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A Riesz-Fredholm Type Theorem on Hilbert C*-Modules Satisfying Property [$\mathbb{H}$]

Zahra Panahi[1] and Kamran Sharifi[2*]

Abstract. Consider a compact modular operator C on a Hilbert C*-module E that fulfills property [$\mathbb{H}$] [*J. Math. Phys.*, 49 (2008), 033519] and define $L := I - C$. We prove the existence of a unique natural number r for which L^r is an EP operator on E. Moreover, we show that the kernel of L^r is a finitely generated submodule of E and that E admits the decomposition $E = \ker(L^r) \oplus \operatorname{ran}(L^r)$. These results provide a framework for analyzing the solvability of the equation $x - Cx = f$ on E.

1. Introduction and Preliminaries.

In Banach spaces theory, compact operators are linear operators on Banach spaces that map bounded sets to relatively compact sets. In the case of a Hilbert C*-modules, the compact operators are the closure of the finite rank operators in the uniform operator topology. In general, operators on Hilbert modules feature properties that do not appear in the Hilbert (or Banach) spaces case, cf. [1, 6, 11, 16]. In this paper, we are going to present the basic theory for an operator equation

$$x - Cx = f$$

with a compact modular operator $C : E \to E$ mapping a Hilbert C*-module E into itself. This work extends the classical framework established by Riesz and Fredholm for integral equations in Hilbert and Banach spaces, cf. [9, 13].

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A *pre-Hilbert module* over a C*-algebra $\mathcal{A}$ is a complex linear space E which is an algebraic right $\mathcal{A}$-module equipped with an $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle : E \times E \to \mathcal{A}$ satisfying the following properties for each $x, y, z \in E$, $\lambda \in \mathbb{C}$ and $a \in \mathcal{A}$:

(i) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$,
(ii) $\langle x, y + \lambda z\rangle = \langle x, y\rangle + \lambda \langle x, z\rangle$,
(iii) $\langle x, ya\rangle = \langle x, y\rangle a$,
(iv) $\langle y, x\rangle = \langle x, y\rangle^*$.

An $\mathcal{A}$-module E attains the structure of a Hilbert C*-module when it is complete in the norm defined by $\|x\| = \|\langle x, x\rangle\|^{1/2}$.

For any $\mathcal{A}$-submodule M of E (which may not be closed), the set $M^\perp := \{y \in E : \langle x, y\rangle = 0, \text{ for all } x \in M\}$ constitutes a closed $\mathcal{A}$-submodule. Moreover, the closure of M, denoted by $\overline{M}$, is contained in its biorthogonal $M^{\perp\perp}$. A closed submodule M of a Hilbert $\mathcal{A}$-module E is considered *topologically complementable* provided there exists another closed submodule N such that E is the direct sum of M and N, i.e., $E = M + N$ and $M \cap N = 0$. In this case, we write $E = M \dot{+} N$. The submodule M is called *orthogonally complemented* if we further have $M \perp N$. In this case $N = M^\perp$ and we write $E = M \oplus N$.

The set of all adjointable operators on E forms a C*-algebra, denoted by $\mathcal{L}(E)$. Recall that an operator $T : E \to E$ is termed adjointable if it is $\mathcal{A}$-linear and there exists a map $T^* : E \to E$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ holds for every $x, y \in E$. For any $x, y \in E$, the rank one operator $\theta_{x,y}$ is defined by

$$\theta_{x,y}(z) = x\langle y, z\rangle, \quad \text{for every } z \in E.$$

Each $\theta_{x,y}$ is a bounded adjointable operator with adjoint $(\theta_{x,y})^* = \theta_{y,x}$. The space of compact operators $\mathcal{K}(E)$ is defined as the norm closure of the linear span of all such rank one operators in $\mathcal{L}(E)$.

We note that some familiar properties of Hilbert spaces, such as Pythagoras' equality, self-duality, the polar decomposition of operators and even the decomposition into orthogonal complements, do not hold in the framework of Hilbertian modules. In general, operators on Hilbert modules feature properties that do not appear in the Hilbert (or Banach) spaces case. The reader is encouraged to study the publications [2, 4, 6–8, 12, 20] and the references therein for more detailed information on modular operators and some recent developments in the theory of Hilbert C*-modules.

We use the notations $\ker(\cdot)$ and $\operatorname{ran}(\cdot)$ to denote the kernel and range of operators, respectively. The modular operator T has closed range if and only if T^* has closed range. In this case, the space E decomposes as $E = \ker(T) \oplus \operatorname{ran}(T^*) = \ker(T^*) \oplus \operatorname{ran}(T)$; see [11, Theorem 3.2].

Every bounded sequence in a Hilbert space has a weakly convergent subsequence. Furthermore, compact operators map weakly convergent sequences to strongly convergent ones. Now, suppose E is a Hilbert $\mathcal{A}$-module. A natural question is whether every bounded sequence (ζ_n) in E admits a subsequence (ζ_{n_k}) and an element $\zeta \in E$ such that

$$\|C\zeta_{n_k} - C\zeta\| \to 0, \quad \text{for every } C \in \mathcal{K}(E).$$

In general, the answer is negative. For example, let $\mathcal{A} = \mathcal{B}(H)$ for some infinite-dimensional Hilbert space H and consider $E = \mathcal{A}$ as a Hilbert C*-module over itself. Then the identity operator on H, which is not a compact operator on H, corresponds to a compact operator in $\mathcal{K}(E)$. However, since H is infinite dimensional, the identity operator cannot map a weakly convergent sequence to a strongly convergent one, demonstrating that the desired property fails. This argument motivates the study of property $[\mathbb{H}]$, initially introduced in [3, 6], which will be discussed in more detail in the next section. It is known that every Hilbert C*-module over a finite dimensional C*-algebra satisfies property $[\mathbb{H}]$.

Let E be a Hilbert C*-module over a unital C*-algebra $\mathcal{A}$ that satisfies property $[\mathbb{H}]$ and let L be the perturbation of the identity by a compact operator C in $\mathcal{K}(E)$, that is, $L = I - C$. In this paper, we study the equation $x - Cx = f$ and prove that there exists a unique natural number r such that L^r is an EP operator on E, that is, the range of L^r is closed and satisfies $\operatorname{ran}(L^r) = \operatorname{ran}(L^{r*})$. In particular, the kernel of L^r is a finitely generated submodule of E and E admits the direct sum decomposition $E = \ker(L^r) \oplus \operatorname{ran}(L^r)$. Some further useful results are also obtained.

2. Hilbert C*-Modules Satisfying Property $[\mathbb{H}]$.

In this section, we review the property $[\mathbb{H}]$ introduced by Chmieliński, Ilišević, Moslehian and Sadeghi [3], along with a characterization of finite dimensional C*-algebras established by Arambašić, Bakić and Rajić [1]. Definition 2.1, Lemma 2.3 and Lemma 2.4 serve as foundational elements for our main results. We adopt the following definition from Proposition 2.1 of [3].

Definition 2.1. Suppose $\mathcal{A}$ is a C*-algebra. The right Hilbert $\mathcal{A}$-module E satisfies property $[\mathbb{H}]$, if for every bounded sequence (ζ_n) in E there are a subsequence (ζ_{n_k}) of (ζ_n) and $\zeta \in E$ such that

$$\langle v, \zeta_{n_k} \rangle \to \langle v, \zeta \rangle, \quad \text{for every } v \in E. \tag{2.1}$$

Frank has shown that every Hilbert C*-module over an arbitrary finite dimensional C*-algebra of coefficients is a self-dual Hilbert module, cf.

[6, Proposition 4.4]. Chmieliński et al. utilized this result to establish the following lemma:

Lemma 2.2 ([3, Proposition 2.1])**.** *Let E be a Hilbert C*-module over a finite dimensional C*-algebra $\mathcal{A}$, then E satisfies property* $[\mathbb{H}]$.

It is known that full Hilbert C*-modules can be described by the property $[\mathbb{H}]$ as follows. Let E be a full right Hilbert $\mathcal{A}$-module. For every bounded sequence (ζ_n) in E there are a subsequence (ζ_{n_k}) of (ζ_n) and $\zeta \in E$ such that (2.2) holds if and only if $\mathcal{A}$ is a finite dimensional C*-algebra, cf. [1, Theorem 2.3].

Lemma 2.3 ([1, Theorem 2.3])**.** *Let E be a Hilbert $\mathcal{A}$-modules that satisfies property* $[\mathbb{H}]$. *Then for every compact operator C in $\mathcal{K}(E)$ we have*

$$\|C\zeta_{n_k} - C\zeta\| \to 0. \tag{2.2}$$

Proof. To derive (2.2), observe that $\mathcal{K}(E)$ equals the closed linear span of $\{\theta_{u,v} : u, v \in E\}$ in $\mathcal{L}(E)$ and

$$\|\theta_{u,v}(\zeta_{n_k}) - \theta_{u,v}(\zeta)\| = \|u\langle v, \zeta_{n_k}\rangle - u\langle v, \zeta\rangle\| \to 0,$$

for every $u, v \in E$. □

The compact operators on Hilbert modules over a finite dimensional C*-algebra are notable because they exhibit as much similarity to matrices as one can expect from a general operator.

Some aspects of weakly convergent bounded sequences in Hilbert C*-modules are presented in [16]; this reference specifically addresses some results on the fundamental properties of compact operators defined on Hilbert modules over an arbitrary finite dimensional C*-algebra.

Lemma 2.4. *Let E be a Hilbert $\mathcal{A}$-module satisfying property* $[\mathbb{H}]$ *and let $L := I - C$, where $C \in \mathcal{K}(E)$. Then the following assertions hold.*

(i) *For every $y \in \overline{\operatorname{ran}(L)}$, there exist sequences (z_n) in E and (u_n) in the closed submodule $\ker(L)$ of E such that $Lz_n = z_n - Cz_n$ converges to y and*

$$\|z_n - u_n\| = \delta_n := \inf\{\|z_n - u\| : u \in \ker(L)\}.$$

(ii) *The sequence $\zeta_n := z_n - u_n$ is a bounded sequence in E.*

Proof. Let y in the closure of $\operatorname{ran}(L)$, then one can find a sequence (z_n) in E for which $Lz_n = z_n - Cz_n$ converges to y. We demonstrate the existence of a sequence (u_n) within the closed submodule $\ker(L) \subset E$ that satisfies $\|z_n - u_n\| = \delta_n$. Utilizing the property of infimum, for any fixed n, there exists a sequence (w_m^n) in $\ker(L)$ such that

$$\lim_m \|z_n - w_m^n\| = \delta_n. \tag{2.3}$$

Consequently, the sequence (w_m^n) is bounded with $\delta_n + 1 + \|z_n\|$, for sufficiently large m. By Lemma 2.3, we may extract a subsequence $\left(w_{m_k}^n\right)$ converging weakly to $u_n \in \ker(L)$ in the sense that

$$\left\langle v, w_{m_k}^n \right\rangle \to \langle v, u_n \rangle, \quad \text{for any } n \in \mathbb{N} \text{ and } v \in E,$$

with the additional property

$$\left\| C w_{m_k}^n - C u_n \right\| \to 0, \quad \text{for each } n \in \mathbb{N}.$$

Since both (u_n) and (w_m^n) are contained $\ker(I - C)$, we obtain the fixed point properties:

$$\begin{aligned} C u_n &= u_n, \\ C w_{m_k}^n &= w_{m_k}^n. \end{aligned}$$

The reverse triangle inequality together with the two earlier equalities imply that the difference $\left| \|z_n - u_n\| - \left\|z_n - w_{m_k}^n\right\| \right|$ is bounded by

$$\left\| w_{m_k}^n - u_n \right\| = \left\| C w_{m_k}^n - C u_n \right\| \to 0, \quad \text{as } k \to \infty.$$

Combining (2.3) with the preceding argument, we obtain the equality $\delta_n = \lim_k \left\|z_n - w_{m_k}^n\right\| = \|z_n - u_n\|$, which establishes assertion (i).

We claim $\zeta_n = z_n - u_n$ is a bounded sequence in E. Suppose otherwise, there is a subsequence (ζ_{n_k}) such that $\|\zeta_{n_k}\| \geq k$. By compactness of C and Lemma 2.3, there exists a subsequence $\left(v_{k_j}\right)$ of $v_k := \frac{\zeta_{n_k}}{\|\zeta_{n_k}\|}$ and v such that $C v_{k_j} \to v$. Since $L\zeta_{n_k} = L z_{n_k} - L u_{n_k} = L z_{n_k} \to y$,

$$\|L v_k\| = \frac{\|L\zeta_{n_k}\|}{\|\zeta_{n_k}\|} \leq \frac{\|L\zeta_{n_k}\|}{k} \to 0, \quad \text{as } k \to \infty.$$

We therefore have $v_{k_j} = L v_{k_j} + C v_{k_j} \to v$, which implies $0 = \lim L v_{k_j} = Lv$. Since $u_{n_k} + \|\zeta_{n_k}\| v$ is in $\ker(L)$ we have

$$\begin{aligned} \|v_k - v\| &= \frac{1}{\|\zeta_{n_k}\|} \left\| z_{n_k} - (u_{n_k} + \|\zeta_{n_k}\| v) \right\| \\ &\geq \frac{1}{\|\zeta_{n_k}\|} \inf \|z_{n_k} - u\| \\ &= 1, \end{aligned}$$

contradicting the convergence $v_{k_j} \to v$. Consequently, the initial sequence (ζ_n) must be bounded, which completes the proof of (ii). □

3. EP operators and Riesz-Fredholm Theorem on Certain Hilbert C*-Modules.

The subsequent lemma is an adaptation of the classical Riesz's lemma for Banach spaces to the setting of Hilbert C*-modules. The argument

for the following lemma is based on the one found in Kreyszig's book [10, Lemma 2.5-4].

Lemma 3.1. *Suppose E is a Hilbert C*-module over a C*-algebra $\mathcal{A}$ and W is a closed proper submodule of E. Then, for every $0 < \gamma < 1$, there exists a unit vector $z \in E$ such that its distance to W is at least γ, i.e.,*

$$\gamma \leq \|z - W\| := \inf \{\|z - w\| : w \in W\}.$$

Lemma 3.2. *Let E be a Hilbert $\mathcal{A}$-module satisfying property $[\mathbb{H}]$ and $C \in \mathcal{K}(E)$. Then $\operatorname{ran}(\lambda I - C)$ is an orthogonal summand for every nonzero $\lambda \in \mathbb{C}$. In particular, $E = Ker(I - C^*) \oplus \operatorname{ran}(I - C)$.*

Proof. Let y in the closure of $\operatorname{ran}(L)$ and $L = I - C$. In view of Lemma 2.4, there exist sequences (z_n) in E and (u_n) in the closed submodule $\ker(L)$ of E such that $Lz_n = z_n - Cz_n$ converges to y, $\zeta_n := z_n - u_n$ defines a bounded sequence in E and

$$\|z_n - u_n\| = \inf \{\|z_n - u\| : u \in \ker(L)\}.$$

By Lemma 2.3, we can extract a subsequence (ζ_{n_k}) from the bounded sequence (ζ_n) for which $(C\zeta_{n_k})$ converges. Consequently, the identity $\zeta_{n_k} = L\zeta_{n_k} + C\zeta_{n_k}$ implies that (ζ_{n_k}) itself converges to some $\zeta \in E$. Since, $Lz_n \to y$ and $Lz_{n_k} - Lu_{n_k} = L\zeta_{n_k} \to L\zeta$, we deduce that $y = L\zeta$, i.e., the range of L is closed. Consequently, the closed submodule $\operatorname{ran}(L)$ is an orthogonal summand by [11, Theorem 3.2]. The range of $\lambda I - C$ is an orthogonal summand, too. □

Lemma 3.3. *A Hilbert C*-module E over a unital C*-algebra $\mathcal{A}$ is finitely generated if and only if the C*-algebra $\mathcal{K}(E)$ of compact operators is unital.*

This result is a direct consequence of the equivalence between statements (6) and (7) in Theorem 15.4.2, as discussed in Remark 15.4.3 of the book [18]. A key observation is that every finite dimensional C*-algebra $\mathcal{A}$ is unital, which follows from its structure as a direct sum of full matrix algebras, each of which is unital.

In the following results, we unify key matrix-theoretic properties and extend them to characterize the spectral behavior of compact operators on Hilbert C*-modules.

Remark 3.4. If E is a Hilbert $\mathcal{A}$-module that is not finitely generated and $C \in \mathcal{K}(E)$ is compact modular operator, then 0 belongs to the spectrum of C, denoted by

$$\sigma(C) := \{\lambda \in \mathbb{C} : \lambda I - C \text{ is not invertible in } \mathcal{L}(E)\}.$$

This fact follows immediately from Lemma 3.3, see also [16, Lemma 2.5].

Lemma 3.5. *Let E be a Hilbert C^*-module over a unital C^*-algebra $\mathcal{A}$ and $C \in \mathcal{K}(E)$. Then $\ker(I-C) = \{x \in E : Cx = x\}$ is a finitely generated Hilbert $\mathcal{A}$-module.*

Proof. The kernel of the bounded linear operator $L = I - C$ is a closed submodule of E. Since $Cx = x$, for every x in the kernel of L, the restriction of C to $\ker(I-C)$ coincides with the identity operator, that is, $I = C_{|_{\ker(L)}} : \ker(L) \to \ker(L)$ is a compact modular operator from $\ker(L)$ onto $\ker(L)$. Hence, the closed $\mathcal{A}$-submodule $\ker(L)$ is a finitely generated Hilbert $\mathcal{A}$-module by Lemma 3.3. □

Theorem 3.6. *Let E be a Hilbert C^*-module over a unital C^*-algebra $\mathcal{A}$ that satisfies property $[\mathbb{H}]$ and $C \in \mathcal{K}(E)$. If $L = I - C$, then there exists a uniquely determined non-negative integer r such that*

$$\{0\} = \ker\left(L^0\right) \subset \ker\left(L^1\right) \subset \cdots \subset \ker\left(L^r\right) = \ker\left(L^{r+1}\right) = \cdots, \tag{3.1}$$

and

$$E = \operatorname{ran}\left(L^0\right) \supset \operatorname{ran}\left(L^1\right) \supset \cdots \supset \operatorname{ran}\left(L^r\right) = \operatorname{ran}\left(L^{r+1}\right) = \cdots. \tag{3.2}$$

Proof. Since for each x with $Lx = 0$ it follows that $L^{n+1}x = 0$, we trivially have

$$\{0\} = \ker\left(L^0\right) \subset \ker\left(L^1\right) \subset \ker\left(L^2\right) \subset \cdots.$$

Suppose, for contradiction, we assume that $\ker\left(L^n\right) \neq \ker\left(L^{n+1}\right)$, for every natural number n. We apply Lemma 3.1 for the finitely generated submodule $\ker\left(L^n\right)$, there exists a sequence (x_n) in $\ker\left(L^{n+1}\right)$ such that $\|x_n\| = 1$ and

$$\|x_n - x\| \geq \|x_n - \ker\left(L^n\right)\| \geq \frac{1}{2}, \tag{3.3}$$

for every $x \in \ker\left(L^n\right)$. For every $n > m$ we have

$$Cx_n - Cx_m = x_n - \left(x_m + Lx_n - Lx_m\right),$$

and

$$\begin{aligned} L^n\left(x_m + Lx_n - Lx_m\right) &= L^{n-m-1}\left(L^{m+1}x_m\right) + L^{n+1}x_n \\ &\quad - L^{n-m}\left(L^{m+1}x_m\right) \\ &= 0, \end{aligned}$$

which imply $x_m + Lx_n - Lx_m \in \ker\left(L^n\right)$. In view of (3.3), we have

$$\begin{aligned} \|Cx_n - Cx_m\| &= \|x_n - \left(x_m + Lx_n - Lx_m\right)\| \\ &\geq \frac{1}{2}, \quad \text{for all } n > m. \end{aligned}$$

Consequently, the sequence Cx_n does not contain a convergent subsequence which is a contradiction to the compactness of C and the property

[$\mathbb{H}$]. Thus, we now conclude that in the sequence $\ker(L^n)$, there exist two consecutive kernels that are equal. Suppose r is the smallest natural number m such that the equality $\ker(L^m) = \ker\left(L^{m+1}\right)$ holds, then

$$\ker(L^r) = \ker\left(L^{r+1}\right) = \ker\left(L^{r+2}\right) = \cdots.$$

To see this, let $x \in \ker\left(L^{r+2}\right)$ then $L^{r+1}(Lx) = 0$, and so $Lx \in \ker\left(L^{r+1}\right) = \ker(L^r)$. We can now conclude inductively that $\ker(L^n) = \ker\left(L^{n+1}\right)$ for every $n \geq r$.

To see the second assertion, let $F_n = \operatorname{ran}(L^n)$. Then the F_n's are closed $\mathcal{A}$-submodules by Lemma 3.2 that satisfy $F_{n+1} \subseteq F_n$ for every nonnegative integer n. Suppose otherwise, we assume that $F_{n+1} \neq F_n$, for every natural number n. We apply Lemma 3.1 for the closed submodule F_{n+1}, there exists a sequence (y_n) in F_n such that $\|y_n\| = 1$ and

$$\|y_n - y\| \geq \|y_n - F_{n+1}\| \geq \frac{1}{2},$$

for every $y \in F_{n+1}$. For every $m > n$ we have

$$Cy_n - Cy_m = y_n - (y_m + Ly_n - Ly_m),$$

and

$$y_m + Ly_n - Ly_m = L^{n+1}\left(L^{m-n-1}x_m + x_n - L^{m-n}x_m\right), \tag{3.4}$$

where $y_n = L^n x_n$. The equality (3.4) implies that $y_m + Ly_n - Ly_m \in F_{n+1}$ and so

$$\|Cy_n - Cy_m\| = \|y_n - (y_m + Ly_n - Ly_m)\| \geq \frac{1}{2}, \quad \text{for all } m > n,$$

which derives a contradiction again to the compactness of C and the property [$\mathbb{H}$]. Hence, we now deduce that in the sequence F_n, there exist two consecutive ranges that are equal. Suppose s is the smallest natural number m such that the equality $\operatorname{ran}(L^m) = \operatorname{ran}\left(L^{m+1}\right)$ holds, then

$$\operatorname{ran}(L^s) = \operatorname{ran}\left(L^{s+1}\right) = \operatorname{ran}\left(L^{s+2}\right) = \cdots.$$

To see this, let $y = L^{s+1}x \in \operatorname{ran}\left(L^{s+1}\right)$ then $L^s x = L^{s+1}x_0$ for some $x_0 \in E$, since $\operatorname{ran}(L^s) = \operatorname{ran}\left(L^{s+1}\right)$. Consequently, $y = L^{s+2}x_0 \in \operatorname{ran}\left(L^{s+2}\right)$ and so $\operatorname{ran}\left(L^{s+1}\right) \subseteq \operatorname{ran}\left(L^{s+2}\right)$. We can now conclude inductively that $\operatorname{ran}(L^n) = \operatorname{ran}\left(L^{n+1}\right)$ for every $n \geq s$.

We claim that $r = s$. Let $r > s$ and $x \in \ker(L^r)$, then $L^{r-1}x \in \operatorname{ran}\left(L^{r-1}\right) = \operatorname{ran}(L^r)$ and so $0 = L^r x = L\left(L^{r-1}x\right) = L\left(L^r x_0\right) = L^{r+1}x_0$. Thus, $x_0 \in \ker\left(L^{r+1}\right) = \ker(L^r)$, which implies $L^{r-1}x = L^r x = 0$. Consequently, $\ker(L^r) = \ker\left(L^{r-1}\right)$, which contradicts the choice of r. We now assume that $r < s$ and $y = L^{s-1}x \in \operatorname{ran}\left(L^{s-1}\right)$.

Then $Ly = L^s x \in \operatorname{ran}(L^s) = \operatorname{ran}\left(L^{s+1}\right)$, we therefore have $Ly = L^{s+1}x_0$ and

$$L^s(x - Lx_0) = Ly - L^{s+1}x_0 = 0,$$

for some $x_0 \in E$. Since $\ker\left(L^{s-1}\right) = \ker(L^s)$, we have $L^{s-1}(x - Lx_0) = 0$ and so $y = L^s x_0 \in \operatorname{ran}(L^s)$. Consequently, $\operatorname{ran}(L^s) = \operatorname{ran}\left(L^{s-1}\right)$, which contradicts the definition of s. □

Let E be a Hilbert $\mathcal{A}$-modules and let $T \in \mathcal{L}(E)$. Then a bounded adjointable operator $T^\dagger \in \mathcal{L}(E)$ is called the *Moore-Penrose inverse* of T if

$$TT^\dagger T = T, \qquad T^\dagger T T^\dagger = T^\dagger, \qquad \left(TT^\dagger\right)^* = TT^\dagger, \qquad \left(T^\dagger T\right)^* = T^\dagger T.$$

Xu and Sheng in [19] have shown that a bounded adjointable operator between two Hilbert C*-modules admits a bounded Moore-Penrose inverse if and only if the operator has closed range. The reader should be aware of the fact that a bounded adjointable operator may admit an unbounded operator as its Moore-Penrose, see [8, 15].

An operator $T \in \mathcal{L}(E)$ is called *EP* if $\operatorname{ran}(T)$ and $\operatorname{ran}(T^*)$ have the same closure. Let $T \in \mathcal{L}(E)$ be an EP operator with closed range, then T commutes with the bounded operator $T^\dagger$; see [14] for details. Products of EP operators on Hilbert C*-modules and some useful examples can be found in [5, 17].

Theorem 3.7. *Let E be a Hilbert C*-module over a unital C*-algebra $\mathcal{A}$ that satisfies property $[\mathbb{H}]$ and $C \in \mathcal{K}(E)$. If $L = I - C$, then there exists a uniquely determined non-negative integer r such that*

$$E = \ker(L^r) \oplus \operatorname{ran}(L^r). \tag{3.5}$$

In particular, there exists a uniquely determined non-negative integer r such that L^r is an EP operator.

Proof. We adhere to the notations and follow the reasoning used in the proof of the previous theorem. Suppose $y \in \ker(L^r) \cap \operatorname{ran}(L^r)$. Then there exists $x \in E$ such that $y = L^r x$ and $L^{2r}x = 0$, that is, $x \in \ker\left(L^2 r\right) = \ker(L^r)$ and so $y = L^r x = 0$. We now suppose x is an arbitrary element in E. Then $L^r x \in \operatorname{ran}(L^r) = \operatorname{ran}\left(L^{2r}\right)$ and so there exists x_0 such that $L^r x = L^{2r}x_0$. We set $y = L^r x_0$ and $z = x - y$, then $L^r z = L^r x - L^{2r}x_0 = 0$, which means $x = z + y \in \ker(L^r) \dot{+} \operatorname{ran}(L^r)$. Hence,

$$E = \ker(L^r) \dot{+} \operatorname{ran}(L^r). \tag{3.6}$$

In view of the closedness of the range L^r and [11, Theorem 3.2], we have

$$E = \ker(L^{*r}) \oplus \operatorname{ran}(L^r) = \ker(L^r) \oplus \operatorname{ran}(L^{*r}). \tag{3.7}$$

According to (3.6) and (3.7) we infer, $\operatorname{ran}(L^{*r}) = \operatorname{ran}(L^r)$, that is L^r is an EP operator with closed range. □

Corollary 3.8. *Let E be a Hilbert C*-module over a unital C*-algebra $\mathcal{A}$ that satisfies property $[\mathbb{H}]$ and $C \in \mathcal{K}(E)$. If $L = I - C$, then the EP operator L^r admits the matrix form*

$$L^r = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \operatorname{ran}(L^r) \\ \ker(L^r) \end{bmatrix} \to \begin{bmatrix} \operatorname{ran}(L^r) \\ \ker(L^r) \end{bmatrix},$$

where, I_r is an invertible in $\mathcal{L}(\operatorname{ran}(L^r), \operatorname{ran}(L^r))$. Indeed, I_r is the identity operator on $\operatorname{ran}(L^r)$.

Proof. The matrix form follows from the closedness of the range of L^r, Theorem 3.7 and Lemma 3.6 of [14]. □

In the following two corollaries, we present a reformulation of the classical results of Riesz and Fredholm [9, 13] for Hilbert modules with property $[\mathbb{H}]$, separately analyzing the cases $r = 0$ and $r > 0$.

Corollary 3.9. *Let E be a Hilbert C*-module over a unital C*-algebra $\mathcal{A}$ that satisfies property $[\mathbb{H}]$ and $C \in \mathcal{K}(E)$. Let $L = I - C$ is injective, then:*

(i) *The modular operator L is surjective.*
(ii) *The homogeneous equation $x - Cx = 0$ only has the trivial solution $x = 0$ and for all $f \in E$, the inhomogeneous equation*

$$x - Cx = f$$

has a unique solution $x \in E$. Moreover, this solution depends continuously on f.

Proof. In view of Theorem 3.6 and the fact that $\ker(L) = \{0\}$, we infer $r = 0$ and $\operatorname{ran}(L) = E$, that is, L is surjective.

To prove the second part, we have to show that the inverse of $L = I - C$ is bounded, i.e. $(I - C)^{-1} \in \mathcal{L}(E)$. Suppose L^{-1} is not bounded. Then there exists a sequence (y_n) with $\|y_n\| = 1$ such that the sequence $x_n := L^{-1}y_n$ is not bounded. We set $u_n := \frac{y_n}{\|x_n\|}$ and $z_n := \frac{x_n}{\|x_n\|}$. Then $\|z_n\| = 1$, $Lz_n = \frac{Lx_n}{\|x_n\|} = u_n$ and $u_n \to 0$ as $n \to \infty$. Utilizing the property $[\mathbb{H}]$, Lemma 2.3 and compactness of C, we can find a subsequence z_{n_k} such that $Cz_{n_k} \to z \in E$ as $k \to \infty$. Then, the equalities

$$Lz_{n_k} = z_{n_k} - Cz_{n_k} = u_{n_k},$$

imply that $z_{n_k} \to z$ with $Lz = 0$. Thus $z \in \ker(L) = \{0\}$, which contradicts $\|z_n\| = 1$ for all $n \in \mathbb{N}$. Therefore, $(I - C)^{-1} f$ is the unique solution of the equation $x - Cx = f$, provided that $(I - C)^{-1}$ exists in $\mathcal{L}(E)$. □

Corollary 3.10. *Let E be a Hilbert C*-module over a unital C*-algebra $\mathcal{A}$ that satisfies property $[\mathbb{H}]$ and $C \in \mathcal{K}(E)$. Let $L = I - C$ is not injective, then:*

(i) *The $\mathcal{A}$-submodule $\ker(L)$ is finitely generated and the range $\operatorname{ran}(L)$ is a proper closed $\mathcal{A}$-submodule.*
(ii) *If the homogeneous equation $x - Cx = 0$ has a nontrivial solution, then the inhomogeneous equation*

$$x - Cx = f$$

is either unsolvable or its general solution is of the form

$$x = x_0 + \sum_{k=1}^{k=m} a_k x_k,$$

where $x_1, \ldots, x_m$ are the generators of finitely generated $\mathcal{A}$-submodule $\ker(L)$, $a_1, \ldots, a_m$ are in $\mathcal{A}$, and x_0 denotes a special solution of the inhomogeneous equation.

Proof. The $\mathcal{A}$-submodule $\ker(L)$ is finitely generated by Lemma 3.5. Since $L = I - C$ is not injective, $\ker(L) \neq \{0\}$ and $r > 0$ and so $\operatorname{ran}(L) \neq E$ by Theorem 3.6.

To prove the second part, suppose y_0 is a nontrivial solution of the homogeneous equation $x - Cx = 0$ and the inhomogeneous equation $x - Cx = f$ is solvable with a solution x_0. Then, the general solution of $x - Cx = f$ is of the form $x = x_0 + y_0$. Since the $\mathcal{A}$-submodule $\ker(L)$ is finitely generated, we can express y_0 as

$$y_0 = \sum_{k=1}^{k=m} a_k x_k,$$

where $x_1, \ldots, x_m$ are the generators of $\mathcal{A}$-submodule $\ker(L)$ and $a_1, \ldots, a_m$ are elements of $\mathcal{A}$. This completes the proof of (2).

□

Theorem 3.11. *Let E be a Hilbert C*-module over a unital C*-algebra $\mathcal{A}$ that satisfies property $[\mathbb{H}]$. Let $C \in \mathcal{K}(E)$ and $L = I - C$.*

(i) *The kernel of L^r is a finitely generated submodule of E and the orthogonal projection*

$$P : E = \ker(L^r) \oplus \operatorname{ran}(L^r) \to \ker(L^r)$$

is a compact projection.
(ii) *The operator $I - C - P$ is a bijection.*

Proof. The operator L^r can be expressed as $L^r = I - C_r$, where

$$C_r = \sum_{j=1}^{j=r} (-1)^{j-1} \frac{r!}{j!\,(r-j)!} C^j$$

is a compact operator on E. According to Lemma 3.5 and Lemma 3.3, the kernel of L^r is a finitely generated submodule of E. If $E_0 = \operatorname{ran}(P) = \ker(L^r)$, then $\mathcal{K}(E_0)$ is a unital C*-algebra. Then the identity map on the Hilbert $\mathcal{A}$-module E_0 can be written as $\sum_{i=1}^{i=m} \theta_{x_i, y_i}$, for some x_i, $y_i \in E_0$ and so we have

$$\begin{aligned} Px &= I_{\mathcal{K}(E_0)}(Px) \\ &= \sum_{i=1}^{i=m} \theta_{x_i,y_i}(Px) \\ &= \sum_{i=1}^{i=m} x_i \,\langle y_i, Px\rangle \\ &= \sum_{i=1}^{i=m} \theta_{x_i,Py_i}(x), \end{aligned}$$

for all $x \in E$. Consequently, P is a compact projection in $\mathcal{K}(E)$.

To prove the second part, we need to show that $\ker(I - C - P) = \{0\}$. Let $x \in \ker(L - P)$, then

$$\begin{aligned} L^{r+1}x + 0 &= L^{r+1}x - L^r(Px) \\ &= L^r(Lx - Px) \\ &= L^r(0) \\ &= 0. \end{aligned}$$

That is, $x \in \ker\left(L^{r+1}\right) = \ker(L^r)$. Hence, $Px = x$, which implies $0 = Lx - Px = Lx - x$, or equivalently, $x = Lx$. It follows that x is a fixed point for all powers of L and therefore

$$x = L^r x = 0.$$

This proves that $\ker(L - P) = \{0\}$. Utilizing Corollary 3.9 for the compact operator $C + P$, we conclude that $L - P = I - (C + P)$ is surjective. □

We end our paper with the remark that the operator $L = I - C$ in our results can be generalized to $L = \lambda I - C$, where $\lambda \in \mathbb{C}$ and $C \in \mathcal{K}(E)$.

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