(s,f(\lambda(f(s)))\right)ds\\
&-\frac{1}{\Gamma(\beta+1)}\int_0^p(p-s)^{\beta}(1-s)^{\alpha-1}h\left(p,f(\lambda(f(s)))\right)ds\Bigg\}\Bigg|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \Bigg| g(p) - \Bigg\{ \frac{\rho}{(1-\rho)\Gamma(\beta+2)} \int_0^1 \left[(1-s)^{\alpha-1} - (1-s)^{\beta+\alpha}\right] h\left(s, f(\lambda f(s))\right) ds \\
&\quad + \frac{1}{\Gamma(\beta+1)} \int_0^1 p^{\beta}(1-s)^{\alpha-1} h\left(s, f(\lambda f(s))\right) ds \\
&\quad - \frac{1}{\Gamma(\beta+1)} \int_0^p (p-s)^{\beta}(1-s)^{\alpha-1} h\left(s, f(\lambda f(s))\right) ds \Bigg\} \Bigg| \\
&\leq \Bigg| \frac{\rho}{(1-\rho)\Gamma(\beta+2)} \int_0^1 \left[(1-s)^{\alpha-1} - (1-s)^{\beta+\alpha}\right] h\left(s, g(\lambda g(s))\right) ds \\
&\quad + \frac{1}{\Gamma(\beta+1)} \int_0^1 p^{\beta}(1-s)^{\alpha-1} h\left(s, g(\lambda g(s))\right) ds \\
&\quad - \frac{1}{\Gamma(\beta+1)} \int_0^p (p-s)^{\beta}(1-s)^{\alpha-1} h\left(s, g(\lambda g(s))\right) ds \\
&\quad - \Bigg\{ \frac{\rho}{(1-\rho)\Gamma(\beta+2)} \int_0^1 \left[(1-s)^{\alpha-1} - (1-s)^{\beta+\alpha}\right] h\left(s, f(\lambda(f(s)))\right) ds \\
&\quad + \frac{1}{\Gamma(\beta+1)} \int_0^1 p^{\beta}(1-s)^{\alpha-1} h\left(s, f(\lambda(f(s)))\right) ds \\
&\quad - \frac{1}{\Gamma(\beta+1)} \int_0^p (p-s)^{\beta}(1-s)^{\alpha-1} h\left(s, f(\lambda(f(s)))\right) ds \Bigg\} \Bigg| \\
&\leq \epsilon \frac{\psi(1)}{\psi(0)} \Upsilon\psi(p) + \frac{\mathcal{L}\rho}{(1-\rho)(\Gamma(\beta+2))} \int_0^1 \left[(1-s)^{\alpha-1} - (1-s)^{\beta+\alpha}\right] \\
&\quad \times |g(\lambda g(s)) - f(\lambda f(s))| ds + \frac{\mathcal{L}\rho}{(1-\rho)(\Gamma(\beta+2))} \\
&\quad \times \int_0^1 \left[(1-s)^{\alpha-1} - (1-s)^{\beta+\alpha}\right] |g(\lambda g(s)) - f(\lambda(f(s))| ds \\
&\quad + \frac{\mathcal{L}}{\Gamma(\beta+1)} \int_0^1 (1-s)^{\alpha-1} |g(\lambda g(s)) - f(\lambda f(s))| ds \\
&\quad + \frac{\mathcal{L}}{\Gamma(\beta+1)} \int_0^1 (1-s)^{\alpha-1} |g(\lambda g(s)) - f(\lambda(f(s))| ds \\
&\quad + \frac{\mathcal{L}}{\Gamma(\beta+1)} \int_0^p (p-s)^{\beta} |g(\lambda g(s)) - f(\lambda f(s))| ds \\
&\quad + \frac{\mathcal{L}}{\Gamma(\beta+1)} \int_0^p (p-s)^{\beta} |g(\lambda g(s)) - f(\lambda(f(s))| ds \\
&\leq \epsilon \frac{\psi(1)}{\psi(0)} \Upsilon\psi(p) + \mathcal{L}S\lambda(\mathcal{N}+1)\Upsilon \left\| g - f \right\|.
\end{aligned}
$$

Hence, we get

$$
\left\| g - f \right\| \leq \epsilon \frac{\psi(1)}{\psi(0)} \Upsilon\psi(t) + \mathcal{L}\lambda(\mathcal{N}+1)\Upsilon \left\| g - f \right\|.
$$

It gives

$$\|g-f\| \leq \frac{\epsilon\psi(1)\Upsilon}{\psi(0)\left[1-\mathcal{L}\lambda(\mathcal{N}+1)\Upsilon\right]}\psi(t),$$

or we can write

$$\|g-f\| \leq d\epsilon\psi(t),$$

where $d = \dfrac{\psi(1)\Upsilon}{\psi(0)\left[1-\lambda\mathcal{L}(\mathcal{N}+1)\Upsilon\right]} > 0$. Hence, the BVP (1.1)-(1.3) is Ulam-Hyers-Rassias stable and therefore generalized Ulam-Hyers-Rassias stable. □

Example 4.8.

$$\begin{cases} D_{1^-}^{\frac{1}{2}}\left({}^{C}D_{0^+}^{\frac{1}{2}}f\right)(p) = -\dfrac{11}{13}p + \dfrac{11}{28}f\left(\dfrac{1}{2}f(p)\right), \quad 0<p<1, \\ f(0) = \dfrac{1}{2}\displaystyle\int_0^1 f(p)dp, \\ {}^{C}D_{0^+}^{\frac{1}{2}}f(1) = 0. \end{cases} \tag{4.6}$$

Solution:

Here, $h(p, f(f(p))) = -\dfrac{11}{13}t + \dfrac{1}{28}f\left(\dfrac{1}{2}f(p)\right)$, $\beta = \alpha = \rho = \dfrac{1}{2}$.

Let us take

$$\begin{aligned} &\left|h\left(p, f_1\left(\frac{1}{2}f_1(p)\right)\right) - h\left(p, f_2\left(\frac{1}{2}f_2(p)\right)\right)\right| \\ &= \left|-\frac{11}{13}p + \frac{1}{28}f_1\left(\frac{1}{2}f_1(p)\right) - \left[-\frac{11}{13}p + \frac{1}{28}f_2\left(\frac{1}{2}f_2(p)\right)\right]\right| \\ &\leq \frac{1}{28}\left|f_1\left(\frac{1}{2}f_1(p)\right) - f_2\left(\frac{1}{2}f_2(p)\right)\right|. \end{aligned}$$

Hence, the Lipschitz constant $\mathcal{L} = \dfrac{1}{28}$.

Also, by using the values of $\beta,\ \alpha,\ \rho$ in Υ, we get

$$\begin{aligned} \Upsilon &= \frac{1}{\Gamma(\beta+1)}\left[\frac{\rho}{(1-\rho)\alpha(\beta+\alpha+1)} + \frac{1}{\alpha} + \frac{1}{\beta+1}\right] \\ &\approxeq 4.137399. \end{aligned}$$

We are interested for solutions in

$$C_\beta = \left\{|f(p_1) - f(p_2)| \leq \frac{1}{2}|p_1 - p_2|,\ \forall\ p_1,\ p_2 \in J\right\},$$

$\mathcal{N} = \dfrac{1}{2}$ and $\lambda(\mathcal{N}+1)\Upsilon\mathcal{L} = 0.5 \times 1.5 \times 4.137399 \times \dfrac{1}{28} = 0.11082318 < 1.$

Hence the BVP (4.6) has a unique solution.

Let $g \in E$ be the solution of the inequality

$$\left| D_{1-}^{\frac{1}{2}} \left({}^{C}D_{0+}^{\frac{1}{2}} f \right)(p) - \left[-\frac{11}{13}p + \frac{11}{28} f\left(\frac{1}{2} f(p) \right) \right] \right| \leq \epsilon, \quad p \in [0,1],$$

$$c = \frac{\Upsilon}{[1 - \lambda \mathcal{L} \Upsilon(\mathcal{N}+1)]} = \frac{4.137399}{1 - 0.11082318} > 0.$$

Hence, the BVP is Ulam-Hyers stable.

Also, let $g \in E$ be the solution of

$$\left| D_{1-}^{\frac{1}{2}} \left({}^{C}D_{0+}^{\frac{1}{2}} f \right)(p) - \left[-\frac{11}{13}p + \frac{11}{28} f\left(\frac{1}{2} f(p) \right) \right] \right| \leq \epsilon \psi(p), \quad p \in [0,1],$$

where $\psi(p) = \dfrac{p+1}{p+2}$ which is increasing in $[0,1]$. Then

$$c = \frac{\psi(1)\Upsilon}{\psi(0)\left[1 - \lambda \mathcal{L}(\mathcal{N}+1)\Upsilon\right]} = \frac{\frac{2}{3}(4.137399)}{\frac{1}{2}[1 - 0.11082318]} > 0.$$

Hence, the BVP (4.6) is Ulam-Hyers-Rassias stable.

Conclusion. The results studied in this paper demonstrate that combining fractional calculus tools with fixed point theorems provides an efficient framework for analyzing mixed type fractional iterative differential equations with deviating arguments subject to boundary conditions.

Also, our analysis of Ulam–Hyers and Ulam–Hyers–Rassias stability provided a more profound insight into the resilience of the suggested problem. The provided example validate the theoretical results, demonstrating the relevance and importance of our discoveries in the context of fractional differential equations.

Future work may consider uniqueness, stability and numerical methods for fractional inclusions as well as extensions to more general fractional operators and systems in higher dimensions.

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