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Some Bounds for the First Eigenvalue of a Generalized Laplace Operator

Seyyed Hamed Fatemi[1], Seyyed Mohammad Bagher Kashani[2] and Shahroud Azami[3*]

Abstract. In this paper, by using extended Reilly formula and extended Bochner formula we find some lower bounds for the first eigenvalue of $\Delta_B(u) := div(B\nabla u) - \langle divB, \nabla u\rangle$, where B denotes a positive definite divergence free self-adjoint $(1,1)$-Codazzi tensor field on a compact Riemannian manifold M.

1. Introduction

Let (M, g) be a compact Riemannian manifold. Computing the Laplace operator's first eigenvalue is a significant and enduring issue in geometric analysis and partial differential equation theory on manifolds. Finding a reasonable lower bound for the first eigenvalue of the Laplace operator is highly interesting, as it is used in the Poincaré inequality. Additionally, it may be used to estimate the values of the heat kernel [2, 4, 5, 12–14]. Comparable outcomes have been obtained for weighted manifolds [11].

On the other hand, elliptic operators are important extensions of the Laplace operator and play a major role in geometric analysis and PDEs on manifolds. Therefore, one of the important and natural questions is how results for the Laplace operator can be extended to elliptic operators on manifolds. Many authors have addressed this problem. For instance, Gomes et al. [8, 10] considered the elliptic operator $\mathcal{L}u = \operatorname{div}(T(\nabla u)) - \langle \nabla\eta, T(\nabla u)\rangle$ and studied the universal inequality for its eigenvalues on

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* Corresponding author.

a compact Riemannian manifold M where T is a positive definite self-adjoint $(1,1)$-tensor field. It is known that an elliptic operator, $L : C^\infty(M) \to C^\infty(M)$ can be described as follows

$$Lu = \operatorname{tr}(T \circ \operatorname{hess} u) + \langle V, \nabla u \rangle,$$

where V is a vector field on M (see [9]). Note that if $T = \mathrm{Id}$ and $V = \nabla f$ for some smooth function f then L becomes the weighted Laplacian Δ_f and its eigenvalues have been investigated in [14]. One of the important steps in the study of elliptic operators is to consider the operator $Lf = \operatorname{tr}(B \circ \operatorname{hess} f)$ which we denote by $\Delta_B f$, where B is a self-adjoint Codazzi tensor field on M. An interesting and important problem is to study the spectrum of Δ_B for a compact manifold M [1, 7, 10, 17] for valuable results in this context. Inspired by the above results, in this paper, we derive some bounds for the first eigenvalue of the operator Δ_B, where B is a Codazzi tensor and satisfies the measure condition of Proposition 2.3. Our approach is similar to the classical one for the Laplace and weighted Laplace operators by using the Reilly formula , which is established by integrating the standard Bochner formula [16]. For the Laplace operator, the Reilly formula reads as follows:

$$\begin{aligned}(1.1)\qquad &\int_M \left((\Delta v)^2 - \operatorname{Ric}(\nabla v, \nabla v) - |\operatorname{hess}(v)|^2\right) dvol_g \\ &= \int_{\partial M} \left(2v_n \Delta^\partial v + II\left(\nabla^\partial v, \nabla^\partial v\right) + H v_n^2\right) dvol_{\bar{g}}\end{aligned}$$

where $v_n = \frac{\partial v}{\partial \overrightarrow{n}}$, II is the second fundamental form of ∂M, H is the mean curvature of ∂M ([16]). Formula (1.1) also offers a wide variety of fascinating geometric applications, such as Heintze-Karcher inequality and Alexandrov's theorem (see [16]). For weighted manifolds, an analogous formula (1.1) was established [18] as follows:

$$\begin{aligned}(1.2)\qquad &\int_M \left((\Delta_f v)^2 - |\operatorname{Hess}(v)|^2 - \operatorname{Ric}_f(\nabla v, \nabla v)\right) e^{-f} dvol_g \\ &= \int_{\partial M} \left(v_n\left(H_f v_n + 2\Delta_f^\partial v\right) + II\left(\nabla^\partial v, \nabla^\partial v\right)\right) e^{-f} dvol_{\bar{g}}\end{aligned}$$

where $H_f = H - \langle \nabla f, \overrightarrow{n} \rangle$ is the weighted mean curvature of ∂M and $\operatorname{Ric}_f = \operatorname{Ric} + \operatorname{Hess} f$ is the classical Bakry-Émery Ricci tensor.

Our results are as follows. For convenience of expression, we use M as a compact n-dimensional Riemannian manifold with boundary ∂M and let B be a $(1,1)$-Codazzi tensor field. B is bounded when $\delta_1 |Y|^2 < \langle Y, BY \rangle < \delta_n |Y|^2$ for some constants $\delta_1, \delta_n > 0$ and any vector field $Y \in \mathfrak{X}(M)$. First, as in the classical case, we integrate both sides of the extended Bochner formula in Theorem 2.5 and extend the Reilly formula as follows in Theorem 1.1.

Theorem 1.1 (Extended Reilly formula)**.** *Let B satisfy $B\nabla f_B = \operatorname{div} B$ for some smooth function f_B. Then*

$$\Omega = \Xi, \tag{1.3}$$

where

$$\begin{aligned}\Omega = &\int_{\partial M} \left(\left\langle \nabla^{\partial}_{\nabla^{\partial} u} \nabla^{\partial} u, B\vec{n} \right\rangle - 2 \left\langle \operatorname{shape}^{\partial}(\nabla^{\partial} u), B\vec{n} \right\rangle \right) d\bar{m}_B \\ &+ \frac{1}{2} \int_{\partial M} \left\langle \nabla^{\partial} u, (\nabla_{\vec{n}} B) \nabla^{\partial} u \right\rangle d\bar{m}_B \\ &- \int_{\partial M} \left(u_n^2 H_B^{\partial} - \left\langle B\vec{n}, \nabla^{\partial} u_n \right\rangle \right) d\bar{m}_B \\ &+ \frac{1}{2} \int_{\partial M} u_n^2 \left\langle \vec{n}, (\nabla_{\vec{n}} B)\vec{n} \right\rangle d\bar{m}_B + \int_{\partial M} \left(\nabla^{\partial} u \cdot u_n \right) \left\langle \vec{n}, B\vec{n} \right\rangle d\bar{m}_B \\ &- \int_{\partial M} u_n \Delta_B^{\partial}(u) d\bar{m}_B + \int_{\partial M} u_n \left\langle \nabla^{\partial} u, (\nabla_{\vec{n}} B)\, \vec{n} \right\rangle d\bar{m}_B,\end{aligned}$$

and

$$\begin{aligned}\Xi = &\int_M \operatorname{tr}\left(B \circ \operatorname{hess}^2(u) \right) dm_B + \frac{1}{2} \int_M \left\langle \nabla u, (\Delta B)\, \nabla u \right\rangle dm_B \\ &- \frac{1}{2} \int_M \left\langle (\nabla_{\nabla \operatorname{tr}(B)} B) \nabla u, \nabla u \right\rangle dm_B - \int_M (\Delta_B u)\, (\Delta u)\, dm_B \\ &+ \int_M \left\langle \nabla u, \nabla \operatorname{tr}(B) \right\rangle dm_B + \int_M \operatorname{Ric}_B(\nabla u, \nabla u)\, dm_B.\end{aligned}$$

Here $H_B^{\partial} := \operatorname{tr}\left(B \circ \operatorname{shape}^{\partial} \right)$ is defined as the B-mean curvature of the boundary ∂M, $\bar{g}$ is the restricted metric on ∂M, $\vec{n}$ is the outward unit vector field on ∂M, ∇^{∂} and Δ_B^{∂} are the gradient and extended Laplacian of $\bar{g}$, $dm_B = e^{-f_B} dvol_g$, $d\bar{m}_B = e^{-f_B} dvol_{\bar{g}}$, and $\bar{g} = g|_{\partial M}$.

Remark 1.2. Note that when $B = \mathrm{Id}$, the formula (1.1) can be derived from the formula in Theorem 1.1.

Using formula (1.3) and following a similar approach as in [6, 14, 16], we obtain an estimate for the first eigenvalue of Δ_B when the extended Ricci tensor is bounded below by a positive constant.

Theorem 1.3. *Let M^n be a closed manifold and B be bounded and satisfy the equation $B\nabla f_B = \operatorname{div} B$ for some smooth function f_B. If K_1, K_2 are constants and the following conditions are satisfied:*

(a) $\int_M |\nabla f_B|^2 e^{-f_B} dvol_g = \|\nabla f_B\|_2^2 < \infty$,
(b) $\langle (\nabla_{\nabla f_B} B)\, X, X \rangle \le K_2$,
(c) $\operatorname{Hess}(\operatorname{tr} B)(X, X) + \operatorname{Ric}_B(X, X) + \operatorname{Ric}(X, BX) \ge K_1$,

(d) *The polynomial*

$$p(x) := \left(\frac{1}{(n-1)\,\delta_n} - \frac{1}{\delta_1}\right) x^3 - \frac{1}{\sqrt{\delta_1}} \|\nabla f_B\|_2 \, x^2 + \left(\frac{K_1}{2\delta_n} - \frac{K_2}{2\delta_1}\right) x - \frac{1}{\sqrt{\delta_1}} \|\nabla f_B\|_2$$

has three distinct real roots $x_1 < x_2 < x_3$*, then for the first positive eigenvalue of* Δ_B*, we have:*

$$\sqrt{\lambda} \leq x_2 \quad or \quad \sqrt{\lambda} \geq x_3.$$

For the special case where $\mathrm{tr}(B) := \mathrm{Trace}(B)$ *is constant and* $\mathrm{Ric}_B(X,X) + \mathrm{Ric}(X, BX) \geq K_1 |X|^2$ *with* $K_1 > 0$*, then:*

$$\lambda \geq \frac{\mathrm{tr}(B) K_1}{2\,(\mathrm{tr}(B) - \delta_1)},$$

where

$$\delta_1 = \min_{|X|=1} \langle X, BX \rangle, \qquad \delta_n = \max_{|X|=1} \langle X, BX \rangle.$$

Remark 1.4. Note that when $B = \mathrm{Id}$, the Lichnerowicz and Obata estimate [16] can be derived from Theorem 1.3.

When the polynomial $p(x)$ has only one root, this root is negative and the estimate in Theorem 1.3 is not useful. Therefore, Theorem 1.6 is derived by adapting the Li and Yau method. It is a variant of Theorem 5.3 of [12], in which we replace the Laplacian by Δ_B and the restriction on the Ricci tensor by new inequalities. The newly obtained estimate is more involved.
The covariant exterior derivative of the tensor field B which we denote by T^B is defined as follows.

Definition 1.5 ([15])**.** Suppose B is a $(1,1)$-tensor field on M. The $(2,1)$-tensor field T^B is defined by

$$T^B(Y,Z) := d^\nabla B(Y,Z) = (\nabla_Y B)\, Z - (\nabla_Z B)\, Y.$$

When $T^B = 0$, B is called a Codazzi tensor. In addition, the continuous function F^B is defined by

$$F^B(x) := \sup_{Y \in T_x M, |Y|=1} \sum_j \langle T^{(\nabla_Y B)}(e_j, Y), e_j \rangle(x),$$

where $\{e_j\}$ is a local frame on M.

Theorem 1.6. *Let* M^n *be closed,* $\mathrm{diam}(M) \leq D$*, and* B *be a Codazzi tensor on* M*. Assume that the following conditions are satisfied:*

(a) $\langle (\nabla_X B)\, X, X \rangle \leq K |X|^3$,
(b) $\mathrm{Ric}(X, BX) - F^B |X|^2 \geq -H^2 |X|^2$.

Then for the first eigenvalue of Δ_B, we have the following estimate:

$$\frac{2\delta_1^2}{D^2\,(n-3)\,\delta_n}\left(\sqrt{1+S}+1\right)\exp\left(-\sqrt{1+S}-K-1\right)\leq\lambda,$$

where

$$S=\frac{\delta_n\,(n-1)}{\delta_1^2}\left(\frac{K^2\delta_n\,(n-1)}{4\delta_1^2}+H^2\right)D^2.$$

When $\mathrm{tr}(B)$ *is constant, we have*

$$\frac{2\delta_1^2}{D^2(2\delta_n+\mathrm{tr}(B))}\left(\sqrt{1+S_1}+1\right)\exp\left(-\sqrt{1+S_1}-K-1\right)\leq\lambda,$$

where

$$S_1=\frac{\mathrm{tr}(B)}{\delta_1^2}\left(\frac{\mathrm{tr}(B)K^2}{4\delta_1^2}+H^2\right)D^2.$$

Notice that if we consider $B=\mathrm{Id}$, then by using the above theorems we obtain an estimate for the first eigenvalue of the Laplace operator. The results presented in this paper represent a significant generalization of the properties of the Laplace operator. While the results of this paper are different from those of [14] in the general setting, they coincide when we specialize to the case of constant h. Specifically, if in this paper we set $B=\mathrm{Id}$ (which corresponds to the standard Laplacian) and in [14] we assume that h is constant, then the same eigenvalue estimates are obtained. Thus, our results provide a generalization of the work in [14] for the case of constant h.

2. Preliminaries

In this section, we summarize some prerequisites.

Definition 2.1. For B, let $\{e_i\}$ be a local frame on M. As generalizations of the Laplacian and the Ricci tensor, we define L_B, Δ_B and Ric_B as follows:

(a) $L_B(u):=\mathrm{div}\,(B\nabla u)=\sum_i\langle\nabla_{e_i}(B\nabla u),e_i\rangle$,
(b) $\Delta_B(u):=\mathrm{tr}\,(B\circ\mathrm{hess}\,u)=\sum_i\langle\nabla_{e_i}\nabla u,Be_i\rangle$,
(c) $\mathrm{Ric}_B(X,Y):=\sum_i\langle R(X,Be_i)e_i,Y\rangle$.

We call Ric_B a generalized Ricci tensor, where u is a smooth function and X,Y are vector fields on M.

The Reilly formulas (1.1) and (1.2) are obtained with respect to the volumes $dvol_g$ and $e^f dvol_g$, which are invariant measures for the Laplace operator Δ and the weighted Laplace operator Δ_f, respectively. These measures satisfy the divergence theorem. The operator Δ_B is not of the divergence form, but if there exists a smooth function f_B such that $B\nabla f_B=\mathrm{div}\,B$, we can obtain the same formula for Δ_B with respect

to the measure $e^{-f_B}dvol_g$. Using these measures, we rewrite some basic formulas for integrations related to Δ_B.

Definition 2.2. We say that B satisfies the f_B-measure condition if there exists a function f_B such that $B\nabla f_B = \operatorname{div} B$, and the B-measure is defined by $dm_B := e^{-f_B}dvol_g$.

Now, we give the following results.

Proposition 2.3. *Let B be a positive definite self-adjoint $(1,1)$-tensor field on M satisfying the f_B-measure condition. Then we have the following results:*

(a) $\int_M (\Delta_B u)\, dm_B = \int_{\partial M} \langle B\nabla u, n\rangle\, d\bar{m}_B$,
(b) $\int_M h\,(\Delta_B u)\, dm_B + \int_M \langle \nabla h, B\nabla u\rangle\, dm_B = \int_{\partial M} h\,\langle \nabla u, Bn\rangle\, d\bar{m}_B$,
(c) $\int_{\partial M} \langle u\nabla v - v\nabla u, Bn\rangle\, d\bar{m}_B = \int_M (u\Delta_B v - v\Delta_B u)\, dm_B$,

where $d\bar{m}_B := e^{-f_B}dvol_{\bar{g}}$ and $\bar{g} = g|_{\partial M}$.

Proof. It is sufficient to consider $(\Delta_B u)\, e^{-f_B} = \operatorname{div}\left(e^{-f_B}B\nabla u\right)$, so the results follow by the usual divergence theorem. □

Example 2.4. When B is a Codazzi tensor on (M,g) and $\operatorname{tr}(B) := \operatorname{trace}(B)$ is constant, then $f_B \equiv 0$ and $dm_B = dvol_g$.

Here, we recall the following important result.

Theorem 2.5 (Extended Bochner formula[10])**.** *Let (M,g) be a Riemannian manifold and T be a self-adjoint $(1,1)$-tensor field on M. Then one has*

$$\frac{1}{2}\Delta_T\left(|\nabla u|^2\right) = \operatorname{tr}\left(T\circ \operatorname{hess}^2(u)\right) + \langle \nabla u, \nabla\left(\Delta_T u\right)\rangle - \Delta_{(\nabla_{\nabla u}T)}u + \operatorname{Ric}_T\left(\nabla u, \nabla u\right), \quad \forall u \in C^\infty(M). \tag{2.1}$$

Remark 2.6. We recall that the second fundamental form of a hypersurface $\Sigma \subset M$ is defined by

$$II(X,Y) = \left(\overline{\nabla}_X Y\right)^\perp = -\langle \nabla_X N, Y\rangle\, N,$$

where N is a local unit normal vector field on Σ. A hypersurface is called "convex" if its second fundamental form is negative definite.

3. Proof of the Extended Reilly Formula

We will prove the extended Reilly formula in this section. To this end, inspired by [14, 16], we integrate both sides of (2.1) to obtain the extended Reilly formula. It is worth mentioning that our proof is coordinate-independent.

Proof of Theorem 1.1. We integrate both sides of (2.1) to obtain an extended Reilly formula. For simplicity of notation, we denote dm_B by dm. We obtain

$$\begin{aligned}(3.1)\qquad &\frac{1}{2}\int_M \Delta_B\left(|\nabla u|^2\right) dm \\ &\quad = \int_M \mathrm{tr}\left(B\circ \mathrm{hess}^2(u)\right) dm + \int_M \langle \nabla u, \nabla\left(\Delta_B u\right)\rangle\, dm \\ &\qquad - \int_M \Delta_{(\nabla_{\nabla u}B)} u dm + \int_M \mathrm{Ric}_B\left(\nabla u, \nabla u\right) dm.\end{aligned}$$

The following is the calculation of each of the terms in the above formula:

$$(3.2)\qquad \int_M \Delta_B\left(|\nabla u|^2\right) dm = \int_{\partial M}\left\langle \nabla\left(|\nabla u|^2\right), B\vec{n}\right\rangle d\bar{m}.$$

Note that $e^{-f_B}\nabla\left(\Delta_B u\right) = \nabla\left(e^{-f_B}\Delta_B u\right) + \langle \nabla u, \nabla f_B\rangle e^{-f_B}$, so we deduce that

$$\begin{aligned}(3.3)\qquad &\int_M \langle \nabla u, \nabla\left(\Delta_B u\right)\rangle\, dm \\ &\quad = \int_M \left\langle \nabla u, \nabla\left(e^{-f_B}\Delta_B u\right)\right\rangle dvol_g + \int_M \langle \nabla u, \nabla f_B\rangle e^{-f_B} dvol_g \\ &\quad = -\int_M \left(\Delta_B u\right)\left(\Delta u\right) e^{-f_B} dvol_g + \int_{\partial M}\left(\Delta_B u\right)\langle \nabla u, \vec{n}\rangle e^{-f_B} dvol_{\bar{g}} \\ &\qquad + \int_M \langle \nabla u, \nabla f_B\rangle e^{-f_B} dvol_g \\ &\quad = -\int_M \left(\Delta_B u\right)\left(\Delta u\right) dm + \int_M \langle \nabla u, \nabla\,\mathrm{tr}(B)\rangle\, dm \\ &\qquad + \int_{\partial M}\left(\Delta_B u\right)\langle \nabla u, \vec{n}\rangle\, d\bar{m}.\end{aligned}$$

To simplify $\int_M \Delta_{(\nabla_{\nabla u}B)} u dm$, let $\{e_i\}$ be a local orthonormal frame field on a normal neighborhood around x such that $\nabla_{e_i} e_j(x) = 0$. By Remark 3.1, we see that

$$\begin{aligned}\Delta_{(\nabla_{\nabla u}B)} u &= L_{(\nabla_{\nabla u}B)} u - \langle \nabla u, \mathrm{div}\left(\nabla_{\nabla u}B\right)\rangle \\ &= L_{(\nabla_{\nabla u}B)} u - \langle \nabla u, \left(\Delta B\right)\nabla u\rangle - \Delta_{(\nabla_{\nabla u}B)} u.\end{aligned}$$

Equivalently,

$$\Delta_{(\nabla_{\nabla u}B)} u = \frac{1}{2}\left(L_{(\nabla_{\nabla u}B)} u - \langle \nabla u, \left(\Delta B\right)\nabla u\rangle\right),$$

thus

(3.4)
$$\begin{aligned}-\int_M \Delta_{(\nabla_{\nabla u}B)}u dm &= -\frac{1}{2}\int_M \left(L_{(\nabla_{\nabla u}B)}u - \langle \nabla u, (\Delta B)\,\nabla u\rangle\right) dm \\ &= \frac{1}{2}\int_M \langle \nabla u, (\Delta B)\,\nabla u\rangle\, dm - \frac{1}{2}\int_M L_{(\nabla_{\nabla u}B)}u dm.\end{aligned}$$

Note that

$$\begin{aligned}\int_M L_{(\nabla_{\nabla u}B)}u dm &= \int_M \operatorname{div}\left((\nabla_{\nabla u}B)\,\nabla u\right) e^{-f_B} dvol_g \\ &= \int_M \operatorname{div}\left(e^{-f_B}\,(\nabla_{\nabla u}B)\,\nabla u\right) dvol_g \\ &\quad + \int_M \langle \nabla \operatorname{tr}(B), (\nabla_{\nabla u}B)\,\nabla u\rangle\, e^{-f_B} dvol_g \\ &= \int_{\partial M} \langle (\nabla_{\nabla u}B)\,\nabla u, n\rangle\, d\bar{m} \\ &\quad + \int_M \langle \nabla \operatorname{tr}(B), (\nabla_{\nabla u}B)\,\nabla u\rangle\, dm.\end{aligned}$$

Hence, we can write (3.4) as

(3.5)
$$\begin{aligned}&-\int_M \Delta_{(\nabla_{\nabla u}B)}u dm \\ &\quad = \frac{1}{2}\int_M \langle \nabla u, (\Delta B)\,\nabla u\rangle\, dm - \frac{1}{2}\int_M \langle (\nabla_{\nabla f_B}B)\,\nabla u, \nabla u\rangle\, dm \\ &\qquad - \frac{1}{2}\int_{\partial M} \langle (\nabla_{\nabla u}B)\,\nabla u, n\rangle\, d\bar{m}.\end{aligned}$$

Thus, from (3.1), (3.2), (3.3) and (3.5), we get

$$\Omega = \Xi,$$

where

(3.6)
$$\begin{aligned}\Omega = &\frac{1}{2}\int_{\partial M} \left\langle \nabla\left(|\nabla u|^2\right), B\vec{n}\right\rangle d\bar{m} - \int_{\partial M} (\Delta_B u)\,\langle \nabla u, \vec{n}\rangle\, d\bar{m} \\ &+ \frac{1}{2}\int_{\partial M} \langle (\nabla_{\nabla u}B)\,\nabla u, \vec{n}\rangle\, d\bar{m},\end{aligned}$$

and

$$\begin{aligned}\Xi = &\frac{1}{2}\int_M \langle \nabla u, (\Delta B)\,\nabla u\rangle\, dm + \int_M \operatorname{tr}\left(B \circ \operatorname{hess}^2(u)\right) dm \\ &- \frac{1}{2}\int_M \langle (\nabla_{\nabla f_B}B)\,\nabla u, \nabla u\rangle\, dm + \int_M \langle \nabla u, \nabla f_B\rangle\, dm \\ &- \int_M (\Delta_B u)\,(\Delta u)\, dm + \int_M \operatorname{Ric}_B(\nabla u, \nabla u)\, dm.\end{aligned}$$

For the boundary conditions, we should compute (3.6) on the boundary of M. So, the following terms shall be used for the extrinsic and intrinsic geometry of ∂M:

$$\left\langle \nabla \left(|\nabla u|^2 \right), B\overrightarrow{n} \right\rangle, \tag{3.7}$$

$$\left(\Delta_B u\right) \left\langle \nabla u, \overrightarrow{n} \right\rangle, \tag{3.8}$$

and

$$\left\langle \left(\nabla_{\nabla u} B\right) \nabla u, \vec{n} \right\rangle. \tag{3.9}$$

We denote the B-Laplacian and gradient of the manifold ∂M by Δ_B^{∂} and ∇^{∂}, respectively. We also denote the gradient and B-Laplacian on M by ∇ and Δ_B, respectively. Now, we compute all terms (3.7)-(3.9).

The computation of (3.7) is as follows:

$$\frac{1}{2}\left\langle \nabla \left(|\nabla u|^2 \right), B\overrightarrow{n} \right\rangle = \frac{1}{2} B\overrightarrow{n} \cdot \langle \nabla u, \nabla u \rangle = \langle \nabla_{B\overrightarrow{n}} \nabla u, \nabla u \rangle .$$

To compute (3.8), we have

$$\begin{aligned} \Delta_B u = \Delta_B^{\partial}(u) + \left\langle \vec{n}, B\left(\mathrm{shape}^{\partial}\left(\nabla^{\partial} u \right) \right) \right\rangle + \left\langle B\vec{n}, \nabla^{\partial} u_n \right\rangle - u_n H_B^{\partial} \\ + \mathrm{Hess}(u)\left(\vec{n}, B\vec{n}\right). \end{aligned}$$

We are aware that (3.6) is the result of integration of $\langle \nabla_{Bn} \nabla u, \nabla u \rangle - u_n \Delta_B u$. So, we have

$$\begin{aligned} &\langle \nabla_{B\vec{n}} \nabla u, \nabla u \rangle - u_n \Delta_B u \\ &\quad = \mathrm{Hess}(u)\left(\nabla u, B\vec{n}\right) - u_n \Delta_B^{\partial}(u) - u_n \left\langle B\vec{n}, \left(\mathrm{shape}^{\partial}\left(\nabla^{\partial} u \right) \right) \right\rangle \\ &\qquad - u_n \left\langle B\vec{n}, \nabla^{\partial} u_n \right\rangle + u_n^2 H_B^{\partial} - \mathrm{Hess}(u)\left(u_n \vec{n}, B\vec{n}\right). \end{aligned}$$

But

$$\begin{aligned} &\mathrm{Hess}(u)\left(\nabla u, B\vec{n}\right) - \mathrm{Hess}(u)\left(u_n \vec{n}, B\vec{n}\right) \\ &\quad = \left\langle \nabla_{\nabla^{\partial} u} \nabla^{\partial} u, B\vec{n} \right\rangle + \left(\nabla^{\partial} u \cdot u_n \right) \langle \vec{n}, B\vec{n} \rangle - u_n \left\langle \mathrm{shape}^{\partial}\left(\nabla^{\partial} u \right), B\vec{n} \right\rangle. \end{aligned}$$

To compute (3.9), by the relation $\nabla u = \nabla^{\partial} u + u_n \overrightarrow{n}$, one has

$$\begin{aligned} \left\langle \left(\nabla_{\nabla u} B\right) \nabla u, \vec{n} \right\rangle = \left\langle \nabla^{\partial} u, \left(\nabla_{\vec{n}} B\right) \nabla^{\partial} u \right\rangle + 2u_n \left\langle \nabla^{\partial} u, \left(\nabla_{\vec{n}} B\right) \vec{n} \right\rangle \\ + u_n^2 \left\langle \vec{n}, \left(\nabla_{\vec{n}} B\right) \vec{n} \right\rangle, \end{aligned}$$

and finally the formula (1.3) is obtained. □

Remark 3.1. In fact, when

$$\sum_i \left\langle \nabla_{e_i} \nabla u, T^B\left(e_i, \nabla u\right) \right\rangle + \sum_i \left\langle \nabla u, T^B\left(\nabla_{e_i} \nabla u, e_i\right) \right\rangle = 0, \tag{3.10}$$

then a similar formula as in Theorem 1.1 can be derived. For example, when B is a Codazzi tensor or B satisfies the following tensorial covariant differential equality:

$$B^q_{q;i} - B^q_{i;q} = 0,$$

the equation (3.10) is satisfied.

4. Lower Bounds for the First Non-Zero Eigenvalue

The following eigenvalue problems are addressed in this section:

(a) The Dirichlet boundary condition:

$$\begin{cases} \Delta_B u + \lambda u = 0 & \text{in } M, \\ u = 0 & \text{on } \partial M. \end{cases} \tag{4.1}$$

(b) The B-Neumann boundary condition:

$$\begin{cases} \Delta_B u + \lambda u = 0 & \text{in } M, \\ \langle \nabla u, Bn \rangle = 0 & \text{on } \partial M. \end{cases} \tag{4.2}$$

Note that, for every $u, v \in C^\infty(M)$, we have

$$\int_M (u\Delta_B v - v\Delta_B u)\, dm_B = \int_{\partial M} (u \langle B\nabla v, n\rangle - v \langle B\nabla u, n\rangle)\, d\bar{m}_B.$$

Thus for the problems (4.2) and (4.1), Δ_B is a self-adjoint operator in $L^2(M, dm_B)$ that satisfies the boundary conditions (4.2) or (4.1). Hence, Δ_B has discrete spectrum:

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to +\infty.$$

We recall from the Mini-Max Theorem:

$$\lambda_1^{ANBC} = \inf_{u\in\mathcal{C}} \left\{ \frac{\int_M \langle \nabla u, B\nabla u\rangle\, dm_B}{\int_M u^2 dm_B} : \langle \nabla u, Bn\rangle = 0 \right\},$$

and

$$\lambda_1^{DBC} = \inf_{u\in\mathcal{C}} \left\{ \frac{\int_M \langle \nabla u, B\nabla u\rangle\, dm_B}{\int_M u^2 dm_B} : u|_{\partial M} = 0 \right\}$$

where $\mathcal{C} = \left\{u \in W^{1,2}(M) \setminus \{0\} : \int_M u dm_B = 0\right\}$.

We wish to obtain an estimate for the first eigenvalue of Δ_B. To do this, we exhibit the relations between Ric_B and other geometric quantities such as the Laplacian of the tensor B and the Ricci tensor. We shall first recall the following result from [3].

Lemma 4.1 ([3])**.** *For any vector field Y, we have:*

$$\langle (\Delta B)\, Y, Y\rangle = \mathrm{Hess}\,(\mathrm{tr}(B))\,(Y,Y) - \mathrm{Ric}_B\,(Y,Y) + \mathrm{Ric}\,(Y, BY)\,.$$

Proof of Theorem 1.3. The original idea of the proof comes from the Laplacian case (see e.g. Theorem 2.3.1 p. 35 in [6]), but our proof is totally different from the original one and more difficult because of the basic changes in the assumptions. We obtain the estimate as follows.

By our assumptions, we have:

$$\delta_1 \langle X, X\rangle \leq \langle BX, X\rangle \leq \delta_n \langle X, X\rangle .$$

Let λ be a positive eigenvalue of Δ_B and let u be the corresponding eigenfunction. Denote dm_B by dm, and assume $u > 0$ and $\int_M u^2 dm = 1$. So,

$$\delta_1 \int_M |\nabla u|^2 \, dm \leq \int_M \langle B\nabla u, \nabla u\rangle \, dm = -\int_M u\Delta_B(u) dm = \lambda.$$

Similarly, $\delta_n \int_M |\nabla u|^2 \, dm \geq \lambda$, hence

$$\frac{\lambda}{\delta_n} \leq \int_M |\nabla u|^2 \, dm \leq \frac{\lambda}{\delta_1}.$$

Since $\partial M = \emptyset$, by Theorem 1.1, we conclude that

$$\begin{aligned} 0 = {} & \frac{1}{2}\int_M \langle \nabla u, (\Delta B)\nabla u\rangle \, dm + \int_M \operatorname{tr}\left(B \circ \operatorname{hess}^2(u)\right) dm \\ & - \frac{1}{2}\int_M \langle (\nabla_{\nabla f_B} B)\nabla u, \nabla u\rangle \, dm + \int_M \langle \nabla u, \nabla f_B\rangle \, dm \\ & - \int_M (\Delta_B u)(\Delta u) \, dm + \int_M \operatorname{Ric}_B(\nabla u, \nabla u) \, dm. \end{aligned} \tag{4.3}$$

By Lemma 4.1, equation (4.3) can be written as

$$\begin{aligned} 0 = {} & -\int_M (\Delta_B u)(\Delta u) \, dm + \int_M \operatorname{tr}\left(B \circ \operatorname{hess}^2(u)\right) dm \\ & - \frac{1}{2}\int_M \langle (\nabla_{\nabla f_B} B)\nabla u, \nabla u\rangle \, dm + \frac{1}{2}\int_M \operatorname{Hess}(\operatorname{tr} B)(\nabla u, \nabla u) \, dm \\ & + \int_M \langle \nabla u, \nabla f_B\rangle \, dm + \frac{1}{2}\int_M \operatorname{Ric}_B(\nabla u, \nabla u) \, dm \\ & + \frac{1}{2}\int_M \operatorname{Ric}(\nabla u, B\nabla u) \, dm. \end{aligned}$$

Since $\Delta_B u = -\lambda u$ and $\operatorname{tr}\left(B \circ \operatorname{Hess}^2(u)\right) \geq \frac{(\Delta_B u)^2}{(n-1)\delta_n}$, the conditions on $\operatorname{Hess}(\operatorname{tr} B)(\nabla u, \nabla u) + \operatorname{Ric}_B(\nabla u, \nabla u) + \operatorname{Ric}(\nabla u, B\nabla u)$ yield:

$$\begin{aligned} 0 \geq {} & \frac{\lambda^2}{(n-1)\delta_n}\int_M u^2 dm + \lambda \int_M u(\Delta u) \, dm \\ & - \frac{1}{2}\int_M \langle (\nabla_{\nabla f_B} B)\nabla u, \nabla u\rangle \, dm + \int_M \langle \nabla u, \nabla f_B\rangle \, dm \end{aligned} \tag{4.4}$$

$$+\frac{K_1}{2}\int_M |\nabla u|^2\, dm.$$

Note that:

$$\begin{aligned}
\int_M u\,(\Delta u)\, dm &= \int_M u\,(\Delta u)\, e^{-f_B} dvol_g \\
&= -\int_M \left\langle \nabla\left(e^{-f_B}u\right), \nabla u\right\rangle dvol_g \\
&= -\int_M e^{-f_B}\,\langle \nabla u, \nabla u\rangle\, dvol_g + \int_M e^{-f_B} u\,\langle \nabla f_B, \nabla u\rangle\, dvol_g \\
&= -\int_M |\nabla u|^2\, dm + \int_M u\,\langle \nabla f_B, \nabla u\rangle\, dm \\
&\geq -\int_M |\nabla u|^2\, dm \\
&\quad -\left(\int_M u^2 dm\right)^{1/2}\left(\int_M |\nabla f_B|^2 dm\right)^{1/2}\left(\int_M |\nabla u|^2\, dm\right)^{1/2} \\
&\geq -\frac{\lambda}{\delta_1} - \|\nabla f_B\|_2 \left(\frac{\lambda}{\delta_1}\right)^{1/2}.
\end{aligned}$$

Also,

$$\int_M \langle (\nabla_{\nabla f_B} B)\,\nabla u, \nabla u\rangle\, dm \leq K_2 \int_M |\nabla u|^2\, dm \leq K_2 \frac{\lambda}{\delta_1},$$

and

$$\begin{aligned}
\int_M \langle \nabla u, \nabla f_B\rangle\, dm &\geq -\left(\int_M |\nabla u|^2\, dm\right)^{1/2}\left(\int_M |\nabla f_B|^2 dm\right)^{1/2} \\
&= -\|\nabla f_B\|_2 \left(\frac{\lambda}{\delta_1}\right)^{1/2},
\end{aligned}$$

thus

$$\begin{aligned}
0 \geq{}& \left(\frac{1}{(n-1)\delta_n} - \frac{1}{\delta_1}\right)\lambda^{3/2} - \frac{1}{\sqrt{\delta_1}}\|\nabla f_B\|_2\,\lambda + \left(\frac{K_1}{2\delta_n} - \frac{K_2}{2\delta_1}\right)\lambda^{1/2} \\
&- \frac{1}{\sqrt{\delta_1}}\|\nabla f_B\|_2.
\end{aligned}$$

The polynomial

$$\begin{aligned}
p(\lambda) :={}& \left(\frac{1}{(n-1)\delta_n} - \frac{1}{\delta_1}\right)\lambda^3 - \frac{1}{\sqrt{\delta_1}}\|\nabla f_B\|_2\,\lambda^2 + \left(\frac{K_1}{2\delta_n} - \frac{K_2}{2\delta_1}\right)\lambda \\
&- \frac{1}{\sqrt{\delta_1}}\|\nabla f_B\|_2
\end{aligned}$$

has one negative root or three distinct roots where two of them are positive (it depends on the values of $K_1, K_2, \|\nabla f_B\|_2, \delta_1$ and δ_n). When (d) is satisfied, we conclude that $\lambda \leq x_2$ or $x_3 \leq \lambda$.

For the special case when $\mathrm{tr}(B)$ is constant, then $\mathrm{Hess}\,(\mathrm{tr}(B)) = 0$ and $f_B = 0$, so (4.4) can be rewritten as:

$$\frac{1}{\mathrm{tr}(B)}\lambda^2 + (K_1/2 - \lambda)\int_M |\nabla u|^2\, dvol_g \leq 0.$$

One can show that $K_1/2 \leq \lambda$, hence

$$\frac{\lambda^2}{\mathrm{tr}(B)} + (K_1/2 - \lambda)\,\frac{\lambda}{\delta_1} \leq 0.$$

And finally:

$$\lambda \geq \frac{\mathrm{tr}(B)K_1}{2\,(\mathrm{tr}(B) - \delta_1)}.$$

□

For the Dirichlet boundary condition (4.1), with a similar argument as in the proof of Theorem 1.3, we have the following result.

Corollary 4.2. *When $\overrightarrow{Bn}$ is parallel to $\overrightarrow{n}$, where $\overrightarrow{n}$ is the outward unit vector field on ∂M, then we have the following estimates for λ_1^{DBC} and λ_1^{BNBC}:*

(a) *If the B-mean curvature of ∂M is negative, then the estimates of Theorem 1.3 hold for λ_1^{DBC}.*
(b) *If $(\nabla_{\vec{n}} B)$ is negative, then the estimates of Theorem 1.3 hold for λ_1^{BNBC}.*

When $p(\lambda)$ in the proof of Theorem 1.3 has a unique negative root, the estimates of Theorem 1.3 are not useful. For the estimation of the first eigenvalue of Δ_B, we use the Li and Yau method [12]. In principle, using the Bochner formula and the maximum principle, the Li-Yau method estimates the gradient of the λ_1-eigenfunction. Hence, by adapting their method, we apply (2.5) more directly. The complexity of our process is due to the existence of the term $\Delta_{\nabla_u B} u$. This term does not have a serious effect in the Reilly formula when B is a Codazzi tensor because it is omitted by the divergence theorem, but when we use the extended Bochner formula directly, the term affects seriously and the situation is very much complicated and depends on the analytic and algebraic properties of B. By the following Proposition, we simplify (2.5) when B is a Codazzi tensor.

Proposition 4.3 ([3])**.** *We have:*

$$\frac{1}{2}\Delta_B\left(|\nabla v|^2\right) = \mathrm{tr}\left(B \circ \mathrm{hess}^2(v)\right) + \nabla v \cdot \Delta_B v \tag{4.5}$$

$$+\sum_i \left\langle T^{(\nabla_{\nabla v}B)}\left(\nabla v, e_i\right), e_i\right\rangle + \mathrm{Ric}\left(\nabla v, B\nabla v\right).$$

One of the complicated terms in (4.5) is $T^{\nabla_X B}(Y,Z)$. This term is tensorial with respect to Y,Z but it is not tensorial with respect to X, so we prove the following lemma.

Lemma 4.4. *Suppose D is a $(1,1)$-tensor field on M and $f\in C^\infty(M)$, then*

$$\begin{aligned} T^{\nabla_{fX}D}(Y,Z) = &\ (Y\cdot f)\,T^D(X,Z) - (Z\cdot f)\,T^D(X,Y) \\ &+ \left(\nabla_{(Y\cdot f)Z-(Z\cdot f)Y}D\right)X + f\cdot T^{\nabla_X D}(Y,Z) \end{aligned}$$

for any $X,Y,Z\in\mathfrak{X}(M)$.

Proof. We compute as follows:

$$\begin{aligned} T^{\nabla_{fX}D}v\,(Y,Z) &= \left(\nabla_Y\left(\nabla_{fX}D\right)\right)Z - \left(\nabla_Z\left(\nabla_{fX}D\right)\right)Y \\ &= \left(\nabla_Y\left(f\nabla_X D\right)\right)Z - \left(\nabla_Z\left(f\nabla_X D\right)\right)Y \\ &= (Y\cdot f)\left(\nabla_X D\right)Z - (Z\cdot f)\left(\nabla_X D\right)Y \\ &\quad + f\cdot T^{\nabla_X D}(Y,Z) \\ &= (Y\cdot f)\left(\nabla_X D\right)Z - (Y\cdot f)\left(\nabla_Z D\right)X \\ &\quad + (Y\cdot f)\left(\nabla_Z D\right)X - (Z\cdot f)\left(\nabla_X D\right)Y \\ &\quad + (Z\cdot f)\left(\nabla_Y D\right)X - (Z\cdot f)\left(\nabla_Y D\right)X \\ &\quad + f\cdot T^{\nabla_X D}(Y,Z) \\ &= (Y\cdot f)\,T^D(X,Z) - (Z\cdot f)\,T^D(X,Y) \\ &\quad + (Y\cdot f)\left(\nabla_Z D\right)X \\ &\quad - (Z\cdot f)\left(\nabla_Y D\right)X + f\cdot T^{\nabla_X D}(Y,Z) \\ &= (Y\cdot f)\,T^D(X,Z) - (Z\cdot f)\,T^D(X,Y) \\ &\quad + \left(\nabla_{(Y\cdot f)Z-(Z\cdot f)Y}D\right)X + f\cdot T^{\nabla_X D}(Y,Z). \end{aligned}$$

□

Corollary 4.5. *For B we have:*

$$T^{\nabla_{fX}B}(Y,Z) = f\cdot T^{\nabla_X B}(Y,Z) + \left(\nabla_{(Y\cdot f)Z-(Z\cdot f)Y}B\right)X.$$

By definition of F^B, we estimate $\sum_i \left\langle T^{(\nabla_{\nabla u}B)}\left(e_i,\nabla u\right),e_i\right\rangle$ for unit vector fields.

Example 4.6. In the following cases, $F^B=0$:

(a) When $\Sigma^n\subset M^{n+1}(c)$ is a totally umbilic hypersurface and B is its shape operator, then $\Delta_{\nabla_{\nabla u}B}u=0$, so $F^B=0$.
(b) For a Codazzi tensor B, if $\nabla^2 B=0$, then $F^B=0$.

The following Proposition relates $\sum_i \left\langle T^{(\nabla_{\nabla u}B)}(e_i,\nabla u),e_i\right\rangle$ to F^B and we use it to get Proposition 4.8, which is used to prove Theorem 1.6.

Proposition 4.7. *Let $v\in C^\infty(M)$, then*

$$\sum_i \left\langle T^{(\nabla_{\nabla v}B)}\left(e_i,\nabla v\right),e_i\right\rangle \le |\nabla v|\left\langle \left(\nabla_{(\nabla v)_1}B\right)(\nabla v)_1,\nabla|\nabla v|\right\rangle - (\nabla v\cdot|\nabla v|)\sum_i\langle(\nabla v)_1,\operatorname{div}B\rangle + |\nabla v|^2F^B,$$

where $(\nabla h)_1$ is defined as follows:

$$(\nabla h)_1 := \begin{cases} 0 & \text{if } \nabla h=0,\\ (\nabla h)_1/|\nabla h| & \text{if } \nabla h\neq 0.\end{cases}$$

Proof. Let $(\nabla v)_1 := \frac{\nabla v}{|\nabla v|}$, then

$$\begin{aligned}
&\sum_i\left\langle T^{(\nabla_{\nabla v}B)}\left(e_i,\nabla v\right),e_i\right\rangle\\
&\quad=\sum_i\left\langle\left(\nabla_{(e_i\cdot|\nabla v|)(\nabla v)-((\nabla v)\cdot|\nabla v|)e_i}B\right)(\nabla v)_1,e_i\right\rangle\\
&\qquad+\sum_i\left\langle|\nabla v|\,T^{\nabla_{(\nabla v)_1}B}\left(e_i,\nabla v\right),e_i\right\rangle\\
&\quad=\sum_i\left\langle\Big[\left(\nabla_{(\nabla v)_1}B\right)\left((e_i\cdot|\nabla v|)(\nabla v)-((\nabla v)\cdot|\nabla v|)e_i\right)\right.\\
&\qquad\left.+|\nabla v|^2\,T^{\nabla_{(\nabla v)_1}B}\left(e_i,(\nabla v)_1\right)\Big],e_i\right\rangle\\
&\quad\le\sum_i\left\langle\left(\nabla_{(\nabla v)_1}B\right)(\nabla v),(e_i\cdot|\nabla v|)e_i\right\rangle\\
&\qquad-\sum_i\left\langle\left(\nabla_{(\nabla v)_1}B\right)((\nabla v)\cdot|\nabla v|)e_i,e_i\right\rangle+|\nabla v|^2F^B\\
&\quad=|\nabla v|\left\langle\left(\nabla_{(\nabla v)_1}B\right)(\nabla v)_1,\nabla|\nabla v|\right\rangle\\
&\qquad-(\nabla v\cdot|\nabla v|)\sum_i\langle(\nabla v)_1,\operatorname{div}B\rangle+|\nabla v|^2F^B.
\end{aligned}$$

□

So we get the following result.

Proposition 4.8. *Let B be a $(1,1)$-self-adjoint Codazzi tensor field on M and $v\in C^\infty(M)$, then*

$$\frac{1}{2}\Delta_B\left(|\nabla v|^2\right)\ge\operatorname{tr}\left(B\circ\operatorname{hess}^2(v)\right)+\nabla v\cdot\Delta_Bv - |\nabla v|\left\langle\left(\nabla_{(\nabla v)_1}B\right)(\nabla v)_1,\nabla|\nabla v|\right\rangle$$

$$+ (\nabla v \cdot |\nabla v|) \langle (\nabla v)_1, \operatorname{div} B\rangle - |\nabla v|^2 F^B + \operatorname{Ric}(\nabla v, B\nabla v).$$

We are going to prove Theorem 1.6 as follows.

Proof of Theorem 1.6. We carefully adapt the proof of Theorem 5.3 of [12] to our case in which the Laplacian is replaced by Δ_B. We use the generalized Bochner formula and the maximum principle. Let u be an eigenfunction of Δ_B, i.e., $\Delta_B u + \lambda u = 0$, where $\lambda > 0$. Let

$$\int_M u dvol_g = 0, \qquad 1 > \max_{x\in M} u > \min_{x\in M} u = -1.$$

By Proposition 4.8, we have:

$$\begin{aligned}\frac{1}{2}\Delta_B\left(|\nabla u|^2\right) \geq{}& \operatorname{tr}\left(B \circ \operatorname{Hess}^2(u)\right) + \nabla u \cdot \Delta_B u \\ &- |\nabla u| \left\langle \left(\nabla_{(\nabla u)_1} B\right)(\nabla u)_1, \nabla |\nabla u|\right\rangle \\ &+ (\nabla u \cdot |\nabla u|) \langle (\nabla u)_1, \operatorname{div} B\rangle \\ &- |\nabla u|^2 F^B + \operatorname{Ric}(\nabla u, B\nabla u).\end{aligned}$$

Let $v := \ln(a+u)$, so $\nabla v = \frac{1}{u+a}\nabla u$. Also note that $\Delta_B(f(u)) = f''(u)|\nabla u|_B^2 + f'(u)\Delta_B u$, thus:

$$\begin{aligned}\frac{1}{(n-1)\delta_n}(\Delta_B v)^2 &= \frac{1}{(n-1)\delta_n}\left(-|\nabla v|_B^2 - \lambda\frac{u}{u+a}\right)^2 \\ &= \frac{1}{(n-1)\delta_n}\left(|\nabla v|_B^4 + \lambda^2\frac{u^2}{(u+a)^2} + 2\lambda\frac{u|\nabla v|_B^2}{u+a}\right),\end{aligned}$$

and

$$\begin{aligned}\nabla v \cdot \Delta_B v &= \nabla v \cdot \left(-|\nabla v|_B^2 - \lambda\frac{u}{u+a}\right) \\ &= -2\operatorname{Hess} v(B\nabla v, \nabla v) - \langle (\nabla_{\nabla v} B)\nabla v, \nabla v\rangle - \lambda\frac{a}{a+u}|\nabla v|^2 \\ &= -B\nabla v \cdot |\nabla v|^2 - \langle (\nabla_{\nabla v} B)\nabla v, \nabla v\rangle - \lambda\frac{a}{a+u}|\nabla v|^2.\end{aligned}$$

Let $Q := |\nabla v|$, so we get:

$$\begin{aligned}\frac{1}{2}\Delta_B\left(Q^2\right) \geq{}& \frac{1}{(n-1)\delta_n}\left(|\nabla v|_B^4 + \lambda^2\frac{u^2}{(u+a)^2} + 2\lambda\frac{u|\nabla v|_B^2}{u+a}\right) \\ &- B\nabla v \cdot |\nabla v|^2 - \langle (\nabla_{\nabla v} B)\nabla v, \nabla v\rangle - \lambda\frac{a}{a+u}|\nabla v|^2 \\ &- |\nabla v| \langle (\nabla_{\nabla v} B)(\nabla v)_1, \nabla |\nabla v|\rangle + (\nabla v \cdot |\nabla v|)\langle (\nabla v)_1, \operatorname{div} B\rangle \\ &- |\nabla v|^2 F^B + \operatorname{Ric}(\nabla v, B\nabla v).\end{aligned}$$

If x_0 is the maximum point of Q, then $\Delta_B Q(x_0) \leq 0$ and $\nabla Q(x_0) = 0$, so at this point, we have:

$$\frac{1}{2}\Delta_B\left(Q^2\right) \geq \frac{1}{(n-1)\delta_n}\left(\delta_1^2 Q^4 + \lambda^2 \frac{u^2}{(u+a)^2} + 2\lambda \frac{u\,|\nabla v|_B^2}{u+a}\right) - \left\langle \left(\nabla_{(\nabla v)_1} B\right)(\nabla v)_1, (\nabla v)_1 \right\rangle Q^3 - \lambda \frac{a}{a+u} Q^2 - H^2 Q^2.$$

Hence at the point x_0, we get that:

$$0 \geq \frac{\delta_1^2 Q^4}{(n-1)\delta_n} - KQ^3 - \left(H^2 + \lambda\left(\frac{(n-1)\delta_n a - 2\delta_1}{(n-1)\delta_n(1+a)}\right)\right) Q^2.$$

Therefore for all $x \in M$, we infer:

$$Q \leq Q(x_0) \leq \frac{(n-1)\delta_n}{2\delta_1^2}\mathcal{K}$$

where $\mathcal{K} = K + \sqrt{K^2 + 4\frac{\delta_1^2}{(n-1)\delta_n}\left(H^2 + \lambda\frac{(n-1)\delta_n a - 2\delta_1}{(n-1)\delta_n(1+a)}\right)}$. By integrating $P = |\nabla \ln(a+u)|$ along a (minimal) geodesic β joining the points $u = -1$ and $u = \max u \geq 0$, we obtain:

$$\begin{aligned} \ln\left(\frac{a}{a-1}\right) &\leq \ln\left(\frac{a+\max u}{a-1}\right) \\ &\leq \int_\beta |\nabla \ln(u+a)|\, ds \\ &\leq D\frac{(n-1)\delta_n}{2\delta_1^2}\mathcal{K} \\ &\leq D\frac{(n-1)\delta_n}{2\delta_1^2}\mathcal{K}_1 \end{aligned}$$

where $\mathcal{K}_1 = K + \sqrt{K^2 + 4\frac{\delta_1^2}{(n-1)\delta_n}\left(H^2 + \frac{(n-3)}{(n-1)}\lambda\frac{a}{a-1}\right)}$. Setting $t = \frac{a-1}{a}$, we see that:

(4.6)

$$\frac{(n-1)}{(n-3)} t\left(\frac{(n-1)\delta_n}{4\delta_1^2}\left(\left(\frac{2\delta_1^2}{D(n-1)\delta_n}\ln t + K\right)^2 - K^2\right) - H^2\right) \leq \lambda.$$

Consequently, maximizing (4.6) yields:

$$\frac{2\delta_1^2}{(n-3)\delta_n D^2}\left(1 + \sqrt{1+S}\right)\exp\left(-1 - \sqrt{1+S} - K\right) \leq \lambda,$$

where

$$S = \frac{(n-1)\delta_n}{\delta_1^2}\left(\frac{(n-1)\delta_n K^2}{4\delta_1^2} + H^2\right) D^2.$$

When $\mathrm{tr}(B)$ is constant, (4.6) can be written as:

$$\frac{\mathrm{tr}(B)}{2\delta_n+\mathrm{tr}(B)}t\left(\frac{\mathrm{tr}(B)}{4\delta_1^2}\left(\left(\frac{2\delta_1^2}{D\,\mathrm{tr}(B)}\ln t-K_2\right)^2-K_2^2\right)-K_1\right)\leq\lambda.$$

Consequently, maximizing yields:

$$\frac{2\delta_1^2}{(2\delta_n+\mathrm{tr}(B))\,D^2}\left(1+\sqrt{1+S_1}\right)\exp\left(-1-\sqrt{1+S_1}-K_2\right)\leq\lambda,$$

where

$$S_1=\frac{\mathrm{tr}(B)}{\delta_1^2}\left(\frac{\mathrm{tr}(B)K_2^2}{4\delta_1^2}+K_1^2\right)D^2. \qquad \square$$

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[1] Department of pure Math, Faculty of Math.science Tarbiat modares university, Tehran, Iran.
Email address: fatemi.shamed@gmail.com

[2] Department of pure Math, Faculty of Math.science Tarbiat modares university, Tehran, Iran.
Email address: kashanim@modares.ac.ir

[3] Department of Pure Mathematics, Faculty of Sciences Imam Khomeini International University, Qazvin, Iran.
Email address: azami@sci.ikiu.ac.ir