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Hyperstability and Hyers-Ulam Stability of Quadratic Mappings Over Restricted Domains

Mohammad Zaeripournia[1] and Abbas Najati[2*]

Abstract. This article is devoted to the analysis of Hyers–Ulam stability for quadratic and mixed-type functional equations on restricted domains, extending several known results. We provide improved bounds that refine the earlier findings of Jung [12] and Rassias [19]. Furthermore, we explore the hyperstability and asymptotic behavior of quadratic and mixed additive-quadratic mappings, illustrating the broader applicability of our approach.

In normed linear spaces, spaces endowed with an inner product hold a uniquely significant place within functional analysis. For such an inner product space E, the norm that arises from this inner product obeys the fundamental parallelogram identity:

$$\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2, \quad x,y\in E.$$

A foundational result states that a norm on a vector space X can be derived from an inner product precisely when this parallelogram law holds [10]. This identity leads us to consider a quadratic functional equation of the form

$$f(x+y)+f(x-y)=2f(x)+2f(y), \tag{0.1}$$

where the function f maps from a vector space V to another vector space W, and is the unknown to be determined.

A mapping $f:V\to W$ is said to be quadratic provided that it satisfies equation (0.1) for all $x,y\in V$. It is a classical fact that every solution of (0.1) can be expressed in the form $f(x)=B(x,x)$, $x\in V$, where

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$B : V \times V \to W$ is a symmetric bi-additive mapping. Moreover, such a mapping B can be recovered from f via the formula

$$B(x, y) = \frac{1}{4} [f(x + y) - f(x - y)], \quad x, y \in V.$$

This representation theorem, originally established by Aczél [1], has been extended to more general algebraic settings. In particular, if $(G, +)$ and $(H, +)$ are abelian groups for which the equation $2x = h$ admits a unique solution in H for every $h \in H$, then a function $f : G \to H$ satisfying (0.1) admits a representation of the form $f(x) = B(x, x)$, where B denotes a symmetric biadditive function from $G \times G$ to H.

A fundamental question regarding the stability of algebraic structures was first raised by Ulam in 1940 [22]. More precisely, let (G, o) and $(H, *)$ be groups and, let $d(\cdot, \cdot)$ be a metric on H . For an arbitrary $\varepsilon > 0$, the question is whether there exists a $\delta > 0$ such that the following property holds: if a mapping $f : G \to H$ approximately preserves the group operation in the sense that

$$d(f(xoy), f(x) * f(y)) \leq \delta, \quad x, y \in G,$$

then there exists an exact homomorphism $g : G \to H$ such that

$$d(f(x), g(x)) \leq \varepsilon, \quad x \in G.$$

An affirmative answer to this question was first provided by Hyers [11] for the case of approximately additive mappings. This result assumed that G and H are both Banach spaces. The stability problem for the quadratic functional equation (0.1) was initially investigated by Skof [20] within the Hyers–Ulam framework (see also [3]). Thereafter, numerous researchers explored the stability of this equation across different frameworks [2, 4–9, 13–18, 21].

1. Preliminary Results

The stability theorem of Hyers–Ulam–Rassias type for the quadratic functional equation was later extended by Czerwik in 1992 [4]. One important special case of this result can be stated as follows:

Theorem 1.1. *Fix a nonnegative constant δ. Suppose that a mapping $f : X \to Y$ from a vector space X to a Banach space Y fulfills*

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \delta \tag{1.1}$$

for every $x, y \in X$. Then there exists exactly one quadratic mapping $Q : X \to Y$ satisfying

$$\|f(x) - Q(x)\| \leq \frac{\delta}{2}, \quad x \in X.$$

Furthermore, assuming measurability of f or the continuity of $t \mapsto f(tx)$ in $\mathbb{R}$ for each fixed $x \in X$, the mapping Q is homogeneous of degree two; that is,

$$Q(tx) = t^2 Q(x), \quad x \in X,\ t \in \mathbb{R}.$$

One should observe that the findings of Skof [20] and Czerwik [4] can be directly derived from a general stability theorem established by Borelli and Forti [2], which is applicable to a broad range of functional equations. Subsequently, in 1998, Jung [12] investigated the Hyers–Ulam stability phenomena for additive as well as quadratic mappings when the defining conditions are imposed only on restricted domains.

Theorem 1.2 ([12, Theorem 2])**.** *Let $d > 0$ and $\delta \geq 0$ be given constants. Consider a function $f : X \to Y$ between a normed space X and a Banach space Y that fulfills* (1.1) *for all $x, y \in X$ satisfying $\|x\| + \|y\| \geq d$. Then one can find a unique quadratic mapping $Q : X \to Y$ such that the estimate*

$$\|f(x) - Q(x)\| \leq \frac{7}{2}\delta \tag{1.2}$$

holds for every $x \in X$. Furthermore, if f is measurable, or if for each fixed $x \in X$ the mapping $t \mapsto f(tx)$ is continuous in $\mathbb{R}$, then Q is homogeneous of degree two; i.e., $Q(tx) = t^2 Q(x)$ for all $x \in X$ and all real numbers t.

Furthermore, an interesting asymptotic behavior is proved: a mapping $\phi : X \to Y$ is quadratic exactly when

$$\|\phi(x+y) + \phi(x-y) - 2\phi(x) - 2\phi(y)\| \to 0 \quad \text{as} \quad \|x\| + \|y\| \to \infty. \tag{1.3}$$

No additional condition is assumed beyond (1.3). Rassias [19] refined the estimate in inequality (1.2), reducing the upper bound from $\frac{7}{2}\delta$ to $\frac{5}{2}\delta$.

A generalized quadratic functional equation

$$f(x+y) + f(y+z) + f(z+x) = f(x+y+z) + f(x) + f(y) + f(z) \tag{1.4}$$

was introduced in [12] and studied under approximately even (or odd) conditions. The solution of (1.4) in fields of characteristic zero, due to Kannappan [13], can be expressed as the sum of an additive mapping A and a symmetric biadditive mapping B, with the representation $f(x) = A(x) + B(x, x)$. The stability of (1.4) was examined for both quadratic and additive mappings, considering both restricted domains (where $\|x\| + \|y\| + \|z\| \geq d$) and asymptotic behavior as $\|x\| + \|y\| + \|z\| \to \infty$.

In this work, we improve the estimates previously established by Jung and Rassias, thereby strengthening their corresponding stability results.

Moreover, we apply our new findings to examine the asymptotic behavior and hyperstability of the functional equations (0.1) and (1.4).

In what follows, the symbols X and Y will denote normed vector spaces, while $\mathcal{Y}$ is defined as a Banach space. Additionally, $d > 0$ and $\delta \geq 0$ are taken as fixed real-valued constants.

2. Principal Results

In this part, we define

$$\mathscr{D} = \{(x, y) \in X \times X : \ \min\{\|x\|, \|y\|\} \geq d\},$$
$$\mathscr{E} = \{(x, y) \in X \times X : \ \|x\| + \|y\| \geq d\}.$$

In the following theorem, we study the stability problem under more restricted domain conditions ($\min\{\|x\|, \|y\|\} \geq d$) and obtain new results.

Theorem 2.1. *Assume $f : X \to \mathcal{Y}$ is a mapping fulfilling* (1.1) *for every $(x, y) \in \mathscr{D}$. Then one can find a unique quadratic mapping $\Phi : X \to \mathcal{Y}$ satisfying*

$$\|\Phi(x) - f(x) + f(0)\| \leq \frac{8}{3}\delta, \quad x \in X. \tag{2.1}$$

If, in addition, f is measurable or, for each fixed x, the function $t \mapsto f(tx)$ is continuous in $\mathbb{R}$, then Φ is homogeneous of degree two; that is, $\Phi(tx) = t^2\Phi(x)$ for all real numbers t and all $x \in X$.

Proof. Setting $y = x$ in (1.1), we derive the inequality:

$$\|f(2x) - 4f(x) + f(0)\| \leq \delta, \qquad \|x\| \geq d. \tag{2.2}$$

Replacing x with $2^k x$ in (2.2) and dividing both sides by 4^{k+1}, we obtain

$$\left\| \frac{f(2^{k+1}x)}{4^{k+1}} - \frac{f(2^k x)}{4^k} + \frac{f(0)}{4^{k+1}} \right\| \leq \frac{\delta}{4^{k+1}}, \quad \|x\| \geq d, \ k \geq 0.$$

Using the triangle inequality, we get

$$\begin{aligned}
&\left\| \frac{f(2^{n+1}x)}{4^{n+1}} - \frac{f(2^m x)}{4^m} + \sum_{k=m}^{n} \frac{f(0)}{4^{k+1}} \right\| \\
&= \left\| \sum_{k=m}^{n} \left[\frac{f(2^{k+1}x)}{4^{k+1}} - \frac{f(2^k x)}{4^k} \right] + \sum_{k=m}^{n} \frac{f(0)}{4^{k+1}} \right\| \\
&\leq \sum_{k=m}^{n} \left\| \frac{f(2^{k+1}x)}{4^{k+1}} - \frac{f(2^k x)}{4^k} + \frac{f(0)}{4^{k+1}} \right\| \\
&\leq \sum_{k=m}^{n} \frac{\delta}{4^{k+1}}, \qquad \|x\| \geq d, \quad n \geq m \geq 0.
\end{aligned}$$

Consequently,

(2.3)
$$\left\| \frac{f(2^{n+1}x)}{4^{n+1}} - \frac{f(2^m x)}{4^m} + \sum_{k=m}^{n} \frac{f(0)}{4^{k+1}} \right\| \leq \sum_{k=m}^{n} \frac{\delta}{4^{k+1}}, \quad \|x\| \geq d,\ n \geq m \geq 0.$$

As the quantity on the right in (2.3) vanishes when both m and n tend to infinity, the sequence $\left\{ \frac{f(2^n x)}{4^n} \right\}_n$ is Cauchy for every x satisfying $\|x\| \geq d$. Furthermore, the sequence can readily be shown to be Cauchy for every $x \in X$. By virtue of the completeness of $\mathcal{Y}$, we introduce a mapping $\Phi : X \to \mathcal{Y}$ via

$$\Phi(x) := \lim_{n\to\infty} \frac{f(2^n x)}{4^n}, \quad x \in X.$$

Clearly, we have $\Phi(0) = 0$. Now, let $x, y \in X \setminus \{0\}$. Applying inequality (1.1), we have

$$\begin{aligned} &\|\Phi(x+y) + \Phi(x-y) - 2\Phi(x) - 2\Phi(y)\| \\ &\quad = \lim_n \left\| \frac{f\left(2^n(x+y)\right)}{4^n} + \frac{f\left(2^n(x-y)\right)}{4^n} - \frac{2f(2^n x)}{4^n} - \frac{2f(2^n y)}{4^n} \right\| \\ &\quad \leq \lim_n \frac{\delta}{4^n} = 0. \end{aligned}$$

Thus

$$\Phi(x+y) + \Phi(x-y) = 2\Phi(x) + 2\Phi(y) \tag{2.4}$$

for all $x, y \in X \setminus \{0\}$. By substituting y with $-y$ in (2.4), we obtain $\Phi(-y) = \Phi(y)$ for every $y \in X \setminus \{0\}$. Noting that $\Phi(0) = 0$, it follows that Φ is an even function. Consequently, (2.4) is valid for all $x, y \in X$, confirming that Φ is a quadratic mapping. Finally, if we take $m = 0$ and let n tend to infinity in (2.3), the result is

$$\left\| \Phi(x) - f(x) + \frac{f(0)}{3} \right\| \leq \frac{\delta}{3}, \qquad \|x\| \geq d. \tag{2.5}$$

Our next aim is to generalize inequality (2.5) to every $x \in X$. For this purpose, let us take an arbitrary element $z \in X$ and select $x, y \in X$ satisfying

$$z = x + y \quad \text{and} \quad \min\{\|x\|, \|y\|, \|x-y\|\} \geq d.$$

For nonzero $z \in X$, one possible choice is $x = (m+1)z$ and $y = -mz$ for sufficiently large m. In the case where $z = 0$, we may take $x = ke$ and $y = -ke$ for some nonzero $e \in X \setminus \{0\}$ and sufficiently large k. Applying (2.5), we obtain

$$\left\| \Phi(x-y) - f(x-y) + \frac{f(0)}{3} \right\| \leq \frac{\delta}{3},$$

$$\left\|2f(x) - 2\Phi(x) - \frac{2}{3}f(0)\right\| \leq \frac{2\delta}{3},$$
$$\left\|2f(y) - 2\Phi(y) - \frac{2}{3}f(0)\right\| \leq \frac{2\delta}{3}.$$

Summing these three inequalities along with (1.1) yields

$$\|f(x+y) + \Phi(x-y) - 2\Phi(y) - 2\Phi(x) - f(0)\| \leq \frac{8\delta}{3}.$$

As Φ is quadratic and $z = x + y$, we derive from the previous inequality that

$$\|f(z) - \Phi(z) - f(0)\| \leq \frac{8\delta}{3},$$

which establishes (2.1). Now, assume there is an additional quadratic mapping $\widetilde{\Phi} : X \to \mathcal{Y}$ fulfilling (2.1). Hence $\widetilde{\Phi}(x) = \lim_n \frac{f(2^n x)}{4^n}$ for each $x \in X$. This shows that Φ is uniquely determined. The final statement follows directly from Theorem 1.2. □

Given the inclusion $\mathscr{D} \subseteq \mathscr{E}$, the following corollary can be obtained immediately from Theorem 2.1. This result provides an improved approximation, reducing the bound from $\frac{7}{2}\delta$ in (1.2) of Theorem 1.2 to the tighter $\frac{8}{3}\delta$.

Corollary 2.2. *Assume $f : X \to \mathcal{Y}$ is a mapping fulfilling* (1.1) *for every $(x, y) \in \mathscr{E}$. Then one can find a unique quadratic mapping $\Phi : X \to \mathcal{Y}$ satisfying* (2.1) *for all $x \in X$. Furthermore, if f is measurable or, for each fixed x, the function $t \mapsto f(tx)$ is continuous in $\mathbb{R}$, then Φ is homogeneous of degree two; that is, $\Phi(tx) = t^2\Phi(x)$ for all real numbers t and all $x \in X$.*

Building upon the work of J. M. Rassias [19], who enhanced the approximation bound in (1.2) of Theorem 1.2 from 1.2 from $\frac{7}{2}\delta$ to $\frac{5}{2}\delta$, we provide an additional refinement in the next theorem. Specifically, we reduce the bound from $\frac{5}{2}\delta$ to $\frac{7}{6}\delta$, achieving a tighter approximation.

Theorem 2.3. *Assume $f : X \to \mathcal{Y}$ is a mapping satisfying inequality* (1.1) *for every $(x, y) \in \mathscr{E}$. Then one can find a unique quadratic mapping $\Phi : X \to \mathcal{Y}$ fulfilling*

$$\|\Phi(x) - f(x)\| \leq \frac{7}{6}\delta, \quad x \in X. \tag{2.6}$$

Moreover, if f is measurable or, for each fixed x, the function $t \mapsto f(tx)$ is continuous in $\mathbb{R}$, then Φ is homogeneous of degree two; that is, $\Phi(tx) = t^2\Phi(x)$ for all real numbers t and all $x \in X$.

Proof. Following an analogous argument to Theorem 2.1, we obtain a quadratic mapping $\Phi : X \to \mathcal{Y}$ that satisfies (2.5) for all x with $\|x\| \geq d$. We now consider an arbitrary $x \in X$ and select $y \in X$ fulfilling

$$\min\{\|y\|, \|x+y\|, \|x-y\|\} \geq d.$$

Applying (2.5), we obtain

$$\left\|\Phi(x+y) - f(x+y) + \frac{f(0)}{3}\right\| \leq \frac{\delta}{3},$$
$$\left\|\Phi(x-y) - f(x-y) + \frac{f(0)}{3}\right\| \leq \frac{\delta}{3},$$
$$\left\|2f(y) - 2\Phi(y) - \frac{2}{3}f(0)\right\| \leq \frac{2\delta}{3}.$$

By combining the three inequalities with equation (1.1), we obtain the estimate

$$\|\Phi(x+y) + \Phi(x-y) - 2\Phi(y) - 2f(x)\| \leq \frac{7\delta}{3}.$$

The quadratic nature of Φ allows us to simplify this inequality to

$$\|f(x) - \Phi(x)\| \leq \frac{7\delta}{6},$$

thereby verifying condition (2.6). To prove uniqueness, consider another quadratic mapping $\widetilde{\Phi} : X \to \mathcal{Y}$ satisfying (2.6). The representation $\widetilde{\Phi}(x) = \lim_n \frac{f(2^n x)}{4^n}$ for all $x \in X$ follows immediately, confirming that Φ is uniquely determined. The final claim follows directly from Theorem 1.2. □

Corollary 2.4. *Consider a mapping $\phi : X \to Y$. The statements below are equivalent.*

(*i*) *ϕ is a quadratic mapping on X.*
(*ii*) *ϕ fulfills* (0.1) *for all $(x, y) \in \mathscr{E}$.*
(*iii*) *ϕ fulfills* (0.1) *for all $(x, y) \in \mathscr{D}$.*

Proof. The directions $(i) \Rightarrow (ii) \Rightarrow (iii)$ are straightforward. To establish $(iii) \Rightarrow (i)$, we make use of Theorem 2.1 with $\delta = 0$. By treating $\phi : X \to \widetilde{Y}$, where $\widetilde{Y}$ denotes the completion of Y, we obtain a quadratic function $\Phi : X \to \widetilde{Y}$ that fulfills

$$\Phi(x) = \phi(x) - \phi(0), \quad x \in X.$$

Since Φ is quadratic, we obtain

$$\phi(x+y) + \phi(x-y) = 2\phi(x) + 2\phi(y) - 2\phi(0), \quad x, y \in X. \tag{2.7}$$

By combining (2.7) with condition (iii), we conclude that $\phi(0) = 0$. As a result, (2.7) implies ϕ is a quadratic mapping on X. □

3. Restricted-domain stability of Eq. (1.4)

We begin by defining two important sets:

$$\mathscr{A} = \{(x,y,z) \in X \times X \times X : \ \min\{\|x\|, \|y\|, \|z\|\} \geq d\},$$
$$\mathscr{B} = \{(x,y,z) \in X \times X \times X : \ \|x\| + \|y\| + \|z\| \geq d\}.$$

For an arbitrary function φ between two vector spaces, we define a difference operator:

$$\begin{aligned} D\varphi(x,y,z) := \varphi(x+y+z) + \varphi(x) + \varphi(y) + \varphi(z) \\ - \varphi(x+y) - \varphi(y+z) - \varphi(x+z). \end{aligned}$$

The next theorem is devoted to the study of stability on a more restricted domain characterized by the condition $\min\{|x|, |y|, |z|\} \geq d$, yielding new conclusions.

Theorem 3.1. *Consider an even mapping $f : X \to \mathcal{Y}$ satisfying*

$$\|Df(x,y,z)\| \leq \delta \tag{3.1}$$

for each $(x,y,z) \in \mathscr{A}$. Then one can find a unique quadratic mapping $\Phi : X \to \mathcal{Y}$ satisfying

$$\|f(x) - \Phi(x) - f(0)\| \leq \frac{4\delta}{3}, \quad x \in X. \tag{3.2}$$

Proof. Putting $y = x$ and $z = -x$ in (3.1), we have

$$\|f(2x) - 4f(x) + 2f(0)\| \leq \delta, \quad \|x\| \geq d.$$

Therefore

$$\left\| \frac{f(2^{n+1})x}{4^{n+1}} - \frac{f(2^m x)}{4^m} + \sum_{k=m}^{n} \frac{2f(0)}{4^{k+1}} \right\| \leq \sum_{k=m}^{n} \frac{\delta}{4^{k+1}}, \quad \|x\| \geq d,\ n \geq m \geq 0. \tag{3.3}$$

Using inequality (3.3), we deduce that the sequence $\left\{ \frac{f(2^n x)}{4^n} \right\}_n$ is Cauchy for each $x \in X$ with $\|x\| \geq d$. A straightforward extension of this argument shows that the sequence remains Cauchy for all $x \in X$. Using the completeness of $\mathcal{Y}$, we deduce pointwise convergence, and thus we can define $\Phi : X \to \mathcal{Y}$ as the pointwise limit.

$$\Phi(x) := \lim_n \frac{f(2^n x)}{4^n}, \quad x \in X.$$

Clearly, $\Phi(0) = 0$ and Φ inherits the evenness property from f. Considering nonzero elements $x, y, z \in X \setminus \{0\}$, we combine condition (3.1) with the definition of Φ to derive:

$$\|D\Phi(x,y,z)\| = \lim_n \frac{1}{4^n} \|Df(2^n x, 2^n y, 2^n z)\|$$

$$\leq \lim_{n} \frac{\delta}{4^n} = 0.$$

This establishes $D\Phi(x, y, z) = 0$ for all $x, y, z \in X \setminus \{0\}$. The initial condition $\Phi(0) = 0$ then extends this result to the full space, proving $D\Phi(x, y, z) = 0$ holds for all $x, y, z \in X$. In particular, $D\Phi(x, y, -y) = 0$ for all $x, y \in X$. This immediately yields

$$\Phi(x + y) + \Phi(x - y) = 2\Phi(x) + 2\Phi(y), \quad x, y \in X.$$

Hence, Φ is a quadratic mapping. Taking $m = 0$ and letting $n \to +\infty$ in (3.3), we obtain

$$\left\| \Phi(x) - f(x) + \frac{2}{3} f(0) \right\| \leq \frac{\delta}{3}, \qquad \|x\| \geq d. \tag{3.4}$$

In order to generalize the inequality to all of X, consider an arbitrary element $w \in X$. We construct elements $x, y \in X$ satisfying $w = x + y$ with the lower bound condition:

$$\min\{\|x\|, \|y\|, \|x + 2y\|\} \geq d.$$

If $w \neq 0$, take $x = (m + 1)w$ and $y = -mw$ with sufficiently large m. When $w = 0$, choose $x = ke$ and $y = -ke$ for some nonzero $e \in X$ and sufficiently large k. It follows from (3.1) and (3.4) the following inequalities

$$\begin{aligned}
&\|Df(x, y, y)\| = \|f(x + 2y) - 2f(x + y) - f(2y) + f(x) + 2f(y)\| \leq \delta, \\
&\left\| \Phi(x + 2y) - f(x + 2y) + \frac{2}{3} f(0) \right\| \leq \frac{\delta}{3}, \\
&\left\| f(2y) - \Phi(2y) - \frac{2}{3} f(0) \right\| \leq \frac{\delta}{3}, \\
&\left\| \Phi(x) - f(x) + \frac{2}{3} f(0) \right\| \leq \frac{\delta}{3}, \\
&\left\| 2\Phi(y) - 2f(y) + \frac{4}{3} f(0) \right\| \leq \frac{2\delta}{3}.
\end{aligned}$$

Adding the above inequalities, we derive the following bound:

$$\|\Phi(x + 2y) - \Phi(2y) + \Phi(x) + 2\Phi(y) - 2f(x + y) + 2f(0)\| \leq \frac{8\delta}{3}.$$

Since $D\Phi(x, y, y) = 0$ and setting $w = x + y$, this inequality leads to:

$$\|\Phi(w) - f(w) + f(0)\| \leq \frac{4\delta}{3}.$$

This confirms the validity of the desired inequality (3.2). □

Since $\mathscr{A} \subseteq \mathscr{B}$, an immediate consequence of Theorem 3.1 yields a sharper estimate. While [19, Theorem 5] establishes a bound of 15δ, our result improves this to $\frac{4\delta}{3}$ (see also [12, Theorem 8]).

Corollary 3.2. *Suppose $f : X \to \mathcal{Y}$ is an even map satisfying* (3.1) *for all triples $(x, y, z) \in \mathscr{B}$. Then a unique quadratic mapping $\Phi : X \to Y$ can be defined so that* (3.2) *holds for every $x \in X$.*

Theorem 3.3. *Assume $f : X \to \mathcal{Y}$ is an odd mapping that fulfills condition* (3.1) *for every triple $(x, y, z) \in \mathscr{A}$. Then a unique additive mapping $A : X \to \mathcal{Y}$ can be defined so that*

$$\|f(x) - A(x)\| \leq \frac{7\delta}{4}, \quad x \in X. \tag{3.5}$$

Proof. Substituting $z = -2x$ and $y = x$ in (3.1), we derive

$$\|f(2x) - 2f(x)\| \leq \frac{\delta}{2}, \qquad \|x\| \geq d.$$

This leads to the key estimate:

$$\left\| \frac{f(2^{n+1})x}{2^{n+1}} - \frac{f(2^m x)}{2^m} \right\| \leq \sum_{k=m}^{n} \frac{\delta}{2^{k+2}}, \qquad \|x\| \geq d,\ n \geq m \geq 0. \tag{3.6}$$

From (3.6), it follows that the sequence $\left\{ \frac{f(2^n x)}{2^n} \right\}_n$ is Cauchy for each $x \in X$ with $\|x\| \geq d$. A natural extension of this reasoning confirms that the sequence is Cauchy for every $x \in X$. Given the completeness of $\mathcal{Y}$, the sequence converges pointwise, permitting the definition of a mapping $A : X \to \mathcal{Y}$ as the limit:

$$A(x) := \lim_n \frac{f(2^n x)}{2^n}, \quad x \in X.$$

The mapping A naturally preserves the oddness property of f. For $x, y, z \in X \setminus \{0\}$, we apply condition (3.1) along with the definition of A to obtain:

$$\begin{aligned} \|DA(x, y, z)\| &= \lim_n \frac{1}{2^n} \|Df(2^n x, 2^n y, 2^n z)\| \\ &\leq \lim_n \frac{\delta}{2^n} \\ &= 0. \end{aligned}$$

This confirms that $DA(x, y, z) = 0$ for all $x, y, z \in X \setminus \{0\}$. Due to the oddness of A, this conclusion extends to the entire space X, proving $DA(x, y, z) = 0$ holds for all $x, y, z \in X$. Specifically, $DA(x, y, -x-y) = 0$ for all $x, y \in X$. Consequently, we deduce

$$A(x + y) = A(x) + A(y), \quad x, y \in X.$$

Thus, A is additive. Finally, setting $m = 0$ and passing to the limit as $n \to +\infty$ in (3.6), we arrive at

$$\|A(x) - f(x)\| \leq \frac{\delta}{2}, \qquad \|x\| \geq d. \tag{3.7}$$

To generalize the inequality to the full space X, let $w \in X$ be any arbitrary element. Following the argument of Theorem 3.1, we can select $x, y \in X$ satisfying $w = x + y$ while ensuring the following lower-bound condition holds:

$$\min\{\|x + 2y\|, \|x\|, \|y\|\} \geq d.$$

Using (3.1) and (3.7), the next inequalities follow:

$$\begin{aligned}
&\|Df(x, y, y)\| = \|f(x + 2y) - 2f(x + y) - f(2y) + f(x) + 2f(y)\| \leq \delta,\\
&\|A(x + 2y) - f(x + 2y)\| \leq \frac{\delta}{2},\\
&\|f(2y) - A(2y)\| \leq \frac{\delta}{2},\\
&\|A(x) - f(x)\| \leq \frac{\delta}{2},\\
&\|2A(y) - 2f(y)\| \leq \delta.
\end{aligned}$$

By summing these inequalities, we arrive at the following estimate:

$$\|A(x + 2y) - A(2y) + A(x) + 2A(y) - 2f(x + y)\| \leq \frac{7\delta}{2}.$$

By additivity of A and setting $w = x + y$, the above reduces to:

$$\|A(w) - f(w)\| \leq \frac{7\delta}{4}.$$

This establishes the desired inequality (3.5). □

Since $\mathscr{A} \subseteq \mathscr{B}$, an immediate consequence of Theorem 3.1 yields a sharper estimate. While [19, Theorem 6] establishes a bound of 15δ, our result improves this to $\frac{7\delta}{4}$ (see also [12, Theorem 11]).

Corollary 3.4. *Suppose $f : X \to \mathcal{Y}$ is an odd mapping that fulfills condition* (3.1) *for every triple $(x, y, z) \in \mathscr{B}$. Then one can find exactly one additive mapping $A : X \to \mathcal{Y}$ fulfilling* (3.5) *for every $x \in X$.*

We now state the principal theorem of this section.

Theorem 3.5. *Suppose $f : X \to \mathcal{Y}$ is a function satisfying condition* (3.1) *for all triples $(x, y, z) \in \mathscr{A}$ (or $(x, y, z) \in \mathscr{B}$). Then there are unique functions $A : X \to \mathcal{Y}$ (additive) and $\Phi : X \to \mathcal{Y}$ (quadratic) satisfying*

$$\|f(x) - A(x) - \Phi(x) - f(0)\| \leq \frac{37}{12}\delta, \quad x \in X. \tag{3.8}$$

Proof. There exists a unique decomposition $f = f_e + f_o$, where

$$2f_e(x) = f(x) + f(-x) \quad \text{and} \quad 2f_o(x) = f(x) - f(-x), \quad x \in X.$$

It follows immediately from the definitions that f_e is even, while f_o is odd, with both satisfying (3.1) for all triples $(x, y, z) \in \mathscr{A}$. Using Theorems 3.1 and 3.3, we derive mappings $\Phi, A : X \to \mathcal{Y}$ satisfying

$$\|f_e(x) - \Phi(x) - f(0)\| \leq \frac{4\delta}{3} \quad \text{and} \quad \|f_o(x) - A(x)\| \leq \frac{7\delta}{4}, \quad x \in X.$$

Consequently, the triangular inequality yields

$$\begin{aligned}\|f(x) - A(x) - \Phi(x) - f(0)\| &\leq \|f_e(x) - \Phi(x) - f(0)\| + \|f_o(x) - A(x)\| \\ &\leq \frac{37}{12}\delta\end{aligned}$$

for all $x \in X$. The representations

$$A(x) = \lim_n \frac{f_o(nx)}{n} \quad \text{and} \quad \Phi(x) = \lim_n \frac{f_e(nx)}{n^2}, \quad x \in X.$$

derived from (3.8) establish the uniqueness A and Φ. □

Corollary 3.6. *Assume $f : X \to Y$ is a function such that $Df(x, y, z) = 0$ holds for every triple $(x, y, z) \in \mathscr{A}$ (or $(x, y, z) \in \mathscr{B}$). Then, f can be expressed as*

$$f(x) = A(x) + \Phi(x) + f(0), \quad x \in X,$$

where $A : X \to Y$ is additive and $\Phi : X \to Y$ is quadratic.

4. Some Asymptotic Behaviors

Jung [12] established an asymptotic characterization of the quadratic functional equation. Specifically, he demonstrated that a function $g : X \to \mathcal{Y}$ is quadratic if and only if

$$\lim_{\|x\|+\|y\|\to\infty} \|g(x + y) + g(x - y) - 2g(x) - 2g(y)\| = 0.$$

In the next proposition, we present an alternative asymptotic property concerning quadratic mappings.

Proposition 4.1. *A mapping $\phi : X \to Y$ is quadratic precisely when the following asymptotic relation is valid*

$$\lim_{\min\{\|x\|,\|y\|\}\to\infty} \|\phi(x + y) + \phi(x - y) - 2\phi(x) - 2\phi(y)\| = 0. \tag{4.1}$$

Proof. Evidently, any quadratic function ϕ satisfies condition (4.1). We now assume that the asymptotic condition (4.1) is valid. Consider an arbitrary $\varepsilon > 0$. The asymptotic property in (4.1) guarantees the existence of a $d_\varepsilon > 0$ for which

$$\|\phi(x + y) + \phi(x - y) - 2\phi(x) - 2\phi(y)\| \leq \varepsilon$$

holds whenever $\min\{\|x\|, \|y\|\} \geq d_\varepsilon$. According to Theorem 2.1, there exists a quadratic mapping $Q_\varepsilon : X \to \mathcal{Y}$, where $\mathcal{Y}$ denotes the completion of Y, such that

$$\|Q_\varepsilon(x) - \phi(x) + \phi(0)\| \leq \frac{8}{3}\varepsilon, \quad x \in X.$$

Hence

$$\begin{aligned}
&\|\phi(x+y) + \phi(x-y) - 2\phi(x) - 2\phi(y) + 2\phi(0)\| \\
&\quad \leq \|\phi(x+y) - Q_\varepsilon(x+y) - \phi(0)\| + \|\phi(x-y) - Q_\varepsilon(x-y) - \phi(0)\| \\
&\qquad + \|2Q_\varepsilon(x) - 2\phi(x) + 2\phi(0)\| + \|2Q_\varepsilon(y) - 2\phi(y) + 2\phi(0)\| \\
&\quad \leq 16\varepsilon
\end{aligned}$$

for all $x, y \in X$. As $\varepsilon > 0$ was arbitrary, we deduce

$$\phi(x+y) + \phi(x-y) - 2\phi(x) - 2\phi(y) + 2\phi(0) = 0, \quad x, y \in X.$$

Applying the asymptotic condition (4.1) yields

$$\phi(x+y) + \phi(x-y) = 2\phi(x) + 2\phi(y), \quad x, y \in X.$$

This proves that ϕ is a quadratic mapping. □

Proposition 4.2. *Let $f : X \to Y$. Then $Df(x, y, z) = 0$ for all $x, y, z \in X$ if and only if the asymptotic condition*

$$\lim_{\min\{\|x\|, \|y\|, \|z\|\} \to \infty} \|Df(x, y, z)\| = 0 \tag{4.2}$$

holds true.

Proof. Clearly, $Df(x, y, z) = 0$ for all $x, y, z \in X$, implies (4.2). Now, let us suppose that the asymptotic condition (4.2) holds. Taking an arbitrary positive number $\varepsilon > 0$, the asymptotic behavior described in (4.2) ensures we can find $d_\varepsilon > 0$ such that

$$\|Df(x, y, z)\| \leq \varepsilon, \qquad \min\{\|x\|, \|y\|, \|z\|\} \geq d_\varepsilon.$$

An application of Theorem 3.5 yields an $A_\varepsilon : X \to \mathcal{Y}$ together with a quadratic mapping $Q_\varepsilon : X \to \mathcal{Y}$, where $\mathcal{Y}$ denotes the completion of Y, such that

$$\|f(x) - A_\varepsilon(x) - Q_\varepsilon(x) - f(0)\| \leq \frac{37}{12}\varepsilon, \quad x \in X.$$

Let $G(x) := A_\varepsilon(x) + Q_\varepsilon(x) + f(0)$ for all $x \in X$. Then

$$\begin{aligned}
&\|Df(x, y, z) - f(0)\| \\
&\quad = \|f(x+y+z) - f(x+y) - f(y+z) - f(x+z) \\
&\qquad + f(x) + f(y) + f(z) - f(0)\| \\
&\quad \leq \|f(x+y+z) - G(x+y+z)\| + \|G(x+y) - f(x+y)\| \\
&\qquad + \|G(y+z) - f(y+z)\| + \|G(x+z) - f(x+z)\|
\end{aligned}$$

$$\begin{aligned} &+ \|f(x) - G(x)\| + \|f(y) - G(y)\| + \|f(z) - G(z)\| \\ &\leq 7 \times \frac{37}{12}\varepsilon < 22\varepsilon \end{aligned}$$

for all $x, y \in X$. Since the choice of $\varepsilon > 0$ was arbitrarily, one may conclude that

$$Df(x, y, z) - f(0) = 0, \quad x, y, z \in X.$$

Using the asymptotic condition given in (4.2), we derive $f(0) = 0$, which immediately implies

$$Df(x, y, z) = 0, \quad x, y, z \in X.$$

This confirms the asserted conclusion, and the proof is finished. □

5. Hyperstabilty Results

Theorem 5.1. *Consider real numbers $\varepsilon, \delta \geq 0$ and $r, s < 0$. A mapping $\phi : X \to Y$ is quadratic on X if and only if*

$$\|\phi(x + y) + \phi(x - y) - 2\phi(x) - 2\phi(y)\| \leq \varepsilon\|x\|^r + \delta\|y\|^s \tag{5.1}$$

for all $x, y \in X \setminus \{0\}$.

Proof. From (5.1) we derive

$$\lim_{\min\{\|x\|,\|y\|\}\to\infty} \|\phi(x + y) + \phi(x - y) - 2\phi(x) - 2\phi(y)\| = 0.$$

Consequently, Proposition 4.1 directly implies the required conclusion. □

Theorem 5.2. *Consider real numbers $\varepsilon, \delta, \theta \geq 0$ and $r, s, t < 0$. A mapping $\psi : X \to Y$ fulfills $D\psi(x, y, z) = 0$ for all $x, y, z \in X$ if and only if*

$$\|D\psi(x, y, z)\| \leq \varepsilon\|x\|^r + \delta\|y\|^s + \theta\|z\|^t \tag{5.2}$$

for all $x, y, z \in X \setminus \{0\}$.

Proof. Starting from condition (5.2), we deduce

$$\lim_{\min\{\|x\|,\|y\|,\|z\|\}\to\infty} \|D\psi(x, y, z)\| = 0.$$

As a result, Proposition 4.2 immediately leads to the desired conclusion. □

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